Hamada's Theorem for a certain type of the operators with double characteristics

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ハマダの定理について

Hamada's Theorem for a certain type of the operators with double characteristics.

(2重特性根をもつある種の作用素に対する浜田の定理)
を取り扱った。Tricomi型とは、

\[ \partial_t^2 - t \partial_x^2 + (\text{first order}) \]

であり、参考論文2では、

\[ L = P(x,D)^2 - x_0 Q(x,D) + R(x,D) \]

の形の方程式を扱っている。P.Qはそれぞれ斎次m, 2m次の作用素、Rは（2m-1）次以下である。申請者は、このとき多重度が一定である場合と異なる様相を示すことを明らかにした。相関数\( \varphi^\pm \)は、

\[ P(x,\nabla \varphi)^2 - x_0 Q(x,\nabla \varphi) = 0 \]

をみたし、\( \varphi^\pm(o,x^1) = x_1, D_0 \varphi^\pm(o,x^1) = \lambda \)をみたすものとして定義されるが、さらに、\( \varphi^\pm = \rho \pm \frac{2}{3} \theta^3 \)という2つの解析的関数（\( \rho, \theta \)）で表示される。\( \lambda = P_m(x,\lambda,1,0,\ldots,0) = 0 \)

の根であり、m個の相異なる\( \lambda_\beta \)をもつとき、Q(x,\( \lambda_\beta,1,\ldots,0) \equiv 0 \)とする。このとき、作用素\( \partial_\theta^2 - \theta \partial_\rho^2 \)の適当な独立解X(\( \theta, \rho \)), Y(\( \theta, \rho \))をとり、

\[ X(\theta(x), \rho(x)), Y(\theta(x), \rho(x)) \]

を展開の基底として採用している。

主論文は、上の後をうけて

\[ L = P(x,D)^2 - x_0^2 Q(x,D) + R(x,D) \]

の場合を論じている。P, Qについては前と同一であるが、さらにsubprinciple symbol L_sが特性面上で定数となる仮定がつけ加えられている。このとき\( u(x) \)は、特性面\( \{ x; \varphi^\pm_\beta(x) = 0 \} \)，\( 1 \leq \beta \leq m \)を除いた集合の上の普遍被覆空間上で1値正則関数として一意的に定まることが示されている。具体的には次の形となる。\( F(\alpha,\beta,r;z) \)をガウスの超幾何関数とし,

\[
\begin{align*}
X^{(q)}_{\alpha,\beta}(\theta, \rho) &= \frac{1}{q!} \partial_\alpha \varphi^q \left[ F(\alpha - q, \frac{1+c}{4}, \frac{1}{2}; 1 - \frac{\varphi_+}{\varphi_-}, \frac{\varphi_-^\alpha q}{\Gamma(\alpha+1)} \right] |_{c=c_\beta},

Y^{(q)}_{\alpha,\beta}(\theta, \rho) &= \frac{1}{q!} \partial_\alpha \varphi^q \left[ F(\alpha - q, \frac{3+c}{4}, \frac{3}{2}; 1 - \frac{\varphi_+}{\varphi_-}, \frac{\varphi_-^\alpha q}{\Gamma(\alpha+1)} \right] |_{c=c_\beta}
\end{align*}
\]
を定義する。ここで $C_\beta$ は特性面上の subprincipal symbolの値である。

\[
u(x) = \sum_{\beta} \sum_{\alpha} \sum_{q} \left\{ u_{\alpha,\beta}^{(q)}(x) Y_{\alpha-1,\beta}^{(q)}(\theta_\beta, \rho_\beta) + r_{\alpha,\beta}^{(q)}(x) \partial_\theta X_{\alpha,\beta}^{(q)}(\theta_\beta, \rho_\beta) + v_{\alpha,\beta}^{(q)}(x) Y_{\alpha-1,\beta}^{(q)}(\theta_\beta, \rho_\beta) + h_{\alpha,\beta}^{(q)}(x) \partial_\theta Y_{\alpha,\beta}^{(q)}(\theta_\beta, \rho_\beta) \right\}
\]

ここで係数はすべて正則である。この結果、特性面上で $u(x)$ は、

\[
(\varphi_\beta^{\pm})^{1}(3 \pm c_\beta) + i + j \left( \log \varphi_\beta^{\pm} \right)^{i}, \quad (\varphi_\beta^{\pm})^{1}(3 \mp c_\beta) + i + j \left( \log \varphi_\beta^{\pm} \right)^{i}
\]

($i = 0, 1, 2, \cdots; \quad j = -\ell, -\ell + 1, \cdots$) の重ね合わせの特異性をもつことが示されている。
Hamada's theorem for a certain type of the operators

with double characteristics.

by

Jiichiroh Urabe
Introduction

We consider non-characteristic Cauchy problem with meromorphic data for a linear partial differential equation with holomorphic coefficients in the complex domain.

This problem, for the operator with constant multiple characteristics, has been investigated by Y. Hamada, J. Leray and C. Wagshal [2] and others in bibliography of [2]. This problem, for the operator with involutive characteristics, has been investigated by Y. Hamada and G. Nakamura [3], [7] and D. Shiltz, J. Vaillant et C. Wagshal [9] and T. Kobayashi [6]. The author treated this problem for a certain class of operators containing Tricomi operator in [10], [11].

We shall treat the most general case but it will be very difficult to solve the problem. In this paper, we treat the limited class of operators originated from \( P_c = \partial_t^2 - t^2 \partial_x^2 - c \partial_x \) (\( c \) is a constant). Our method to solve this problem, is to const-
ruct the formal solution as the series of the auxiliary functions with the holomorphic coefficients. Those auxiliary functions, described precisely in the appendix, are composed mainly of hypergeometric functions, whose monodromy theory makes the ramification of the solution around the characteristic surfaces clear, and the convergence of the formal solution valid.

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§ 1. Assumptions and results.

Let $\mathcal{O}$ be a neighbourhood of the origin of $\mathbb{C}^{n+1}$, with the coordinates $x=(x_0, x_1, \cdots, x_n)$. By $L^k(\mathcal{O})$, we mean the set of all linear partial differential operators of order $k$ of which coefficients are holomorphic in $\mathcal{O}$. We shall be studying a linear partial differential operator $L(x, D) \in L^{2m}(\mathcal{O})$ with the principal symbol $\tilde{L}(x, \xi)$ of the following form:

$$
\tilde{L}(x, \xi) = P(x, \xi)^2 - x_0^2Q(x, \xi)
$$

where $\xi=(\xi_0, \xi_1, \cdots, \xi_n)$ and $D=(D_0, D_1, \cdots, D_n)$, $D_i=\frac{\partial}{\partial x_i}$.

We shall impose on $P(x, \xi)$ and $Q(x, \xi)$ the following conditions:

Assumptions

(P) (i) $P(x, \xi)$ is a homogeneous polynomial in $\xi$ of degree $m$.

(ii) $P(x, 1, 0, \cdots, 0) \neq 0$.

(iii) The equation $P(0, \xi_0, 1, 0, \cdots, 0)=0$ has mutually distinct $m$ roots $\lambda_\beta$ ($\beta=1, \cdots, m$).

(Q) (i) $Q(x, \xi)$ is a homogeneous polynomial in $\xi$ of degree $2m$. 
Then there exist $2m$ characteristic surfaces $K^\pm_\beta$ ($\beta = 1, \ldots, m$) issuing from $(n-1)$-plane $x_0 = x_1 = 0$. $K^\pm_\beta$ are defined by the equations $\varphi^\pm_\beta(x) = 0$. Here $\varphi^\pm_\beta(x)$ are the solutions of the following eikonal equation:

$$\left\{ \begin{array}{l}
\mathcal{L}(x, \varphi^\pm_\beta(x)) = 0 \\
\varphi^\pm_\beta(0, x') = x_1, \quad \text{and} \quad \varphi^\pm_\beta(x_0) = \lambda^\beta_\rho, \quad (x' = (x_1, \ldots, x_n))
\end{array} \right.$$  

(In § 4, we shall study phase functions $\varphi^\pm_\beta(x)$ precisely.)

We write $K = \bigcup_{\beta = 1}^m K^\pm_\beta$. And according to assumptions (P)(i), (ii) and (iii), $P(x, \xi)$ is decomposed in the following

$$P(x, \xi) = P(x, 1, 0, \ldots, 0) \prod_{\rho = 1}^m (\xi_0 - \lambda^\rho_\rho(x, \xi')), $$

where $\lambda^\rho_\rho(x, \xi')$ are holomorphic in $(x, \xi') = (x, \xi_1, \ldots, \xi_n)$ near $(x, \xi') = (0, 1, 0; \ldots, 0)$ and mutually distinct and satisfy $\lambda^\rho_\rho(0, 1, 0, \ldots, 0) = \lambda^\rho_\rho$. We put $\lambda^\rho_\rho(x') = \lambda^\rho_\rho(0, x', 1, 0, \ldots, 0)$ and $\xi^\rho_\rho = (0, x', \lambda^\rho_\rho(x'), 1, 0, \ldots, 0)$.

Furthermore, we shall impose on the symbol $L_s(x, \xi)$ of
$L_s(x, D)$ which is the homogeneous part of order $2m-1$ of $L(x, D)$.

the following condition:

Assumption $L_s$ (i) \[
\left\{ P(0, 0)(e_\beta) \sqrt{Q(e_\beta)} \right\}^{-1} \left[ -2L_s(e_\beta)P(0)(e_\beta) + P(0, 0)(e_\beta)P_{x_0}(e_\beta) + \sum_{j=1}^{n} \lambda_{x_j}(x) \{ P(0, 0)(e_\beta)P(j)(e_\beta) - 2P(0)(e_\beta) \} P(0, j)(e_\beta) \right] = c_\beta \quad (c_\beta \text{ is a constant depending only on } \beta),
\]

where $P(j)(x, \xi) = D_{\xi_j}^j P(x, \xi)$ and $P(i, j)(x, \xi) = D_{\xi_i}\xi_j^j P(x, \xi)$

Note: We give below three simple examples of $L(x, D)$ which satisfy Assumptions $(P), (Q)$ and $(L_s)$.

(Ex.0) $L = D_0^2 - x_0^2 D_1^2 - cD_1 \quad (c \text{ is a constant.})$

We call this operator $P_c$ henceforth.

(Ex.1) $L = D_0^2 - x_0^2 D_1^2 - b(x)D_1$, where $b(0, x') = c \quad (c \text{ is a constant.}) \quad (\Leftrightarrow (L_s)(i))$.

(Ex.2) $L = D_0^2 - x_0^2 Q(x, D') + R(x, D)$, where $Q(x, D') = \sum_{i, j=1}^{n} q_{i, j}(x) D_i D_j$ and $q_{11}(0) \not= 0 \quad (\Leftrightarrow (Q))$,

$R(x, D) = \sum_{i=0}^{n} r_i(x) D_i s(x)$ and $\frac{-r_1(0, x')}{\sqrt{q_{11}(0, x')}} = c \quad (c \text{ is a constant.}) \quad (\Leftrightarrow (L_s)(i))$. 


Now, we consider the non-characteristic Cauchy problem with singular data

\[
\begin{align*}
L(x,D)u(x) &= 0 \\
D_0^h u(0,x') &= \mathcal{W}_h(x') \quad (h=0,\ldots,2m-1)
\end{align*}
\]

where at least one of \( \mathcal{W}_h(x') \) has poles along \( x_0=x_1=0 \).

To study this Cauchy problem, and its solution, namely its singularities, we need the auxiliary functions \( X^{(j)}_\alpha \) and \( Y^{(j)}_\alpha \). First we introduce the so-called wave forms \( f_\alpha(\rho) \) and \( k_\alpha(\rho) \):

\[
f_\alpha(\rho) = \frac{\rho^\alpha}{r^{(\alpha+1)}}
\]

and

\[
k_\alpha(\rho) = \frac{\partial}{\partial \rho} f_\alpha(\rho) = \begin{cases} 
\frac{\rho^\alpha}{r^{(\alpha+1)}} (\log \rho - \psi(\alpha+1)) \\
\text{especially } |\alpha+1|! (-1)^{\alpha-1} \rho^\alpha \text{ for } \alpha = -1, -2, \ldots
\end{cases}
\]

where \( \psi(\alpha) \) is a di- \( \Gamma \) function, namely \( \frac{d}{d \alpha} \Gamma(\alpha) \) and \( \alpha \) is a complex parameter.

Next we introduce the fundamental auxiliary functions \( U_\alpha(\theta,p,C) \) and \( V_\alpha(\theta,p,C) \) as the solutions of the Cauchy problems for the operator \( P_c = \partial_\theta^2 - \theta^2 \partial_p^2 - c \partial_p \) (\( c \) is a complex parameter).
respectively: \[ P_c U_\alpha(\theta, \rho, c) = 0 \]

with initial data
\[
\begin{align*}
U(0, \rho, c) &= f_\alpha(\rho) \\
U_\alpha(0, \rho, c) &= 0
\end{align*}
\]

and
\[ P_c V_\alpha(\theta, \rho, c) = 0 \]

with initial data
\[
\begin{align*}
V_\alpha(0, \rho, c) &= 0 \\
V_\alpha(0, \rho, c) &= f_\alpha(\rho)
\end{align*}
\]

We remark that the following explicit representation of \( U_\alpha \) and \( V_\alpha \) are known:
\[
\begin{align*}
U_\alpha(\theta, \rho, c) &= \frac{(\frac{\rho}{\rho'})^{-\alpha}}{1^{(\alpha+1)}} F(-\alpha, \frac{1+c}{4}, \frac{1}{2}, z) \\
V_\alpha(\theta, \rho, c) &= \frac{(\frac{\rho}{\rho'})^{-\alpha}}{1^{(\alpha+1)}} F(-\alpha, \frac{3+c}{4}, \frac{3}{2}, z)
\end{align*}
\]

where \( \rho' = \rho + \frac{1}{2} \theta^2 \) and \( z = 1 - \frac{\rho'}{\rho} \).

We define the auxiliary functions \( X_\alpha(\theta, \rho, c) \) and \( Y_\alpha(\theta, \rho, c) \):
\[
\begin{align*}
X_\alpha(\theta, \rho, c) &= \sum_\alpha U_\alpha(\theta, \rho, c), \\
Y_\alpha(\theta, \rho, c) &= \sum_\alpha V_\alpha(\theta, \rho, c).
\end{align*}
\]

Therefore \( X_\alpha \) satisfies the Cauchy problem \( P_c X_\alpha = 0 \) with the initial data \( X_\alpha(0, \rho, c) = k_\alpha(\rho) \) and \( X_\alpha(0, \rho, c) = 0 \).
And \( Y_\alpha \) satisfies the Cauchy problem \( P_c Y_\alpha = 0 \) with the initial
data \( Y_\alpha(0, \rho, c) = 0 \) and \( Y_\beta(0, \rho, c) = k_\omega(\rho) \).

We introduce the auxiliary functions \( U^{(q)}(\theta, \rho, c) \) and \( V^{(q)}(\theta, \rho, c) \)
as the solutions of the Cauchy problem inductively \((q=0, 1, 2, \cdots)\):

First we set \( U^{(0)}(\theta, \rho, c) = U^{(0)}(\theta, \rho, c) \) and \( V^{(0)}(\theta, \rho, c) = V^{(0)}(\theta, \rho, c) \).

\[
P_c U^{(q)}(\theta, \rho, c) = U^{(q-1)}(\theta, \rho, c) \quad \text{(for } q \geq 1)\]

with null initial data \( U^{(q)}(0, \theta, c) = 0 \)

\[
U^{(q)}(0, \theta, c) = 0.
\]

\[
P_c V^{(q)}(\theta, \rho, c) = V^{(q-1)}(\theta, \rho, c) \quad \text{(for } q \geq 1)\]

with null initial data \( V^{(q)}(0, \theta, c) = 0 \)

\[
V^{(q)}(0, \theta, c) = 0.
\]

Finally we reach the definition of \( X^{(q)}(\theta, \rho, c) \) and \( Y^{(q)}(\theta, \rho, c) \)
which play important role in this paper.

For \( q=0 \), we set \( X^{(0)}(\theta, \rho, c) = X^{(0)}(\theta, \rho, c) \) and \( Y^{(0)}(\theta, \rho, c) = Y^{(0)}(\theta, \rho, c) \). For \( q \geq 1 \), we define \( X^{(q)} \) and \( Y^{(q)} \) as the solutions
of the Cauchy problems:
\[ P_c X^{(q)}(\theta, \rho, c) = X^{(q-1)}(\theta, \rho, c) \quad \text{for } q \geq 1 \]

with null initial data

\[
\begin{align*}
X^{(q)}(0, \rho, c) &= 0 \\
X^{(q)}(0, \rho, c) &= 0.
\end{align*}
\]

\[ P_c Y^{(q)}(\theta, \rho, c) = Y^{(q-1)}(\theta, \rho, c) \quad \text{for } q \geq 1 \]

with null initial data

\[
\begin{align*}
Y^{(q)}(0, \rho, c) &= 0 \\
Y^{(q)}(0, \rho, c) &= 0.
\end{align*}
\]

Therefore \( X^{(q)}(\theta, \rho, c) = \partial^a u^{(q)}(\theta, \rho, c) \) and \( Y^{(q)}(\theta, \rho, c) = \partial^a v^{(q)}(\theta, \rho, c) \) hold.

These auxiliary functions \( f^\alpha, k^\alpha, u^{(q)}_\alpha, v^{(q)}_\alpha, x^{(q)}_\alpha, y^{(q)}_\alpha \) satisfy the relations

\[
\begin{align*}
\frac{d}{d\rho} f^\alpha &= f^\alpha_{\alpha - 1}, \\
\frac{d}{d\rho} k^\alpha &= k^\alpha_{\alpha - 1}, \\
\partial^\alpha u^{(q)}_\alpha &= u^{(q)}_\alpha, \\
\partial^\alpha v^{(q)}_\alpha &= v^{(q)}_\alpha, \\
\partial^\alpha x^{(q)}_\alpha &= x^{(q)}_\alpha, \\
\partial^\alpha y^{(q)}_\alpha &= y^{(q)}_\alpha,
\end{align*}
\]

respectively.

To describe the multi-valued functions \( u^{(q)}_\alpha, v^{(q)}_\alpha, x^{(q)}_\alpha \) and \( y^{(q)}_\alpha \) precisely, we need the following lemma.

**Lemma 1.1.** We have the following explicit representations of these multi-valued functions

\[
\begin{align*}
u^{(q)}_\alpha(\theta, \rho, c) &= \frac{1}{q!} \partial^q c^\alpha \nu^{(q)}_\alpha(\theta, \rho, c), \\
v^{(q)}_\alpha(\theta, \rho, c) &= \frac{1}{q!} \partial^q c^\alpha \nu^{(q)}_\alpha(\theta, \rho, c),
\end{align*}
\]
\[ x^{(q)}_{\alpha}(\theta, p, c) = \frac{1}{q!} \partial_c^{\alpha} q_{\alpha+q}(\theta, p, c) = \frac{1}{q!} \partial_c^{\alpha} q_{\alpha+q}(\theta, p, c), \]

\[ y^{(q)}_{\alpha}(\theta, p, c) = \frac{1}{q!} \partial_c^{\alpha} q_{\alpha+q}(\theta, p, c) = \frac{1}{q!} \partial_c^{\alpha} q_{\alpha+q}(\theta, p, c). \]

Therefore

\[ u^{(q)}_{\alpha} = \frac{1}{q!} \partial_c^{\alpha} \left[ F(-\alpha - q, 1 + c, 1, z) \right], \]

\[ v^{(q)}_{\alpha} = \frac{1}{q!} \partial_c^{\alpha} \left[ F(-\alpha - q, 3 + c, 3, z) \right], \]

\[ x^{(q)}_{\alpha} = \frac{1}{q!} \partial_c^{\alpha} \left[ F(-\alpha - q, 1 + c, 1, z) \right], \]

\[ y^{(q)}_{\alpha} = \frac{1}{q!} \partial_c^{\alpha} \left[ F(-\alpha - q, 3 + c, 3, z) \right]. \]

Now we describe our theorem. We put

\[ x^{(q)}_{\alpha}(\theta, p, c_{\beta}) = x^{(q)}_{\alpha, \beta}(\theta, p), \]

and

\[ y^{(q)}_{\alpha}(\theta, p, c_{\beta}) = y^{(q)}_{\alpha, \beta}(\theta, p). \]

**THEOREM 1.1.** Under Assumptions (P), (Q) & (L_s), for \( r > 0 \) sufficiently small, the Cauchy problem (1.1) has a unique holomorphic solution on the universal covering space over \( \mathbb{D}_r \setminus K \), where \( \mathbb{D}_r = \left\{ x \in \mathbb{C}; |x| < r \right\} \). More precisely speaking, the solution \( u(x) \) is given in the following form:

\[
\begin{align*}
  u(x) &= \sum_{\alpha = -1}^{\infty} \sum_{\beta = 1}^{\infty} \sum_{q = 0}^{\infty} \left\{ u^{(q)}_{\alpha, \beta}(x) X^{(q)}_{\alpha}(\theta_{\beta}(x), f_{\beta}(x)) +
  e^{(q)}_{\alpha, \beta}(x) D_x^{q} X^{(q)}_{\alpha}(\theta_{\beta}(x), f_{\beta}(x)) +
  v^{(q)}_{\alpha, \beta}(x) Y^{(q)}_{\alpha}(\theta_{\beta}(x), f_{\beta}(x)) +
  h^{(q)}_{\alpha, \beta}(x) D_x^{q} Y^{(q)}_{\alpha}(\theta_{\beta}(x), f_{\beta}(x)) \right\}
\end{align*}
\]
where 1 is the highest order of poles of the initial data and
\[ u_{\alpha, \beta}^{(q)}(x), \ g_{\alpha, \beta}^{(q)}(x), \ v_{\alpha, \beta}^{(q)}(x), \ h_{\alpha, \beta}^{(q)}(x), \ \theta_\beta(x) \text{ and } \rho_\beta(x) \] are holomorphic in \( D_r \) (as for \( \theta_\beta(x) \) and \( \rho_\beta(x) \) such that \( \varphi^+(x) = \rho_\beta(x) + \frac{1}{2} \left[ \Theta_\beta(x) \right]^2 \), see § 4).

For the proof of this theorem, we construct the formal solution of the Cauchy problem (1,1) in the above form, and then confirm the convergence of the formal solution. This theorem shows that the singularities of the solution \( u(x) \) are reduced to the singularities of the auxiliary functions \( X_{\alpha, \beta}^{(q)} \) and \( Y_{\alpha, \beta}^{(q)} \) which are to be studied in Appendix in detail. To construct the formal solution, first we are to prepare some calculations and some properties of operator \( L(x, D) \) and the auxiliary functions, with which we start in the next section.
§ 2. Preliminary calculation

To construct the formal solution of the Cauchy problem (1.1), we substitute series of the formal solution in \( L(x,D)u(x) \) and calculate it. To do so, we need to represent \( \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial x^j} \phi(1) \) and \( \frac{\partial^1}{\partial x^1} \frac{\partial^j}{\partial x^j} \psi(1) \) in terms of \( \frac{\partial^k}{\partial x^k} \phi(1-q) \) and \( \frac{\partial^{k-1}}{\partial x^{k-1}} \phi(1-q) \), \( \frac{\partial^{k}}{\partial x^{k}} \psi(1-q) \), respectively. So we employ the following formula:

\[
\begin{align*}
\frac{\partial^{2r}}{\partial \theta^r} u(j) & = \sum_{k=0}^{r-1} \sum_{s=2k}^{r+k} c_{s,k} \frac{2r}{\alpha-2r+s} U(j-k) \\
& \quad + \sum_{k=0}^{r-1} \sum_{s=2k+1}^{r+k} d_{s,k} \frac{2r}{\alpha-2r+s+1} \frac{\partial}{\partial \theta} U(j-k) \\
\frac{\partial^{2r+1}}{\partial \theta^r} u(j) & = \sum_{k=0}^{r} \sum_{s=2k}^{r+k} c_{s,k} \frac{2r+1}{\alpha-2r+s} U(j-k) \\
& \quad + \sum_{k=0}^{r} \sum_{s=2k}^{r+k} d_{s,k} \frac{2r+1}{\alpha-2r+s} \frac{\partial}{\partial \theta} U(j-k)
\end{align*}
\]

where \( c_{s,k} \) and \( d_{s,k} \) are polynomials of \( \theta \) with integer coefficients, especially.

\[
\begin{align*}
c_{2k,k} & = r_k \theta^{2r-2k}, \quad c_{2r,0} = r \theta^{2r-2}, \quad c_{2r+1,0} = 2r \theta^{2r-1} \\
d_{2k,k} & = r_k \theta^{2r-2k}, \quad d_{2r+1,0} = r \theta^{2r-2}, \quad d_{2r,0} = 2r(r-1) \theta^{2r-3}, \\
( r_k & = \frac{r'}{k!/(r-k)!} )
\end{align*}
\]
In this formula, we may replace \( U \) by \( V, X, \) or \( Y. \)

Let \( K(x, \xi) \) be a homogeneous polynomial of degree \( t \) in \( \xi = (\xi_0, \xi_1, \ldots, \xi_n) \). We shall write \( K(x, \xi) = D_x^t K(x, \xi) \).

We define \( K_j(x, \xi, \eta) = \frac{1}{j!} \sum_{i=0}^{j} \xi_i K(x, \eta) \), where

\[
\eta = (\eta_0, \eta_1, \ldots, \eta_n), \ K(\xi) = (\xi_0, \xi_1, \ldots, \xi_n), \text{ and}
\]

\[
|\eta| = \eta_0 + \eta_1 + \cdots + \eta_n.
\]

So, \( K(x, r^2 + s \eta) = \sum_{i=0}^{t} K_i(x, r^2, s \eta) \) holds, where \( r, s \in \mathbb{C} \). We shall use

\[
\partial_{i} = \partial_{x_i} + \rho \partial_{p_i} \quad (i = 0, \ldots, n), \ \partial = (\partial_0, \partial_1, \ldots, \partial_n)
\]

and

\[
D^i = \partial_{x_i} + \rho \partial_{p_i}.
\]

The relation \( K(x, \partial) = (x, 0 \partial + \rho \partial) = \sum_{i=0}^{t} K_i(x, 0 \partial, \rho \partial) \partial_{x} \partial_{p}^{t-i} \) and

(F.1) lead to the following formula.

(F.2)

\[
\begin{align*}
K(x, \partial) U^{(1)}_{\alpha} & = \left[ \frac{1}{(t+1)} \right] \sum_{k=0}^{t-1} \sum_{s=t-k} K_{k, s}^{t} (x, \partial, \rho) U_{\alpha}^{(1-k)} \alpha - s + k + 1 \\
& + \left[ \frac{1}{2} \right] \left( \frac{t}{2} \right) \left[ \frac{1}{2} \right] - 1 \\
& + \sum_{k=0}^{t-1} \sum_{s=t-k} K_{k, s}^{t} (x, 0 \partial, \rho) \partial_{\alpha} U_{\alpha}^{(1-k)} \alpha - s + k + 1 \\
K(x, \partial) \partial_{\alpha} U^{(1)}_{\alpha} & = \left[ \frac{t+1}{2} \right] \sum_{k=0}^{t} \sum_{s=t-k} K_{k, s}^{t} (x, \partial, \rho) \partial_{\alpha} U_{\alpha}^{(1-k)} \alpha - s + k + 1
\end{align*}
\]
\[
\begin{align*}
&+ \sum_{s=0}^{\lfloor \frac{t}{2} \rfloor} \sum_{k=0}^{s} K_{k,s}(x, \theta, \rho) \partial_x^{l-k}, \\
&\text{( \([A]\) is a integer part of } A. \text{ )}
\end{align*}
\]

where
\[
\begin{align*}
K_{t,1}^{t}(x, \theta, \rho) &= \sum_{i=0}^{t} \theta_{2i} K_{2i}(x, \theta_x, \rho_x), \\
K_{0,t-1}(x, \theta, \rho) &= \sum_{i=1}^{t-1} \theta_{2i-2} K_{2i+1}(x, \theta_x, \rho_x) + \sum_{i=1}^{\lfloor \frac{t}{2} \rfloor} \theta_{2i} K_{2i+1}(x, \theta_x, \rho_x), \\
K_{t,2}^{t}(x, \theta, \rho) &= \sum_{i=0}^{t-1} K_{2i+1}(x, \theta_x, \rho_x), \\
K_{t,1}^{t}(x, \theta, \rho) &= \sum_{i=1}^{t} \theta_{2i-2} K_{2i+1}(x, \theta_x, \rho_x) + \sum_{i=2}^{t-1} \theta_{2i} K_{2i+1}(x, \theta_x, \rho_x), \\
K_{k,k-t}(x, \theta, \rho) &= \sum_{i=0}^{t-k} \theta_{2i} K_{2i}(x, \theta_x, \rho_x), \\
K_{0,t-1}(x, \theta, \rho) &= \sum_{i=1}^{t} \theta_{2i-2} K_{2i+1}(x, \theta_x, \rho_x), \\
K_{t,3}^{t}(x, \theta, \rho) &= \sum_{i=1}^{t} \theta_{2i-2} K_{2i-1}(x, \theta_x, \rho_x), \\
K_{0,t-1}(x, \theta, \rho) &= \sum_{i=1}^{t} \theta_{2i-2} K_{2i-1}(x, \theta_x, \rho_x) + \sum_{i=2}^{t-1} \theta_{2i} K_{2i+1}(x, \theta_x, \rho_x), \\
K_{t,4}^{t}(x, \theta, \rho) &= \sum_{i=0}^{t} \theta_{2i} K_{2i}(x, \theta_x, \rho_x), \\
K_{0,t-1}(x, \theta, \rho) &= \sum_{i=1}^{t} \theta_{2i-2} K_{2i+1}(x, \theta_x, \rho_x) + \sum_{i=2}^{t-1} \theta_{2i} K_{2i+1}(x, \theta_x, \rho_x), \\
K_{t,3}^{t}(x, \theta, \rho) &= \sum_{i=1}^{t} \theta_{2i-2} K_{2i-1}(x, \theta_x, \rho_x) + \sum_{i=2}^{t-1} \theta_{2i} K_{2i+1}(x, \theta_x, \rho_x), \\
K_{k,k-t}(x, \theta, \rho) &= \sum_{i=0}^{t-k} \theta_{2i} K_{2i}(x, \theta_x, \rho_x).
\end{align*}
\]
\[ k_{k,t-k}(x,\theta,P) = \sum_{i=k}^{\left\lfloor \frac{k}{2} \right\rfloor} C_k \theta^{2i-2k} K_{2i}(x,\theta,P) \]

We shall sometimes use the following formula by chain rule:

\[
(F.3) \quad K(x,D)\left[ w(x)W(\theta(x),f(x)) \right] = w \cdot K(x,\partial)W \\
+ \sum_{i,j=0}^{n} \frac{1}{2} K(i,j)(x,\partial)(D_i \partial_j)W + \sum_{i=0}^{n} D_i w \cdot K(i)(x,\partial)W \\
+ ( \text{lower order term} ).
\]

Now using (F.2) and (F.3), we calculate \( L(x,D)u(x)U(j)_{\alpha-1}(\theta(x),\bar{P}(x)) \) and \( L(x,D)g(x)\partial_{\alpha}U(j)(\theta(x),\bar{P}(x)) \). We have,

\[
(F.4) \quad L(x,D)uU(j)_{\alpha-1} = \sum_{\nu=0}^{2m-2k} \frac{1}{\nu!} D^\nu u \cdot L(\sigma)_{\alpha}L_{\alpha-1} \\
= \sum_{\nu=0}^{2m-2k} \frac{1}{\nu!} D^\nu u \cdot \left[ L(\sigma)_{\alpha}U(j)_{\alpha-1} + \sum_{i,j=0}^{n} \frac{1}{2} L(\sigma)(i,j)(x,\partial)(D_i \partial_j) + L_s(x,\partial) + \cdots \right] U(j)_{\alpha-1} \\
= \sum_{k=0}^{m} \sum_{\nu=0}^{2m-2k} 1_{\nu,k}(u)U(j-k)_{\alpha-\nu-1} + \sum_{k=0}^{m} \sum_{\nu=1}^{2m-2k} 2_{\nu,k}(u)\partial_{\alpha}U(j-k)_{\alpha-\nu-1} + \\
\sum_{k=0}^{m} \sum_{\nu=0}^{2m-2k} 4_{\nu,k}(g)\partial_{\alpha}U(j-k)_{\alpha-\nu-1} \]

where \( h_{\nu,k} = h_{\nu,k}(x,\theta,f,D) \in L^{2m-2k-\nu}(\mathbb{D}) \) (\( h=1,2,3,4 \)) and especially \( 3_{L-1,0} = 0 \).

We see also the following relations from the above.
\[
\begin{align*}
\left\{ \begin{array}{l}
1^0_{L^k, k} = \sum_{\nu = 2m - 2k - \nu} L^0_{2k} (x, \theta, x, P_x) \frac{D^\nu}{\delta!} \pmod{\theta^2}, \\
2^0_{L^k, k} = \sum_{\nu = 2m - 2k - \nu} L^0_{2k+1} (x, \theta, x, P_x) \frac{D^\nu}{\delta!} \pmod{\theta^2}, \\
3^0_{L^k, k} = \sum_{\nu = 2m - 2k - \nu} L^0_{2k-1} (x, \theta, x, P_x) \frac{D^\nu}{\delta!} \pmod{\theta^2} \quad (k \geq 1), \\
4^0_{L^k, k} = \sum_{\nu = 2m - 2k - \nu} L^0_{2k} (x, \theta, x, P_x) \frac{D^\nu}{\delta!} \pmod{\theta^2},
\end{array} \right.
\]
\[ (F.5) \]

where \( h^0_{L^k, k} \) are the principal part of \( h^0_{L, k} \).

More precisely for \( k = 0 \) and \( \nu = 2m \) or \( 2m - 1 \), we have:

\[
\begin{align*}
1^0_{L^2, 0} = \sum_{i = 0}^{2m} L^0_{2i} (x, \theta, x, P_x) \theta^{2i}, \\
2^0_{L^2, 0} = \sum_{i = 0}^{2m-1} L^0_{2i+1} (x, \theta, x, P_x) \theta^{2i}, \\
3^0_{L^2, 0} = \sum_{i = 0}^{2m-1} L^0_{2i+1} (x, \theta, x, P_x) \theta^{2i+2}, \\
4^0_{L^2, 0} = \sum_{i = 0}^{2m} L^0_{2i} (x, \theta, x, P_x) \theta^{2i},
\end{align*}
\]
\[ (F.6) \]

and

\[
\begin{align*}
1_{L^{2m-1}, 0} (x, \theta, P, D) = M + R_0 + N + R_1 + \frac{1}{2} \left\{ \sum_{\mu, \nu = 0}^{n} \rho_{x, \mu x, \nu} s(\mu, \nu) \\
+ \Theta_{x, \mu} \Theta_{x, \nu} \theta^{2} s(\mu, \nu) \right\} , \\
2_{L^{2m-1}, 0} (x, \theta, P, D) = L + R_1 + N + R_1 + \frac{1}{2} \left\{ \sum_{\mu, \nu = 0}^{n} \Theta_{x, \mu} \Theta_{x, \nu} s(\mu, \nu) \\
+ \Theta_{x, \mu} \Theta_{x, \nu} s(\mu, \nu) \right\} , \\
3_{L^{2m-1}, 0} (x, \theta, P, D) = \theta^2 L + \Theta_{x, \mu} \Theta_{x, \nu} s(\mu, \nu) + \frac{1}{2} \sum_{\mu, \nu = 0}^{n} \Theta_{x, \mu x, \nu} s(\mu, \nu).
\end{align*}
\]
\[ (F.7) \]
\[ L_{2m-1,0}(x, \theta, \rho, \beta) = M+R_0+N_c+R_3+\frac{1}{2}\left\{ \sum_{\alpha, \nu=0}^{\infty} \int_{x, x, x} S(\alpha, \nu) \right\}, \]

where

\[ L = \sum_{i=1}^{\infty} \frac{m-1}{2i+1}(x, \theta_x, \rho_x) \theta^{2i} \left\{ D^\theta \right\} \]

\[ M = \sum_{i=1}^{\infty} \frac{m-1}{2i}(x, \theta_x, \rho_x) \theta^{2i} \left\{ D^\theta \right\} \]

\[ R_0 = \sum_{i=0}^{\infty} s_{2i}(x, \theta_x, \rho_x) \theta^{2i} \]

\[ R_1 = \sum_{i=0}^{\infty} s_{2i+1}(x, \theta_x, \rho_x) \theta^{2i} \]

\[ R_2 = \sum_{i=2}^{\infty} s_{2i-1}(x, \theta_x, \rho_x) \cdot 2(i-1)^2 \theta^{2i-3} \]

\[ R_3 = \sum_{i=2}^{\infty} s_{2i-1}(x, \theta_x, \rho_x) \cdot 2(i-1) \theta^{2i-3} \]

\[ R_4 = \sum_{i=2}^{\infty} s_{2i-1}(x, \theta_x, \rho_x) \cdot 2i \theta^{2i-1} \]

\[ N_c = \sum_{i=1}^{\infty} L_{2i}(x, \theta_x, \rho_x) c_i \theta^{2i-2} \]

\[ N_1 = \sum_{i=1}^{\infty} L_{2i-1}(x, \theta_x, \rho_x) c_i \theta^{2i-2} \]

\[ N_2 = \sum_{i=1}^{\infty} L_{2i-1}(x, \theta_x, \rho_x) c(i-1) \theta^{2i-4} \]

\[ S(\alpha, \nu) = \sum_{i=0}^{\infty} L_{2i}(x, \theta_x, \rho_x) \theta^{2i} \]

\[ S(\alpha, \nu) = \sum_{i=1}^{\infty} L_{2i-1}(x, \theta_x, \rho_x) \theta^{2i-2} \]
We remark, in the above, next relations:

\[ L_0(x, \theta_x, p_x) = L(x, p_x) \quad \text{and} \quad L_1(x, \theta_x, p_x) = \sum_{\mu=0}^{N} \theta_x^{\mu} L^{(\mu)}(x, p_x), \quad \text{and} \]

\[ L_2(x, \theta_x, p_x) = \frac{1}{2} \sum_{\mu, \nu=0}^{n} \theta_x^{\mu} \theta_x^{\nu} L^{(\mu, \nu)}(x, p_x). \]

We remark also that in these formulae (F.1), (F.2), (F.3) and (F.4) obtained above, we may replace U by V, X and Y.
§ 3 Construction of the formal solution.

Taking account of the principle of the superposition, we have only to solve the following Cauchy problem with the special initial data:

\[
\begin{align*}
L(x,D)u(x) &= 0 \\
D_0^hu(0,x') &= \omega_h(x^n)k_{-1}(x_1) \quad (h=0,\ldots,2m-1)
\end{align*}
\]

where \( x''=(x_2,\ldots,x_n) \) and \( \omega_h(x^n) \) are holomorphic functions of \( x^n \) in the neighbourhood of \( 0\in\mathbb{C}^{n-1} \).

We seek the formal solution of the form in Theorem 1.1. Namely we determine the coefficients \( u^{(q)}_{\alpha,\beta}, v^{(q)}_{\alpha,\beta}, w^{(q)}_{\alpha,\beta} \), \( h^{(q)}_{\alpha,\beta} \) and the auxiliary phase functions \( \beta_{\phi}, \phi_{\beta} \), and show the convergence of this formal solution. First to determine these coefficients and auxiliary phase functions, we substitute this formal solution in \( L(x,D)u(x)=0 \), and using the formulae obtained in the preceding section and the relations \( \gamma(q) \)

\[
= \gamma_{\alpha-1,\beta} \quad \text{and} \quad \delta(q) = \delta_{\alpha-1,\beta},
\]

we have,
\[ L(x, D)u(x) = \sum_{\beta=1}^{m} \sum_{\gamma=1}^{m} \sum_{j=0}^{2m-2k} \left\{ \sum_{k=0}^{m} \sum_{\nu=-1}^{2m-2k} \left( 1_{L_{\nu,k,\beta}}(u(j+k)) \right) \right\} \]

\[ + \sum_{\beta=1}^{m} \sum_{\gamma=1}^{m} \sum_{j=0}^{2m-2k} \left\{ \sum_{k=0}^{m} \sum_{\nu=-1}^{2m-2k} \left( 2_{L_{\nu,k,\beta}}(u(j+k)) \right) \right\} \]

\[ + \sum_{\beta=1}^{m} \sum_{\gamma=1}^{m} \sum_{j=0}^{2m-2k} \left\{ \sum_{k=0}^{m} \sum_{\nu=-1}^{2m-2k} \left( 3_{L_{\nu,k,\beta}}(u(j+k)) \right) \right\} \]

\[ + \sum_{\beta=1}^{m} \sum_{\gamma=1}^{m} \sum_{j=0}^{2m-2k} \left\{ \sum_{k=0}^{m} \sum_{\nu=-1}^{2m-2k} \left( 4_{L_{\nu,k,\beta}}(u(j+k)) \right) \right\} \]

where \( h_{L_{\nu,k,\beta}} = h_{L_{\nu,k}}(x, \xi_\beta, \zeta_\beta, D) \) \( (h=1,2,3,4) \) and \( 1_{L_{\nu,k,\beta}}^{1_{L_{\nu,k,\beta}}} = 2_{L_{\nu,k,\beta}}^{2_{L_{\nu,k,\beta}}} = 0 \).

We set the coefficients of \( X^{(j)}_{\alpha-1,\beta}, \partial_y X^{(j)}_{\alpha-1,\beta}, Y^{(j)}_{\alpha-1,\beta} \) and \( \partial_y Y^{(j)}_{\alpha-1,\beta} \) equal to zero. And especially we set \( 1_{L_{2m,0,\beta}}^{1_{L_{2m,0,\beta}}} = 2_{L_{2m,0,\beta}}^{2_{L_{2m,0,\beta}}} = 3_{L_{2m,0,\beta}}^{3_{L_{2m,0,\beta}}} = 4_{L_{2m,0,\beta}}^{4_{L_{2m,0,\beta}}} = 0 \) which are non-linear partial differential equations of first order in \( \xi_\beta \) and \( \zeta_\beta \), namely so-called eikonal equations, so that we can determine these auxiliary phase functions \( \xi_\beta \) and \( \zeta_\beta \), (we shall study these non-linear partial differential equations in the next section). Thus we have reached the systems of the transport equations which determine the coefficients \( u^{(q)}_{\alpha,\beta}, \xi^{(q)}_{\alpha,\beta}, \zeta^{(q)}_{\alpha,\beta} \) and \( h^{(q)}_{\alpha,\beta} \).
\[
\begin{align*}
(\text{i}) \quad & 2_{L_2 m-1,0,\beta}^0 (u_{\alpha+2m-1,\beta}(j)) + 4_{L_2 m-1,0,\beta}^0 (g_{\alpha+2m-1,\beta}(j)) \\
& = - \sum_{k=1}^{m} \sum_{\nu=-1}^{2m-2} \left\{ 2_{L_2 m-1,0,\beta}^0 (u_{\alpha+\nu,\beta}(j+k)) + 4_{L_2 m-1,0,\beta}^0 (g_{\alpha+\nu,\beta}(j+k)) \right\} \\
(\text{TE}) \quad & 3_{L_2 m-1,0,\beta}^0 (h_{\alpha+2m-1,\beta}(j)) + 1_{L_2 m-1,0,\beta}^0 (v_{\alpha+2m-1,\beta}(j)) \\
& = - \sum_{k=1}^{m} \sum_{\nu=-1}^{2m-2} \left\{ 3_{L_2 m-1,0,\beta}^0 (h_{\alpha+\nu,\beta}(j+k)) + 1_{L_2 m-1,0,\beta}^0 (v_{\alpha+\nu,\beta}(j+k)) \right\} \\
(\text{ii}) \quad & 2_{L_2 m-1,0,\beta}^0 (v_{\alpha+2m-1,\beta}(j)) + 4_{L_2 m-1,0,\beta}^0 (h_{\alpha+2m-1,\beta}(j)) \\
& = - \sum_{k=1}^{m} \sum_{\nu=-1}^{2m-2} \left\{ 2_{L_2 m-1,0,\beta}^0 (v_{\alpha+\nu,\beta}(j+k)) + 4_{L_2 m-1,0,\beta}^0 (h_{\alpha+\nu,\beta}(j+k)) \right\}
\end{align*}
\]

On the other hand, from the initial data we substitute the formal solution \( u(x) \) in \( D_0^h u(x)|_{x_0=0} \) and calculate this, using the formulae (F.4), (F.5) and (F.6) which are valid under the replacement of \( L(x,D) \) by \( D_0^h \), we have,
$D_0^h u(x)|_{x=0} = \sum_{\beta=1}^{m} \sum_{j=0}^{m} \left[ \sum_{k=0}^{h-2k} 1_{M, h, \nu, k, \beta} (u(\alpha, \beta)) \right] + \sum_{j=0}^{h-1} 2_{M, h, \nu, k, \beta} (g(\alpha, \beta)) \partial_x x(j-k) + \sum_{j=0}^{h-1} 3_{M, h, \nu, k, \beta} (g(\alpha, \beta)) \partial_x x(j-k) + \sum_{j=0}^{h-1} 4_{M, h, \nu, k, \beta} (g(\alpha, \beta)) \partial_x x(j-k)

\text{where } P_{\nu, h, 0, \beta} (p=1, 2, 3, 4) \text{ is a linear ordinary differential operators in } D_0 \text{ of order } h-\nu-2k \text{ and } 1_{M, h, 0, \beta} = 2_{M, h, 0, \beta} = h(0, x')^h - \Omega_{h, 0, \beta} (0, x') \text{ and } 3_{M, h, 0, \beta} = 4_{M, h, 0, \beta} = 0 \text{ (as for } \Omega_{h, 0, \beta} (0, x'), \text{ see } \S 4 \text{ ).}

These ordinary differential operators are determined only by

$h$ and $\theta_{\nu, h, 0, \beta}$, and have holomorphic coefficients in $x'$. We have:

$D_0^h u(x)|_{x=0} = \sum_{\beta=1}^{m} \sum_{j=0}^{h-2k} \left[ \sum_{k=0}^{h-2k} 1_{M, h, \nu, k, \beta} (u(\alpha, \beta)) \right] + \sum_{j=0}^{h-1} 2_{M, h, \nu, k, \beta} (g(\alpha, \beta)) \partial_x x(j-k) + \sum_{j=0}^{h-1} 3_{M, h, \nu, k, \beta} (g(\alpha, \beta)) \partial_x x(j-k) + \sum_{j=0}^{h-1} 4_{M, h, \nu, k, \beta} (g(\alpha, \beta)) \partial_x x(j-k)

\text{Setting the coefficients of } x(0) \text{ and } \partial_x x(0) \text{ equal to zero,}
we obtain the following systems of linear equations:

\[(I.E) \sum_{\beta=1}^{2m} \left[ \sum_{k=0}^{[\frac{h+1}{2}]} \sum_{\nu=-1}^{2k-1} \begin{bmatrix} h \cdot k \cdot (u(k)_{\nu, \kappa, \beta} + (\nu + 1, \beta) + 3 \cdot h \cdot (h(k)_{\nu, \kappa, \beta}) \end{bmatrix} + \sum_{k=0}^{\left[ \frac{h}{2} \right]} \sum_{\nu=0}^{2k-1} \begin{bmatrix} h \cdot k \cdot (v(k)_{\nu, \kappa, \beta} + 4 \cdot h \cdot (h(k)_{\nu, \kappa, \beta}) \end{bmatrix} \right] x_0 = 0 \]

\[
\begin{cases} 
  w_h(x^\nu) & \text{for } \alpha = -1 + h + 1, \\
  0 & \text{otherwise} 
\end{cases}
\]

Remark that the determinant of the following $2m \times 2m$ matrix

\[
\begin{vmatrix}
1 & 0 & \cdots & 1 & 0 \\
\gamma_1 & 1 & \cdots & \gamma_m & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\gamma_1^{2m-1} & (2m-1)\gamma_1^{2m-2} & \cdots & \gamma_m^{2m-1} & (2m-1)\gamma_m^{2m-2}
\end{vmatrix}
\]

\(( \gamma_\beta = \int_{0}^{x_0} (0, x') \ ) \)

does not vanish if $\gamma_\beta (\beta = 1, \ldots, m)$ are mutually distinct,

( in §4 we shall see $\gamma_\beta$ are mutually distinct ).

From the remark described above, we can solve the system of

$2m$ linear equations with respect to $(u_{\nu+2m-1, \beta}^{(0)} + h_{\nu+2m-2, \beta}^{(0)})(0, x')$

and $v_{\nu+2m-2, \beta}^{(0)}(0, x') (\beta = 1, \ldots, m)$ and then we obtain
Lemma 3.1 \((u(0), h(0), v(0))\) and \(v(0)\)

are represented in the following form:

\[
\sum_{\beta=1}^{2m-2} \left[ \sum_{k=1}^{2m-2} 1\gamma_{\mu-2k, \beta}(u(k)) + 4\gamma_{\mu-2k, \beta}(h(k) + \frac{1}{2} - 2k, \rho - \frac{1}{2} - 2k, v^{+2m-2+p, \beta}) + \right. \\
\left. 3\gamma_{\mu-2k, \beta}(\epsilon^{+2m-2+p, \beta}) + 2\gamma_{\mu-2k, \beta}(v^{+2m-2+p, \beta}) \right]
\]

\[
\sum_{\mu=1}^{2m-2} \left[ \sum_{k=1}^{2m-2} 1\gamma_{\mu, \beta}(u(0)) + 4\gamma_{\mu, \beta}(h(0) + \frac{1}{2} - 2m-1+p, \rho + \frac{1}{2} - 2m-1+p, v^{+2m-2+p, \beta}) + \right. \\
\left. 3\gamma_{\mu, \beta}(\epsilon^{+2m-2+p, \beta}) + 2\gamma_{\mu, \beta}(v^{+2m-2+p, \beta}) \right]_{x=0=0}
\]

where \(\gamma_{\mu, \beta}\) are linear ordinary differential operators in \(D_0\) of order \(\mu\), and \(2\gamma_{\mu, \beta}\) are linear ordinary differential operators in \(D_0\) of order \(\mu-1\). These linear ordinary differential operators are determined only by \(\epsilon_{\beta}(x)\) and \(f_{\beta}(x)\), and have holomorphic coefficients in \(x^1\).

We are to determine \(\epsilon_{\beta}\) and \(f_{\beta}\) from the so-called eikonal equations that are to be studied in the next section, and these coefficients from the transport equations that are to be studied in \(\S 5\) precisely.
§4. Phase functions and auxiliary phase functions.

In this section, we study eikonal equations

\( h_{L_{2m},0,\beta}(x, \theta_{x}, \rho_{x}) = 0 \) \( (\beta = 1, 2, 3, 4) \)

and some properties of \( \theta_{x}(x) \) and \( \rho_{x}(x) \).

From the definition of \( h_{L_{2m},0,\beta}(F, \xi) \), we write again:

\[
\begin{align*}
1_{L_{2m},0,\beta} &= \sum_{i=0}^{m} L_{2i} (x, \theta_{x}, \rho_{x}) \xi^{2i} = 0 \\
2_{L_{2m},0,\beta} &= \sum_{i=0}^{m-1} L_{2i+1} (x, \theta_{x}, \rho_{x}) \xi^{2i} = 0 \\
3_{L_{2m},0,\beta} &= \theta_{x}^{2} \cdot 2_{L_{2m},0,\beta} = 0.
\end{align*}
\]

By the homogeneity of \( L_{j}(x, \xi, \eta) = r \cdot L_{j}(x, \xi, \eta) \),

the above equations lead to

\[
\begin{align*}
1_{L_{2m},0,\beta} &= \sum_{i=0}^{m} L_{2i} (x, \xi \theta_{x}, \xi \rho_{x}) = 0 \\
\theta_{x}^{2} 2_{L_{2m},0,\beta} &= \sum_{i=0}^{m-1} L_{2i+1} (x, \xi \theta_{x}, \xi \rho_{x}) = 0.
\end{align*}
\]

Adding these two equations and taking account of the relations

\[ L(x, \xi, \eta) = \sum_{j=0}^{m} L_{j}(x, \xi, \eta) \] and \( \theta_{x} \cdot \theta_{x} = \frac{1}{2} (\theta_{x})^{2} \),

we obtain the ordinary eikonal equations.
\[1_{L_2m,0,\beta} + \Theta_\beta \cdot 2_{L_2m,0,\beta} = L(x, (p_\beta \pm \Theta_\beta \frac{2}{x})_x) = 0\]

Setting \(\Phi_{\beta}^\pm(x) = p_\beta(x) \pm \frac{1}{2}(\Theta_\beta(x))^2\), we rewrite this equation in the familiar form: \(L(x, \Phi_{\beta}^\pm_x) = 0\).

First we solve this Cauchy problem \(L(x, \Phi_{\beta}^\pm_x) = 0\) with the initial data \(\Phi_{\beta}^\pm(0,x^1) = x_1\), and then we get \(\Phi_{\beta}(x)\) and \(p_\beta(x)\) by \(\Phi_{\beta}^\pm(x) = p_\beta(x) \pm \frac{1}{2}(\Theta_\beta(x))^2\) and some properties of these functions.

**Proposition 4.1**

In a certain neighbourhood of \(0 \in C^{n+1}\), there exist the holomorphic solutions \(\Phi_{\beta}^\pm(x)\) (\(\beta = 1, \ldots, m\)) of the Cauchy problem:

\[
\begin{align*}
L(x, \Phi_{\beta}^\pm_x) = 0 \\
\Phi_{\beta}^\pm(0,x^1) = x_1 \quad \text{and} \quad \Phi_{\beta}^\pm(x_0) = \lambda_\beta
\end{align*}
\]

Precisely speaking, \(\Phi_{\beta}^\pm(x)\) are represented in the form:

\(\Phi_{\beta}^\pm(x) = p_\beta(x) \pm \frac{1}{2}(\Theta_\beta(x))^2\).
\( \Theta_\beta(x) \) and \( \rho_\beta(x) \) are holomorphic functions in a neighbourhood of \( 0 \in \mathbb{C}^{n+1} \), and satisfy (4.1) respectively.

Moreover \( \Theta_\beta(x) \) are expressed as follows,

\[
\Theta_\beta(x) = x_0 \Theta_\beta(x) \quad ( \Theta_\beta(0, x') = \left( \frac{\sqrt{Q(e)}}{p(0)(e_\beta)} \right)^{1/2} \neq 0 )
\]

And \( \rho_\beta(x) \) are expressed as follows,

\[
\rho_\beta(x) = x_1 + x_0 \lambda_\beta(x'') + x_0^2 \tau_\beta(x)
\]

\[
( \tau_\beta(0, x') = \frac{-1}{p(0)(e_\beta)} \left\{ p_x (e_\beta) + \sum_{i=1}^{n} \lambda_\beta x_i p_i (e_\beta) \right\} )
\]

We call \( \Theta_\beta(x) \) phase functions and \( \Theta_\beta(x) \), \( \rho_\beta(x) \) auxiliary phase functions, respectively.
§ 5 Some properties of $h_{L_{\nu,k,\beta}}$.

For the construction of the formal solution, we have to determine the coefficients $u_{\alpha,\beta}^{(q)}$, $g_{\alpha,\beta}^{(q)}$, $v_{\alpha,\beta}^{(q)}$ and $h_{\alpha,\beta}^{(q)}$ by solving systems of the transport equations (T.E.). In this section we study transport operators $h_{L_{2m-1,0,\beta}}$ ($h=1,2,3,4$) which govern (T.E.), and other important operators, $3_{L_{\nu,0,\beta}}$ and $1_{L_{2m-2,1,\beta}}$. First we recall the definition of the transport operators $h_{L_{2m-1,0,\beta}}$ ($h=1,2,3,4$). We rewrite (F.7) again in the following form:

$$
1_{L_{2m-1,0,\beta}} = M_{\beta} + \theta_{\beta}^{m_{1,1}}(x),
$$

$$
2_{L_{2m-1,0,\beta}} = L_{\beta} + m_{2,1}(x),
$$

$$
3_{L_{2m-1,0,\beta}} = \theta_{\beta}^{2,1} L_{\beta} + \theta_{\beta} m_{0,1}(x') + \theta_{\beta}^{2,1} m_{3,1}(x),
$$

$$
4_{L_{2m-1,0,\beta}} = M_{\beta} + \theta_{\beta} m_{4,1}(x),
$$

where

$$
L = \sum_{|\gamma|=1} \sum_{i=0}^{m} \frac{1}{L_{2i+1}(x,\partial_{\beta})} \partial_{\beta}^{2i} D^\gamma \\
M = \sum_{|\gamma|=1} \sum_{i=0}^{m-1} \frac{1}{L_{2i}(x,\partial_{\beta})} \partial_{\beta}^{2i} D^\gamma
$$

These transport operators have next properties which play
an important role in the determinations of the coefficients.

Noting \( e_\beta = (0, x', P_{x}(0, x')) \), we have:

**Lemma 5.1** (i) \( L_\beta \bigg|_{x_0=0} = 2 \int_{0}^{x'} P(0)(e_\beta) \left\{ \sum_{j=0}^{n} P(j)(e_\beta) D_j \right\} \),

(ii) \( M_\beta = x_0 P_\beta(x') \left\{ \sum_{j=0}^{n} P(j)(e_\beta) D_j \right\} + x_0^2 \cdot M'_\beta \),

(\text{where } P(x, \int_{0}^{x}(x)) = x_0 P_2(x) \text{ and } P_{x}(0, x') = P_{x}(x') \)

(iii) \( m_{0\beta}(x') = 2 \left\{ \int_{0}^{x'} P(0)(e_\beta) \right\}^2 \cdot 0 \)

(iv) \( 1_{L_{2m-1,0},\beta} \bigg|_{x_0=0} = 4_{L_{2m-1,0},\beta} \bigg|_{x_0=0} = 0 \)

In this lemma, Assumptions (P) and (Q) guarantee that the initial surface \( x_0=0 \) is non-characteristic for \( L_\beta \), and Assumption (Ls) is (iv) itself. By this lemma we can express transport operators in the following form:

\[
\begin{cases}
1_{L_{2m-1,0},\beta} = \theta_\beta \cdot \tilde{\imath}_\beta \\
2_{L_{2m-1,0},\beta} = L_\beta + m_{2\beta}(x) \\
3_{L_{2m-1,0},\beta} = \theta_\beta \cdot \tilde{\imath}_\beta \\
4_{L_{2m-1,0},\beta} = \theta_\beta \cdot \tilde{\imath}_\beta
\end{cases}
\]

(\text{where } \tilde{\imath}_\beta = \theta_\beta \cdot L_\beta + m_{0\beta}(x') + \theta_\beta \cdot m_{3\beta}(x) \)
On the other hand, $^{1}_{L_{2m-2,1}}$ are functions which play an important role in the determination of the initial data of the transport equations (T.E.). We know the next property of $^{1}_{L_{2m-2,1}}$.

Lemma 5.2 $^{1}_{L_{2m-2,1}} |_{x_0=0} = m_0^\beta(x') = [0_\beta(0,x')P(0)(e_\beta)]^{2,\alpha}$. 0.

In the estimation of the right hand sides of the transport equations (T.E.) for the proof of the convergence of the formal solution, $^{3}_{L_{\nu,0}}$ plays an important role. We see the following expression of $^{3}_{L_{\nu,0}}$.

Lemma 5.3 $^{3}_{L_{\nu,0}} = \sum_{|\gamma|=2m-\nu-1} c_\alpha^\delta L_1(\xi,x,\xi_\beta x,\xi_\beta x') \frac{D_\gamma}{\delta_1} + (\text{operators of order } 2m-\nu-2) \mod. x_0$.
§.6 Determination of $u^{(q)}$, $g^{(q)}$, $v^{(q)}$ and $h^{(q)}$.

In §.3, we obtained the transport equations (T.E.), that is the first order system by which $u^{(q)}$, $g^{(q)}$, $v^{(q)}$ and $h^{(q)}$ are determined. First we remark these first order system are composed of the first order systems of the same form:

\[
(F.S.) \left\{ \begin{array}{l}
\partial_{p_{\beta}} \tilde{I}_{\beta} E + \partial_{p_{\beta}} \tilde{M}_{\beta} u = D(x) \\
(L_{\beta} + m_{2\beta}) u + \partial_{p_{\beta}} \tilde{M}_{\beta} g = E(x)
\end{array} \right.
\]

We note that for the solvability of this first order system of $u$ and $g$, the following condition is necessary:

\[ D(x) \big|_{x_{0}=0} = 0 \]

We shall use next notations.

\[ R[f(x)] = \frac{1}{x_{0}} ( f(x) - f(0,x') ) \]

\[ R_{\beta}[f(x)] = \frac{1}{\tilde{\sigma}_{\beta}(x)} ( f(x) - f(0,x') ) = \frac{1}{\tilde{\sigma}_{\beta}(x)} R[f(x)] \]

\[ S[f(x)] = f(0,x') \quad , \quad S_{\beta}[f(x)] = \frac{1}{m_{0\beta}(x')} f(0,x') \]

Now we apply this necessary condition of the solvability of the above first order system (F.S.) to the first order system
of the transport equations (T.E.). Taking account of lemma 5.2, we can write this necessary condition in the following form:

\[
\begin{align*}
    u^{(j+1)}_{\alpha+2m-2,\beta}(0,x^1) &= -S_\beta \left[ \sum_{k=2}^{m-2} \sum_{\nu=-1}^{2m-2k} 1_{L,\nu,k,\beta} \left( u^{(j+1)}_{\alpha+\nu,\beta} \right) + \\
    & \quad 3_{L,\nu,k,\beta} \left( u^{(j)}_{\beta+\nu,\beta} \right) + \sum_{\nu=-1}^{2m-2} 1_{L,\nu,0,\beta} \left( u^{(j)}_{\alpha+\nu,\beta} \right) + 3_{L,\nu,0,\beta} \left( g^{(j)}_{\alpha+\nu,\beta} \right) \\
    & \quad + \sum_{\nu=-1}^{2m-4} 1_{L,\nu,1,\beta} \left( u^{(j+1)}_{\alpha+\nu,\beta} \right) + 3_{L,\nu,1,\beta} \left( g^{(j+1)}_{\alpha+\nu,\beta} \right) \right] (j \geq 0),
\end{align*}
\]

\[
\begin{align*}
    v^{(j+1)}_{\alpha+2m-2,\beta}(0,x^1) &= -S_\beta \left[ \sum_{k=2}^{m-2} \sum_{\nu=-1}^{2m-2k} 1_{L,\nu,k,\beta} \left( v^{(j+1)}_{\alpha+\nu,\beta} \right) + \\
    & \quad 3_{L,\nu,k,\beta} \left( v^{(j)}_{\beta+\nu,\beta} \right) + \sum_{\nu=-1}^{2m-2} 1_{L,\nu,0,\beta} \left( v^{(j)}_{\alpha+\nu,\beta} \right) + 3_{L,\nu,0,\beta} \left( h^{(j)}_{\alpha+\nu,\beta} \right) \\
    & \quad + \sum_{\nu=-1}^{2m-4} 1_{L,\nu,1,\beta} \left( v^{(j+1)}_{\alpha+\nu,\beta} \right) + 3_{L,\nu,1,\beta} \left( h^{(j+1)}_{\alpha+\nu,\beta} \right) \right] (j \geq 0)
\end{align*}
\]

Under the necessary condition of the solvability of (F.S.) , (F.S.) is reduced to the next ordinary Fuchsian system of the first order:

\[
\begin{align*}
    \left\{ \begin{array}{l}
    \tilde{L}_\beta \bar{G} + \tilde{M}_\beta \bar{u} = R_\beta[D(x)] \\
    (L_\beta + m_{2\beta})\bar{u} + \delta_\beta \tilde{M}_\beta \bar{G} = E(x)
    \end{array} \right.
\]

Taking account of the remarks described above, We can reduce the systems of transport equations (T.E.) to the following form.
\[
\tilde{L}_\beta(g_{\alpha+2m-1,\beta}) + \tilde{M}_\beta(u_{\alpha+2m-1,\beta})
\]

\[
= R_\beta \left[ - \sum_{k=1}^{\frac{m}{2}} \sum_{\nu=-1}^{2m-2k} \left\{ 1L_{\nu,k,\beta}(u_{\alpha+\nu,\beta}) + 3L_{\nu,k,\beta}(g_{\alpha+\nu,\beta}) \right\} \right]
\]

\[
(i) \quad - \sum_{\nu=-1}^{2m-2} \left\{ 1L_{\nu,0,\beta}(u_{\alpha+\nu,\beta}) + 3L_{\nu,0,\beta}(g_{\alpha+\nu,\beta}) \right\}
\]

\[
(L_\beta + m_2\beta)(u_{\alpha+2m-1,\beta}) + \beta_\beta \tilde{M}_\beta(g_{\alpha+2m-1,\beta})
\]

\[
= - \sum_{k=0}^{m} \sum_{\nu=0}^{2m-2k} \left\{ 2L_{\nu,k,\beta}(u_{\alpha+\nu,\beta}) + 4L_{\nu,k,\beta}(g_{\alpha+\nu,\beta}) \right\}
\]

\[
(ii) \quad - \sum_{\nu=-1}^{2m-2} \left\{ 2L_{\nu,0,\beta}(u_{\alpha+\nu,\beta}) + 4L_{\nu,0,\beta}(g_{\alpha+\nu,\beta}) \right\}
\]

\[\text{(T.E.) is a linear first order system of } u_{\alpha+2m-1,\beta}, g_{\alpha+2m-1,\beta}, \text{ and } h_{\alpha+2m-1,\beta}.\]
Initial data which we impose on this (T.E.) is obtained in lemma 3.1 for $j=0$, and for $j \geq 1$ is obtained, as follows:

(I.D.)

\[
\begin{align*}
\mathbf{u}_{\alpha+2m-1, \beta}^{(j)}(0, x') &= \mathbf{S}_{\beta} \left[ \sum_{k=2}^{m} \sum_{\nu=-1}^{2m-2k} \left\{ \mathbf{L}_{\nu, k, \beta} \left( \mathbf{u}_{\alpha+1+\nu, \beta}^{(j)}, \mathbf{E}_{\alpha+1+\nu, \beta}^{(j+k-1)} \right) \right\} \right] \\
&\quad + \sum_{\nu=-1}^{2m-4} \left\{ \mathbf{L}_{\nu, 0, \beta} \left( \mathbf{u}_{\alpha+1+\nu, \beta}^{(j-1)}, \mathbf{E}_{\alpha+1+\nu, \beta}^{(j-1)} \right) \right\}(j \geq 1),
\end{align*}
\]

We remark above problems take the same form

\[
\begin{align*}
\mathbf{L}_{\beta}^g + \mathbf{M}_{\beta}^g u &= \mathbf{R}_\beta \left[ D(x) \right] \\
(\mathbf{L}_{\beta}^g + m_{2\beta}) u + \mathbf{E}_{\beta}^g \cdot \mathbf{M}_{\beta}^g \mathbf{g} &= \mathbf{E}(x)
\end{align*}
\]

with the initial data $u(0, x') = u_0(x')$, where $\mathbf{L}_{\beta}$, $\mathbf{L}_{\beta}$, $\mathbf{M}_{\beta}$, and $\mathbf{M}_{\beta}'$ have holomorphic coefficients and $m_{2\beta}$, $D(x)$ and $E(x)$, $u_0(x')$
are holomorphic in a neighbourhood of $0 \in \mathbb{C}^{n+1}$. It is known that for this Cauchy problem there exist unique holomorphic solutions $u(x)$ and $g(x)$ in a neighbourhood of $0 \in \mathbb{C}^{n+1}$.

By this fact, holomorphic coefficients $u_{\alpha, \beta}^{(q)}$, $g_{\alpha, \beta}^{(q)}$, $v_{\alpha, \beta}^{(q)}$ and $h_{\alpha, \beta}^{(q)}$ can be determined inductively.

First, we suppose all $u_{\delta, \beta}^{(p)}$, $g_{\delta, \beta}^{(p)}$, $v_{\gamma, \beta}^{(p)}$ and $h_{\delta, \beta}^{(p)}$ ($0 \leq p$, $\beta = 1, 2, \ldots, m$ for $\gamma \leq \delta + 2m - 2$, and $0 \leq p \leq j$, $\beta = 1, \ldots, m$ for $\gamma = d + 2m - 1$) are determined and then the right hand side of (T.E.) are known.

For $j = 0$, using lemma 3.1 and for $j \geq 1$, using (I.D.), we can solve the Cauchy problem for (T.E.) and then we can determine $u_{\alpha + 2m - 1, \beta}^{(j)}$, $g_{\alpha + 2m - 1, \beta}^{(j)}$, $v_{\alpha + 2m - 1, \beta}^{(j)}$, and $h_{\alpha + 2m - 1, \beta}^{(j)}$ inductively.

We shall prove that these coefficients $u_{\alpha, \beta}^{(q)}$, $g_{\alpha, \beta}^{(q)}$, $v_{\alpha, \beta}^{(q)}$, and $h_{\alpha, \beta}^{(q)}$ have a common existence domain and suitable estimates.
§.7 New coordinates and $h_{L_\nu,k,\beta}$.

To make (T.E.) easier in the treatment in the estimation of the coefficients, we introduce the new coordinates $y_\beta = (y_0, \beta, \ldots, y_n, \beta)$ ($\beta = 1, \ldots, m$), as follows.

We set

$$
\begin{cases}
  y_0, \beta = x_0 \\
y_i, \beta = \psi_i, \beta(x) \quad (i = 1, \ldots, n)
\end{cases}
$$

where $\psi_i, \beta(x)$ are defined as the solutions of the Cauchy problem:

$$
\sum_{j=1}^{n} p(j)(e_\beta)D_j \psi_i, \beta(x) = 0
$$

with the initial data $\psi_i, \beta(0, x') = x_1$.

Considering the transformation of the coordinates $x$ into the new coordinates $y_\beta$, we have

$$
\begin{cases}
  D_0 = D_0, \beta + \sum_{i=1}^{n} \psi_i, \beta(x)D_i, \beta \\
  D_\beta = \sum_{i=1}^{n} \psi_i, \beta(x)D_i, \beta
\end{cases}
$$

where $D_i, \beta = \frac{\partial}{\partial y_i, \beta}$ and $\psi_i, \beta(0, x') = \rho_i(x)$.

We are to see some properties of $h_{L_\nu,k,\beta}$ in terms of the new coordinates $y_\beta$. $L_\beta$ and $M_\beta$ are expressed as follows:
\[
\begin{align*}
L_\beta &= a_\beta[y_\beta] D_0,\beta + y_0,\beta \left\{ \sum_{i=0}^{n} a_{\beta, i}[y_\beta] D_i,\beta \right\} + c_\beta[y_\beta], \\
M_\beta &= b_\beta[y_\beta] y_0,\beta D_0,\beta + y_0,\beta \left\{ \sum_{i=0}^{n} b_{\beta, i}[y_\beta] D_i,\beta \right\} + c_\beta'[y_\beta],
\end{align*}
\]

where \( a_\beta, a_{\beta, i}, b_\beta, b_{\beta, i}, c_\beta \) and \( c_\beta' \) are holomorphic functions of \( y_\beta \) in a neighbourhood of the origin. And so we have the next lemma with respect to the transport operators \( h_{L_{2m-1,0,\beta}} \).

**Lemma 7.1** \( h_{L_{2m-1,0,\beta}} \) are expressed, in terms of the new coordinates \( y_\beta \), as follows.

(i) \( h_{L_{2m-1,0,\beta}} = y_0,\beta b_\beta[y_\beta] D_0,\beta + y_0,\beta \sum_{i=0}^{n} b_{\beta, i}[y_\beta] D_i,\beta \) \\
(ii) \( h_{L_{2m-1,0,\beta}} = a_\beta[y_\beta] D_0,\beta + y_0,\beta \sum_{i=0}^{n} a_{\beta, i}[y_\beta] D_i,\beta + c_2,\beta[y_\beta] \) \\
(iii) \( h_{L_{2m-1,0,\beta}} = c_\beta[y_\beta] \left\{ a_\beta'[y_\beta] \{ y_0,\beta D_0,\beta + 1 \} + y_0,\beta \sum_{i=0}^{n} a_{\beta, i}[y_\beta] D_i,\beta \right\} \) \\
(iv) \( h_{L_{2m-1,0,\beta}} = y_0,\beta b_\beta[y_\beta] D_0,\beta + y_0,\beta \sum_{i=0}^{n} b_{\beta, i}[y_\beta] D_i,\beta \) \\
where \( a_\beta[y_\beta] = 2 (P(0)(e_\beta))^2 \mathcal{O}_\beta(0, x') \approx 0 \), \\
a_\beta'[y_\beta] = 2 m_0 \beta (x') \approx 0, \]

\[ b_\beta[y_\beta] = P_\beta( x' ) P(0)(e_\beta) \approx 0. \]
Among other operators $h_{L, k, \beta}$, we see $h_{L, 0, \beta}$ and $3_{L, 1, \beta}$.

Taking account of (F.5) and lemma 5.3, we have.

Lemma 7.2 Exchanging $x$ for $y_{\beta}$, we get the following representations of $h_{L, 0, \beta}$ ($h=1, 2, 3, 4$), $3_{L, 0, \beta}$ and $3_{L, 1, \beta}$.

(i) The principal part of $h_{L, 0, \beta} \left[ y_{\beta}, D_{y_{\beta}} \right]$ equal to $h_{L, 0, \beta} \left[ y_{\beta}, D_{y_{\beta}} \right]$, can be expressed in the form:

$$H \left[ y_{\beta}, D_{y_{\beta}} \right] D_{0, \beta}^2 + y_{0, \beta} K \left[ y_{\beta}, D_{y_{\beta}} \right] D_{0, \beta} + y_{0, \beta}^2 J \left[ y_{\beta}, D_{y_{\beta}} \right].$$

(ii) The principal part of $h_{L, 0, \beta} \left[ y_{\beta}, D_{y_{\beta}} \right]$ and the principal part of $3_{L, 1, \beta} \left[ y_{\beta}, D_{y_{\beta}} \right]$ can be expressed in the form:

$$y_{0, \beta}^2 K \left[ y_{\beta}, D_{y_{\beta}} \right] D_{0, \beta} + y_{0, \beta} J \left[ y_{\beta}, D_{y_{\beta}} \right].$$

(iii) The principal part of $3_{L, 0, \beta} \left[ y_{\beta}, D_{y_{\beta}} \right]$ can be expressed in the form:

$$y_{0, \beta}^2 K \left[ y_{\beta}, D_{y_{\beta}} \right] D_{0, \beta} + y_{0, \beta}^3 J \left[ y_{\beta}, D_{y_{\beta}} \right].$$

(iv) $3_{L, 0, \beta} \left[ y_{\beta}, D_{y_{\beta}} \right] = I \left[ y_{\beta}, D_{y_{\beta}} \right] D_{0, \beta} + (\text{operators of order } 2m-\gamma-2) \mod. x_0.$

where $I \left[ y_{\beta}, D_{y_{\beta}} \right]$ is a linear partial differential operator of order $2m-\gamma-2$. 
§.8 Convergence of the formal solution.

After the construction of the formal solution which has been done in §.3 and §.7, it remains for us to verify the convergence of the formal solution. To do so, we studied how the holomorphic coefficients were determined, that is, in §.6 they were determined successively by solving the Cauchy problem for the systems of the transport equations \( (T.E.) \) with the initial data by (I.D.) or lemma 3.1. Moreover we study the systems of the transport equations \( (T.E.) \), especially, transport operators precisely and other operators appearing in right hand sides of the transport equations, in §.5 and §.7. In this section, we are to prove the convergence of the formal solution by the majorization method. To do so, we begin it with the introduction of a family of the scale functions \( \{ \phi_j(t, s) \} \).

We define \( \phi_j(t, s) \) as follows.

\[
\phi_j(t, s) = \partial_s^j \phi(t, s)
\]

, where \( \phi(t, s) = (\sqrt{2(R-s)} - \rho t)^{-1} \) \( (\rho > 1) \).
We put \( R[f(t,s)] = t^{-1}(f(t,s) - f(0,s)) \). The following proposition can be easily verified.

**Proposition 6.1** \( \{ \Phi_j(t,s) \} \) have following properties.

1. \( \partial_t \Phi_j \gg t^j \Phi_{j+1} \)
2. \( \partial_t^2 \Phi_j \gg t^j \Phi_{j+1} \)
3. \( \partial_t^2 \Phi_j \gg t^j \Phi_{j+2} \)
4. \( (\partial_t + 1) \Phi_{j+1} \gg \partial_t \Phi_j \), \( R[\partial_t \Phi_j] \), \( \partial_t^2 \Phi_j \)
5. \( 8 \partial_t \Phi_{j+1} \gg \partial_t^3 \Phi_j \), \( R[\partial_t^3 \Phi_j] \)
6. \( 2 \Phi_{j+1}(0,s) \gg \partial_t^2 \Phi_j(0,s) \)
7. \( 4 \partial_t^2 \Phi_{j+1}(0,s) \gg \partial_t^4 \Phi_j(0,s) \)
8. \( (R^l - s)(R^n - t) \partial_t^k \Phi_j \ll (R^l - s)(R^n - t) \partial_t^k \Phi_j \left( t \leq 1, R \leq 1, l \right) \)

The majoration method applied to the proof of the convergence of the formal solution is based on the next proposition.

**Proposition 6.2** For the Cauchy problem

\[
\begin{cases}
(x_0 D_0 + 1)g + \left( \sum_{i=0}^{n} x_0 \alpha_i D_i + x_0 \gamma_1 \right) g + \\
\sum_{i=0}^{n} x_0 \beta_i D_i + \gamma_2 u = T(x)
\end{cases}
\]
\[ \begin{align*}
D_0 u + \left( \sum_{i=0}^{n} x_0 \alpha_i D_i + \gamma_3 \right) u + x_0 \left( \theta_0 D_0^+ \sum_{i=0}^{n} x_0 \beta_i D_i + \gamma_4 \right) g &= T(x),
\end{align*} \]

with the initial data \( u(0, x') = u_0(x') \),

where \( \alpha_i, \beta_i, \gamma_i, \delta_i, S, T \) and \( u_0 \) are holomorphic functions in a neighbourhood of the origin, there exist unique holomorphic solutions \( u(x) \) and \( g(x) \) in a neighbourhood of the origin.

Moreover, assuming \( \alpha_i, \beta_i, \gamma_i, \delta_i, S, T, u_0 \ll \), we can verify that \( u(x) \ll \tilde{u}(x) \) and \( g(x) \ll \tilde{g}(x) \), if \( \tilde{u}(x) \) and \( \tilde{g}(x) \) satisfy

\[ \begin{align*}
(x_0 D_0 + 1) \tilde{g} &\gg \left( \sum_{i=0}^{n} x_0^2 \alpha_i + x_0 \beta_i \right) \tilde{g} + \\
&\quad (\tilde{\gamma}_0 D_0^+ \sum_{i=0}^{n} x_0 \tilde{\beta}_i D_i + \tilde{\gamma}_2) \tilde{u} + \tilde{S},
\end{align*} \]

\[ \begin{align*}
D_0 \tilde{u} &\gg \left( \sum_{i=0}^{n} x_0 \tilde{\delta}_i + \tilde{\delta}_3 \right) \tilde{u} + \\
&\quad x_0 \left( \tilde{\gamma}_0 D_0^+ \sum_{i=0}^{n} x_0 \tilde{\beta}_i D_i + \tilde{\gamma}_4 \right) \tilde{g} + \tilde{T},
\end{align*} \]

\( \tilde{u}(0, x') \gg \tilde{u}_0(x') \).

(For the proof see \[10\].)

From these propositions and lemmata obtained in the preceding
sections, we have reached the following proposition.

**Proposition 6.3** There exist positive constants $A, B, C, D, E, F, K, L, R$ and $\mathcal{F}$ independent of $\alpha$ and $j$, such that

1. $u^{(j)}_{\alpha, \beta} \ll \frac{A^j}{L^j} \partial_t^{\alpha+j+1}(y_0, \beta, y_1^j)$
2. $e^{(j)}_{\alpha, \beta} \ll \frac{B^j}{L^j} \partial_t^{\alpha+j+1}(y_0, \beta, y_1^j)$
3. $\nu^{(j)}_{\alpha, \beta} \ll \frac{C^j}{L^j} \partial_t^{\alpha+j+1}(y_0, \beta, y_1^j)$
4. $h^{(j)}_{\alpha, \beta} \ll \frac{D^j}{L^j} \partial_t^{\alpha+j+1}(y_0, \beta, y_1^j)$
5. $(u^{(0)}_{\alpha, \beta} + h^{(0)}_{\alpha-1, \beta})(0, x^1) \ll \mathcal{F}^j \partial_t^{\alpha+j+1}(0, x^1)$
6. $u^{(0)}_{\alpha, \beta}(0, x^1) \ll \mathcal{F}^j \partial_t^{\alpha+j+1}(0, x^1)$

where $y_1^j = \sum_{i=1}^n y_i, \beta$ and $x^1 = \sum_{i=1}^n x_i$.

From this proposition, we know that holomorphic coefficients $u^{(j)}_{\alpha, \beta}, e^{(j)}_{\alpha, \beta}, \nu^{(j)}_{\alpha, \beta}$ and $h^{(j)}_{\alpha, \beta}$ have a common existence domain that is a neighbourhood of $0 \in \mathbb{C}^{n+1}$ and the estimates $|u^{(j)}_{\alpha, \beta}|, |e^{(j)}_{\alpha, \beta}|, |\nu^{(j)}_{\alpha, \beta}|, |h^{(j)}_{\alpha, \beta}| < C(\alpha+j+1)! \frac{K^j}{L^j}$ in this common existence domain, where $C, L$ and $K$ are positive constants independent of $\alpha$ and $j$. On the other hand, in Appendix we have
the estimates $|X^{(j)}_{\alpha, \beta}|, |Y^{(j)}_{\alpha, \beta}| \leq C_K \left( \frac{\nu_{\alpha+j+1}}{(\alpha+j+1)!} r^{\alpha+j}(\log r)^j C^{\alpha+j} \right)$ on any compact set $K$ in the universal covering space $\widetilde{D_r \setminus K}$ over $D_r \setminus K$, where $\alpha > 0$, and $C_K$ is a constant which depends only on $K$. Thus choosing $r > 0$ sufficiently small, we can prove the convergence of the formal solution on $\widetilde{D_r \setminus K}$. Uniqueness of the solution is due to the Cauchy-Kobalevskaya theorem. We remark $u(x)$ does not ramify on $x_0=0 \wedge x_1=0$.

Correction. In §6 of the author's preceding paper [11], we must replace $\phi_{\alpha}(z, \zeta, y)$ by the new $\phi_{\alpha}(z, \zeta, y) = \phi_{\alpha}(z, \zeta + y, 0)$, (where $\phi_{\alpha}$ of the right hand side of this identity is one defined in §6 of [11]). With this replacement of the old $\phi_{\alpha}$ by the new $\phi_{\alpha}$, we have only to replace $[R'-(2/3)] [R^n-fz-y]$ by $[R'-(2/3)(\zeta+y)][R^n-\rho z]$ in (5) of Proposition 6.1 and

$M(R'-(2/3)y_1, \beta)^{-1}(R^n-y_0, \beta - \sum_{\nu=2}^{n} y_{\nu, \beta})^{-1}$ by $M(R'-(2/3)\sum_{\nu=1}^{n} y_{\nu, \beta})^{-1}$ taking account of the fact that the principal part of $3^0_{\nu, \beta}$ is expressed in the form $y_0, \rho \left[ y_\beta, D_\beta \right]_{D_0, \rho}^+$. 

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$y_{0,\beta}^2 J[y_\beta, D_\psi_0]$, and that the principal part of $h^0_{L,\alpha}(h=1,2,4)$ are expressed in the form $K[y_\beta, D_\psi_0]D_\psi_0 + y_{0,\beta}J[y_\beta, D_\psi_0]$.

(for the proof of this fact, see the proof of lemma 7.2 of this paper).
§.9 Proof.

Proof of Lemma 1.1 We differentiate by c both sides of
\[ P_c(U_{\alpha+q})=0 \] q times and then we get the first relation inductively

We prove other relations in the similar way.

Proof of Proposition 4.1 Consider the Cauchy problem
\[ L(x,\xi_x)=P(x,\xi_x)^2 - x_0^2 q(x,\xi_x)=0 \]
with the initial data \( \phi(0,x')=x_1 \)

From this, we have the new Cauchy problem
\[ (*) P(x,\xi_x)=x_0^2 Q(x,\xi_x) \]
with the initial data \( \phi(0,x')=x_1 \)

Taking account of (P)(ii),(iii) and (Q)(ii), Cauchy-Kowalevskaya
theorem and implicit function theorem guarantee that this Cauchy
problem (*) has \( 2m \) solutions \( \psi_{\beta}(x) \) which are holomorphic.

Differentiating the equation (*) by \( x_0 \) and restricting it on
the initial surface, we have \( \phi_{x_0}^+ (0,x')=\lambda_\beta (x') \) and \( \phi_{x_0}^+ (0,x') \)
\[ =\left[ p(0) e_\beta \right]^{-1} \left( \right) \left( \right) \sum_{i=1}^n \lambda_\beta (x') p(i)(e_\beta) \}

Then we
have \( f_\beta(x) = \frac{1}{2}(\psi_\beta^+(x) + \psi_\beta^-(x)) = x_0 + x_0 \lambda_\beta(x') + x_0^2 \gamma_\beta(x) \) and

\[\psi_\beta^+(x) = (\psi_\beta^+(x) - \psi_\beta^-(x))^{1/2} = x_0 \sigma_\beta(x)\]

where \( \gamma_\beta(0,x') = \left[ \frac{\sigma(\epsilon_\beta)}{p(0)(\epsilon_\beta)} \right]^{1/2} \)

\[\{ p_{x_0}^j(\epsilon_\beta) + \sum_{i=1}^n \lambda_i x_i \} p(1)(\epsilon_\beta) \]

and \( \sigma_\beta(0,x') = \left[ \frac{\sigma(\epsilon_\beta)}{p(0)(\epsilon_\beta)} \right]^{1/2} \)

Proof of Lemma 5.1

\( (i) \quad S[L_\beta] = \left[ \sum_{i=0}^n L_1^0(j,0,x', \epsilon_\beta x', \beta x')] D_j \right] = \sum_{i,j=0}^n \epsilon_\beta x_i p_{x_0}^j(\epsilon_\beta) D_j x_0 = 0 \)

\( = \sigma_\beta(0,x') \sum_{i=0}^n L^0(0,j)(\epsilon_\beta) D_j \) and \( L(0,j)(\epsilon_\beta) = 2 p(0)(\epsilon_\beta) p(j)(\epsilon_\beta) \)

\( (ii) \quad M_\beta = \sum_{j=0}^{m-1} \left[ \sum_{i=0}^n L_2^0(i,x, \epsilon_\beta x', \beta x')] \epsilon_\beta^{2i} D_j \right] \)

\( = \sum_{j=0}^n L_0^0(j,x, \epsilon_\beta x', \beta x') D_j \quad \text{mod. } \epsilon_\beta^2 \)

\( = \sum_{j=0}^n L_0^0(j,x, \beta x') D_j \quad \text{mod. } \epsilon_\beta^2 \)

\( = \sum_{j=0}^n P(x, \beta x') P(j)(x, \beta x') D_j \quad \text{mod. } \epsilon_\beta^2 \)

On the other hand by \( S[P(x, \beta x')] = S[P(x, \beta x')] = 0 \), we can put \( P(x, \beta x') = x_0 p_\beta(x) \). We proved (ii).

\( (iii) \quad \text{Note } S\left[ \frac{1}{\epsilon_\beta} L_{2m-1,0,\beta} \right] = S\left[ \frac{1}{\epsilon_\beta} (N_{c,\beta}^* + R_{i,\beta}^*) \right] \)

\( N_{c,\beta}^* = \sum_{i=0}^m L_2^0 i(-1)(x, \epsilon_\beta x', \beta x') c_i \epsilon_\beta^{2i-2} \)

\( = c L_1^0(x, \epsilon_\beta x', \beta x') \quad \text{mod. } \epsilon_\beta^2 \)

On the other hand from the eikonal equation \( 2L_{2m,0,\beta} = \)
\[ L_1(x, \theta x, \rho x) + \sum_{i=1}^{m-1} L_{2i+1}(x, \theta x, \rho x) \theta^{2i} = 0, \quad \text{we get} \]
\[ L_1(x, \theta x, \rho x) = 0 \mod \theta^2. \quad \text{R} \begin{array}{l} = \sum_{i=0}^{m} L_{2i}(x, \theta x, \rho x) 2^i \theta^{2i-1} \\
= L_2(x, \theta x, \rho x) 2 \theta, \mod \theta^2 \end{array} \]
\[ \sum_{i=0}^{n} \theta^{i} \theta x_{i, j} \theta x_{i, j} = 0, \quad \text{lead to (iii).} \]

(iv) \[ S[1L_{2m-1,0}, \beta] = S[4L_{2m-1,0}, \beta] \]
\[ = S[12 \sum_{i=0}^{n} (\theta x_{i, j} L_0^0(i, j)(x, \theta x, \rho x)] + c_\beta L_2^0(x, \theta x, \rho x) + L_s^0(x, \theta x, \rho x) \]
\[ = S[12 \sum_{i=0}^{n} (\theta x_{i, j} L_0^0(i, j)(x, \theta x, \rho x)] + L_s^0(x, \theta x, \rho x) \]
\[ = \left( \sum_{r=1}^{n} (\theta x_{r, j} L_0^0(i, j)(x, \theta x, \rho x)] + c_\beta L_s^0(x, \theta x, \rho x) \right) \]
\[ = \frac{1}{2} \left( \sum_{r=1}^{n} \theta x_{r, j} L_0^0(i, j)(x, \theta x, \rho x) + c_\beta L_s^0(x, \theta x, \rho x) \right) \]
\[ = 12 L_0^0(x, \theta x, \rho x) + 2 \sum_{j=1}^{n} \theta x_{j} \theta x_{j} \theta x_{j} \]
\[ + L_s^0(\rho) = 0. \quad \text{This is (L}_s^0) (i) \text{ itself.} \]

Proof of Lemma 5.2
\[ S[1L_{2m-2,1, \beta}] = S[1L_{2m-2,1, \beta}] = S[12 \sum_{i=0}^{n} \theta x_{i, j} L_0^0(i, j)(x, \theta x, \rho x) \]
\[ = m_0^0(\theta x) \neq 0. \]

Proof of Lemma 5.3 (F.5) and (F.2) lead to Lemma 5.3.

Proof of Lemma 7.2

(i) \[ L_{\nu}^0 = 2^0 L_{\nu}^0 = \sum_{|\alpha| = 2m-\nu} L_0^0(\alpha)(x, \theta x, \rho x) D_{\alpha}^x \quad \mod x^2_0 \]
\[ \equiv \sum_{|\alpha| = 2m-\nu} L_0^0(\alpha) D_{\alpha}^x \quad \mod x^2_0. \]
On the other hand, by Leibniz's formula

\[ L^{(\alpha)}(x, \xi) = \sum_{|\beta+\gamma|=2m-\nu} \frac{\alpha!}{\beta! \gamma!} P^{(\beta)}(x, \xi) P^{(\gamma)}(x, \xi) \mod. x^2 \]

we get

\[ \sum_{|\beta+\gamma|=2m-\nu} \frac{\alpha!}{\beta! \gamma!} P^{(\beta)}(x, \xi) \frac{D^\beta}{\beta!} \left\{ \sum_{|\delta|=2m-\nu-k} P^{(\gamma)}(x, \xi) \frac{D^\gamma}{\gamma!} \right\} \mod. x^2 \]

\[ = \text{the principal part of} \]

\[ \sum_{k=0}^{2m-k} \frac{\alpha!}{\beta! \gamma!} P^{(\beta)}(x, \xi) \frac{D^\beta}{\beta!} \left\{ \sum_{|\delta|=2m-\nu-k} P^{(\gamma)}(x, \xi) \frac{D^\gamma}{\gamma!} \right\} \mod. x^2 \]

Therefore it suffices to prove that the principal part of the operator obtained by exchanging \( x \) for \( y_\beta \) from

\[ \sum_{|\alpha|=k} P^{(\alpha)}(x, \xi) \frac{D^\alpha}{\alpha!} \left( \frac{\xi}{\beta} - \lambda_\beta(x, \xi) \right) \] is expressed in the form \( K(y_\beta, D_{y_\beta}) y_0, y_\beta \)

\[ + y_0, y_\beta J(y_\beta, D_{y_\beta}) \]. On the other hand, \( P(x, \xi) = R(x, \xi) r(x, \xi) \)

\( (r(x, \xi) = \xi_0 - \lambda_\beta(x, \xi) \) leads to

\[ P^{(\alpha)}(x, \xi) = \sum_{|\alpha|=k} \frac{\alpha!}{\gamma!} R^{(\gamma)}(x, \xi) r^{(\gamma)}(x, \xi). \] So, we get

\[ \sum_{|\alpha|=k} \frac{\alpha!}{\gamma!} P^{(\alpha)}(x, \xi) \frac{D^\alpha}{\alpha!} = \text{the principal part of} \]

\[ \left\{ \sum_{|\gamma|=j} R^{(\gamma)}(x, \xi) \frac{D^\gamma}{\gamma!} \right\} \left\{ \sum_{|\delta|=k-j} r^{(\delta)}(x, \xi) \frac{D^\delta}{\delta!} \right\} \]

And so, it is sufficient for us to prove that the principal part of the operator obtained by exchanging \( x \) for \( y_\beta \) from
\[
\sum_{|\mathbf{k}|=k} r^{(\mathbf{\ell})}(x, y) \frac{D^{\mathbf{\ell}}}{n!} \quad (k=0, \ldots, m, y = \rho_{x}) \text{ is expressed in the form } K(y, D)D_{y}\beta_{0}+y_{0}J(y_{\rho}, D_{y}) \quad . \text{ On the other hand, } r(x, \rho_{x}) = r(e_{\rho}) \mod. x_{0} .
\]

For k=0, by r(e_{\rho})=0, we get r(x, \rho_{x})=y_{0}J(y_{\rho}) .

For k=1, we get \[\sum_{i=0}^{n} r^{(i)}(e_{\rho})D_{i} = D_{y}\beta_{0} \quad .\]

For k\geq 2, Euler's identity with respect to homogeneous polynomials leads to \[r^{(\mathbf{\ell})}(e_{\rho}) = 0 \quad .\]

We proved (i) and then we can prove (ii),(iii) and (iv) in the similar way.

Proof of Proposition 6.3 We use M as a suitable positive constant in this proof. Let all coefficients of \[h_{L_{\nu}, k, \rho} < M(R' - y_{\rho}^{1})^{-1} R'' - \rho y_{0, \rho} \quad . \text{ We prove this proposition by induction of } q \text{ and } j . \text{ Assume that these estimates are valid for } q=1 \ldots , n+2m-2 \text{ and } j\geq 0 \quad . \text{ First we treat (5).}
\]

Lemma 3.1 leads to \[S_{L_{w}+2m-1, k, \rho}^{(0)} + h_{\alpha}^{(0)} \quad \left|_{x_{0}=0} \right\].

\[\sum_{\mu=1}^{2m-2} \sum_{\mu=1}^{2m-2} \mu_{\mu, \beta}^{(0)}(u_{\alpha+2m-1-\mu, \beta}^{(0)} + h_{\alpha+2m-2-\mu, \beta}^{(0)}) + \text{ (rest) } \left|_{x_{0}=0} \right\].
Making use of the fact that $D_0^\mu = S_\beta(y_\beta, D_y) D_0, \beta + T_\beta(y_\beta, D')$ where $S_\beta$ is a linear partial differential operator of order $\mu - 1$ in $D_y$ and $T_\beta$ is a linear partial differential operator of order $\mu$ in $D_y^p = (D_y, \cdots, D_y^n)$, we have

$$S\left[ D_0^\mu(u(0) + h(0)) + h(0) + h(0) \right] =$$

$$S(0, y_\beta^1, D_y) D_0, \beta (u(0) + h(0)) + T_\beta(y_\beta, D_y) (u(0) + h(0)) \Big|_{x_0 = 0} = 0 \ .$$

To the former part of the right hand side of this identity, we apply the estimates of (1), (4) of the assumption of the induction, and to the latter part of the right hand side of this identity, we apply the estimates of (5) of the assumption of the induction. Then we have

$$S[u(0) + h(0)] < K^{2n+3} M(KD+A+B+C+D) \delta + 2M+1 \ .$$

For (5), it suffices that the inequality

(i) $EK^2 > MKD+(A+B+C+D)M$

is valid.
Secondly we treat (6). We restrict the transport equation
\[ \sim L_\beta(h_{x+2m-2,\beta}(0)) = -N_\beta(v_{x+2m-2,\beta}(0)) + R_\beta \cdots \] on the initial surface, and then we have
\[ h_{x+2m-2,\beta}(0, x') \ll K^{x+2m-2} \left[ C_t \phi_{x+2m+1} + R[\phi_t] \right] + R[\delta_t \phi_{x+2m+1} - 1] \mid_{x=0} . \]

On the other hand, \( S[u_{x+2m-1,\beta}(0)] = S[u_{x+2m-1,\beta} + h_{x+2m-2,\beta}(0)] - S[h_{x+2m-2,\beta}] \).

To the former part of the right hand side of this identity, we apply the estimates of (5) and to the latter part, we apply the estimates obtained above. For (6), it suffices that the inequality
\[ (ii) \quad F > (E+D)M \]
is valid.

Thirdly for (1) and (2), by (T.E.) it is known, it suffices that the following system of the inequalities (iii) is valid.
\[ (iii) \quad \left\{ \begin{array}{c}
F_B(1 - \frac{M}{f})K > M(A+B) + \frac{M}{LK}(A+B)
\end{array} \right\} \]
Lastly we treat (3) and (4) lead to

\[ v_{\alpha+2m-1}^{(0)}(0, x') \ll EK^{\alpha+2m-1}x_{\alpha+2m-1+2+1}(0, x', \#) \]

For (3) and (4), it suffices that the following system of the inequalities (iv) is valid.

(iv) \[
\begin{align*}
\rho D(1 - \frac{M}{F})K &> M(C+D) + \frac{M}{LK}(C+D) \\
C(1 - \frac{M}{F}) &> M\frac{K}{F}D + \frac{M}{LK}(C+D) \\
C &> E
\end{align*}
\]

For \( j \geq 1 \), we treat them in the similar way, but lemma 3.1 is to be replaced by (I.D.). So the following system of inequalities must be satisfied.

(v) \[
\begin{align*}
A &> M(K^{-2}A+K^{-1}B+L^{-1}B) \\
C &> M(K^{-2}C+K^{-1}D+L^{-1}D)
\end{align*}
\]

Our problem is reduced to the problem of the existence of positive constants \( A, B, C, D, E, F, K, L, R \) and \( P \) which satisfy the system of inequalities (i), (ii), (iii), (iv) and (v). In fact,
In fact, making $P$, $K$, and $L$ sufficiently large and $R$ is sufficiently small, we can make these constant satisfy the system of inequalities (i)(ii)(iii)(iv) and (v).
Appendix

In this appendix, we study the multi-valuedness or the singularity of the auxiliary functions $U_\alpha$, $V_\alpha$, $U^{(j)}_\alpha$, $V^{(j)}_\alpha$, $X_\alpha$, $Y_\alpha$, $X^{(j)}_\alpha$ and $Y^{(j)}_\alpha$ which are the analytic functions of $x$, $p$, $\lambda$ and $c$. We use as independent variables $(t,x) \in \mathbb{C}^2$ instead of $(\theta, \phi) \in \mathbb{C}^2$ in this appendix and treat these auxiliary functions in the more general form.

To be precise, we put

$$P_c = \partial^2_t - t^{2b} \partial^2_x - ct^{b-1} \partial_x, \quad (b \geq 0 \text{ is an integer})$$

Newly we introduce $U_\alpha$ and $V_\alpha$ as the solutions of the following Cauchy problems respectively:

$$P_c U_\alpha = (\partial^2_t - t^{2b} \partial^2_x - ct^{b-1} \partial_x) U_\alpha = 0$$

with the initial data

$$U_\alpha(0,x) = f_\alpha(x) \quad U_{\alpha t}(0,x) = 0$$

$$P_c V_\alpha = (\partial^2_t - t^{2b} \partial^2_x - ct^{b-1} \partial_x) V_\alpha = 0$$

with the initial data

$$V_\alpha(0,x) = 0$$
\[ V_{\alpha t}(0,x) = f_{\alpha}(x) \]

We denote \( \frac{A}{2(b+1)} \) by \( A^* \).

The following explicit representation are known, (see\([1]\)).

\[
U_{\alpha}(t,x) = \frac{\xi^\alpha}{\Gamma(\alpha+1)} F(-\alpha, b^*+c^*, 2b^*, z) \]
\[
= \frac{\eta^\alpha}{\Gamma(\alpha+1)} F(-\alpha', b^*-c^*, 2b^*, \zeta) ,
\]
\[
V_{\alpha}(t,x) = \frac{\xi^\alpha t}{\Gamma(\alpha+1)} F(-\alpha, c^*-b^*+1, 1+2^* z) \]
\[
= \frac{\eta^\alpha t}{\Gamma(\alpha+1)} F(-\alpha', 1-b^*-c^*, 1+2^*, \zeta),
\]

where \( \xi = x - \frac{1}{b+1}t^{b+1} \) and \( \eta = x + \frac{1}{b+1}t^{b+1} \) are so-called characteristic coordinates used instead of \( \psi^- \) and \( \psi^+ \) respectively in this appendix and \( z = 1 - \frac{\eta}{\xi} , \zeta = 1 - \frac{\xi}{\eta} \).

Now we define \( U_{\alpha}^{(j)}(t,x) \) and \( V_{\alpha}^{(j)}(t,x) \) as the solutions of the following Cauchy problems respectively:

For \( j=0 \), we set \( U_{\alpha}^{(0)}(t,x) = U_{\alpha}(t,x) \) and \( V_{\alpha}^{(0)}(t,x) = V_{\alpha}(t,x) \).

For \( j \geq 1 \),

\[ P_{c} U_{\alpha}^{(j)} = t^{b-1}U_{\alpha}^{(j-1)} \]

with the null initial data \[
\begin{align*}
U_{\alpha}^{(j)}(0,x) &= 0 \\
U_{\alpha t}^{(j)}(0,x) &= 0
\end{align*}
\]
and \( p_c \alpha (j) = t^{b-1} \alpha (j-1) \)

with the null initial data \[
\begin{align*}
V(0,x) &= 0 \\
V_t(0,x) &= 0
\end{align*}
\]

And then we define \( X_\alpha (t,x), Y_\alpha (t,x), X_\alpha (j)(t,x) \) and \( Y_\alpha (j)(t,x) \) as follows.

\[
\begin{align*}
X_\alpha (t,x) &= \partial_\alpha U_\alpha (t,x) \quad \text{and} \quad X(j)(t,x) = \partial_\alpha U(j)(t,x) \\
Y_\alpha (t,x) &= \partial_\alpha V_\alpha (t,x) \quad \text{and} \quad Y(j)(t,x) = \partial_\alpha V(j)(t,x)
\end{align*}
\]

Therefore \( X_\alpha \) and \( Y_\alpha \) satisfy the following Cauchy problem respectively.

\[
\begin{align*}
P_c X_\alpha &= 0 \quad \text{and} \quad X_\alpha (0,x) = k_\alpha (x) \quad \text{and} \quad X_{\alpha t}(0,x) = 0 . \\
P_c Y_\alpha &= 0 \quad \text{and} \quad Y_\alpha (0,x) = 0 \quad \text{and} \quad Y_{\alpha t}(0,x) = k_\alpha (x) .
\end{align*}
\]

\( X(j) \) and \( Y(j) \) satisfy the following Cauchy problem respectively for \( j \geq 1 \).

\[
\begin{align*}
P_c X(j)(\alpha) &= t^{b-1} X(j-1)(\alpha) \quad \text{and} \quad X(j-1)(0,x) = X(j-1)(0,x) = 0 . \\
P_c Y(j)(\alpha) &= t^{b-1} Y(j-1)(\alpha) \quad \text{and} \quad Y(j-1)(0,x) = Y(j-1)(0,x) = 0 .
\end{align*}
\]
Remark that these auxiliary functions depend on $c$ analytically but we omit $c$ for brevity. For example we write $U_\alpha(t,x)$ instead of $U_\alpha(t,x,c)$ in this appendix.

The explicit representations of $U_\alpha$ and $V_\alpha$ lead to the explicit representations of $U^{(j)}_\alpha$, $V^{(j)}_\alpha$, $X^{(j)}_\alpha$ and $Y^{(j)}_\alpha$, as follows.

Lemma A.1

\[
U^{(j)}_\alpha(t,x) = \frac{1}{j! \partial \mathcal{C}_\alpha} \frac{\partial^j}{\partial \mathcal{C}_\alpha} U^{(j)}_\alpha(t,x) = \frac{1}{j! \partial \mathcal{C}_\alpha} \frac{\partial^j}{\partial \mathcal{C}_\alpha} F(-j, b^* + c^*, 2b^*, z),
\]

\[
V^{(j)}_\alpha(t,x) = \frac{1}{j! \partial \mathcal{C}_\alpha} \frac{\partial^j}{\partial \mathcal{C}_\alpha} V^{(j)}_\alpha(t,x) = \frac{1}{j! \partial \mathcal{C}_\alpha} \frac{\partial^j}{\partial \mathcal{C}_\alpha} F(-j, c^* - b^* + 1, 1 + 2b^*, z),
\]

\[
X^{(j)}_\alpha(t,x) = \frac{1}{j! \partial \mathcal{C}_\alpha} \frac{\partial^j}{\partial \mathcal{C}_\alpha} X^{(j)}_\alpha(t,x) = \frac{1}{j! \partial \mathcal{C}_\alpha} \frac{\partial^j}{\partial \mathcal{C}_\alpha} X^{(j)}_\alpha(t,x)
\]

\[
Y^{(j)}_\alpha(t,x) = \frac{1}{j! \partial \mathcal{C}_\alpha} \frac{\partial^j}{\partial \mathcal{C}_\alpha} Y^{(j)}_\alpha(t,x) = \frac{1}{j! \partial \mathcal{C}_\alpha} \frac{\partial^j}{\partial \mathcal{C}_\alpha} Y^{(j)}_\alpha(t,x)
\]

Lemma A.1 shows that $U^{(j)}_\alpha$, $V^{(j)}_\alpha$, $X^{(j)}_\alpha$ and $Y^{(j)}_\alpha$ being represented as the derivatives by $\alpha$ and $c$ of $U_\alpha$ and $V_\alpha$, first we must study the auxiliary functions $U_\alpha$ and $V_\alpha$. 
Since $U_\alpha$ and $V_\alpha$ have hypergeometric functions as important factors respectively, the study of the multi-valuedness and the singularities of these auxiliary functions are reduced to those of hypergeometric functions. The next lemma about the analytic continuation, that is the monodromy theory of hypergeometric functions, plays a fundamental role in the study of the behaviour of these auxiliary functions around the branching surfaces.

Lemma A.2  Let $S$ be the Riemann sphere. Put $D=S\setminus\{0,1,\infty\}$. Set $F=F(A,B,C,z)$ and $\tilde{F}=z^{1-C}F(A-C+1,B-C+1,2-C,z)$ which constitute a fundamental system of solutions for the hypergeometric ordinary differential equation

$$\left[ z(1-z)\frac{d^2}{dz^2} + \left( -\frac{\alpha + \beta + 1}{z} + \frac{\gamma - 1}{1-z} \right) \frac{d}{dz} - \frac{\alpha \beta}{z(1-z)} \right] u(z) = 0.$$  

Now, the monodromy representation $P$ of the hypergeometric differential ordinary equation with respect to the fundamental system $\{F, \tilde{F}\}$ is defined in the following way:

Denote by $l_0$, $l_1$, $l_\infty$ respectively a loop which encircles
the point 0 ( 1, oo respectively ) once in the positive sense .

We denote by the same letter \( l_0 \) ( \( l_1 \), \( l_\infty \) respectively ) a homotopy class containing \( l_0 \) ( \( l_1 \), \( l_\infty \) respectively ). Let \( \pi_1 = \pi_1(D) \) be the fundamental group of \( D \). Then we can define a homomorphism \( \rho \) of the group \( \pi_1 \) onto the group \( G \subset GL(2, \mathbb{C}) \) which is called the monodromy representation of the hypergeometric differential equation with respect to the fundamental system \( \{ F, \tilde{F} \} \), where \( G \) is the sub-group of \( GL(2, \mathbb{C}) \), generated by \( g_0 \) and \( g_1 \).

\[
\begin{align*}
\rho(l_0) &= g_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi i C} \end{pmatrix} \\
\rho(l_\infty) &= g_\infty = (g_0 g_1)^{-1} \\
\rho(l_1) &= g_1 = D^{-1} C^{-1} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i (C-B-A)} \end{pmatrix} C D
\end{align*}
\]

where

\[
C = \begin{pmatrix} e^{-2\pi i A} & e^{-2\pi i C} \\ e^{-2\pi i B} & 1 \end{pmatrix}, \quad D = \begin{pmatrix} \frac{\Gamma(B)\Gamma(C-B)}{\Gamma^2(C)} & 0 \\ 0 & \frac{\Gamma(A-C+1)\Gamma(1-A)}{\Gamma^2(2-C)}e^{\pi i (C+A-B-1)} \end{pmatrix}
\]
\[
C^{-1} = \frac{1}{|C|} \begin{pmatrix}
1 - e^{-2\pi i (C-B)} & 1 - e^{-2\pi i B} \\
e^{-2\pi i (C-B)}(1 - e^{-2\pi i A}) & e^{-2\pi i A} - e^{-2\pi i C}
\end{pmatrix}
\]

\[|C| = e^{-2\pi i A}(1 - e^{-2\pi i C})(1 - e^{-2\pi i (C-B-A)})\]

Namely, if \{F, \tilde{F}\} is continued analytically along \(l_D, l_1, l_{\infty}\) respectively), then

\[
\left(\begin{array}{c}
\tilde{F} \\
F
\end{array}\right) \quad \text{goes to} \quad \left(\begin{array}{c}
g_0(\tilde{F}) \\
g_1(\tilde{F}) \\
g_{\infty}(\tilde{F})
\end{array}\right) \quad \text{respectively}.
\]

(Correction: In Lemma 2.1 of the author's preceding paper [10], D of the above lemma was dropped.)

We apply this lemma to the study of \(U_\alpha\) and \(V_\alpha\). In our case, \(F\) is \(F(\alpha; \xi, \eta) = F(-\alpha, b^* + c^*, 2b^*, z)\) and so \(\tilde{F}\) is \(\tilde{F}(\alpha; \xi, \eta) = z^{1 - 2b^*}F(-\alpha - 2b^* + 1, c^* - b^* + 1, 2 - 2b^*, z)\). To explain the multi-valuedness of \(U_\alpha\) and \(V_\alpha\) in terms of the monodromy theory of hypergeometric functions, we introduce the new functions \(\tilde{U}_\alpha(t, x)\) and \(\tilde{V}_\alpha(t, x)\):

\[
\tilde{U}_\alpha(t, x) = \frac{\xi^\alpha}{\Gamma(\alpha + 1)} \tilde{F}(\alpha; \xi, \eta)
\]
Let \( D_r \) be any point belonging to \( D_r \) and keep \( P \) fixed. We consider any loop \( l \) starting and terminating at \( P \) in the domain \( D_r \). Especially we denote by \( l(P) \) the terminating point in order to distinguish the terminating point from the starting point \( P \). We denote by the same letter \( l \) the homotopy class containing \( l \), too. Let \( \pi = \pi_l(D_r, P) \) be the fundamental group of \( D_r \) with the base point \( P \). Denote by \( l_\infty \in \pi \) (\( l_1 \in \pi \), \( l_0 \in \pi \) respectively) a loop which encircles \( \xi = 0 \) (\( \eta = 0 \), \( \xi = \eta \) respectively) once in the positive sense, where by "in the positive sense" we mean that the loop which is transformed by the mapping \( z = 1 - \frac{\eta}{\xi} \) encircles the corresponding point in the positive sense in \( z \)-plane. We note that this mapping \( z = 1 - \frac{\eta}{\xi} \) transforms \( \xi = 0 \), \( \eta = 0 \), \( \xi = \eta \) to \( z = \infty \), 1, 0 respectively and \( l_0 \), \( l_1 \), \( l_\infty \) to
the corresponding loops in Lemma A.2

Using Lemma A.2, we have

\[
\begin{align*}
(F(\alpha; l_0(\mu))) &= \xi_0(F(\alpha; P)), \\
(F(\alpha; l_0(\mu))) &= \xi_1(F(\alpha; P)),
\end{align*}
\]

\[
\begin{align*}
(F(\alpha; l_\infty(\mu))) &= \xi_0(F(\alpha; P)), \\
(F(\alpha; l_\infty(\mu))) &= \xi_\infty(F(\alpha; P)).
\end{align*}
\]

Generally we have

\[
\begin{align*}
(F(\alpha; l(\mu))) &= \rho(1) F(\alpha; P)) \\
(F(\alpha; l(\mu))) &= \rho(1) F(\alpha; P)).
\end{align*}
\]

Therefore we have

\[
\begin{align*}
(U_{\alpha}(l_0(P))) &= \xi_0(U_{\alpha}(P)) \\
(U_{\alpha}(l_0(P))) &= \xi_1(U_{\alpha}(P)) \\
(U_{\alpha}(l_1(P))) &= \xi_1(U_{\alpha}(P)) \\
(U_{\alpha}(l_\infty(P))) &= e^{2\pi i \alpha} \xi_\infty(U_{\alpha}(P)).
\end{align*}
\]

We introduce the new monodromy representation \( \rho^* \) with respect to the pair \( \{ U_{\alpha}, \tilde{U}_{\alpha} \} \) as a homomorphism \( \rho^* \) of the group \( \pi \)
onto the group $G^* \subset GL(2, \mathbb{C})$, where $G^*$ is the sub-group of $GL(2, \mathbb{C})$
generated by $g_0$, $g_1$ and $g_\infty = e^{2\pi i g_\infty}$;

$$\rho^* (l_0) = g_0, \quad \rho^* (l_1) = g_1, \quad \rho^* (l_\infty) = g_\infty.$$

Then generally we have

$$\left( \begin{array}{c}
U_\alpha (1(P)) \\
\tilde{U}_\alpha (1(P))
\end{array} \right) = \rho^* (1) \left( \begin{array}{c}
U_\alpha (P) \\
\tilde{U}_\alpha (P)
\end{array} \right).$$

Remark: we need only $U_\alpha (1(P))$, but we do not need $\tilde{U}_\alpha (1(P))$.

For the study of the multi-valuedness of $U_\alpha (1(P))$, we have to
study $\rho^* (1)$ namely $g_0$, $g_1$, $g_\infty$, $C$ and $D$. Seeing that
both $g_0$ and $D$ are diagonal matrices, they are commutative each
other, and so $g_0 = D^{-1} E_0 D$. On the other hand $g_1$ and $g_\infty$
are also in the form $D^{-1} GD$, that is, $g_1 = D^{-1} E_1 D$ and $g_\infty =
D^{-1} E_\infty D$. Therefore, $\rho^* (1) = D^{-1} \rho^{\#} (1) D$ holds where $\rho^{\#} (1)$
is a homomorphism $\pi_\alpha$ onto $G^{\#} \subset GL(2, \mathbb{C})$, the sub-group of $GL(2, \mathbb{C})$
generated by $g_0$, $E_1$ and $E_\infty$. $\rho^{\#} (1)$, the principal part of
$\rho^* (1)$, is to be investigated precisely.
$E_{\infty} = E_1^{-1}g_0^{-1}$ and $E_1$ generating $g^\#$ with the simple matrix $g_0$.

we have only to examine $E_1$ and $E_1^{-1}$.

Put $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ and then we have $C = \frac{1}{|C|} \begin{pmatrix} C_{22} & -C_{12} \\ -C_{21} & C_{11} \end{pmatrix}$

Put $G = C^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \kappa \end{pmatrix} C$, where $G = E_1$ or $E_1^{-1}$, and then we have

$$G = \frac{1}{|C|} \begin{pmatrix} C_{11}C_{22} - \kappa C_{21}C_{12} & (1-\kappa)C_{12}C_{22} \\ (k-1)C_{11}C_{21} & \kappa C_{11}C_{22} - C_{21}C_{12} \end{pmatrix}$$

$$= \frac{1}{|C|} \begin{pmatrix} |C| + (1-\kappa)C_{12}C_{21} & (1-\kappa)C_{12}C_{22} \\ -(1-\kappa)C_{11}C_{21} & |C| - (1-\kappa)C_{11}C_{22} \end{pmatrix}$$

where $\kappa = \left\{ \begin{array}{ll} e^{2\pi i(C-B-A)} & \text{in the case } G = E_1 \\ e^{-2\pi i(C-B-A)} & \text{in the case } G = E_1^{-1} \end{array} \right.$

So, it is possible that $G$ has the singularity on

$$|C| = e^{-2\pi iA}(1-e^{-2\pi iC})(1-e^{-2\pi i(C-B-A)}) = 0.$$  

We choose $b$ such that Re $b > 0$ and keep fixed, so that we have $1-e^{-2\pi iC} \neq 0$. There is possibility that $G$ has the singularities on $1 - e^{-2\pi i(C-B-A)} = 0$. 

\[5\]
1 - e^{-2\pi i(C-B-A)} = \begin{cases} \frac{1}{k}(1-k) & \text{in the case } G = E_1 \\ (1-k) & \text{in the case } G = E_1^{-1} \end{cases}.

Put \( G = \frac{1}{|G|} \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \) and then from the above, every component \( G_{ij} \) of \( G \) has the factor \( 1-k \). Therefore, the singularity on \( 1 - e^{-2\pi i(C-B-A)} = 0 \) of \( G \) is removable. So we see \( E_1 \) and \( E_1^{-1} \) are entire functions of \( c \) and \( \alpha' \).

Lemma A.3 \( \rho^*(1) \) is an entire function of \( c \) and \( \alpha' \).

\( \rho^*(1) \) is a periodic function of \( \alpha' \) with period 1 and \( \rho^*(1) \) is a periodic function of \( c \) with period \( 2(b+1) \), too.

Put \( \rho^*(1) = \begin{pmatrix} \rho^*_{11} & \rho^*_{12} \\ \rho^*_{21} & \rho^*_{22} \end{pmatrix} \) and \( D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \).

Then we have
\[
\rho^*(1) = \begin{pmatrix} \rho^*_{11} & d_1^{-1}d_2 \rho^*_{12} \\ d_1d_2^{-1} & \rho^*_{22} \end{pmatrix}.
\]

By \( \begin{pmatrix} U_{\alpha}(1(P)) \\ \tilde{U}_{\alpha}(1(P)) \end{pmatrix} = \rho^*(1) \begin{pmatrix} U_{\alpha}(P) \\ \tilde{U}_{\alpha}(P) \end{pmatrix} \), we have reached the next
formula which links $U_\alpha$ on the universal covering space with $U_\alpha$ and $\tilde{U}_\alpha$ on the base space.

(F.A) $U_\alpha(l(P)) = \rho_1^\#(l(1)U_\alpha(P) + \frac{\Gamma(C)\Gamma(A-C+1)\Gamma(1-A)}{\Gamma(B)\Gamma(C-B)\Gamma(2-C)} e^{-\pi i (C+A-B-1)} \rho_1^\#(l(1)\tilde{U}_\alpha(P))$. 

Note (1) Putting $A=-\alpha$, $B=-\beta+c^*$, $C=1+2^*$, instead of $A=-\alpha, B=b^*+c^*$, $C=2b^*$, we have the formula of $V_\alpha(l(P))$.

Note (2) In this formula, $\Gamma$ -factor of the right hand side has no pole if $\alpha$ is a positive integer.

Secondly after expressing $\tilde{u}_\alpha$ (V_\alpha$ on the universal covering space with $U_\alpha$ and $\tilde{U}_\alpha$ (V_\alpha$ and $\tilde{V}_\alpha$ on the base space we investigate $U_\alpha$ and $\tilde{U}_\alpha$ (V_\alpha$ and $\tilde{V}_\alpha$ on the base space with means of the connection formulae of the hypergeometric functions which is described below.

(C.F.) (1) $F(A,B,C,z) = \frac{\Gamma(A+B-C)(C)\Gamma(1-z)}{\Gamma(A)\Gamma(B)} C-A-B F(C-A,C-B,C-A-B+1,1-z) + \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)} F(A,B,A+B-C+1,1-z)$ in $|1-z| = |\frac{\eta}{\xi}| < 1$.

(2) $F(A,B,C,z) = \frac{\Gamma(B-A)(C)}{\Gamma(C-A)\Gamma(B)} (1-z)^{-A} F(A,C-B,A-B+1,1-z)$.
\[
+ \frac{\Gamma(A-B)\Gamma(C)}{\Gamma(C-B)\Gamma(A)}(1-z)^{-B}F(B, C-A, B-A+1, 1 - z) \quad \text{in} \quad \left| \frac{1}{1-z} \right| = \left| \frac{z}{1} \right| < 1.
\]

Using these connection formulae by the help of Kummer's transformations, we list up the representations of \( U_\alpha \), \( \widehat{U}_\alpha \), \( V_\alpha \) and \( \widehat{V}_\alpha \) in \( \left| \frac{\eta}{\frac{1}{2}} \right| \leq 1 \) and \( \left| \frac{z}{\frac{1}{2}} \right| \leq 1 \) respectively. These following formulae (F.B) are fundamental formulae by which we describe the singularities of \( U_\alpha \), \( U_\alpha^{(j)} \), \( V_\alpha \), \( V_\alpha^{(j)} \), \( X_\alpha \), \( X_\alpha^{(j)} \), \( Y_\alpha \) and \( Y_\alpha^{(j)} \).

(F.B) (\( U_\alpha \), \( \widehat{U}_\alpha \), \( V_\alpha \), \( \widehat{V}_\alpha \) on the base space)

\[
U_\alpha = \gamma_1(\alpha, c)F_1(\alpha, c, \frac{\eta}{\frac{1}{2}}) \frac{\eta^\alpha}{\Gamma(\alpha+1)} \left( \frac{\eta}{\frac{1}{2}} \right)^{b^* - c^*}
+ \delta_2(\alpha, c)F_2(\alpha, c, \frac{\eta}{\frac{1}{2}}) \frac{\eta^\alpha}{\Gamma(\alpha+1)} \quad \text{in} \quad \left| \frac{\eta}{\frac{1}{2}} \right| \leq 1
\]

\[
= \delta_2(\alpha, -c)F_2(\alpha, -c, \frac{\eta}{\frac{1}{2}}) \frac{\eta^\alpha}{\Gamma(\alpha+1)}
+ \gamma_1(\alpha, -c)F_1(\alpha, -c, \frac{\eta}{\frac{1}{2}}) \frac{\eta^\alpha}{\Gamma(\alpha+1)} \left( \frac{\eta}{\frac{1}{2}} \right)^{b^* + c^*}
\]

\[
= \delta_3(\alpha, c)F_1(\alpha, c, \frac{\eta}{\frac{1}{2}}) \frac{\eta^\alpha}{\Gamma(\alpha+1)} \left( \frac{\eta}{\frac{1}{2}} \right)^{b^* - c^*}
+ \delta_4(\alpha, c)F_2(\alpha, c, \frac{\eta}{\frac{1}{2}}) \frac{\eta^\alpha}{\Gamma(\alpha+1)} \left( \frac{\eta}{\frac{1}{2}} \right)^{b^* + c^*}
\]

\[
= \delta_4(\alpha, -c)F_2(\alpha, -c, \frac{\eta}{\frac{1}{2}}) \frac{\eta^\alpha}{\Gamma(\alpha+1)}
+ \delta_3(\alpha, -c)F_1(\alpha, -c, \frac{\eta}{\frac{1}{2}}) \frac{\eta^\alpha}{\Gamma(\alpha+1)} \left( \frac{\eta}{\frac{1}{2}} \right)^{b^* + c^*}
\]

\[
\text{in} \quad \left| \frac{\eta}{\frac{1}{2}} \right| \leq 1
\]
where

\[ \gamma_1(\alpha, c) = \frac{\Gamma(-\alpha - b^* + c^*) \Gamma(2b^*)}{\Gamma(-\alpha) \Gamma(b^* + c^*)} \]

\[ \gamma_2(\alpha, c) = \frac{\Gamma(-\alpha - b^* - c^*) \Gamma(2b^*)}{\Gamma(-\alpha - 2b^*) \Gamma(b^* - c^*)} \]

\[ \gamma_3(\alpha, c) = \frac{\Gamma(-\alpha - b^* + c^*) \Gamma(1 + 2b^*)}{\Gamma(-\alpha - 2b^*) \Gamma(c^* - b^* + 1)} \]

\[ \gamma_4(\alpha, c) = \frac{\Gamma(\alpha + b^* - c^*) \Gamma(1 + 2b^*)}{\Gamma(\alpha + 1) \Gamma(b^* - c^* + 2)} \]

\[ F_1(\alpha, c, z) = F(\alpha + 2b^*, b^* - c^*, \alpha + 1 + b^* - c^*, z) \]

\[ F_2(\alpha, c, z) = F(-\alpha', b^* + c^*, -\alpha + c^* - b^* + 1, z) \]

and \( K = (-4^*)^{-\frac{1}{b+1}} \).

Thus we have seen the multivaluedness of \( U_\alpha \) and \( V_\alpha' \).
For the proof of the convergence of the formal solution, we need the estimates of $U_\alpha$ and $V_\alpha$ on the universal covering space, which are reduced to the estimates of $U_\alpha$ and $V_\alpha$ on the base space. By (F.B), those are reduced to the estimates of the hypergeometric series $F_1(\alpha, c, z)$ and $F_2(\alpha, c, z)$, which are based on the following lemmata. Let $b$ be a real positive number and keep fixed henceforth.

Lemma A.4 \[ F_1(\nu, c, z) = F(\alpha + 2b^*, b^* - c^*, \alpha + b^* - c^* + 1, z) \]

\[ KF(\nu + 2b^*, b^* + |\text{Re } c^*| + \varepsilon |\text{Im } c^*|, \alpha + 1 + b^* + |\text{Re } c^*| + \varepsilon |\text{Im } c^*|, z) \]

for $|z| \leq 1$, $\nu \geq 0$, $\varepsilon \geq 1$, where $K$ is a constant depending only on $c$.

Lemma A.5 \[ F_2(\alpha, c, z) = F(-\alpha, b^* + c^*, -\alpha - b^* + c^* + 1, z) \]

\[ T^\alpha \left[ |\text{Re } c^*| \right] + F(\nu, b^* + |\text{Re } c^*| + \varepsilon |\text{Im } c^*|, \alpha + 1 - b^* + |\text{Re } c^*| + \varepsilon |\text{Im } c^*|, z) \]

for $\varepsilon \geq 0$, $\alpha > 0$, where $T$ is a constant depending only on $c$ and $b$, and independent of $\alpha$. 

\[ \frac{1}{16} \]
Remark: the proof of Lemma A.4 is easy, but the proof of Lemma A.5 is done with only tedious calculations. So we omit them. The result about the exceptional values of parameters of the hypergeometric series are to be described later.

Thirdly we are to study $U^{(j)}$, $V^{(j)}$, $X^{(j)}$, and $Y^{(j)}$. By Lemma A.1, we need to differentiate $U_\alpha$ and $V_\alpha$ by $c$ several times and by $\alpha'$ at most once. By (F.A), the differentiations of $U_\alpha$ and $V_\alpha$ are reduced mainly to the differentiations of the hypergeometric series. And so we need the following lemma about the differentiation of the product of $\Gamma$-functions.

Lemma A.6 Let $H(c)$ be an infinitely differentiable function and put $\frac{d}{dc} H(c) = H'(c) = H(c) \gamma(c)$. Then we have the next formula about $\frac{d}{dc}^n H(c)$

$$\frac{d}{dc}^n H(c) = H^{(n)}(c) = \sum_{(s_1, \ldots, s_n) \in P_n} \frac{n!}{s_1! \cdots s_n!} \frac{\gamma_1}{1!}^{s_1} \frac{\gamma_2}{2!}^{s_2} \cdots \frac{\gamma^{(n)}}{n!}^{s_n}$$

where $(s_1, \ldots, s_n)$ varies in $P_n$, that is, the set of all $(s_1, \ldots, s_n)$ such that $\sum_{k=1}^n ks_k = n$ and $s_1 \geq 0, \ldots, s_n \geq 0$. (See [8].)
We explain how to use this lemma. For example, in the case
\[ H(c) = \frac{\Gamma(c+p)}{\Gamma(c+q)} \], we put \[ H'(c) = \frac{\Gamma'(c+p)}{\Gamma(c+p)} - \frac{\Gamma'(c+q)}{\Gamma(c+q)} = H(c) \psi(c) \]

Namely \[ \psi'(c) = \psi(c+p) - \psi(c+q) \], where \( \psi \) of the right hand side
of this identity is \( \Gamma \)-function. Taking account of the fact
that the formula about the \( \Gamma \)-function \( \psi(n)(z+1) = \psi(n)(z) + (-1)^n \frac{n!}{z^{n+1}} \) holds except the nonpositive integer \( z \), we are able
to estimate \( \psi'(n)(c+p) - \psi(n)(c+q) \) for \( n \geq 1 \).

The next lemma about the cardinal number of \( P_n \), which we denote by \( p(n) \), are to be used for the estimation of derivatives
of \( F_1 \) and \( F_2 \).

Lemma A.7  \( p(n) \) is called the partition number of \( n \). The

following estimates of \( p(n) \) are known.
\[ Hn^{-1} e^{2\sqrt{n}} < p(n) < Kn^{-1} e^{2\sqrt{2n}} \]

where \( H \) and \( K \) are absolute constants. (See [4].)

To see \( X^{(j)}(P) \) and \( Y^{(j)}(P) \) on the base space, we have only to
see \( \partial_c^{(1)} F_1(\alpha', c, z) \) and \( \partial_c^{(1)} F_2(\omega', c, z) \) in \( |z| \leq 1 \). We put...
\[ F_1(\alpha, c, z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+2b^2+n)}{\Gamma(\alpha+2b^2)\Gamma(b^2-c^2)\Gamma(\alpha+2b^2-c^2+1+n)} z^n = \sum_{n=0}^{\infty} H_n(c) \frac{z^n}{n!} . \]

\[ \partial_c^r F_1 = \sum_{n=0}^{\infty} H_n(c) \left[ \frac{1}{\sum_{r=0}^{n} \frac{1}{\Gamma(\alpha+2b^2+r)} - \frac{1}{\Gamma(\alpha+2b^2-c^2+1+r)}} \right] \frac{z^n}{n!} , \]

\[ \partial_c \partial_c^r F_1 = \sum_{n=0}^{\infty} H_n(c) \left[ \frac{1}{\sum_{r=0}^{n} \left( \frac{1}{\Gamma(\alpha+2b^2+r)} - \frac{1}{\Gamma(\alpha+2b^2-c^2+1+r)} \right)} \right] \frac{z^n}{n!} , \]

Differentiating by \( c \) this identity several times and applying Lemma A.6 to the coefficients of these power series, we are to see \( \partial_c \partial_c^r F_1 \) and \( \partial_c \partial_c^r F_2 \) inductively and to estimate them by the help of Lemma A.7. In fact, for example, put \( \psi_n(c) = \sum_{r=0}^{n-1} \frac{1}{\Gamma(\alpha+2b^2+r)} - \frac{1}{\Gamma(\alpha+2b^2-c^2+1+r)} \) and then we have \( |\psi_n(c)| < C \) and \( |\psi_{n+1}(c)| < \frac{1}{c!} \), where \( \psi \) is a non-negative integer. As for \( \partial_c \partial_c^r F_2 \), we can treat it in the similar way. By the above lemmata and the formulae with respect to poly-\( \Gamma \)-functions, we have

\[ \partial_c \partial_c^r F_1 \ll B! \cdot \Gamma^r \left( \alpha+2b^2, b^2+b^2 + |\Re c^2| + |\Im c^2|, \alpha+1+b^2+b^2 + |\Re c^2| + |\Im c^2| \right) , \]

for \( \alpha \geq 0, \xi \geq 1 \), and

\[ \partial_c \partial_c^r F_2 \ll B! \cdot \Gamma^r \left( \alpha+2b^2, b^2+b^2 + |\Re c^2| + |\Im c^2|, \right) , \]

for \( \alpha', \xi ' \geq 1 \).
\[\alpha + 1 - b^* + |\Re c^*| + \varepsilon |\Im c^*|, z) \quad \text{for } \alpha > 0, \quad \varepsilon \geq 0,\]

where \(B, C\) and \(K\) are suitable constants independent of \(\alpha\) and \(l\).

To estimate the hypergeometric series in the convergent circle, we need the following lemma.

**Lemma A.8** If \(A, B, C\) and \(C-B-A\) are positive, we have

\[
F(A, B, C, z) = F(A, B, C, 1) = \frac{\Gamma(C)\Gamma(C-B-A)}{\Gamma(C-A)\Gamma(C-B)} \quad \text{in } |z| \leq 1
\]

(see [10]).

Therefore we have the following estimates of \(\frac{\partial}{\partial c} F_1\) and \(\frac{\partial}{\partial c} F_2\) in \(|z| \leq 1\).

\[
\left|\frac{\partial}{\partial c} F_1(\alpha, c, z)\right| = B! C! K \left|\frac{\Gamma(\alpha + 1 + b^* + |\Re c^*| + \varepsilon |\Im c^*|)}{\Gamma(\alpha + 1)}\frac{\Gamma(1-2b^*)}{\Gamma(1-b^* + |\Re c^*| + \varepsilon |\Im c^*|)}\right| \quad \text{in } |z| \leq 1
\]

\[
\left|\frac{\partial}{\partial c} F_2(\alpha, c, z)\right| = B! C! K \left|\frac{\Gamma(\alpha + 1 - b^* + |\Re c^*| + \varepsilon |\Im c^*|)}{\Gamma(\alpha + 1 - 2b^*)\Gamma(1-b^* + |\Re c^*| + \varepsilon |\Im c^*|)}\right| \quad \text{in } |z| \leq 1
\]

for \(\alpha > 0\) and \(\varepsilon \geq 1\).

These estimates lead to the estimates of \(X^{(j)}(P), Y^{(j)}(P), \) and then it remains to estimate \(X^{(j)}(P)\) and \(Y^{(j)}(P)\) on the universal covering space. With respect to \(\gamma\)

\[
\gamma = \frac{\Gamma(C)\Gamma(A-C+1)\Gamma(1-A)}{\Gamma(B)\Gamma(C-B)\Gamma(2-C)}
\]
(F.A), we rewrite as follows:

\[
\Gamma = \begin{cases} 
\gamma_5(\alpha, c) = \frac{\Gamma(2b^*) \Gamma(1-\alpha-2b^*+1)}{\Gamma(b^*+c^*) \Gamma(b^*-c^*) \Gamma(2-2b^*)} & \text{in the case } U_{\omega}(l(p)), \\
\gamma_6(\alpha, c) = \frac{\Gamma(2-2b^*) \Gamma(1-\alpha+1+2b^*) \Gamma(1+\alpha)}{\Gamma(1-b^*+c^*) \Gamma(1-b^*-c^*) \Gamma(2b^*)} & \text{in the case } V_{\omega}(l(P)).
\end{cases}
\]

(Note: \(\gamma_5\) and \(\gamma_6\) have the meaning in the case \(b^*+c^*\) or \(b^*-c^*\) is an integer.)

Applying Lemma A.6 and A.7 to the estimations of \(\gamma_i(\alpha', c)\) \(i=1, \ldots, 6\), we have

\[
\|\partial^1_{\alpha} \gamma_1(\alpha', c)\| \leq B1!C^1 \left| \frac{\Gamma(1+\alpha) \Gamma(2b^*)}{\Gamma(1+c^*+b^*) \Gamma(b^*+c^*)} \right| \left| \gamma_1(1+\alpha+b^*+c^*) \right|,
\]

\[
\|\partial^1_{\alpha} \gamma_2(\alpha', c)\| \leq B1!C^1 \left| \gamma_2(\alpha', c) \right| \left| \gamma_2(1+\alpha+b^*+c^*) \right|,
\]

\[
\|\partial^1_{\alpha} \gamma_3(\alpha', c)\| \leq B1!C^1 \left| \frac{\Gamma(\alpha+2b^*) \Gamma(2-2b^*)}{\Gamma(\alpha+1+b^*-c^*) \Gamma(1+c^*-b^*)} \right| \left| \gamma_3(1+\alpha+b^*+c^*) \right|,
\]

\[
\|\partial^1_{\alpha} \gamma_4(\alpha', c)\| \leq B1!C^1 \left| \gamma_4(\alpha', c) \right| \left| \gamma_4(1+\alpha+b^*+c^*) \right|,
\]

\[
\|\partial^1_{\alpha} \gamma_5(\alpha', c)\| \leq B1!C^1 \left| \gamma_5(\alpha', c) \right|,
\]

\[
\|\partial^1_{\alpha} \gamma_6(\alpha', c)\| \leq B1!C^1 \left| \gamma_6(\alpha', c) \right|.
\]

Using these estimates, we obtain the following estimates of

\[
\partial^1_{\alpha} U_\omega, \, \partial^1_{\alpha} U_\omega, \, \partial^1_{\alpha} V_\omega \, \text{and} \, \partial^1_{\alpha} \gamma_\alpha \, \text{on the base space}.
\]

\[
\| \partial^1_{\alpha} U_\omega(p) \| \leq K^{\alpha} \left( \frac{1}{\Gamma(1+\alpha)} \right) \frac{\gamma_1(1+\alpha+b^*+c^*)}{1!} \left( \frac{1}{\Gamma(1+\alpha)} \right).
\]
\begin{align*}
\max(\vert \gamma_1(\alpha, c), \vert \gamma_2(\alpha, c) \vert) r^{\alpha+1-(\Re c^*)+(\log r)^{1+1}}
\end{align*}

in \( |\beta| \leq r \) and \( |\eta| \leq r \),

\begin{align*}
\left| \frac{\partial}{\partial c} \phi_\alpha(\bar{\alpha}, \bar{\eta}) \right| & \leq K^{\alpha+1-(\Re c^*)+1}B^{1+2} \frac{1}{1+(1+\alpha)} \left( \frac{(\alpha^2+1+b^2+|c^*|)}{1+(\alpha^2+2^2)} \right) \\
\text{Max}(\vert \gamma_3(\alpha, c), \vert \gamma_4(\alpha, c) \vert) r^{\alpha+1-(\Re c^*)+(\log r)^{1+1}}
\end{align*}

in \( |\beta| \leq r \) and \( |\eta| \leq r \),

\begin{align*}
\left| \frac{\partial^2}{\partial c^2} \phi_\alpha(\bar{\alpha}, \bar{\eta}) \right| & = K^{\alpha+1-(\Re c^*)+1}B^{1+2} \frac{1}{1+(1+\alpha)} \left( \frac{(\alpha^2+1+b^2+|c^*|)}{1+(\alpha^2+2^2)} \right) \\
\text{Max}(\vert \gamma_3(\alpha+2^2, c), \vert \gamma_4(\alpha+2^2, c) \vert) r^{\alpha+1-(\Re c^*)+(\log r)^{1+1}}
\end{align*}

in \( |\beta| \leq r \) and \( |\eta| \leq r \),

where \( (\Re A)^+ = \max(0, A) \)

Taking account of the fact that \( \phi_{ij}^\# \) are entire functions of \( \alpha \) and \( c \) and periodic functions of \( \alpha \) with period 1, we have reached, by the help of all estimates obtained above,

\begin{align*}
(\text{E.X}) \quad \left| X_\alpha^{(j)}(1(P)) \right| & \leq C(1(P))K^{\alpha+1}C^{\frac{1}{1+(\alpha+1)}} \left( \frac{(\alpha^2+1+b^2+|c^*|+1)}{1+(\alpha+1)} \right) \chi
\end{align*}
\[ \max(|y_1(\alpha + j, c)|, |y_2(\alpha + j, c)|, |y_3(\alpha + j, c)|), \]
\[ |y_4(\alpha + j, c)| \] \( r^{\alpha + j - \Re c}(\log r)^{j+1} \)

for \(|\beta| < r\) and \(|\eta| < r\), and

\[ (E.Y) \quad |y(\alpha)(1(P))| \leq C(1(P))K^{\alpha + j} \frac{r^{\alpha + 2 \xi + b + \log r}}{r^{\alpha + j + 1}} \]
\[ \max(|y_1(\alpha + 2 \xi + j, c)|, |y_2(\alpha + 2 \xi + j, c)|, |y_3(\alpha + 2 \xi + j, c)|), \]
\[ |y_4(\alpha + 2 \xi + j, c)| \] \( r^{\alpha + 2 \xi + j - \Re c}(\log r)^{j+1} \)

for \(|\beta| < r\) and \(|\eta| < r\),

where \(C(1(P))\) is a constant depending only on \(1(P)\) and \(\alpha + j\) is a non-negative integer.

Lastly we obtain the following estimates, by Stirling's formula and the other formulae with respect to \(\Gamma\)-function.

\[ |X(\alpha)|, |Y(\alpha)| < C K^{\frac{\alpha + j + 1}{\Gamma(\alpha + j + 1)}} \frac{r^{\alpha + j} r^{\alpha + j}(\log r)^j}{r^{\alpha + j + 1}} \]

on any compact set \(K\) of the universal covering space over \(D_r \setminus \{x=0, \eta = 0, \beta = \gamma\}\), where \(\alpha + j\) is a positive integer and \(C_K\) is a constant which depends only on \(K\), and \(D_r = \{(t, x) : |t| r, |\eta| < r\}\), and \(C\) is a constant independent of \(\alpha, j\) and \(r\).
Remark:

The above treatment and estimates of hypergeometric series, that is, $U_{\omega}$, $V_{\omega}$, $V_{\omega}^{(j)}$, $V_{\omega}^{(j)}$, $X_{\omega}^{(j)}$, $Y_{\omega}^{(j)}$ on the base space, are not valid in the case $c$ is an exceptional value of parameters of hypergeometric series. To be precise, as to $U_{\omega}$, in two following cases,

Case I $c^*-b^* = m \in \mathbb{Z}$ (in $|\frac{m}{5}| \leq 1$)

Case II $c^*+b^* = n \in \mathbb{Z}$ (in $|\frac{n}{3}| \leq 1$)

\[
\begin{align*}
\text{Case I} & \quad c^*-b^*-2* = m \in \mathbb{Z} \quad (\text{in } |\frac{m}{5}| \leq 1) \\
\text{Case II} & \quad c^*+b^*+2* = n \in \mathbb{Z} \quad (\text{in } |\frac{n}{3}| \leq 1)
\end{align*}
\]

, (F.B), especially (C.F.) have troubles.

We explain how to overcome these difficulties, getting an example in Case II. Put $c^*+b^*=n$. We rewrite the following connection formula

\[
F(-\alpha,n,2b^*,z)=\frac{\Gamma(n+\alpha)}{\Gamma(2b^*+\alpha)}\frac{\Gamma(2b^*)}{\Gamma(n)}(1-z)^{\alpha}F(-\alpha,2b^*-n,-\alpha+n+1,\frac{1}{1-z}) + \]

\[
\frac{\Gamma(-\alpha-n)}{\Gamma(2b^*-n)}\frac{\Gamma(2b^*)}{\Gamma(-\alpha)}(1-z)^{-n}F(n,2b^*+\alpha,n+d+1,\frac{1}{1-z}) \quad \text{in } |\frac{3}{\eta}| \leq 1,
\]
when \( n \) is positive, into the form,

\[
\sin(n+a)F(-d,n,2b^*,z) = \frac{\Gamma(2b^*) \Gamma(1+a) \Gamma(1+n-2b^*)}{\Gamma(2b^*+a)} (1-z)^{\alpha} F^{\alpha}_{1} (z) \\
+ \frac{\Gamma(2b^*) \Gamma(1-2b^*-a)}{\Gamma(2b^*-n)} (1-z)^{-n} F^{\alpha}_{2} (z) \quad \text{in } \left| \frac{z}{q} \right| \leq 1
\]

where \( F^{\alpha}_{1} (z) = \sum_{k=0}^{+\infty} (1-z)^{-k}[k! \Gamma(1+a-k) \Gamma(1+n-2b^*-k) \Gamma(-\alpha-n+k+1)]^{-1} \)

\( F^{\alpha}_{2} (z) = \sum_{k=0}^{+\infty} (1-z)^{-k}[k! \Gamma(1-2b^*-a-k) \Gamma(n+\alpha+k+1)]^{-1} \)

and when \( n \) is non-positive, into the form

\[
\frac{\sin(n+a)}{n} F(-d,n,2b^*,z) = \frac{\Gamma(2b^*) \Gamma(1+a) \Gamma(1+n-2b^*)}{\Gamma(2b^*+a)} (1-z)^{\alpha} F^{\alpha}_{3} (z) \\
+ \frac{\Gamma(2b^*) \Gamma(1-2b^*-a)}{\Gamma(2b^*-n)} (1-z)^{-n} F^{\alpha}_{4} (z) \quad \text{in } \left| \frac{z}{q} \right| \leq 1
\]

where \( F^{\alpha}_{3} (z) = \sum_{k=0}^{+\infty} (1-z)^{-k}[k! \Gamma(1+a-k) \Gamma(1+n-k-2b^*) \Gamma(1-\alpha-n+k)]^{-1} \)

\( F^{\alpha}_{4} (z) = \sum_{k=0}^{+\infty} (1-z)^{-k}[k! \Gamma(1-n-k) \Gamma(n-2b^*-a-k) \Gamma(1+n+\alpha+k)]^{-1} \).

After operating \( \partial_c^{j+1} \) and \( \partial_{c}^{j+1} \) to the both sides of this new connection formula, we set \( b^*+c^*=n \) to be an integer,

and we can know and estimate \( \partial_{c}^{j+1} \partial_{c}^{j+1} F(-d,n,2b^*,z) \) in \( \left| \frac{z}{q} \right| \leq 1 \)

inductively. We can treat other cases in the similar way.

Remark:

The initial surface \( t=0 \) is corresponded to \( \frac{z}{q} = 0 \). The
characteristic roots of hypergeometric ordinary differential operator at \( z=0 \) are 0 and \( 1-C = \frac{1}{b+1} \). On the other hand,
\[
\gamma - \xi = \frac{2}{b+1} \cdot t^{b+1}.
\]
Therefore the initial surface \( t=0 \) is not a branching surface.

We need the results in Appendix in the case \( b=1 \) for this paper, and in the case \( b=\frac{1}{2} \) for the author's preceding papers [10] and [11].

We finish this paper with the following example.

(Example)

\[
L = \partial^2_t - t^{2b} \partial^2_x - ct^{b-1} \partial_x - at^{b-1} \quad (a \text{ is a constant})
\]

We consider the following Cauchy problems with the singular data.

(1) \( Lu(t,x) = 0 \)

with the initial data
\[
\begin{align*}
&u(0,x) = k_\alpha(x) \\
&u_t(0,x) = 0
\end{align*}
\]

\[
u(t,x) = \sum_{k=0}^{+\infty} a_\alpha \frac{k}{x} (k)(t,x) .
\]
(2) \( L v(t,x) = 0 \)

with the initial data \[
\begin{align*}
v(0,x) &= 0 \\
v_t(0,x) &= k_\eta(x)
\end{align*}
\]

\[
v(t,x) = \sum_{k=0}^{+\infty} a_k^\eta(t,x)
\]

\( u(t,x) \) is a unique holomorphic solution of the Cauchy problem (1) \((2)\) on the universal covering space over \( \mathbb{D} \setminus \{ \zeta = 0, \, \eta = 0 \} \)

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