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STANDING PULSE-LIKE SOLUTIONS
OF A SPATIALLY AGGREGATING POPULATION MODEL

池 田 勉

学 位 審 査 報 告

氏 名	池 田 勉
学 位 の 種 類	理 学 博 士
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論文調査委員	主 査 山 口 昌 哉 池 部 晃 生 溝 畑 茂

(論文内容の要旨)

生物個体群の数理モデルの一つに個体群のパターン形成を数学的に説明しようとしてつくられた非線型微分積分方程式がある。この方程式は退化型の拡散項と生物の個体の求心本能よりくる集中過程をあらわす項とからなり、 u を時刻 t と位置 x の関数としての個体群密度として、次のように書ける。

$$u_t(x, t) = (u^m)_{xx}(x, t) + \left[u(x, t) \left\{ \int_{x-r}^x u(y, t) dy - \int_x^{x+r} u(y, t) dy \right\} \right]_x$$

ここで r は生物の認識範囲を示す正パラメーターであり、 m は 1 より大きな数である。

申請者はこの方程式の定常解について次のような結果を得た。ここで $\| \cdot \|_{\infty}$ は L^{∞} ノルム、 $\| \cdot \|_1$ は L^1 ノルムをあらわすこととする。

定理 I (i) 上の方程式は $r^2 \| u \|_{\infty}^{2-m} \leq 2$ となるような 0 でない定常解をもたない。(ii) $F(m)$ を次のように定義する：

$F(m) = 2^{1-2/m} \int_0^1 \{ \eta(1-\eta) \}^{-1/m} d\eta$ と、各正数 $q \geq F(m)^2$ に対して、上の方程式は $r^2 \| u \|_{\infty}^{2-m} = q$ であり、 u の台の長さが r 以下の 0 でない定常解を必ずしもつ。この定常解は台の内部で正値であり台は連結である。このような定常解を申請者は Standing Solitary Pulse とよんだ。(iii) 関数 $u(x)$ が上の方程式の 0 でない定常解ならば、 $u(x)$ は上述の Standing Solitary Pulse によって展開できる。すなわち、

$$u(x) = \sum_{i \in A} u_i(x) \quad (1)$$

ここに $\{ u_i \}_{i \in A}$ は次の条件をみたす有限又は可算無限個の Standing Solitary Pulse 列である：

$$\left\{ \begin{array}{l} \sup_{i \in A} \{ \| u_i \|_{\infty} \} < +\infty, \quad \sum_{i \in A} \| u_i \|_1 < +\infty \\ i \neq j \quad i, j \in A \quad (u_i \text{ と } u_j \text{ の台の間の距離}) \geq r \end{array} \right. \quad (2)$$

逆に $\{u_j\}$ を(2)をみたす Standing Solitary Pulse の列とすれば(1)で定義される $u(x)$ は上の方程式の 0 でない定常解である。

更に $m=2$ の場合には更に詳しい次の結果が得られている。定理 I は次のように述べられる。

(i) $r^2 \leq 2$ ならば 0 以外に定常解をもたない。

(ii) $r^2 > 2$ のとき方程式は $u_r(0) = 1$ 且つ $u_r(x)$ は連続な偶関数という条件をみたす唯一つの Standing Solitary Pulse $u_r(x)$ を定常解としてもつ。この定常解 u_r の台はコンパクトであり、その長さを $s(r)$ としておく。一方関数 $u(x)$ が方程式の 0 でない定常解となるための必要十分条件は、上の Standing Solitary Pulse u_r 有限個又は可算個の平行移動からなるパルス列が定理 I の(2)の条件をみたしそれによって(1)のように展開できることである。 $u_r(x)$ と $s(r)$ は次の性質をもつ：

- (a) $u_r(x)$ は $(-\infty, 0)$ で単調非減少、 $s(r)$ は r は r の単調非増加連続関数、(b) $s(r) = \pi$ ($r \geq \pi$), $s(r) > \pi$ ($r < \pi$), $r^2 \rightarrow 2$ のとき $s(r)$ は増加して無限大となる。(c) $r \geq \pi$ ならば、 $u_r(x) = \cos x$, $(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2})$
(d) $r < \pi$ ならば $(-s(r)/2, s(r)/2)$ 上で $u_r(x)$ は C^2 級であるが C^3 級ではない。

なお定理 I は定理 II とともに $r = \infty$ のときにも有効である。

(論文審査の結果の要旨)

申請者の主論文は集中過程を含む、非線型拡散方程式で表現された生物個体群のパターン形成についての数学的研究である。この種の方程式の研究が単独方程式であるにもかかわらず困難であった理由はこの方程式が非線型拡散項（密度依存の拡散）と集中過程をあらわす積分であらわされた移流項をもつ微分積分方程式であり、取り扱いの手がかりが見出せなかったことによっている。特別な場合 $r = 0$ の場合には、方程式は多孔物質中を拡散する流れの方程式となり、積分項は消えるので Aronson の数篇の論文となった。また $r = \infty$ の場合には、このモデルの提唱者三村-永井の結果があるが、このモデルとしては現実的な $0 < r < \infty$ の場合には、現在まで全く結果は無かった。申請者は $0 < r \leq \infty$ の場合について、定常解に関する限り、きわめて詳細な、大局的構造を含めた数学的研究をなして大きくこの方面の研究を前進せしめた。得られた2つの定理の特色を次のようにまとめることができる。

- 1) 非自明な定常解の存在の条件を明示したこと。
- 2) 台が連結で台の内では正值な定常解を Standing Solitary Pulse となづけ、これを定常解の最小単位としてとりあげ、その存在の条件と、任意の非自明定常解はこのような Standing Solitary Pulse の列によって展開できることを示したこと。又逆にそのような展開ができる関数は定常解であることを示した。
- 3) $m = 2$ の場合には更に詳細に上に述べた展開のための列は唯一つの特別な形の Standing Solitary Pulse の有限又は可算個の平行移動によって得られることを示した。
- 4) $m = 2$ のときはパルスの間隔と大きさについて、正確な定量的結果を得ていること。
- 5) 証明の方法も新しく、たとえば2階の微分と、2階の差分とのバラ

ンスを研究するなど独創的なところが多い。

上の結果はこの種のモデル方程式の解の性質としては、予想外の詳細なものであり、申請者の考察がきわめて徹底したものであったことを示している。

参考論文 1, 2, 3 は非線型偏微分方程式の差分化の研究であり、4, 5, 6 は申請者の得意とする移流項の加った拡散方程式の差分化の研究であり、特に 6 は重要なモノグラフである。7 は主論文の離散化の研究であり、いづれも申請者の学識の深さを示すものである。

よって本論文は理学博士の学位論文として価値あるものと認める。

なお、主論文及び参考論文に報告されている研究業績を中心とし、これに関連した研究分野について試問した結果、合格と認めた。

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Tsutomu Ikeda

**Department of Mathematics
Faculty of Science
Kyoto University
Kyoto, Japan**

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**Department of Mathematics
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Abstract. The present paper is devoted to the study of stationary solutions of a nonlinear degenerate diffusion equation involving a nonlocal convection term, which represents a mathematical model for spatially aggregating phenomena of populations. The equation has two ecological parameters: $m > 1$ for the diffusion process and $0 \leq r \leq \infty$ for the aggregating process expressed by the convection term. For the special case of $m = 2$, this paper gives all stationary solutions of the one-parameter family $\{P(2,r)\}$ of the equations. The result asserts that $P(2,r)$ has no non-trivial stationary solution when $0 \leq r \leq \sqrt{2}$, while $P(2,r)$ has many pulse-like stationary solutions when $r > \sqrt{2}$. The paper also states a partial result for the general case of m , and offers a view of the global structure of stationary solutions of $P(m,r)$.

Key words and phrases. Stationary solution, Nonlocal interaction, Density-dependent dispersal, Free boundary problem, Eigenvalue problem of differential-difference type.

Running head. A spatially aggregating population model.

1. Introduction

In the present paper, we study the stationary solution of the following nonlinear degenerate diffusion equation involving a non-local convection term

$$u_t(x,t) = (u^m)_{xx}(x,t) + \left(u(x,t) \left(\int_{x-r}^x u(y,t) dy - \int_x^{x+r} u(y,t) dy \right) \right)_x$$

in $\mathbb{R} \times (0, \infty)$,

where $m > 1$ and $0 \leq r \leq \infty$ are parameters and $u(x,t) \geq 0$ denotes the population density at position $x \in \mathbb{R}$ and time $t > 0$. We denote the above equation by $P(m,r)$, by taking into account the parameters m and r .

Before the mathematical study of $P(m,r)$, we consider an ecological meaning of the equation.

Ecology has become a mathematical subject in recent years. There have been proposed many spatially spreading population models for the study of the spatial distribution of organisms, dispersal of insects, migration of animals, plankton patchiness, coexistence of competing species, etc. (Okubo [36] among others). These spatially spreading population models, which are usually given by partial differential equations, include biological interactions and diffusion processes due to a spatial inhomogeneity in the density distribution; convection effects caused by a spatial inhomogeneity in environment also are included in several models.

We here note that some animals in nature congregate and form

groups having clear interfaces (insect swarms and bird flocks for instance), and we proceed to the study of a mathematical model for spatially aggregating phenomena of organisms. Among others (for instance, a mathematical model for the aggregation of amoebae (Keller and Segel [24]) and the spatial pattern of amoeba aggregation (Sperb [42])), we are concerned with a class of spatially aggregating population models, proposed by Nagai and Mimura [32],

$$(1.1) \quad u_t(x,t) = (u^m)_{xx}(x,t) + \left[u(x,t) \int_{-\infty}^{\infty} K(x-y)u(y,t)dy \right]_x$$

in $R \times (0, \infty)$,

where each $K(x)$ is an odd function such that

$$K(x) \geq 0 \quad \text{for } x > 0.$$

The second term on the right-hand side of (1.1) ecologically shows an aggregating mechanism of individuals, which is motivated by the notion of "centripetal instincts" (Hamilton [19]). In fact, the term provides a mechanism that moves individuals at position x to the right (resp. left) direction when

$$\int_0^{\infty} K(y) \{u(x+y,t) - u(x-y,t)\} dy > 0 \quad (\text{resp. } < 0)$$

since $\int_0^{\infty} K(y) \{u(x+y,t) - u(x-y,t)\} dy = -\int_{-\infty}^{\infty} K(x-y)u(y,t)dy$ represents the velocity of drift at position x . The first term on the right-hand side of (1.1) corresponds to the transport of population through a nonlinear diffusion process called density-dependent dispersal (Gurney and Nisbet [16] and Gurtin and MacCamy [17]). We note that the diffusion speed mu^{m-1} decreases with the density u and vanishes at position x where $u(x) = 0$. Consequently, the first term yields a homogenizing process while the second term yields a

dehomogenizing process; a delicate balance between these two processes may give rise to a spatial pattern that shows an aggregation of individuals.

The equation $P(m,r)$ is a special case of (1.1), where the integral kernel $K(x)$ is given by

$$(1.2) \quad K(x) = \begin{cases} 1 & \text{for } 0 < x < r, \\ -1 & \text{for } -r < x < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus a non-trivial stationary solution of $P(m,r)$ ecologically exhibits a spatially aggregating pattern of individuals.

As stated in Nagai [31], mathematical models providing nonlocal interactions such as (1.1) are utilized to explain physical phenomena in other fields (Kuramoto [26] and Munakata [30]).

We now return to the mathematical discussion on the stationary solution of $P(m,r)$. (Later in this section, we will briefly review mathematical works related to the Cauchy problem (1.1) subject to a non-negative initial condition.)

The trivial function $u \equiv 0$ is always a stationary solution of $P(m,r)$, and we naturally are concerned with the non-trivial stationary solutions.

All stationary solutions of $P(m,0)$ and $P(m,\infty)$ have already been obtained. The equation $P(m,0)$ agrees to the porous medium equation

$$(1.3) \quad u_t = (u^m)_{xx} \quad \text{in } \mathbb{R} \times (0, \infty),$$

which appears in the theory of fluid flow through a porous medium

(Bear [8] and Scheidegger [40]). Clearly, (1.3) has no non-trivial stationary solution. On the other hand, $P(m, \infty)$ has non-trivial standing solitary wave solutions with connected compact support (Mimura and Yamaguti [29]). These solutions are determined by the total population and the position of the center of support. In Section 8, we shall discuss these stationary solutions in detail.

In the present paper, we shall study the stationary solution of $P(m, r)$ for $0 < r < \infty$.

First we shall deal with the special case of $m = 2$. The result is summarized as

Theorem 1.1 (Stationary solutions of $P(2, r)$).

(I) Let $r \leq \sqrt{2}$. Then $P(2, r)$ has no non-trivial stationary solution.

(II) Let $r > \sqrt{2}$. Then $P(2, r)$ has a unique stationary solution u_r such that

$$(1.4) \quad \begin{cases} u_r \text{ is a continuous even function,} \\ u_r(0) = 1, \\ u_r \text{ has a connected compact support and is positive on the} \\ \text{interior of the support.} \end{cases}$$

Moreover, a function u becomes a non-trivial stationary solution of $P(2, r)$ if and only if u is expressed in the form

$$(1.5) \quad u(x) = \sum_{i \in \Lambda} a_i u_r(x - c_i),$$

where $\{a_i\}_{i \in \Lambda}$ and $\{c_i\}_{i \in \Lambda}$ (Λ : an index set) are finite or countable sequences of real numbers satisfying the conditions

$$(1.6) \quad a_i > 0 \quad \text{for all } i \in \Lambda, \quad \sum_{i \in \Lambda} a_i < \infty,$$

$$(1.7) \quad |c_i - c_j| \geq r + (\text{the length of the support of } u_r) \\ \text{for all } i \in \Lambda, j \in \Lambda, i \neq j.$$

(See Figure 1.1.) The length $s(r)$ of the support of u_r has the properties

- (1) $s(r) = \pi$ for $r \geq \pi$ and $s(r) > \pi$ for $r < \pi$,
- (2) $s(r)$ increases with decreasing r ,
- (3) $s(r)$ is continuous in $(\sqrt{2}, \infty)$,
- (4) $s(r)$ tends to infinity as r tends to $\sqrt{2}$.

The stationary solution u_r has the properties

- (i) u_r is non-decreasing on $(-\infty, 0)$,
- (ii) for $r \geq \pi$, $u_r(x) = \cos x$ on $(-\frac{\pi}{2}, \frac{\pi}{2}) \equiv (-\frac{1}{2}s(r), \frac{1}{2}s(r))$,
- (iii) for $\sqrt{2} < r < \pi$, u_r is twice continuously differentiable on $(-\frac{1}{2}s(r), \frac{1}{2}s(r))$ but u_r is not thrice continuously differentiable on $(-\frac{1}{2}s(r), \frac{1}{2}s(r))$. \square

We need the condition (1.6) since the stationary solution of $P(m, r)$ is defined as a non-negative valued function belonging to $L^\infty(\mathbb{R})$ and $L^1(\mathbb{R})$ (Section 2). The condition (1.7) asserts that the distance between two arbitrary connected components of the support of a stationary solution is greater than or equal to r . For instance, $d_1 \geq r$ and $d_2 \geq r$ in Figure 1.1. We note that Theorem 1.1 is valid for all $0 \leq r \leq \infty$; when $r = \infty$, the support of a stationary solution should be connected by (1.7) and its length equals π by the property (1) of $s(r)$.

We shall show, in Section 3, that all stationary solutions of $P(m, r)$ are decomposed into simple ones, each of which has a connected support and is positive on the interior of its support. In accordance with Nagai and Mimura [33], we refer to these simple

stationary solutions as standing solitary pulses. The stationary solution u_r is a standing solitary pulse. A standing solitary pulse becomes a solution of a free boundary problem of differential-difference type (Section 4). This free boundary problem is reduced to a linear one when $m = 2$, and we can characterize the stationary solution u_r by using a principal eigenfunction of an eigenvalue problem derived from the free boundary problem (Sections 5 to 7).

Furthermore, we shall deal with the general case of $m \neq 2$ (Section 8). The above method of proof cannot apply to the general case since the free boundary problem becomes a nonlinear one when $m \neq 2$. We have not yet obtained a satisfactory result. However, we can extend, to the general case, the first assertion in Theorem 1.1 and the fact that $P(2,r)$ has a stationary solution such that the length of its support is less than or equal to r when $r \geq \pi$. (Observe the property (1) of $s(r)$.)

Theorem 1.2 (Stationary solutions of $P(m,r)$ for $m \neq 2$).

(I) $P(m,r)$ has no non-trivial stationary solution u such that

$$(1.8) \quad r^2 \|u\|_{\infty}^{2-m} \leq 2,$$

where $\|\cdot\|_{\infty}$ denotes the supremum norm.

(II) Let $F(m) = 2^{1-2/m} \int_0^1 (\eta(1-\eta))^{-1/m} d\eta$ for $m > 1$. For each positive number q such that

$$(1.9) \quad q \geq F(m)^2,$$

$P(m,r)$ has a stationary solution u such that

$$(1.10) \quad r^2 \|u\|_{\infty}^{2-m} = q$$

and the length of the support of u is less than or equal to r .
(See Figure 8.1.) |

The conditions (1.8) and (1.9) with (1.10) are reduced to

$$r \leq \sqrt{2} \quad \text{and} \quad r \geq \pi,$$

respectively, when $m = 2$. Hence Theorem 1.2 is a partial extension of Theorem 1.1. We shall also give a conjecture for the global structure of stationary solutions of $P(m,r)$ in Section 8.

We here review mathematical works related to the Cauchy problem (1.1) subject to a non-negative initial condition.

When the integral kernel $K = 0$, (1.1) is reduced to the porous medium equation (1.3). The distinctive feature of (1.3), which is caused by the degeneracy of diffusion at $u = 0$, is that an initial distribution with compact support spreads out at a finite speed (Oleinik et al. [37]) and loses its initial smoothness (Aronson [4] and Kalashnikov [22]). We can observe this feature in an exact solution obtained by Barenblatt [7] and Pattle [38]. A similar exact solution of $u_t = (u^2)_{xx} + u(1-u)$ was found by Newman [35]. A solution of the Cauchy problem (1.3) is therefore defined in a generalized sense, and the regularity of the generalized solution is discussed in Aronson [3], Caffarelli and Friedman [9], Gilding and Peletier [14], etc. Because of the finite propagation property, there appear interface curves that divide the half plane $\mathbb{R} \times (0, \infty)$ into two regions $\{(x,t); u(x,t) > 0\}$ and $\{(x,t); u(x,t) = 0\}$ (for instance, Aronson [5], Caffarelli and Friedman [10] and Knerr [25]). Graveleau and Jamet [15] and Tomoeda and Mimura [43] have proposed

finite difference schemes where the degeneracy of diffusion is taken into account. Schemes proposed by DiBenedetto and Hoff [12], Mimura et al. [28] and Hoff [20] approximate the interface curves as well as the value of the unknown function.

The Cauchy problem (1.1) has the same distinctive property as that of (1.3). For a class of integral kernels K including the simple one given by (1.2), Nagai [31] has shown the unique existence and the regularity of the generalized solution of (1.1), and he has proved the finite propagation property. Nagai and Mimura [33] have obtained the asymptotic behavior of the solution of $P(m, \infty)$: as time t tends to infinity, a solution of $P(m, \infty)$ tends to a stationary solution uniquely determined by the initial condition. Recently, Nagai and Mimura [34] have derived an equation that describes the motion of the interfaces to $P(m, \infty)$, and have obtained the asymptotic behavior of the interfaces. Ikeda [21] has devised a finite difference scheme for $P(2, \infty)$ and has obtained the asymptotic behavior of the discrete solution. (Figure 1.2 displays a solution of the Cauchy problem $P(2, \infty)$ obtained by this scheme.) A modification of this scheme has been useful in the study of the stationary solution of $P(2, r)$.

Alt ([1] and [2]) has studied stable patterns reproduced by another spatially aggregating population model providing both a degenerate diffusion process and an aggregating process expressed by a nonlocal interaction.

2. Regularity of a stationary solution

In this section, we first define a stationary solution of $P(m,r)$ and then consider its regularity.

A stationary solution u of $P(m,r)$ is defined to be a non-negative valued function belonging to $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ that satisfies

$$(2.1) \quad \left\{ (u^m)_x + u \left\{ \int_{x-r}^x u dy - \int_x^{x+r} u dy \right\} \right\}_x = 0$$

in the distribution sense. (We denote by $C_0^\infty(\Omega)$ the space of infinitely differentiable functions defined on an open subset Ω of \mathbb{R} having compact support in Ω .)

Let u be a stationary solution of $P(m,r)$. An application of the distribution theory yields that the distribution

$$F[u] \equiv (u^m)_x + u \left\{ \int_{x-r}^x u dy - \int_x^{x+r} u dy \right\}$$

is a constant C , and that this constant C is represented by

$$C = F[u](\phi) \equiv \langle F[u], \phi \rangle$$

with $\phi \in C_0^\infty(\mathbb{R})$ such that $\int_{-\infty}^{\infty} \phi(x) dx = 1$ (Schwartz [41, pp.51-53]).

The constant C equals zero by the following proposition.

Proposition 2.1 (Regularity of a stationary solution). If u is a stationary solution of $P(m,r)$, then

$$u \in C(\mathbb{R}) \quad \text{and} \quad u^m \in C^1(\mathbb{R}),$$

and u satisfies

$$(2.2) \quad F[u] \equiv (u^m)_x + u \left\{ \int_{x-r}^x u dy - \int_x^{x+r} u dy \right\} = 0 \quad \text{in } \mathbb{R}$$

in the classical sense. Moreover, u is twice continuously differentiable on the region $D_u = \{x \in \mathbb{R}; u(x) > 0\}$.

Proof. We fix a function $\phi \in C_0^\infty(\mathbb{R})$ satisfying $\int_{-\infty}^{\infty} \phi(x) dx = 1$, and define by $\phi_n(x) = \frac{1}{n} \phi(\frac{x}{n})$ a sequence $\{\phi_n\}_{n=1,2,\dots}$, $\phi_n \in C_0^\infty(\mathbb{R})$ and $\int_{-\infty}^{\infty} \phi_n(x) dx = 1$. Since u belongs to $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$, the constant $C = \langle F[u], \phi \rangle = \langle F[u], \phi_n \rangle$ is estimated by

$$\begin{aligned}
 |C| &\leq |\langle u^m, (\phi_n)_x \rangle| + |\langle u \{ \int_{x-r}^x u dy - \int_x^{x+r} u dy \}, \phi_n \rangle| \\
 (2.3) \quad &\leq \|u^m\|_\infty \|(\phi_n)_x\|_1 + \|u\|_1^2 \|\phi_n\|_\infty \\
 &= \frac{1}{n} \{ \|u^m\|_\infty \|\phi_x\|_1 + \|u\|_1^2 \|\phi\|_\infty \} + 0 \quad \text{as } n \rightarrow \infty,
 \end{aligned}$$

which implies that C equals zero. (The symbol $\|\cdot\|_1$ denotes the usual norm of $L^1(\mathbb{R})$.) Since $F[u] \equiv 0$, $(u^m)_x$ is represented by

$$(2.4) \quad (u^m)_x = -u \{ \int_{x-r}^x u dy - \int_x^{x+r} u dy \}.$$

The right-hand side of (2.4) belongs to $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$, hence $u^m \in C(\mathbb{R})$, and $u \in C(\mathbb{R})$. We know that $u^m \in C^1(\mathbb{R})$ by using (2.4) once again. Hence u satisfies (2.2) in the classical sense.

In general,

$$(2.5) \quad \begin{cases} \text{if } f \in C^1(G) \text{ and } f > 0 \text{ on } G \text{ (} G: \text{ an open set),} \\ \text{then } f^\alpha \in C^1(G) \text{ for all } \alpha \in \mathbb{R}. \end{cases}$$

Applying (2.5) to the case where $G = D_u$, $f = u^m$ and $\alpha = 1/m$, we know that $u = (u^m)^{1/m}$ is continuously differentiable on D_u . Now (2.4) yields that u^m is twice continuously differentiable on D_u since $\int_{x-r}^x u dy - \int_x^{x+r} u dy \in C^1(\mathbb{R})$. Hence u is twice continuously differentiable on D_u since $u_x = \frac{1}{m} u^{1-m} (u^m)_x$ on D_u and u^{1-m} is continuously

differentiable on D_u by (2.5). \square

A stationary solution u of $P(m, \infty)$ is infinitely differentiable on $D_u = \{x \in \mathbb{R}; u(x) > 0\}$ (Mimura and Yamaguti [29]). When $0 < r < \infty$, however, a stationary solution u of $P(m, r)$ is not always thrice continuously differentiable on D_u . In fact, a non-trivial stationary solution u of $P(2, r)$ is never thrice continuously differentiable on D_u when $r < \pi$, while u is infinitely differentiable on D_u when $r \geq \pi$ (Theorems 1.1 and 5.9).

As stated in Section 1, a stationary solution u of $P(m, r)$ is called "a standing solitary pulse" if u has a connected support and is positive on the interior of its support.

3. Decomposition of a stationary solution

In this section, we discuss the support of a stationary solution of $P(m,r)$ and decompose a stationary solution into the standing solitary pulses. The support of a continuous function v is denoted by $\text{supp}[v]$. A standing solitary pulse is abbreviated to an "ss-pulse".

Even if both u and v are ss-pulses of $P(m,r)$, $u + v$ is not always a stationary solution of $P(m,r)$, of course. However, if the distance between $\text{supp}[u]$ and $\text{supp}[v]$ is greater than or equal to r , then $F[u+v] = F[u] + F[v]$ ((2.2) in Proposition 2.1) and $u + v$ becomes a stationary solution of $P(m,r)$.

The following theorem asserts that all non-trivial stationary solutions of $P(m,r)$ are obtained by adding ss-pulses in the above way:

Theorem 3.1 (Decomposition of a stationary solution). All non-trivial stationary solutions u of $P(m,r)$, $r > 0$, are expressed in the form

$$u = \sum_{i \in \Lambda} u_i,$$

where $\{u_i\}_{i \in \Lambda}$ (Λ : an index set) is a finite or countable sequence of ss-pulses such that

$$\text{dis}(\text{supp}[u_i], \text{supp}[u_j]) \geq r \quad \text{for all } i \in \Lambda, j \in \Lambda, i \neq j,$$

where $\text{dis}(A,B)$ denotes the distance between subsets A and B of R .

Proof. The proof is given later in this section. \square

By virtue of Theorem 3.1, we have only to study the ss-pulse in the forthcoming sections.

We begin with a change of the unknown function. With a stationary solution u of $P(m,r)$, we associate a function U defined by

$$(3.1) \quad U(x) = \int_{-\infty}^x u(y)dy \quad \text{for } x \in \mathbb{R}.$$

This function U has the following properties:

$$(3.2) \quad \left\{ \begin{array}{l} U \in C^1(\mathbb{R}), \quad U \text{ is non-decreasing on } \mathbb{R}, \\ U(x) \rightarrow 0 \text{ as } x \rightarrow -\infty \quad \text{and} \quad U(x) \rightarrow \|u\|_1 \text{ as } x \rightarrow \infty, \\ U \text{ is thrice continuously differentiable on } D_u, \end{array} \right.$$

where $D_u = \{x \in \mathbb{R}; u(x) > 0\}$. By the function U , (2.2) is rewritten as

$$(3.3) \quad \{U_x(x)^m\}_x - U_x(x)\{U(x+r)+U(x-r)-2U(x)\} = 0 \quad \text{in } \mathbb{R}.$$

A primitive function $J[U]$ of the function on the left-hand side of (3.3) is given by

$$(3.4) \quad \begin{aligned} J[U](x) &= U_x(x)^m + \int_x^{x+r} U_y(y)\{U(y-r)-U(x)\}dy \\ &= u(x)^m + \int_x^{x+r} u(y)\{U(y-r)-U(x)\}dy \quad \text{for } x \in \mathbb{R}. \end{aligned}$$

By (3.3), $J[U]$ equals a constant C . Moreover, since $u \in L^\infty(\mathbb{R})$ by the definition of a stationary solution,

$$\int_x^{x+r} u(y)\{U(y-r)-U(x)\}dy \rightarrow 0 \quad \text{as } x \rightarrow \infty$$

by (3.2). Hence $\lim_{x \rightarrow \infty} u(x)^m$ exists and equals C . Since $u^m \in L^1(\mathbb{R})$ by the definition of a stationary solution, $C = \lim_{x \rightarrow \infty} u(x)^m = 0$, and

$$(3.5) \quad J[U](x) = 0 \quad \text{for } x \in \mathbb{R}.$$

By using the function $J[U]$, we obtain

Lemma 3.2. Let u be a stationary solution of $P(m,r)$. If $u(y) = 0$ at a point $y \in R$, then there exists a closed interval I such that $y \in I$, (the length of I) $\geq r$ and $u(x) = 0$ for $x \in I$.

Proof. Let $a = \inf\{x \in [y-r, y]; u = 0 \text{ on } [x, y]\}$ and $b = \sup\{x \in [y, y+r]; u = 0 \text{ on } [y, x]\}$. It suffices for the proof to show that $r \leq b - a$. Suppose that $r > b - a$. Let U be a non-decreasing function defined by (3.1). By the assumption,

$$U(x) < U(a) \quad \text{for } x < a$$

and there exists a real number c such that

$$b < c < a+r \quad \text{and} \quad U_x(x) = u(x) > 0 \quad \text{for } b < x < c.$$

These together with (3.2) lead to the estimate

$$\begin{aligned} J[U](a) &= u(a)^m + \int_a^{a+r} u(x) \{U(x-r) - U(a)\} dx \\ &= \int_b^{a+r} u(x) \{U(x-r) - U(a)\} dx \leq \int_b^c u(x) \{U(x-r) - U(a)\} dx < 0, \end{aligned}$$

which contradicts (3.5). Thus the proof is completed. \square

Lemma 3.2 enables us to prove Theorem 3.1.

Proof of Theorem 3.1. Let u be a non-trivial stationary solution of $P(m,r)$, $r > 0$. Lemma 3.2 shows that the distance between two arbitrary connected components of $\text{supp}[u]$ is greater than or equal to r . This implies that the number of connected components of $\text{supp}[u]$ is finite or countable. Let $\{S_i\}_{i \in \Lambda}$ be the sequence of connected components of $\text{supp}[u]$ (Λ : an index set). With each S_i ($i \in \Lambda$), we associate a function u_i given by

$$u_i(x) = u(x) \quad \text{for } x \in S_i \quad \text{and} \quad u_i(x) = 0 \quad \text{for } x \notin S_i.$$

Then, $u(x) = \sum_{i \in \Lambda} u_i(x)$ for $x \in R$, and each u_i is positive on the interior of S_i by Lemma 3.2. Now it suffices to show that each u_i becomes a stationary solution of $P(m,r): F[u_i] = 0$ on R ((2.2) in Proposition 2.1). Since $u_i = 0$ on $R \setminus S_i$, $F[u_i](x) = 0$ for $x \notin S_i$. Lemma 3.2 implies

$$u(y) = u_i(y) \quad \text{for } y \in R \text{ such that } \text{dis}((y), S_i) \leq r;$$

hence $F[u_i](x) = F[u](x) = 0$ for $x \in S_i$. We thus complete the proof of Theorem 3.1. \blacksquare

Remark 3.3. Theorem 3.1 and Lemma 3.2 are valid for all $0 \leq r \leq \infty$; hence a stationary solution of $P(m, \infty)$ has a connected support (Mimura and Yamaguti [29]). \blacksquare

Remark 3.4. Let $u(x,t)$ be a solution of the Cauchy problem $P(m,r)$ subject to a non-negative initial condition. Then the total population $\int_{-\infty}^{\infty} u(y,t) dy = \|u(\cdot, t)\|_1$ is kept constant (Gilding [13]). Hence $P(m,r)$ is transformed into

$$U_t(x,t) = \{U_x(x,t)^m\}_x - U_x(x,t)\{U(x+r,t)+U(x-r,t)-2U(x,t)\}$$

through the same change of the unknown function as (3.1). By $J[U]$, this equation is expressed in the divergence form

$$U_t(x,t) = J[U]_x(x,t). \quad \blacksquare$$

Remark 3.5. Let u be a stationary solution of $P(m,r)$ and define a non-decreasing function U by (3.1). The function $J[U]$ is rewritten as

$$J[U](x) = U_x(x)^m - U(x)(U(x+r)-U(x)) + \int_x^{x+r} U_y(y)U(y-r)dy.$$

Since $u = U_x \geq 0$ on R , $J[U] = 0$ on R and the third term on the

right-hand side of the above equality is non-negative by (3.2),

$$\begin{aligned}
 \|u\|_{\infty}^m &= \sup_{x \in \mathbb{R}} \{u(x)^m\} = \sup_{x \in \mathbb{R}} \{U_x(x)^m\} \\
 (3.6) \quad &\leq \sup_{x \in \mathbb{R}} \{U(x)(U(x+r)-U(x))\} \\
 &\leq \sup_{x \in \mathbb{R}} \{U(x)(\|u\|_1 - U(x))\} = (\|u\|_1/2)^2.
 \end{aligned}$$

When $r = \infty$, $J[U](x)$ is reduced to

$$J[U](x) = U_x(x)^m - U(x)(\|u\|_1 - U(x)),$$

and (3.6) is replaced by the equality

$$\begin{aligned}
 \|u\|_{\infty}^m &= \sup_{x \in \mathbb{R}} \{u(x)^m\} = \sup_{x \in \mathbb{R}} \{U_x(x)^m\} \\
 (3.7) \quad &= \sup_{x \in \mathbb{R}} \{U(x)(\|u\|_1 - U(x))\} = (\|u\|_1/2)^2. \quad \blacksquare
 \end{aligned}$$

4. Standing solitary pulses

In Sections 4 to 6, we study the ss-pulse of $P(2,r)$.

The present section gives a linear differential-difference equation that the ss-pulse satisfies in the interior of its support. By using this linear equation, we show that $P(2,r)$ has no non-trivial stationary solution when $r \leq \sqrt{2}$.

Let u be an ss-pulse of $P(m,r)$. Since $(u^m)_x = \frac{m}{m-1}u(u^{m-1})_x$ on the interior $\text{int}(\text{supp}[u])$ of $\text{supp}[u]$, dividing (2.2) by $u > 0$, we obtain

$$(4.1) \quad \frac{m}{m-1}(u^{m-1})_x + \left\{ \int_{x-r}^x u dy - \int_x^{x+r} u dy \right\} = 0 \quad \text{in } \text{int}(\text{supp}[u]).$$

Hence an ss-pulse u of $P(m,r)$ satisfies the differential-difference equation (Hale [18] and Lange and Miura [27])

$$(4.2) \quad \frac{m}{m-1}(u^{m-1})_{xx} - \{u(x+r)+u(x-r)-2u(x)\} = 0 \quad \text{in } \text{int}(\text{supp}[u]),$$

which is obtained by differentiating (4.1). We note that

- (1) both (4.1) and (4.2) are valid for all $m > 1$,
- (2) $u^{m-1} \in H^1(\mathbb{R})$ for all $m > 1$,
- (3) (4.2) is reduced to the linear equation when $m = 2$.

We denote by Δ and ∇ , respectively, the Laplacian operator and the gradient operator:

$$\Delta v \equiv v_{xx} \equiv (d/dx)^2 v \quad \text{and} \quad \nabla v \equiv v_x \equiv \frac{dv}{dx}.$$

We introduce a discrete Laplacian operator Δ_b and a discrete gradient operator ∇_b given by the central finite difference approximation for Δ and ∇ , respectively:

$$\begin{cases} (\Delta_b v)(x) \equiv \left(\frac{1}{b}\right)^2 (v(x+b) + v(x-b) - 2v(x)), \\ (\nabla_b v)(x) \equiv \frac{1}{b} (v(x+\frac{b}{2}) - v(x-\frac{b}{2})) \end{cases} \quad \text{for } b > 0.$$

For $m = 2$, the equation (4.2) is now rewritten as

$$(4.3) \quad \Delta u - \frac{1}{2}r^2 \Delta_r u = 0 \quad \text{in } \text{int}(\text{supp}[u]).$$

The first and second terms on the left-hand side of (4.3) correspond to the nonlinear diffusion process $(u^2)_{xx}$ and the aggregating process $\left[u \left(\int_{x-r}^x u dy - \int_x^{x+r} u dy \right) \right]_x$, respectively. We note that the problem of the balance between the diffusion process and the aggregating process is reduced to that of the balance between the Laplacian operator and the discrete Laplacian operator when $m = 2$. We also note that if $u(x)$ is an ss-pulse of $P(2,r)$, then so is $au(x)$ (a : an arbitrary positive number) as well as any translation of $u(x)$.

By (4.3), an ss-pulse u of $P(2,r)$ satisfies

$$(\Delta u - \frac{1}{2}r^2 \Delta_r u)v = 0 \quad \text{in } \mathbb{R}$$

for all $v \in H^1(\mathbb{R})$ such that $v = 0$ on $\mathbb{R} \setminus \text{int}(\text{supp}[u])$. Integrating this equation on \mathbb{R} , we obtain

$$(4.4) \quad (\nabla u, \nabla v) - \frac{1}{2}r^2 (\nabla_r u, \nabla_r v) = 0$$

for all $v \in H^1(\mathbb{R})$ such that $v = 0$ on $\mathbb{R} \setminus \text{int}(\text{supp}[u])$, where (\cdot, \cdot) denotes the inner product of $L^2(\mathbb{R})$. (For brevity the norm of $L^2(\mathbb{R})$ is denoted by $\|\cdot\|$ in the present paper.)

Lemma 4.1. The Laplacian operator is stronger than the discrete Laplacian operator in the sense that

$$(\nabla v, \nabla v) > (\nabla_b v, \nabla_b v) \quad \text{for all } v \in H^1(\mathbb{R}) \setminus \{0\} \text{ and } b > 0.$$

Proof. Hölder's inequality yields

$$|v(x+b)-v(x)|^2 = \left| \int_x^{x+b} v_y(y) dy \right|^2 \leq b \int_x^{x+b} |v_y(y)|^2 dy.$$

In the above, the equality sign holds only when v is constant on the interval $(x, x+b)$. Since $v \in H^1(\mathbb{R}) \setminus \{0\}$, v is not constant on \mathbb{R} and

$$\begin{aligned} (\nabla_b v, \nabla_b v) &= \left(\frac{1}{b}\right)^2 \int_{-\infty}^{\infty} |v(x+\frac{b}{2})-v(x-\frac{b}{2})|^2 dx = \left(\frac{1}{b}\right)^2 \int_{-\infty}^{\infty} |v(x+b)-v(x)|^2 dx \\ &< \frac{1}{b} \int_{-\infty}^{\infty} dx \int_x^{x+b} |v_y(y)|^2 dy = \frac{1}{b} \int_{-\infty}^{\infty} dy \int_{y-b}^y |v_y(y)|^2 dx \\ &= \int_{-\infty}^{\infty} |v_y(y)|^2 dy = (\nabla v, \nabla v), \end{aligned}$$

which completes the proof. \square

An ss-pulse u of $P(2, r)$ must satisfy (4.4). Hence Lemma 4.1 implies that $P(2, r)$ has no non-trivial ss-pulse when $r \leq \sqrt{2}$. Combining this fact and Theorem 3.1, we obtain

Theorem 4.2 (Non-existence of a stationary solution for $r \leq \sqrt{2}$). Let $r \leq \sqrt{2}$. Then the equation $P(2, r)$ has no non-trivial stationary solution. \square

Lemma 4.3. Let b and c be positive numbers ($b \leq c$). Then,

$$\|\nabla_b v - \nabla_c v\| \leq \frac{2}{c} |c-b| \|\nabla v\| \quad \text{for all } v \in H^1(\mathbb{R}).$$

Proof. We put $\beta = b/2$ and $\gamma = c/2$. By Cauchy's inequality,

$$\begin{aligned} (bc |\nabla_b v - \nabla_c v|)^2 &= \left| c \int_{x-\beta}^{x+\beta} v_y(y) dy - b \int_{x-\gamma}^{x+\gamma} v_y(y) dy \right|^2 \\ &= \left| (c-b) \int_{x-\beta}^{x+\beta} v_y(y) dy - b \left(\int_{x-\gamma}^{x-\beta} v_y(y) dy + \int_{x+\beta}^{x+\gamma} v_y(y) dy \right) \right|^2 \\ &\leq 2(c-b)^2 \left| \int_{x-\beta}^{x+\beta} v_y dy \right|^2 + 2b^2 \left| \int_{x-\gamma}^{x-\beta} v_y dy + \int_{x+\beta}^{x+\gamma} v_y dy \right|^2 \\ &\leq 2(c-b)^2 \left| \int_{x-\beta}^{x+\beta} v_y dy \right|^2 + 4b^2 \left| \int_{x-\gamma}^{x-\beta} v_y dy \right|^2 + 4b^2 \left| \int_{x+\beta}^{x+\gamma} v_y dy \right|^2. \end{aligned}$$

Then Hölder's inequality yields

$$\begin{aligned} \{bc|\nabla_b v - \nabla_c v|\}^2 &\leq 2b|c-b|^2 \int_{x-\beta}^{x+\beta} |v_y|^2 dy \\ &\quad + 2b^2|c-b| \int_{x-\gamma}^{x-\beta} |v_y|^2 dy + 2b^2|c-b| \int_{x+\beta}^{x+\gamma} |v_y|^2 dy. \end{aligned}$$

Now, by the same method as in the proof of Lemma 4.1, $\{bc\|\nabla_b v - \nabla_c v\|\}^2$ is estimated by

$$\begin{aligned} \{bc\|\nabla_b v - \nabla_c v\|\}^2 &= \int_{-\infty}^{\infty} \{bc|\nabla_b v - \nabla_c v|\}^2 dx \\ &\leq 2b^2|c-b|^2 \|\nabla v\|^2 + b^2|c-b|^2 \|\nabla v\|^2 + b^2|c-b|^2 \|\nabla v\|^2 \\ &= \{2b|c-b|\|\nabla v\|\}^2. \end{aligned}$$

This completes the proof. \square

Lemma 4.4. Let $\{v, V\}$ be a pair of functions on \mathbb{R} such that $v \in C(\mathbb{R})$, $V \in C^1(\mathbb{R})$ and both v and V are twice continuously differentiable on a bounded interval (α, β) . If $V = 0$ on $(-\infty, \alpha]$, $v = 0$ on $\mathbb{R} \setminus (\alpha, \beta)$ and $-\Delta v + \frac{1}{2}r^2 \Delta_b v = 0$ in (α, β) , then

$$(4.5) \quad \int_{\alpha}^{\beta} (-\Delta V + \frac{1}{2}r^2 \Delta_b V) v dx = V(\beta) v_x(\beta-0) + \frac{1}{2} \left(\frac{r}{b}\right)^2 \int_{\beta-b}^{\beta} V(x+b) v(x) dx.$$

Proof. Let ϵ and δ be small positive numbers. By integration by parts,

$$\begin{aligned} \int_{\alpha+\epsilon}^{\beta-\delta} (-\Delta V) v dx &= -[V_x v]_{\alpha+\epsilon}^{\beta-\delta} + \int_{\alpha+\epsilon}^{\beta-\delta} V_x v_x dx \\ &= [V v_x - v_x V]_{\alpha+\epsilon}^{\beta-\delta} + \int_{\alpha+\epsilon}^{\beta-\delta} (-\Delta v) V dx \\ &= [V v_x - v_x V]_{\alpha+\epsilon}^{\beta-\delta} - \int_{\alpha+\epsilon}^{\beta-\delta} \left(\frac{1}{2}r^2 \Delta_b v\right) V dx. \end{aligned}$$

Letting $\epsilon \rightarrow 0$ and $\delta \rightarrow 0$, and noting that $V(\alpha) = v(\alpha) = v(\beta) = 0$, we

obtain

$$(4.6) \quad \int_{\alpha}^{\beta} (-\Delta V) v dx = \int_{\alpha}^{\beta} \left(-\frac{1}{2}r^2 \Delta_b v\right) v dx + V(\beta) v_x(\beta-0).$$

Noting that $v = 0$ on $\mathbb{R} \setminus (\alpha, \beta)$ and $V = 0$ on $(-\infty, \alpha]$, we obtain

$$(4.7) \quad \begin{aligned} \int_{\alpha}^{\beta} \left(\frac{1}{2}r^2 \Delta_b v\right) v dx &= \int_{-\infty}^{\infty} \left(\frac{1}{2}r^2 \Delta_b v\right) v dx = \int_{-\infty}^{\infty} \left(\frac{1}{2}r^2 \Delta_b v\right) v dx \\ &= \int_{\alpha}^{\beta} \left(\frac{1}{2}r^2 \Delta_b v\right) v dx + \frac{1}{2} \left(\frac{r}{b}\right)^2 \int_{\beta}^{\beta+b} v(x-b) v(x) dx \\ &= \int_{\alpha}^{\beta} \left(\frac{1}{2}r^2 \Delta_b v\right) v dx + \frac{1}{2} \left(\frac{r}{b}\right)^2 \int_{\beta-b}^{\beta} v(x+b) v(x) dx. \end{aligned}$$

Now (4.5) is obtained by adding (4.6) and (4.7). \square

We here classify ss-pulses into four groups according to support and symmetry. An ss-pulse u is called a finite ss-pulse when $\text{supp}[u]$ is compact. Finite ss-pulses are classified into two groups; one is the group of symmetric finite ss-pulses that are symmetric with respect to the center of support and the other is the group of non-symmetric ones. An ss-pulse u is called an infinite ss-pulse when $\text{supp}[u] = \mathbb{R}$, while u is called a semi-infinite ss-pulse when $\text{supp}[u]$ is a semi-infinite interval. Thus we have classified ss-pulses into four groups:

- {symmetric finite ss-pulses},
- {non-symmetric finite ss-pulses},
- {infinite ss-pulses},
- {semi-infinite ss-pulses}.

For the equation $P(2, r)$, $r > \sqrt{2}$, we shall show the existence of a symmetric finite ss-pulse in Section 5, and show the non-existence of other three kinds of ss-pulses in Section 6.

5. Symmetric finite standing solitary pulses

In this section, we study the symmetric finite ss-pulse of $P(2,r)$.

Let u be a symmetric finite ss-pulse of $P(2,r)$. We may assume $\text{supp}[u]$ to equal an interval $[-\frac{s}{2}, \frac{s}{2}]$ (s : an unknown positive number) since any translation of u also is a symmetric finite ss-pulse. In general, (4.1) is not equivalent to (4.2), which is obtained by differentiating (4.1). As for even functions however, (4.1) and (4.2) are mutually equivalent: if an even function is a solution of (4.2), then it also fulfils (4.1). Thus the problem to find a symmetric finite ss-pulse is described in the form of the following free boundary problem:

$$(5.1) \quad \begin{cases} \text{Find an even function } u \in C(\mathbb{R}) \text{ and } s > 0 \text{ such that} \\ -\Delta u + \frac{1}{2}r^2 \Delta_r u = 0 & \text{in } (-\frac{s}{2}, \frac{s}{2}), \\ u > 0 & \text{on } (-\frac{s}{2}, \frac{s}{2}), \quad u = 0 & \text{on } \mathbb{R} \setminus (-\frac{s}{2}, \frac{s}{2}). \end{cases}$$

By the coordinate transformation $x \rightarrow \frac{\pi}{s}x$, (5.1) is transformed into

$$(5.2) \quad \begin{cases} \text{Find an even function } w \in C(\mathbb{R}) \text{ and } b > 0 \text{ such that} \\ -\Delta w + \frac{1}{2}r^2 \Delta_b w = 0 & \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}), \\ w > 0 & \text{on } (-\frac{\pi}{2}, \frac{\pi}{2}), \quad w = 0 & \text{on } \mathbb{R} \setminus (-\frac{\pi}{2}, \frac{\pi}{2}). \end{cases}$$

Here is the following one-to-one correspondence between the solutions $\{u, s\}$ of (5.1) and $\{w, b\}$ of (5.2):

$$(5.3) \quad sb = \pi r \quad \text{and} \quad w(x) = u\left(\frac{s}{\pi}x\right) \quad \text{for } x \in \mathbb{R}.$$

We now consider the following eigenvalue problem:

$$E(r,b) \begin{cases} \text{For } r \geq 0 \text{ and } b > 0 \\ \text{find an even function } v \in C(\mathbb{R}) \setminus \{0\} \text{ and } \lambda \in \mathbb{R} \text{ such that} \\ A_{r,b} v = \lambda v \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}), \quad v = 0 \quad \text{on } \mathbb{R} \setminus (-\frac{\pi}{2}, \frac{\pi}{2}), \end{cases}$$

where $A_{r,b}$ denotes the operator $-\Delta + \frac{1}{2}r^2 \Delta_b$. Clearly, (u,s) is a solution of (5.1) if and only if $(v(x)=u(\frac{s}{\pi}x), 0)$ is a solution of $E(r, \pi r/s)$ that satisfies the positivity condition

$$(5.4) \quad v > 0 \quad \text{on } (-\frac{\pi}{2}, \frac{\pi}{2}).$$

In the following four subsections, we shall show that when $r > \sqrt{2}$, $P(2,r)$ has a symmetric finite ss-pulse characterized as a principal eigenfunction of $E(r,b)$ through the coordinate transformation $x \rightarrow \frac{\pi}{s}x$. We shall also show that the symmetric finite ss-pulse of $P(2,r)$ is unique up to the coordinate translation and the multiplication by a positive number. In practice, we shall prove the following properties of the family of eigenvalue problems $E(r,b)$, $r \geq 0$ and $b > 0$:

- (1) each $E(r,b)$ has an increasing sequence of eigenvalues (Subsection 5.1),
- (2) for each $r > \sqrt{2}$, there exists a unique positive number $b(r)$ such that the principal eigenvalue of $E(r,b(r))$ equals zero (Subsection 5.2),
- (3) the principal eigenvalue of $E(r,b(r))$ is simple and a principal eigenfunction is positive on $(-\frac{\pi}{2}, \frac{\pi}{2})$ or negative on $(-\frac{\pi}{2}, \frac{\pi}{2})$ (Subsection 5.3),
- (4) no n -th eigenfunction ($n > 1$) is of one sign on $(-\frac{\pi}{2}, \frac{\pi}{2})$ when the corresponding eigenvalue equals zero (Subsection 5.3).

5.1. Variational formulation of the eigenvalue problem

We practically consider the variational formulation of $E(r,b)$. For this purpose, we introduce a function space H_S and a symmetric continuous bilinear form $a_{r,b}$ in H_S associated with the operator $A_{r,b}$. The function space H_S is a subspace of $H^1(\mathbb{R})$ given by

$$H_S = \{v \in H^1(\mathbb{R}); v = 0 \text{ on } \mathbb{R} \setminus (-\frac{\pi}{2}, \frac{\pi}{2}), v(x) = v(-x) \text{ for } x \in \mathbb{R}\}$$

equipped with the norm

$$\|v\|_{2,1} = \{(v,v) + (\nabla v, \nabla v)\}^{1/2}.$$

The space H_S is identified with the space that consists of even functions belonging to $H_0^1((-\frac{\pi}{2}, \frac{\pi}{2}))$ $\left[H_0^1((-\frac{\pi}{2}, \frac{\pi}{2})) \right]$: the closure of $C_0^\infty((-\frac{\pi}{2}, \frac{\pi}{2}))$ in $H^1((-\frac{\pi}{2}, \frac{\pi}{2}))$. The bilinear form $a_{r,b}: H_S \times H_S \rightarrow \mathbb{R}$ is given by

$$a_{r,b}(v,w) = (\nabla v, \nabla w) - \frac{1}{2}r^2(\nabla_b v, \nabla_b w).$$

We thus obtain the variational formulation of $E(r,b)$:

$$e(r,b) \begin{cases} \text{For } r \geq 0 \text{ and } b > 0, \text{ find } v \in H_S \setminus \{0\} \text{ and } \lambda \in \mathbb{R} \text{ such that} \\ a_{r,b}(v,w) = \lambda(v,w) \quad \text{for all } w \in H_S. \end{cases}$$

A solution of $E(r,b)$ becomes a solution of $e(r,b)$, of course. Let $\{v, \lambda\}$ be a solution of $e(r,b)$. We can identify v with the solution w of the equation

$$-\Delta w = -\frac{1}{2}r^2 \Delta_b v + \lambda v \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}), \quad w = 0 \quad \text{on } \mathbb{R} \setminus (-\frac{\pi}{2}, \frac{\pi}{2}).$$

The solution w is twice continuously differentiable on $(-\frac{\pi}{2}, \frac{\pi}{2})$ since the right-hand side of the above equation belongs to $H^1(\mathbb{R})$. Conse-

quently, $e(r,b)$ is equivalent to $E(r,b)$.

We associate a symmetric continuous bilinear form in H_S

$$\tilde{a}_{r,b}(v,w) = a_{r,b}(v,w) + 2\left(\frac{r}{b}\right)^2(v,w)$$

with $a_{r,b}$, and consider another eigenvalue problem

$$\tilde{e}(r,b) \begin{cases} \text{For } r \geq 0 \text{ and } b > 0, \text{ find } v \in H_S \setminus \{0\} \text{ and } \tilde{\lambda} \in \mathbb{R} \text{ such that} \\ \tilde{a}_{r,b}(v,w) = \tilde{\lambda}(v,w) \quad \text{for all } w \in H_S. \end{cases}$$

Clearly, (v,λ) is a solution of $e(r,b)$ if and only if $(v,\lambda+2\left(\frac{r}{b}\right)^2)$ is a solution of $\tilde{e}(r,b)$. By the estimate

$$\|\nabla_b v\| = \frac{1}{b} \left(\int_{-\infty}^{\infty} |v(x+\frac{b}{2}) - v(x-\frac{b}{2})|^2 dx \right)^{1/2} \leq \frac{2}{b} \|v\| \quad \text{for all } v \in H_S$$

and the Poincaré inequality, the bilinear form $\tilde{a}_{r,b}$ is H_S -elliptic:

$$\begin{aligned} \tilde{a}_{r,b}(v,v) &= \|\nabla v\|^2 - \frac{1}{2}r^2 \|\nabla_b v\|^2 + 2\left(\frac{r}{b}\right)^2 \|v\|^2 \\ (5.5) \quad &\geq \|\nabla v\|^2 = \{2\|\nabla v\|^2 + \pi^2 \|\nabla v\|^2\} / (2+\pi^2) \\ &\geq 2\|v\|_{2,1}^2 / (2+\pi^2) \quad \text{for all } v \in H_S. \end{aligned}$$

Proposition 5.1 (Characterization of principal eigenvalue).

(1) The eigenvalue problem $e(r,b)$ has an increasing sequence of eigenvalues

$$-\infty < \lambda_1(r,b) \leq \lambda_2(r,b) \leq \dots, \quad \text{with } \lim_{i \rightarrow \infty} \lambda_i(r,b) = \infty.$$

(2) Eigenfunctions v_i associated with λ_i ($i=1,2,\dots$) can be orthonormalized so that

$$a_{r,b}(v_i, v_j) = \lambda_i \delta_{ij} \quad \text{and} \quad (v_i, v_j) = \delta_{ij} \quad \text{for all } i \text{ and } j,$$

where δ denotes Kronecker's delta.

(3) The principal eigenvalue $\lambda_1(r,b)$ is characterized by

$$\lambda_1(r,b) = \inf\{R_{r,b}(v); v \in H_S \setminus \{0\}\} = R_{r,b}(v_1)$$

with the aid of the Rayleigh quotient

$$R_{r,b}(v) \equiv a_{r,b}(v,v) / \|v\|^2 \quad \text{for } v \in H_S \setminus \{0\}.$$

(4) A function $v \in H_S \setminus \{0\}$ becomes a principal eigenfunction of $e(r,b)$ if $R_{r,b}(v) = \lambda_1(r,b)$.

Proof. The assertions of the proposition have been proved for the symmetric continuous H_S -elliptic bilinear forms. (For instance, refer to Ciarlet [11, pp.283-286], Riesz and Nagy [39, pp.231-235] and Kato [23, pp.260-262].) This together with the foregoing argument on the relation between $e(r,b)$ and $\tilde{e}(r,b)$ completes the proof. \square

The operator Δ_b is reduced to $-2/b^2$ when $b \geq \pi$, hence the solutions of $e(r,b)$, $b \geq \pi$, are given by

$$v_i(x) = \cos(2i-1)x \quad \text{on } (-\frac{\pi}{2}, \frac{\pi}{2}) \quad \text{and} \quad \lambda_i = (2i-1)^2 - (\frac{\pi}{b})^2.$$

Thus, when $b \geq \pi$, the principal eigenfunction of $e(r,b)$ is positive on $(-\frac{\pi}{2}, \frac{\pi}{2})$ or negative on $(-\frac{\pi}{2}, \frac{\pi}{2})$, the support of the eigenfunction equals $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and

$$\|\nabla v_i\|^2 / \|v_i\|^2 = (2i-1)^2 \quad \text{for } i = 1, 2, \dots$$

When $0 < b < \pi$, a principal eigenfunction of $e(r,b)$ is not always of one sign on $(-\frac{\pi}{2}, \frac{\pi}{2})$, which will be discussed in Subsection 5.3. But we obtain the following two propositions about the support of an eigenfunction and the H_S -norm of a principal eigenfunction.

Proposition 5.2 (Support of an eigenfunction). The support of an eigenfunction of $e(r,b)$ equals $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Proof. We omit the complete proof, and show only that the support of an eigenfunction is never included in $(-\frac{\pi}{2}, \frac{\pi}{2})$. We have already noted that the support of an eigenfunction of $e(r, b)$ equals $[-\frac{\pi}{2}, \frac{\pi}{2}]$ when $b \geq \pi$. Suppose that $e(r, b)$, $b < \pi$, has a solution $\{v, \lambda\}$ such that $\text{supp}\{v\}$ is included in an interval $[-\frac{\pi}{2}+d, \frac{\pi}{2}-d]$ (d : a positive number). We may assume that π/b is not an integer since we can replace $\{v, \lambda\}$ by some $\{w(x)=v(cx), c^2\lambda\}$ ($1-2d/\pi < c < 1$) if necessary, which is a solution of $e(r, \frac{b}{c})$ such that $\text{supp}\{w\}$ is included in $(-\frac{\pi}{2}, \frac{\pi}{2})$. Let $n = \max\{i \in \mathbb{Z}; ib < \pi\}$. We may assume d to be less than $\frac{1}{2}(\pi-nb)$. We partition $[-\frac{\pi}{2}, \frac{\pi}{2}]$ into the intervals

$$\begin{cases} [y_i, y_i+d], & (y_i+d, z_i-d), & [z_i-d, z_i] & (i=0, 1, \dots, n), \\ (z_i, y_{i+1}) & (i=0, 1, \dots, n-1), \end{cases}$$

where $y_i = -\frac{\pi}{2}+ib$ and $z_i = \frac{\pi}{2}-(n-i)b$ ($i=0, 1, \dots, n$). (See Figure 5.1.)

Since $\{v, \lambda\}$ is regarded as a solution of $E(r, b)$ and $v = 0$ on $[y_0, y_0+d]$,

$$0 = A_{r,b} v(x) - \lambda v(x) = \frac{1}{2} \left(\frac{r}{b}\right)^2 v(x+b) \quad \text{for } x \in (y_0, y_0+d).$$

This implies that $v = 0$ on $[y_1, y_1+d]$ since $v \in C(\mathbb{R})$. By mathematical induction, we know that $v = 0$ on $[y_i, y_i+d]$ ($i=0, \dots, n$). The fact that $v = 0$ on $[z_i-d, z_i]$ ($i=0, \dots, n$) is shown by the same method of proof.

Let v_i be the restriction of v on (y_i+d, z_i-d) ($i=0, \dots, n$). We regard v_i 's as functions defined on the interval $(0, \pi-nb-2d)$ through suitable coordinate translations. Since $\{v, \lambda\}$ is regarded as a solution of $E(r, b)$ and $v = 0$ on $[y_i, y_i+d]$ and $[z_i-d, z_i]$ ($i=0, \dots, n$), v_i 's ($i=0, 1, \dots, n$) satisfy

$$\begin{cases} -\Delta v_i + \frac{1}{2}\left(\frac{r}{b}\right)^2(v_{i-1} - 2v_i + v_{i+1}) = \lambda v_i & \text{in } (0, \pi - nb - 2d), \\ v_i(0) = (v_i)_x(0) = v_i(\pi - nb - 2d) = (v_i)_x(\pi - nb - 2d) = 0, \end{cases}$$

where $v_{-1} \equiv v_{n+1} \equiv 0$. This system of linear equations has no non-trivial solution for any $\lambda \in \mathbb{R}$. Hence $v = 0$ on $[y_i + d, z_i - d]$ ($i=0, \dots, n$). The fact that $v = 0$ on $[z_i, y_{i+1}]$ ($i=0, \dots, n-1$) is shown by the same method of proof. Thus,

$$v = 0 \quad \text{on } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

which contradicts the assumption that v is an eigenfunction of $e(r, b)$. \square

Proposition 5.3 (H_S -norm of a principal eigenfunction). A principal eigenfunction v of $e(r, b)$ satisfies the estimate

$$\|\nabla v\|^2 / \|v\|^2 \leq 1 + 2\left(\frac{r}{b}\right)^2 \equiv K(r, b).$$

Proof. Let v be a principal eigenfunction of $e(r, b)$, associated with $\lambda_1(r, b)$. Then v is a principal eigenfunction of $\tilde{e}(r, b)$, associated with $\tilde{\lambda}_1(r, b) = \lambda_1(r, b) + 2\left(\frac{r}{b}\right)^2$. Using (5.5) and applying (3) of Proposition 5.1 to $\tilde{e}(r, b)$, we obtain

$$\begin{aligned} \|\nabla v\|^2 / \|v\|^2 &\leq \tilde{a}_{r, b}(v, v) / \|v\|^2 = \inf\{\tilde{a}_{r, b}(w, w) / \|w\|^2; w \in H_S \setminus \{0\}\} \\ &\leq \inf\left\{\left[\|\nabla w\|^2 + 2\left(\frac{r}{b}\right)^2 \|w\|^2\right] / \|w\|^2; w \in H_S \setminus \{0\}\right\}. \end{aligned}$$

This together with the equality

$$\begin{aligned} \inf\{\|\nabla w\|^2 / \|w\|^2; w \in H_S \setminus \{0\}\} &= \inf\{R_{0, b}(w); w \in H_S \setminus \{0\}\} \\ &= \lambda_1(0, b) = 1 \end{aligned}$$

(an application of (3) of Proposition 5.1 to $e(0, b)$) completes the proof. \square

5.2. Zero points of the principal eigenvalue

The purpose of the present subsection is to prove

Theorem 5.4 (Unique existence of a zero point). Let $r > \sqrt{2}$. Then there exists a unique positive number $b = b(r)$ such that the principal eigenvalue $\lambda_1(r, b)$ of $e(r, b)$ just equals zero. The function $b(r)$ has the following properties:

- (1) $b(r) = r$ for $r \geq \pi$ and $b(r) < \pi$ for $r < \pi$,
- (2) $b(r)$ increases with r ,
- (3) $b(r)$ is a continuous function of r ,
- (4) $b(r)$ tends to zero as r tends to $\sqrt{2}$.

(Figure 5.2 shows the graph of $b(r)$ obtained by a finite difference approximation for $e(r, b)$. The function $b(r)$ has the property

$$b(r) < r \quad \text{for } r < \pi,$$

which will be proved in the next subsection.) \square

We begin with the uniqueness of $b(r)$. Let us show that if $\lambda_1(r, b) = \lambda_1(r, c) = 0$ for $0 < b \leq c$, then $b = c$. Let w be a principal eigenfunction of $e(r, c)$, associated with $\lambda_1(r, c)$. Define a function \tilde{w} by $\tilde{w}(x) = w(\frac{c}{b}x)$ for $x \in \mathbb{R}$. Then $\tilde{w} \in H_S$ and

$$R_{r, b}(\tilde{w}) = (\frac{c}{b})^2 R_{r, c}(w) = (\frac{c}{b})^2 \lambda_1(r, c) = 0.$$

This together with (4) of Proposition 5.1 implies that \tilde{w} is a principal eigenfunction of $e(r, b)$, associated with $\lambda_1(r, b)$. Hence b should equal c since if $b < c$, then $\text{supp}[\tilde{w}]$ is included in $(-\frac{\pi}{2}, \frac{\pi}{2})$ and \tilde{w} is never an eigenfunction of $e(r, b)$ by Proposition 5.2. We

thus obtain the uniqueness of $b(r)$.

We next discuss the existence of $b(r)$. As stated in Subsection 5.1,

$$\lambda_1(r,b) = 1 - \left(\frac{r}{b}\right)^2 \quad \text{for } b \geq \pi,$$

which implies

$$(5.6) \quad \lambda_1(r,b) \geq 0 \quad \text{for } b \geq \max\{\pi, r\}.$$

On the other hand,

$$(5.7) \quad \lambda_1(r,b) < 0 \quad \text{for } 0 < b < B(r) \quad \text{and } r > \sqrt{2},$$

where $B(r)$ denotes the smallest positive solution of the equation

$$1 - \frac{1}{2r} \frac{2\pi-x}{\pi} \left(\frac{\sin(x/2)}{x/2}\right)^2 = 0.$$

This is because the principal eigenvalue is estimated by

$$\lambda_1(r,b) \leq R_{r,b}(w) < \frac{\pi}{2} \left[1 - \frac{1}{2r} \frac{2\pi-b}{\pi} \left(\frac{\sin(b/2)}{b/2}\right)^2 \right]$$

((3) of Proposition 5.1), where $w \in H_S$ is given by $w(x) = \cos x$ for $|x| \leq \frac{\pi}{2}$. By (5.6) and (5.7), for $r > \sqrt{2}$, there exists a positive number $b(r)$ such that $\lambda_1(r, b(r)) = 0$ if $\lambda_1(r, b)$ is continuous with respect to b . Consequently, the following proposition completes the proof of the first assertion of Theorem 5.4.

Proposition 5.5 (Continuity of the principal eigenvalue). The principal eigenvalue $\lambda_1(r, b)$ of $e(r, b)$ is a locally Lipschitz continuous function of $(r, b) \in (0, \infty) \times (0, \infty)$.

Proof. We fix arbitrary positive numbers R and β . Let p, q, b and c be positive numbers such that

$$p < R, \quad q < R \quad \text{and} \quad \beta < b \leq c.$$

We first estimate $|\lambda_1(q, b) - \lambda_1(p, b)|$. Let v_q and v_p be principal eigenfunctions associated with $\lambda_1(q, b)$ and $\lambda_1(p, b)$, respectively.

Using the Rayleigh quotient, we obtain

$$\left\{ \begin{array}{l} \lambda_1(q,b) - \lambda_1(p,b) = R_{q,b}(v_q) - R_{p,b}(v_p) \leq R_{q,b}(v_p) - R_{p,b}(v_p) \\ \qquad \qquad \qquad = \frac{1}{2}(p+q)(p-q) \|\nabla_b v_p\|^2 / \|v_p\|^2, \\ \lambda_1(p,b) - \lambda_1(q,b) \leq \frac{1}{2}(p+q)(q-p) \|\nabla_b v_q\|^2 / \|v_q\|^2. \end{array} \right.$$

From the above estimates, Lemma 4.1 and Proposition 5.3, it follows that

$$\begin{aligned} |\lambda_1(q,b) - \lambda_1(p,b)| &\leq \frac{1}{2}(p+q)|q-p| \max\{\|\nabla_b v_p\|^2 / \|v_p\|^2, \|\nabla_b v_q\|^2 / \|v_q\|^2\} \\ (5.8) \qquad \qquad \qquad &\leq \frac{1}{2}(p+q)|q-p| \max\{\|\nabla v_p\|^2 / \|v_p\|^2, \|\nabla v_q\|^2 / \|v_q\|^2\} \\ &\leq \frac{1}{2}(p+q)|q-p|K(R,b) \leq R|q-p|K(R,\beta). \end{aligned}$$

We next estimate $|\lambda_1(p,b) - \lambda_1(p,c)|$. By Lemmas 4.1 and 4.3,

$$\begin{aligned} \|v\|^2 |R_{p,b}(v) - R_{p,c}(v)| &= \frac{1}{2}p^2 |\|\nabla_b v\|^2 - \|\nabla_c v\|^2| \\ (5.9) \qquad \qquad \qquad &\leq \frac{1}{2}p^2 (\|\nabla_b v\| + \|\nabla_c v\|) \|\nabla_b v - \nabla_c v\| \leq \frac{2}{c}p^2 \|\nabla v\|^2 |c-b| \quad \text{for all } v \in H_S. \end{aligned}$$

Let v_b and v_c be principal eigenfunctions associated with $\lambda_1(p,b)$ and $\lambda_1(p,c)$, respectively. By (3) of Proposition 5.1,

$$\left\{ \begin{array}{l} \lambda_1(p,b) - \lambda_1(p,c) = R_{p,b}(v_b) - R_{p,c}(v_c) \leq R_{p,b}(v_c) - R_{p,c}(v_c), \\ \lambda_1(p,c) - \lambda_1(p,b) \leq R_{p,c}(v_b) - R_{p,b}(v_b). \end{array} \right.$$

Applying (5.9) to $v = v_c$ and $v = v_b$ and using Proposition 5.3, we obtain

$$\begin{aligned}
|\lambda_1(p,b) - \lambda_1(p,c)| &\leq \max\{|R_{p,b}(v_c) - R_{p,c}(v_c)|, |R_{p,c}(v_b) - R_{p,b}(v_b)|\} \\
&\leq \frac{2}{c} p^2 |c-b| \max\{\|\nabla v_c\|^2 / \|v_c\|^2, \|\nabla v_b\|^2 / \|v_b\|^2\} \\
&\leq \frac{2}{c} p^2 |c-b| K(p, \beta) \leq \frac{2}{\beta} R^2 |c-b| K(R, \beta).
\end{aligned}$$

This together with (5.8) completes the proof:

$$\begin{aligned}
|\lambda_1(q,b) - \lambda_1(p,c)| &\leq |\lambda_1(q,b) - \lambda_1(p,b)| + |\lambda_1(p,b) - \lambda_1(p,c)| \\
&\leq R\left(\frac{2}{\beta} R |c-b| + |q-p|\right) K(R, \beta) \\
&\text{for } 0 < p < R, 0 < q < R \text{ and } 0 < \beta < b \leq c. \quad \square
\end{aligned}$$

We now proceed to the discussion on the properties of the function $b(r)$.

The first property follows from the uniqueness of $b(r)$ and the fact that $\lambda_1(r,b) = 1 - (\frac{r}{b})^2$ for $b \geq \pi$.

Let us show the monotonicity (2):

$$b(p) < b(q) \quad \text{for } \sqrt{2} < p < q.$$

Let v be a principal eigenfunction associated with $\lambda_1(p, b(p))$. By (3) of Proposition 5.1 and the fact that $\nabla_{b(p)} v \not\equiv 0$,

$$\begin{aligned}
0 = \lambda_1(p, b(p)) = R_{p, b(p)}(v) &= \{ \|\nabla v\|^2 - \frac{1}{2} p^2 \|\nabla_{b(p)} v\|^2 \} / \|v\|^2 \\
&> \{ \|\nabla v\|^2 - \frac{1}{2} q^2 \|\nabla_{b(p)} v\|^2 \} / \|v\|^2 = R_{q, b(p)}(v),
\end{aligned}$$

which implies that $\lambda_1(q, b(p)) < 0$. This together with (5.6) and Proposition 5.5 implies that $b(p) < b(q)$.

We next show the continuity of $b(r)$. Let $\{r_n\}$ be an arbitrary monotone sequence of positive numbers $r_n > \sqrt{2}$ that converges to $r > \sqrt{2}$. Since $\{b(r_n)\}$ becomes a bounded monotone sequence by the property (2), $b(r_n)$ converges to a positive number $\bar{b} \geq b(\inf\{r_n\}) >$

0. Proposition 5.5 yields

$$|\lambda_1(r, \bar{b})| = |\lambda_1(r, \bar{b}) - \lambda_1(r_n, b(r_n))| \leq K(|r - r_n| + |\bar{b} - b(r_n)|),$$

where K is a positive constant independent of n . The right-hand side of the above inequality tends to zero as n tends to infinity. Hence $\lambda_1(r, \bar{b})$ equals zero, and $\lim_{n \rightarrow \infty} b(r_n) = \bar{b} = b(r) = b(\lim_{n \rightarrow \infty} r_n)$ since $b(r)$ uniquely exists.

Let us show the last property. Let $\{r_n\}$ be an arbitrary monotone decreasing sequence of positive numbers $r_n > \sqrt{2}$ that converges to $\sqrt{2}$. By the property (2) of $b(r)$, $b(r_n)$ converges to a non-negative number \bar{b} . Suppose that $\bar{b} > 0$. Then, using Proposition 5.5, we obtain the estimate

$$\begin{aligned} |\lambda_1(\sqrt{2}, \bar{b})| &= |\lambda_1(\sqrt{2}, \bar{b}) - \lambda_1(r_n, b(r_n))| \\ &\leq K(|\sqrt{2} - r_n| + |\bar{b} - b(r_n)|) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

where K is a positive constant independent of n . The above estimate implies that $\lambda_1(\sqrt{2}, \bar{b}) = 0$, which contradicts Lemma 4.1. Thus we have obtained the property (4). Now, the proof of Theorem 5.4 is completed except for the parenthesized assertion.

5.3. Sign of an eigenfunction

In the previous subsection, we have shown that for each $r > \sqrt{2}$, there exists a unique positive number $b(r)$ such that the principal eigenvalue $\lambda_1(r, b(r))$ of $e(r, b(r))$ equals zero. The present subsection is devoted to showing

- (1) a principal eigenfunction associated with $\lambda_1(r, b(r))$ is positive on $(-\frac{\pi}{2}, \frac{\pi}{2})$ or negative on $(-\frac{\pi}{2}, \frac{\pi}{2})$,
- (2) when the n -th eigenvalue $\lambda_n(r, b)$ ($n > 1$) equals zero, no eigenfunction associated with $\lambda_n(r, b)$ is of one sign on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

When $b \geq \pi$, a principal eigenfunction of $e(r, b)$ is positive on $(-\frac{\pi}{2}, \frac{\pi}{2})$ or negative on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and no n -th eigenfunction ($n > 1$) is of one sign on the interval, as stated in Subsection 5.1. When $b < \pi$, however, this is not always true. See Figure 5.3, which depicts eigenvalues and typical patterns of eigenfunctions obtained by a finite difference approximation for $e(r, b)$ with a fixed r and various values of b . The number of nodes of a principal eigenfunction v , which we here define by the number of zero points of v on $(-\frac{\pi}{2}, \frac{\pi}{2})$, increases with decreasing b , and n -th eigenfunctions ($n > 1$) can be of one sign on $(-\frac{\pi}{2}, \frac{\pi}{2})$ for suitable values of b .

To understand the behavior of eigenvalues and eigenfunctions of $e(r, b)$, let us consider another eigenvalue problem:

$$e^a(r, b) \begin{cases} \text{For } r \geq 0 \text{ and } b > 0, \text{ find } w \in H_a \setminus \{0\} \text{ and } \mu \in \mathbb{R} \text{ such that} \\ a_{r, b}(w, z) = \mu(w, z) \quad \text{for all } z \in H_a, \end{cases}$$

where H_a denotes the orthogonal complement of H_s in the space

$$H = \{v \in H^1(\mathbb{R}); v = 0 \text{ on } \mathbb{R} \setminus (-\frac{\pi}{2}, \frac{\pi}{2})\} = H_0^1((-\frac{\pi}{2}, \frac{\pi}{2})),$$

that is,

$$H_a = \{w \in H^1(\mathbb{R}); w = 0 \text{ on } \mathbb{R} \setminus (-\frac{\pi}{2}, \frac{\pi}{2}), w(x) = -w(-x) \text{ for } x \in \mathbb{R}\}.$$

The same assertions as those in Proposition 5.1 hold for $e^a(r,b)$.

Figure 5.4 depicts eigenvalues μ_n and typical patterns of eigenfunctions w_n obtained by a finite difference approximation for $e^a(r,b)$.

See Figure 5.5, which is obtained by putting together Figures 5.3 and 5.4.

Figure 5.5 reveals that the curves of n -th eigenvalues ($n=1,2,\dots$) of $e(r,\cdot)$ and $e^a(r,\cdot)$ intersect with each other many times, and that the number of nodes of an n -th eigenfunction of $e(r,\cdot)$ increases at points b such that

$$\lambda_n(r,b+y) < \mu_n(r,b+y) \quad \text{and} \quad \lambda_n(r,b-y) > \mu_n(r,b-y) \quad \text{for } 0 < y < \epsilon,$$

where ϵ is a small positive number. At points (r,b) such that

$$\lambda_n(r,b) = \mu_n(r,b),$$

$$w_n = (v_n)_x$$

between the eigenfunctions v_n and w_n of $e(r,b)$ and $e^a(r,b)$, respectively.

The behavior of eigenvalues and eigenfunctions of $e(r,b)$ is complicated as observed in the above. Nevertheless, the behavior of an eigenfunction is simple on the region where the corresponding eigenvalue is non-negative, and we can arrive at the following two theorems, Theorems 5.6 and 5.8, that imply the properties (1) and (2) described in the beginning of the present subsection.

Theorem 5.6 (Sign of a principal eigenfunction). If the principal eigenvalue $\lambda_1(r,b)$ of $e(r,b)$ is non-negative, then a principal eigenfunction v associated with $\lambda_1(r,b)$ is positive on $(-\frac{\pi}{2}, \frac{\pi}{2})$ or negative on $(-\frac{\pi}{2}, \frac{\pi}{2})$. More precisely, if $\lambda_1(r,b) \geq 0$, then

$v > 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and v is non-decreasing on $(-\infty, 0)$

or

$v < 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and v is non-increasing on $(-\infty, 0)$.

Proof. Let v be a principal eigenfunction associated with a non-negative principal eigenvalue $\lambda_1(r, b)$ of $e(r, b)$. If v is non-decreasing (resp. non-increasing) on $(-\infty, 0)$, then $v(x) > 0$ (resp. $v(x) < 0$) for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ by Proposition 5.2. Hence it suffices for the proof to show that v is unimodal: v is non-decreasing on $(-\infty, 0)$ or v is non-increasing on $(-\infty, 0)$.

We begin by showing that a unimodal function $w \in H_g$ given by

$$\begin{cases} w(x) = \int_{-\pi/2}^x |v_y(y)| dy & \text{for } x \leq 0, \\ w(x) = w(-x) & \text{for } x > 0 \end{cases}$$

is a principal eigenfunction associated with $\lambda_1(r, b)$. Clearly,

$$(5.10) \quad \|\nabla w\| = \|\nabla v\|.$$

It is easy to show that

$$(5.11) \quad w \geq |v| \quad \text{on } \mathbb{R} \quad \text{and} \quad |\nabla_b w| \geq |\nabla_b v| \quad \text{on } \mathbb{R},$$

which imply

$$(5.12) \quad \|w\| \geq \|v\| \quad \text{and} \quad \|\nabla_b w\| \geq \|\nabla_b v\|,$$

respectively. By (3) of Proposition 5.1, the assumption on the sign of $\lambda_1(r, b)$, (5.10) and (5.12),

$$\begin{aligned} \lambda_1(r, b) &= R_{r, b}(v) = \{ \|\nabla v\|^2 - \frac{1}{2}r^2 \|\nabla_b v\|^2 \} / \|v\|^2 \\ (5.13) \quad &\geq \{ \|\nabla v\|^2 - \frac{1}{2}r^2 \|\nabla_b v\|^2 \} / \|w\|^2 \geq \{ \|\nabla w\|^2 - \frac{1}{2}r^2 \|\nabla_b w\|^2 \} / \|w\|^2 \\ &= R_{r, b}(w). \end{aligned}$$

Hence w is a principal eigenfunction by (3) and (4) of Proposition 5.1.

Now, suppose that v is not unimodal. Then v_x is not of one sign on $(-\frac{\pi}{2}, 0)$. Fix a small positive number d such that $-\frac{\pi}{2}+d < 0$ and $-\frac{\pi}{2}-b+d < -\frac{\pi}{2}$. We first deal with the case where v_x is not of one sign on $(-\frac{\pi}{2}, -\frac{\pi}{2}+d)$. In this case,

$$\begin{aligned} |\nabla_b v(-\frac{\pi}{2}-\frac{b}{2}+d)| &= \frac{1}{b} |v(-\frac{\pi}{2}+d) - v(-\frac{\pi}{2}-b+d)| = \frac{1}{b} |v(-\frac{\pi}{2}+d) - v(-\frac{\pi}{2})| \\ &= \frac{1}{b} \left| \int_{-\pi/2}^{-\pi/2+d} v_x(x) dx \right| < \frac{1}{b} \int_{-\pi/2}^{-\pi/2+d} |v_x(x)| dx = |\nabla_b w(-\frac{\pi}{2}-\frac{b}{2}+d)|. \end{aligned}$$

This together with (5.11) implies that $\|\nabla_b w\| > \|\nabla_b v\|$ since both $\nabla_b w$ and $\nabla_b v$ are continuous. Hence (5.13) is replaced by

$$\lambda_1(r, b) = R_{r, b}(v) > R_{r, b}(w),$$

which contradicts (3) of Proposition 5.1. We next deal with the other case where v_x is of one sign on $(-\frac{\pi}{2}, -\frac{\pi}{2}+d)$. We may assume that $v_x \geq 0$ on $(-\frac{\pi}{2}, -\frac{\pi}{2}+d)$. Then, $w-v = 0$ on $(-\frac{\pi}{2}, -\frac{\pi}{2}+d)$, while $w(0)-v(0) = \int_{-\pi/2}^0 \{|v_x(x)| - v_x(x)\} dy > 0$ since v_x is not of one sign on $(-\frac{\pi}{2}, 0)$. Hence $w-v$ is a principal eigenfunction whose support is included in $[-\frac{\pi}{2}+d, \frac{\pi}{2}-d]$. This contradicts Proposition 5.2. \blacksquare

Corollary 5.7 (Simplicity of the principal eigenvalue). The principal eigenvalue of $e(r, b)$ is simple when it is non-negative.

Proof. Suppose that a non-negative principal eigenvalue $\lambda_1(r, b)$ of $e(r, b)$ is not simple. Then, by (2) of Proposition 5.1, there exist two principal eigenfunctions v_1 and v_2 associated with $\lambda_1(r, b) = \lambda_2(r, b) \geq 0$ such that $(v_1, v_2) = 0$. This contradicts Theorem 5.6. \blacksquare

Theorem 5.8 (Sign of an n-th eigenfunction). When the n-th eigenvalue $\lambda_n(r,b)$ of $e(r,b)$ is non-negative ($n > 1$), no eigenfunction associated with $\lambda_n(r,b)$ is of one sign on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Proof. Let n be an arbitrary integer greater than the unity. When $b \geq b(r)$, the principal eigenvalue of $e(r,b)$ is non-negative by the uniqueness of $b(r)$, and $e(r,b)$ has a principal eigenfunction \bar{v} such that $\bar{v} > 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ by Theorem 5.6. Hence, when $b \geq b(r)$, no n-th eigenfunction v_n of $e(r,b)$ is of one sign on $(-\frac{\pi}{2}, \frac{\pi}{2})$ since $(\bar{v}, v_n) = 0$ by (2) of Proposition 5.1. Suppose that there exists a positive number $b < b(r)$ such that $\lambda_n(r,b) \geq 0$ and an n-th eigenfunction v_n associated with $\lambda_n(r,b)$ is of one sign on $(-\frac{\pi}{2}, \frac{\pi}{2})$:

$$(5.14) \quad A_{r,b} v_n = \lambda_n v_n \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}), \quad v_n \geq 0 \quad \text{on } (-\frac{\pi}{2}, \frac{\pi}{2}) \quad \text{and} \quad \lambda_n \geq 0.$$

We may assume that $(v_n, 1) = 2$. Let \bar{v} be a principal eigenfunction associated with $\lambda_1(r, b(r))$:

$$A_{r,b(r)} \bar{v} = 0 \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}).$$

The function $v(x) = \bar{v}(\frac{b(r)x}{b})$ satisfies

$$(5.15) \quad A_{r,b} v = 0 \quad \text{in } (-\frac{d}{2}, \frac{d}{2}),$$

where $d = \pi b/b(r)$. We may assume that $(v, 1) = 2$ by Theorem 5.6.

With v_n and v , we associate non-decreasing functions

$$V_n(x) = \int_0^x v_n(y) dy \quad \text{and} \quad V(x) = \int_0^x v(y) dy,$$

respectively. Integrating (5.14) and (5.15), we obtain

$$A_{r,b} V_n = \lambda_n V_n \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \quad \text{and} \quad A_{r,b} V = 0 \quad \text{in } (-\frac{d}{2}, \frac{d}{2}),$$

respectively, since v_n and v are even functions. Hence $W_n(x) = 1 + V_n(x - \frac{\pi}{2})$ and $W(x) = 1 + V(x - \frac{d}{2})$ satisfy

$$(5.16) \quad A_{r,b} W_n = \lambda_n W_n - \lambda_n \quad \text{in } (0, \pi) \quad \text{and} \quad A_{r,b} W = 0 \quad \text{in } (0, d),$$

respectively, and

$$(5.17) \quad W_n = W = 0 \quad \text{on } (-\infty, 0], \quad W_n = 2 \quad \text{on } [\pi, \infty) \quad \text{and} \quad W = 2 \quad \text{on } [d, \infty).$$

Let $w(x) = v(x - \frac{d}{2})$. Then, since v satisfies (5.15), w satisfies

$$A_{r,b} w = 0 \quad \text{in } (0, d),$$

and $w = 0$ on $\mathbb{R} \setminus (0, d)$. Applying Lemma 4.4 to (w, W) , we obtain

$$0 = \int_0^d (A_{r,b} W) w dx = 2w_x(d-0) + \left(\frac{r}{b}\right)^2 \int_{d-b}^d w(x) dx$$

by the second equation in (5.16) and (5.17). We next apply Lemma 4.4 to (w, W_n) . This results in

$$(5.18) \quad \begin{aligned} \int_0^d (A_{r,b} W_n) w dx &= W_n(d) w_x(d-0) + \frac{1}{2} \left(\frac{r}{b}\right)^2 \int_{d-b}^d W_n(x+b) w(x) dx \\ &> \frac{1}{2} W_n(d) \{2w_x(d-0) + \left(\frac{r}{b}\right)^2 \int_{d-b}^d w(x) dx\} = 0 \end{aligned}$$

since $d = \pi b / b(r) < \pi$ and W_n is strictly increasing on $(0, \pi)$ by the assumption on the sign of v_n and Proposition 5.2. On the other hand, noting the symmetry of w with respect to $x = \frac{d}{2}$ and the anti-symmetry of $W_n - 1$ with respect to $x = \frac{\pi}{2}$, we obtain

$$\begin{aligned} \int_0^d (\lambda_n W_n - \lambda_n) w dx &= \lambda_n \left\{ \int_0^{d/2} (W_n - 1) w dx + \int_{d/2}^d (W_n - 1) w dx \right\} \\ &= \lambda_n \int_0^{d/2} (W_n(x) + W_n(d-x) - 2) w(x) dx \\ &\leq \lambda_n \int_0^{d/2} (W_n(x) + W_n(\pi-x) - 2) w(x) dx = 0 \end{aligned}$$

since $\lambda_n \geq 0$, $d < \pi$ and W_n is strictly increasing on $(0, \pi)$. This and (5.18) contradict the first equation in (5.16). \square

Using Theorem 5.6, we can prove

$$(5.19) \quad b(r) < r \quad \text{for } r < \pi$$

(Theorem 5.4). Let v be a principal eigenfunction associated with $\lambda_1(r, b(r))$, $r < \pi$. We may assume v to be positive on $(-\frac{\pi}{2}, \frac{\pi}{2})$ by Theorem 5.6. Then, since $b(r) < \pi$ by Theorem 5.4,

$$(5.20) \quad \int_{-\infty}^{\infty} (v(x+b(r))+v(x-b(r)))v(x)dx > 0.$$

By (3) of Proposition 5.1 and (5.20),

$$\begin{aligned} 0 &= \|v\|_{R_{r,b(r)}}^2(v) \\ &= \|\nabla v\|^2 + \frac{1}{2}(r/b(r))^2 \int_{-\infty}^{\infty} (v(x+b(r))+v(x-b(r))-2v(x))v(x)dx \\ &> \|\nabla v\|^2 - (r/b(r))^2 \int_{-\infty}^{\infty} v(x)^2 dx = \|\nabla v\|^2 - (r/b(r))^2 \|v\|^2, \end{aligned}$$

which implies (5.19) since $\|\nabla w\| \geq \|w\|$ for all $w \in H_S$, as shown in the proof of Proposition 5.3.

5.4. Existence and uniqueness

We now prove

Theorem 5.9 (Symmetric finite ss-pulses of $P(2,r)$ for $r > \sqrt{2}$).

Let $r > \sqrt{2}$. Then $P(2,r)$ has a unique symmetric finite ss-pulse u_r such that

$$(5.21) \quad u_r \text{ is an even function and } u_r(0) = 1.$$

Moreover, a function u is a symmetric finite ss-pulse of $P(2,r)$ if and only if u is expressed in the form

$$(5.22) \quad u(x) = au_r(x-c) \quad \text{for } x \in \mathbb{R}$$

with a positive number a and a real number c . The length $s(r)$ of $\text{supp}[u_r]$ has the properties (1) to (4) stated in Theorem 1.1, and the symmetric finite ss-pulse u_r has the properties (i) to (iii) stated in Theorem 1.1.

Proof. By Theorem 5.4, for each $r > \sqrt{2}$, there exists a unique positive number $b(r)$ such that the principal eigenvalue of $e(r,b(r))$ equals zero. Theorem 5.6 and Corollary 5.7 assure that $e(r,b(r))$ has a unique principal eigenfunction v_r such that $v_r(0) = 1$. (See Figure 5.6). The principal eigenfunction v_r is unimodal and $v_r > 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ by Theorem 5.6.

We define a function u_r by

$$(5.23) \quad u_r(x) = v_r\left(\frac{b(r)x}{r}\right) \quad \text{for } x \in \mathbb{R}.$$

We can regard $\{v_r, 0\}$ as a solution of $E(r,b(r))$, which satisfies the positivity condition (5.4). Hence, by the argument in the beginning of this section, $\{u_r, \pi r/b(r)\}$ is a solution of (5.1) and u_r becomes a symmetric finite ss-pulse of $P(2,r)$ that satisfies (5.21).

Suppose that $P(2,r)$ has another symmetric finite ss-pulse u that satisfies (5.21). Let s be the length of $\text{supp}[u]$. Then, by the argument in the beginning of this section, $\{u,s\}$ is a solution of (5.1) and the function $v(x) = u(\frac{s}{\pi}x)$ becomes an eigenfunction of $E(r, \pi r/s)$, associated with a zero eigenvalue. Since v is positive on $(-\frac{\pi}{2}, \frac{\pi}{2})$, v should be a principal eigenfunction of $E(r, \pi r/s)$ by Theorem 5.8, and $\pi r/s = b(r)$ by Theorem 5.4. Now, by Corollary 5.7 and the condition $v(0) = u(0) = 1$, $v = v_r$ and $u = u_r$. Thus we complete the proof of the unique existence of u_r .

Since $m = 2$, (4.1) is linear, and it is clear that a function given by (5.22) is a symmetric finite ss-pulse.

Let u be an arbitrary symmetric finite ss-pulse, and denote by c the center of $\text{supp}[u]$. We associate a function w given by

$$w(x) = u(x+c)/u(c) \quad \text{for } x \in \mathbb{R}$$

with u . Then w becomes a symmetric finite ss-pulse that satisfies (5.21). Hence $w = u_r$ by the unique existence of u_r , and

$$u(x) = u(c)w(x-c) = u(c)u_r(x-c) \quad \text{for } x \in \mathbb{R}.$$

The length $s(r)$ of $\text{supp}[u_r]$ equals $\pi r/b(r)$ by (5.23). Hence the properties (1) to (4) of $s(r)$ follow from those of $b(r)$ stated in Theorem 5.4 and (5.19).

Before the proof of the properties of u_r , we show

$$(5.24) \quad (u_r)_x(-\frac{1}{2}s(r)+0) = \frac{1}{2} \int_{s(r)/2-r}^{s(r)/2} u_r(x) dx > 0 \quad \text{for } r > \sqrt{2}.$$

The symmetric finite ss-pulse u_r satisfies

$$A_{r,r} u_r = -\Delta u_r + \frac{1}{2r^2} \Delta_r u_r = 0 \quad \text{in } (-\frac{1}{2}s(r), \frac{1}{2}s(r)),$$

and the function $U_r(x) = \int_{-\infty}^x u_r(y) dy$ also satisfies

$$A_{r,r}U_r = 0 \quad \text{in } (-\frac{1}{2}s(r), \frac{1}{2}s(r))$$

since u_r satisfies (4.1) with $m = 2$. An application of Lemma 4.4 to (u_r, U_r) yields

$$\begin{aligned} 0 &= \int_{-s(r)/2}^{s(r)/2} (A_{r,r}U_r)u_r dx \\ &= U_r(\frac{1}{2}s(r))(u_r)_x(\frac{1}{2}s(r)-0) + \frac{1}{2} \int_{s(r)/2-r}^{s(r)/2} U_r(x+r)u_r(x) dx \\ &= -\|u_r\|_1 (u_r)_x(-\frac{1}{2}s(r)+0) + \frac{1}{2} \|u_r\|_1 \int_{s(r)/2-r}^{s(r)/2} u_r(x) dx, \end{aligned}$$

which implies (5.24). The property (i) of u_r follows from the fact that v_r is unimodal (Theorem 5.6). The second property is obtained by noting that

$$v_r(x) = \cos x \quad \text{on } [-\frac{\pi}{2}, \frac{\pi}{2}] \quad \text{and} \quad b(r) = r \quad \text{for } r \geq \pi$$

(Subsections 5.1 and 5.2). For $r > \sqrt{2}$, u_r is twice continuously differentiable on $(-\frac{1}{2}s(r), \frac{1}{2}s(r))$ by Proposition 2.1. Let $\sqrt{2} < r < \pi$. Then, the point $x_- = -\frac{1}{2}s(r)+r$ belongs to $(-\frac{1}{2}s(r), \frac{1}{2}s(r))$ by the property (1) of $s(r)$, and

$$(\Delta u_r)(x) = \frac{1}{2}r^2 (\Delta_r u_r)(x) = \frac{1}{2}(u_r(x+r) - 2u_r(x) + u_r(x-r))$$

is not differentiable at x_- by (5.24). Hence u_r is not thrice continuously differentiable on $(-\frac{1}{2}s(r), \frac{1}{2}s(r))$ when $\sqrt{2} < r < \pi$. \square

6. Non-existence of other kinds of standing solitary pulses

We show in this section that the equation $P(2,r)$ has no other kind of ss-pulse but the symmetric finite ss-pulse.

Non-existence of a non-symmetric finite ss-pulse. Suppose that u is a non-symmetric finite ss-pulse of $P(2,r)$. We may assume that $\text{supp}[u] = [-\frac{s}{2}, \frac{s}{2}]$ (s : a positive number). We define an even function u_s by

$$u_s(x) = \frac{1}{2}(u(x)+u(-x)) \quad \text{for } x \in \mathbb{R}.$$

Since u is an ss-pulse, $u_s > 0$ on $(-\frac{s}{2}, \frac{s}{2})$ and u_s satisfies

$$(6.1) \quad -\Delta u_s + \frac{1}{2}r^2 \Delta_r u_s = 0 \quad \text{in } (-\frac{s}{2}, \frac{s}{2}).$$

Hence u_s is a symmetric finite ss-pulse of $P(2,r)$, and $s = s(r) = \pi r/b(r)$ and $u_s(x) = u_s(0)u_r(x)$ by Theorem 5.9, where u_r is the unique symmetric finite ss-pulse of $P(2,r)$ that satisfies (5.21). The function $v_s(x) = u_s(\frac{s}{\pi}x)$ becomes a principal eigenfunction of $e(r, \pi r/s)$, associated with $\lambda_1(r, \pi r/s) = 0$, as shown in the proof of Theorem 5.9 ($\pi r/s = b(r)$). We define an odd function u_a and an even function U_a by

$$u_a(x) = \frac{1}{2}(u(x)-u(-x)) \quad \text{and} \quad U_a(x) = \int_{-\infty}^x u_a(y)dy \quad \text{for } x \in \mathbb{R}.$$

Then, since u is a non-symmetric finite ss-pulse, $u_a \not\equiv 0$, $U_a \not\equiv 0$ and $U_a(x) = 0$ for $x \in \mathbb{R} \setminus (-\frac{s}{2}, \frac{s}{2})$. Moreover, U_a satisfies

$$-\Delta U_a + \frac{1}{2}r^2 \Delta_r U_a = 0 \quad \text{in } (-\frac{s}{2}, \frac{s}{2})$$

since $u_a(x) = u(x) - u_s(x)$ satisfies (4.1) with $m = 2$ in $(-\frac{s}{2}, \frac{s}{2})$.

Hence $V_a(x) = U_a(\frac{s}{\pi}x)$ also is a principal eigenfunction associated with $\lambda_1(r, \pi r/s)$. By (5.24) and the definition of V_a ,

$$(v_s)_x(-\frac{\pi}{2}+0) = \frac{s}{\pi}(u_s)_x(-\frac{s}{2}+0) = \frac{s}{\pi}u_s(0)(u_r)_x(-\frac{s}{2}+0) > 0,$$

$$(v_a)_x(-\frac{\pi}{2}+0) = \frac{s}{\pi}(U_a)_x(-\frac{s}{2}+0) = \frac{s}{\pi}u_a(-\frac{s}{2}) = 0,$$

which implies that v_s and v_a are linearly independent. This contradicts the fact that $\lambda_1(r, \pi r/s)$ is simple (Corollary 5.7).

Non-existence of an infinite ss-pulse. An infinite ss-pulse u of $P(2,r)$ must satisfy

$$-\Delta u + \frac{1}{2}r^2 \Delta_r u = 0 \quad \text{in } \mathbb{R}.$$

Through the Fourier transformation, the above equation is transformed into

$$(\xi^2 - 1 + \cos r\xi)\bar{u}(\xi) = 0,$$

where \bar{u} denotes the Fourier transformation of u . The function

$$\xi^2 - 1 + \cos r\xi$$

of ξ vanishes at most at a finite number of points on \mathbb{R} , therefore, $\bar{u} \equiv 0$ and $u \equiv 0$. This means that $P(2,r)$ has no infinite ss-pulse.

Non-existence of a semi-infinite ss-pulse. Suppose that $P(2,r)$, $r > \sqrt{2}$, has a semi-infinite ss-pulse w . We may assume $\text{supp}[w]$ to equal $[0, \infty)$ without loss of generality. By Theorem 5.9, $P(2,r)$ has a unique symmetric finite ss-pulse u such that $(u,1) = 2$ and $\text{supp}[u] = [0,s]$ ($s = s(r) = \pi r/b(r)$). We define non-decreasing functions U and W by

$$U(x) = \int_0^x u(y)dy \quad \text{and} \quad W(x) = \int_0^x w(y)dy.$$

Then U and W satisfy

$$A_{r,r}U = 0 \quad \text{in } (0,s) \quad \text{and} \quad A_{r,r}W = 0 \quad \text{in } (0,\infty),$$

respectively, since u and w satisfy (4.1) with $m = 2$. Applying Lemma 4.4 to $\{u, U\}$ and $\{u, W\}$, respectively, we obtain

$$\left\{ \begin{array}{l} 0 = \int_0^s (A_{r,r} U) u dx = 2u_x(s-0) + \int_{s-r}^s u(x) dx, \\ 0 = \int_0^s (A_{r,r} W) u dx = W(s)u_x(s-0) + \frac{1}{2} \int_{s-r}^s W(x+r)u(x) dx \\ > \frac{1}{2}W(s) \{2u_x(s-0) + \int_{s-r}^s u(x) dx\} \end{array} \right.$$

since $W(x)$ is strictly increasing on $(0, \infty)$. The second inequality in the above contradicts the first equality. Hence $P(2, r)$ has no semi-infinite ss-pulse.

7. Stationary solutions of $P(2,r)$

In the present section, we complete the proof of Theorem 1.1 by combining the theorems obtained in the previous sections.

Proof of Theorem 1.1. The first assertion is the same as that of Theorem 4.2.

A stationary solution of $P(2,r)$ that satisfies (1.4) is a symmetric finite ss-pulse of $P(2,r)$ that satisfies (5.21). Hence Theorem 5.9 implies the unique existence of u_r , the properties (1) to (4) of $s(r)$ and the properties (i) to (iii) of u_r .

Let u be a function given by (1.5). Then u belongs to $L^\infty(\mathbb{R})$ and $L^1(\mathbb{R})$ since

$$\begin{cases} \|u\|_\infty = \|u_r\|_\infty \sup_{i \in \Lambda} (a_i) \leq \|u_r\|_\infty \sum_{i \in \Lambda} a_i < \infty, \\ \|u\|_1 \leq \|u_r\|_1 \sum_{i \in \Lambda} a_i < \infty \end{cases}$$

by (1.6). For each $i \in \Lambda$, let

$$u_i(x) = a_i u_r(x - c_i) \quad \text{for } x \in \mathbb{R}.$$

Theorem 5.9 assures that each u_i is a symmetric finite ss-pulse of $P(2,r)$. Since $\text{dis}(\text{supp}[u_i], \text{supp}[u_j]) \geq r$ for all $i \in \Lambda, j \in \Lambda, i \neq j$ by (1.7), $F[u] = \sum_{i \in \Lambda} F[u_i] = 0$ ((2.2) in Proposition 2.1). Hence u is a stationary solution of $P(2,r)$.

Let u be an arbitrary non-trivial stationary solution of $P(2,r)$. By Theorem 3.1, u is decomposed into

$$u(x) = \sum_{i \in \Lambda} u_i(x),$$

where $\{u_i\}_{i \in \Lambda}$ (Λ : an index set) is a finite or countable sequence of ss-pulses such that

$$(7.1) \quad \text{dis}(\text{supp}[u_i], \text{supp}[u_j]) \geq r \quad \text{for all } i \in \Lambda, j \in \Lambda, i \neq j.$$

The equation $P(2,r)$ has no other kind of ss-pulse but the symmetric finite ss-pulse (Section 6). Hence each u_i is a symmetric finite ss-pulse, and by Theorem 5.9, u_i is expressed in the form

$$u_i(x) = u_i(c_i)u_r(x-c_i),$$

where c_i is the center of $\text{supp}[u_i]$. Now u is expressed in the form

$$(7.2) \quad u(x) = \sum_{i \in \Lambda} u_i(c_i)u_r(x-c_i).$$

Let us show how the sequences $\{u_i(c_i)\}_{i \in \Lambda}$ and $\{c_i\}_{i \in \Lambda}$ satisfy the conditions (1.6) and (1.7), respectively. Since each u_i is an ss-pulse and

$$\infty > \|u\|_1 = \|u_r\|_1 \sum_{i \in \Lambda} u_i(c_i)$$

by (7.1) and (7.2), $\{u_i(c_i)\}_{i \in \Lambda}$ satisfies (1.6). By (7.1) and (7.2),

$$\begin{aligned} |c_i - c_j| &= \text{dis}(\text{supp}[u_i], \text{supp}[u_j]) \\ &\quad + \frac{1}{2}(\text{the length of } \text{supp}[u_i]) \\ &\quad + \frac{1}{2}(\text{the length of } \text{supp}[u_j]) \end{aligned}$$

$$\geq r + (\text{the length of } \text{supp}[u_r]) \quad \text{for all } i \in \Lambda, j \in \Lambda, i \neq j,$$

which implies (1.7). We thus complete the proof of Theorem 1.1. \square

8. Stationary solutions of $P(m,r)$ for $m \neq 2$

The present section discusses the stationary solution of $P(m,r)$ for $m \neq 2$.

We could prove Theorem 1.1 by fully utilizing the speciality of the case $m = 2$ when the equation (4.2) is reduced to the linear differential-difference equation when $m = 2$. We have not yet obtained a satisfactory result for the general case of $m \neq 2$. However, we can show a partial result and clarify the difference between the cases of $m = 2$ and $m \neq 2$.

The results obtained in Sections 2 and 3 are valid for all $m > 1$, hence we have only to study the ss-pulse.

We first review the case of $r = \infty$. A stationary solution of $P(m,\infty)$ is always an ss-pulse (Remark 3.3). Mimura and Yamaguti [29] have shown that for each positive number c , $P(m,\infty)$ has a symmetric finite ss-pulse u_c such that $\|u_c\|_1 = c$, and that u_c is unique up to the coordinate translation.

Let u be an ss-pulse of $P(m,\infty)$ such that

$$u = 0 \quad \text{on } (-\infty, 0] \quad \text{and} \quad u > 0 \quad \text{on } (0, \varepsilon)$$

where ε is a small positive number, and define a non-decreasing function U by (3.1). When $r = \infty$, (3.5) is reduced to

$$J[U](x) = U_x(x)^m - U(x)\{\|u\|_1 - U(x)\} = 0 \quad \text{on } \mathbb{R}$$

since $U(-\infty) = 0$ and $U(\infty) = \|u\|_1$. Hence $U(x)$ is determined by the relation

$$x = \int_0^{U(x)} \{\xi(\|u\|_1 - \xi)\}^{-1/m} d\xi \quad \text{for } x > 0.$$

and $u(x)$ is obtained by differentiating $U(x)$. The length $s(u)$ of $\text{supp}[u] \equiv \text{supp}[U_x]$ is given by

$$(8.1) \quad s(u) = \int_0^{\|u\|_1} \{\xi(\|u\|_1 - \xi)\}^{-1/m} d\xi = \|u\|_1^{1-2/m} \int_0^1 \{\eta(1-\eta)\}^{-1/m} d\eta < \infty.$$

Consequently, $s(u)$ is independent of $\|u\|_1$ when $m = 2$, however, $s(u)$ increases with $\|u\|_1$ when $m > 2$, while $s(u)$ decreases with increasing $\|u\|_1$ when $1 < m < 2$. This is a difference between the cases of $m = 2$ and $m \neq 2$.

Combining (3.7) and (8.1), we can rewrite the above fact as

Proposition 8.1 (Support of an ss-pulse of $P(m, \infty)$). Let u be an ss-pulse of $P(m, \infty)$. The length $s(u)$ of $\text{supp}[u]$ is given by

$$(8.2) \quad s(u)^2 = F(m)^2 \|u\|_\infty^{m-2},$$

where $F(m)$ is the same as that given in Theorem 1.2. Hence, $s(u)$ is independent of $\|u\|_\infty$ when $m = 2$, however, $s(u)$ increases with $\|u\|_\infty$ when $m > 2$, while $s(u)$ decreases with increasing $\|u\|_\infty$ when $1 < m < 2$. \square

We now return to the case of $0 < r < \infty$. By the definition of a stationary solution, we obtain

Proposition 8.2 (Transformation of an ss-pulse of $P(m, r)$ for $m \neq 2$). Assume u to be an ss-pulse of $P(m, r)$ for $m \neq 2$. Then, for all $\sigma > 0$,

$$(8.3) \quad u_\sigma(x) \equiv \sigma^{2/(m-2)} u(x/\sigma) \quad \text{for } x \in \mathbb{R}$$

becomes an ss-pulse of $P(m, \sigma r)$. Moreover,

$$(8.4) \quad (\sigma r)^2 \|u_\sigma\|_\infty^{2-m} = r^2 \|u\|_\infty^{2-m} \quad \text{for all } \sigma > 0,$$

$$(8.5) \quad (\sigma r)^m \|u_\sigma\|_1^{2-m} = r^m \|u\|_1^{2-m} \quad \text{for all } \sigma > 0. \quad \square$$

Observing Proposition 8.2, we infer

$$(8.6) \left\{ \begin{array}{l} \text{"there exists a positive number } Q(m) \text{ such that} \\ \text{(a) } P(m,r) \text{ has no ss-pulse } u \text{ such that } 0 < r^2 \|u\|_{\infty}^{2-m} \leq Q(m), \\ \text{(b) for each } q > Q(m), P(m,r) \text{ has an ss-pulse } u \text{ such that} \\ \quad r^2 \|u\|_{\infty}^{2-m} = q", \end{array} \right.$$

which is true for $m = 2$ as stated in Theorem 1.1.

We have not yet proved (8.6) in the general case. However, we obtain Theorem 1.2.

Proof of Theorem 1.2. Let u be a non-trivial stationary solution of $P(m,r)$ and define a non-decreasing function U by (3.1).

Let $X = \{x \in \mathbb{R}; u(x) = \|u\|_{\infty}\}$. We first consider the case where X is not empty. Fix a point $x_0 \in X$, and let

$$x = \inf\{y \leq x_0; z \in X \text{ for all } z \in [y, x_0]\}.$$

Then $x > -\infty$ since $u \in L^1(\mathbb{R})$, and $\int_y^x u(z) dz < (x-y) \|u\|_{\infty}$ for $y < x$.

Noting that $J[U] = 0$ and $u = U_x \geq 0$ on \mathbb{R} , we obtain

$$(8.7) \quad \begin{aligned} \|u\|_{\infty}^m &= u(x)^m = U_x(x)^m = \int_x^{x+r} u(y) (U(x) - U(y-r)) dy \\ &\leq \|u\|_{\infty} \int_x^{x+r} (U(x) - U(y-r)) dy = \|u\|_{\infty} \int_x^{x+r} \int_{y-r}^x u(z) dz dy \\ &< \|u\|_{\infty} \int_x^{x+r} (x-y+r) \|u\|_{\infty} dy = \frac{1}{2} r^2 \|u\|_{\infty}^2. \end{aligned}$$

In the other case where X is empty, we also obtain the same estimate as (8.7). The first assertion of Theorem 1.2 follows from (8.7).

Let q be a positive number that satisfies (1.9), and let $s = rF(m)/\sqrt{q} \leq r$. Combining (8.1) and the argument on the stationary solution of $P(m,\infty)$, we know that $P(m,\infty)$ has a symmetric finite ss-

pulse u_s such that the length of $\text{supp}[u_s]$ equals s . Let us show that u_s is an ss-pulse of $P(m,r)$ satisfying (1.10). The ss-pulse u_s of $P(m,\infty)$ satisfies (4.1):

$$\frac{m}{m-1}(u_s^{m-1})_x + \left\{ \int_{-\infty}^x u_s dy - \int_x^{\infty} u_s dy \right\} = 0 \quad \text{in the interior of } \text{supp}[u_s].$$

We can replace the second term on the left-hand side of the above equation by $\left\{ \int_{x-r}^x u_s dy - \int_x^{x+r} u_s dy \right\}$ since $r \geq s =$ (the length of $\text{supp}[u_s]$). Then the resulting equation implies that u_s is an ss-pulse of $P(m,r)$. Since u_s is an ss-pulse of $P(m,\infty)$, it follows from (8.2) that

$$r^2 \|u_s\|_{\infty}^{2-m} = \left(\frac{r}{s}\right)^2 F(m)^2 = q,$$

which implies (1.10). Thus the proof is completed. \square

We may appropriately conclude the present paper with the following conjecture.

Conjecture (Stationary solutions of $P(m,r)$ for $m \neq 2$).

(I) $P(m,r)$ has no non-trivial stationary solution u such that

$$r^2 \|u\|_{\infty}^{2-m} \leq 2.$$

(II) For each real number $q \geq F(m)^2$, $P(m,r)$ has an ss-pulse u such that

$$r^2 \|u\|_{\infty}^{2-m} = q$$

and the length of $\text{supp}[u]$ is less than or equal to r . The ss-pulse u is unique up to the coordinate translation.

(III) For each real number q such that $2 < q < F(m)^2$, $P(m,r)$ has an ss-pulse u such that

$$r^2 \|u\|_{\infty}^{2-m} = q$$

and the length of $\text{supp}[u]$ is greater than r . The ss-pulse u is unique up to the coordinate translation. (See Figure 8.1.) |

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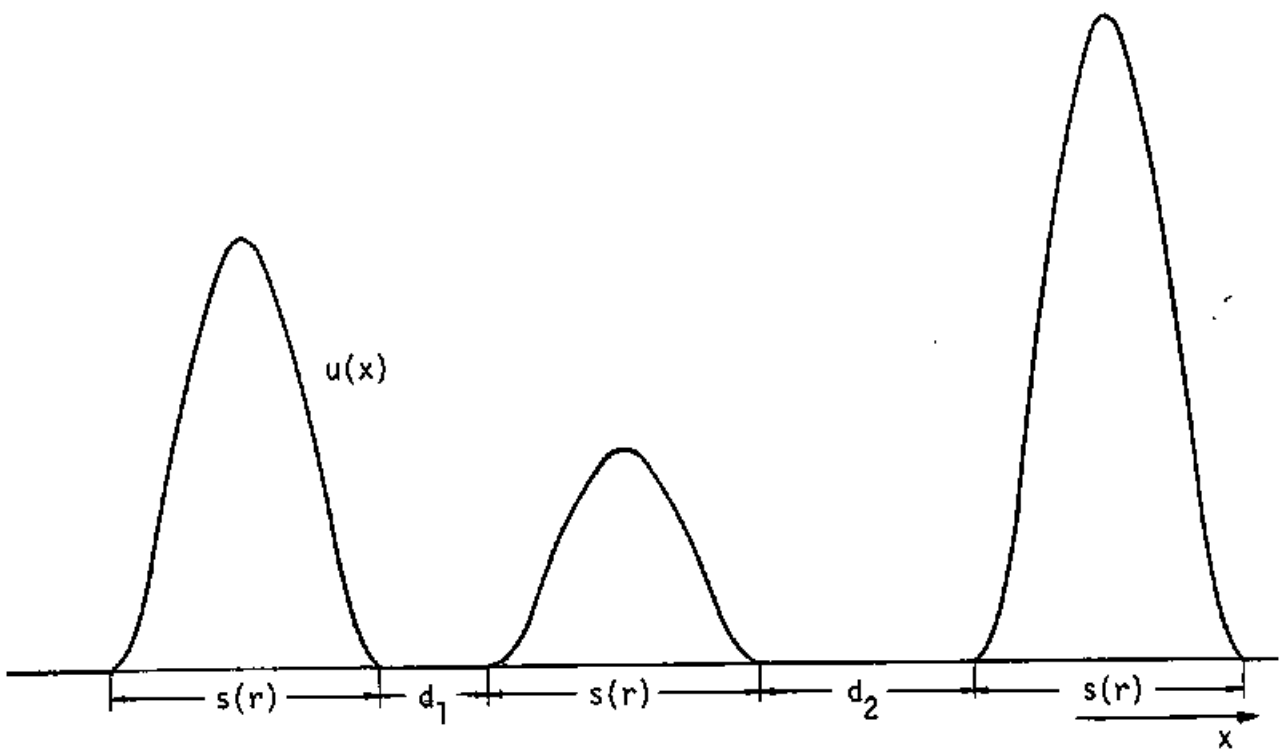
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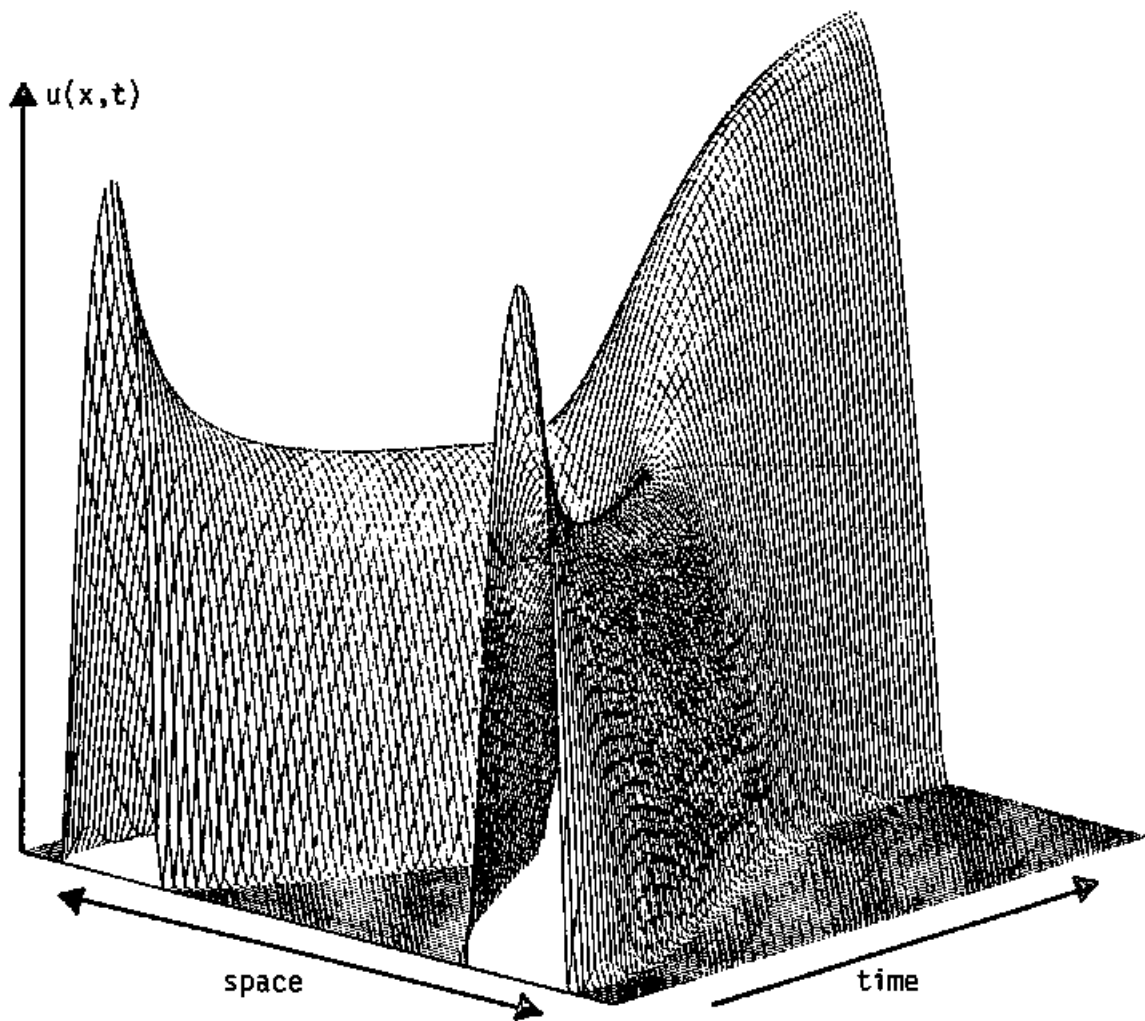
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Figure captions.

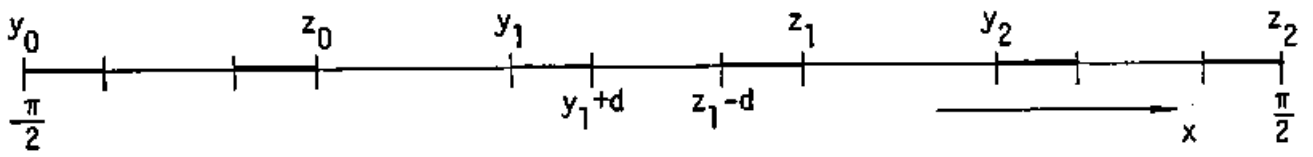
- Figure 1.1. A pulse-like stationary solution of $P(2,r)$ ($d_1 \geq r$ and $d_2 \geq r$).
- Figure 1.2. A solution of the Cauchy problem $P(2,\infty)$ obtained by a finite difference method.
- Figure 5.1. A partition of the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ ($n = 2$).
- Figure 5.2. The graph of the function $b(r)$.
- Figure 5.3. Eigenvalues λ and typical patterns of eigenfunctions of $e(r,b)$ ($r = 3$).
- Figure 5.4. Eigenvalues μ and typical patterns of eigenfunctions of $e^a(r,b)$ ($r = 3$).
- Figure 5.5. Eigenvalues λ and typical patterns of eigenfunctions of $e(r,b)$ and eigenvalues μ of $e^a(r,b)$ ($r = 3$).
- Figure 5.6. Principal eigenfunctions v_r of $e(r,b(r))$ ($r = \pi, 2.0, 1.5$ and 1.42).
- Figure 8.1. Existence and non-existence of ss-pulse of $P(m,r)$ ($s(u) = (\text{the length of } \text{supp}[u])$).



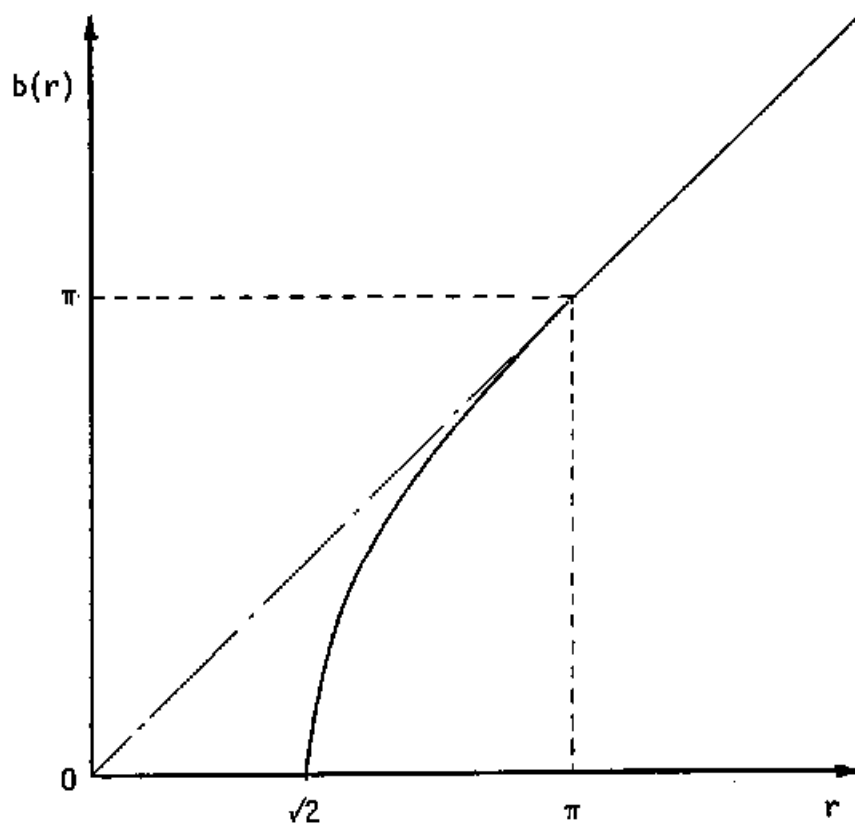
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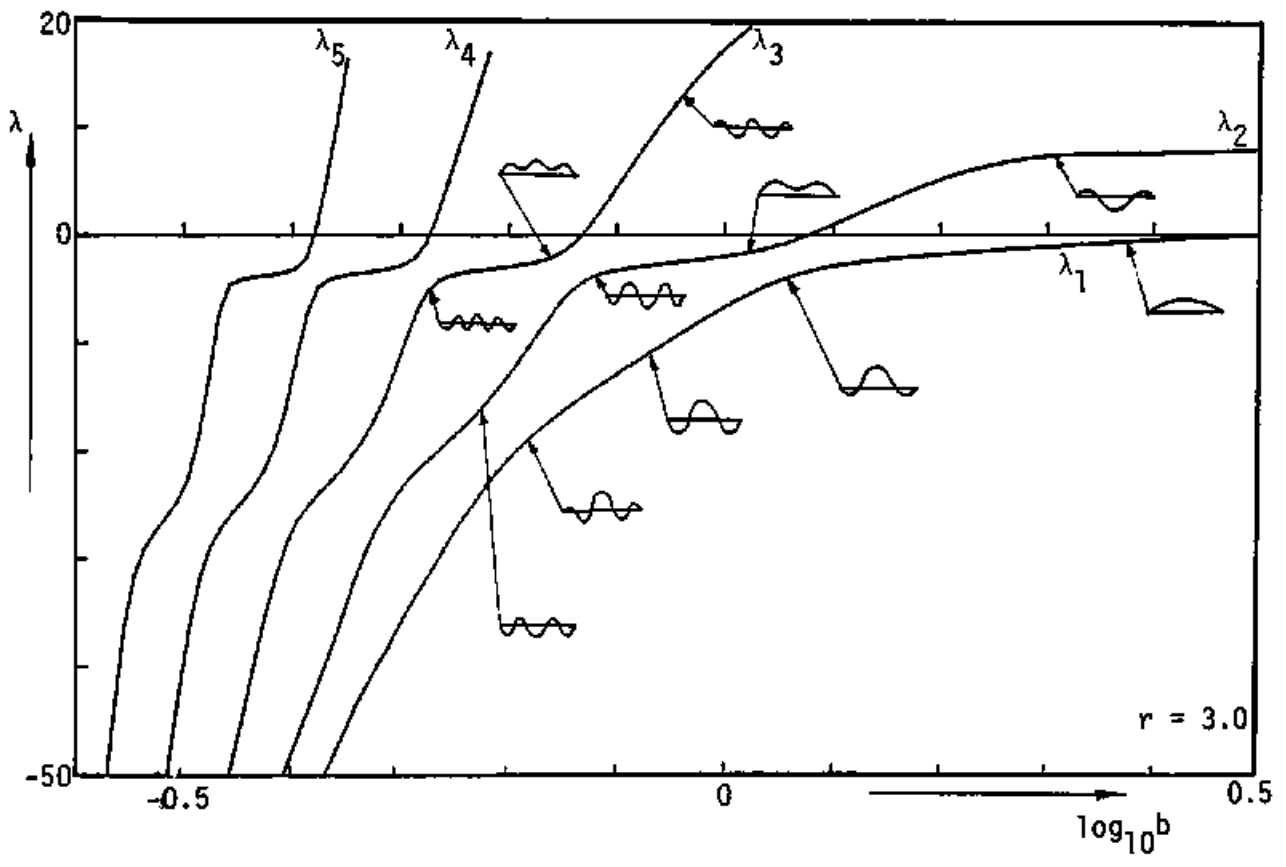
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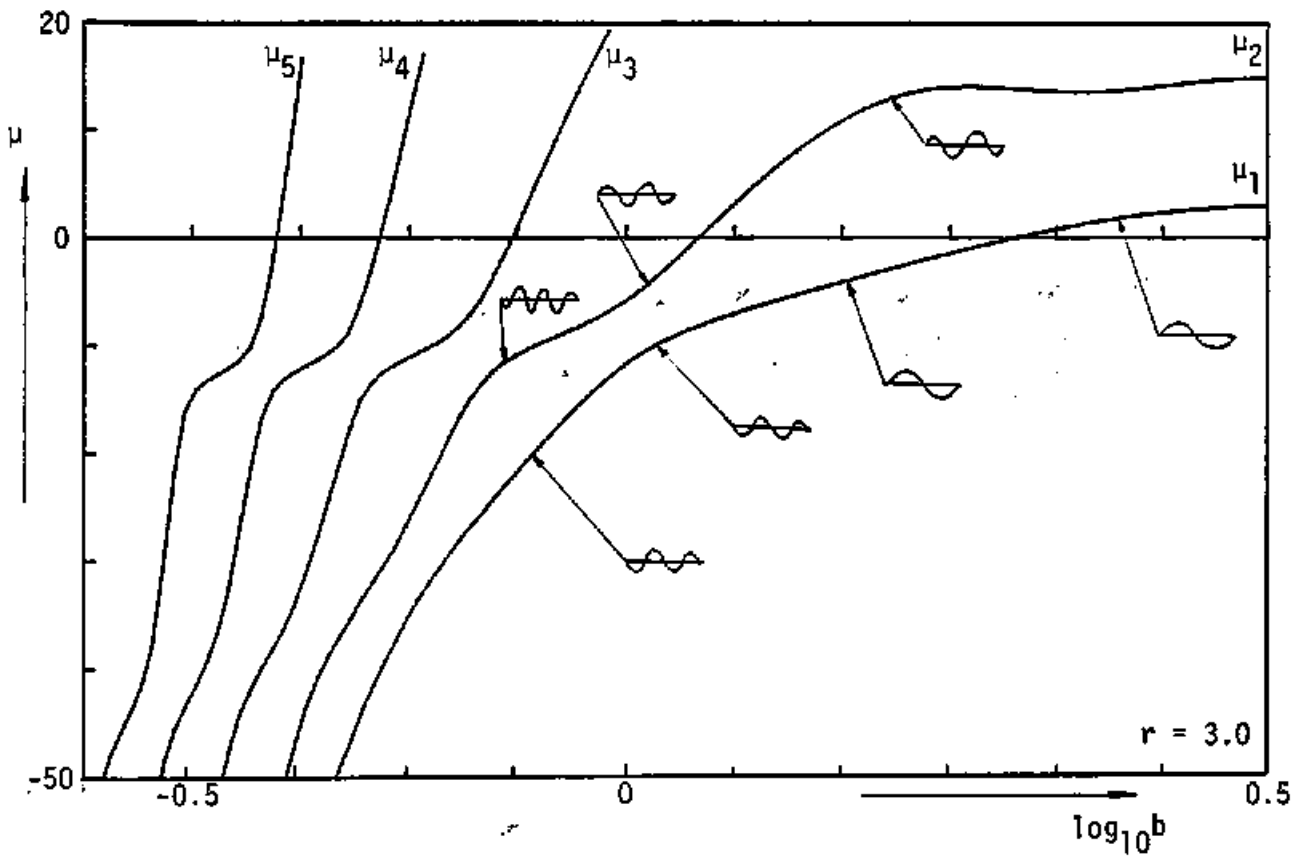
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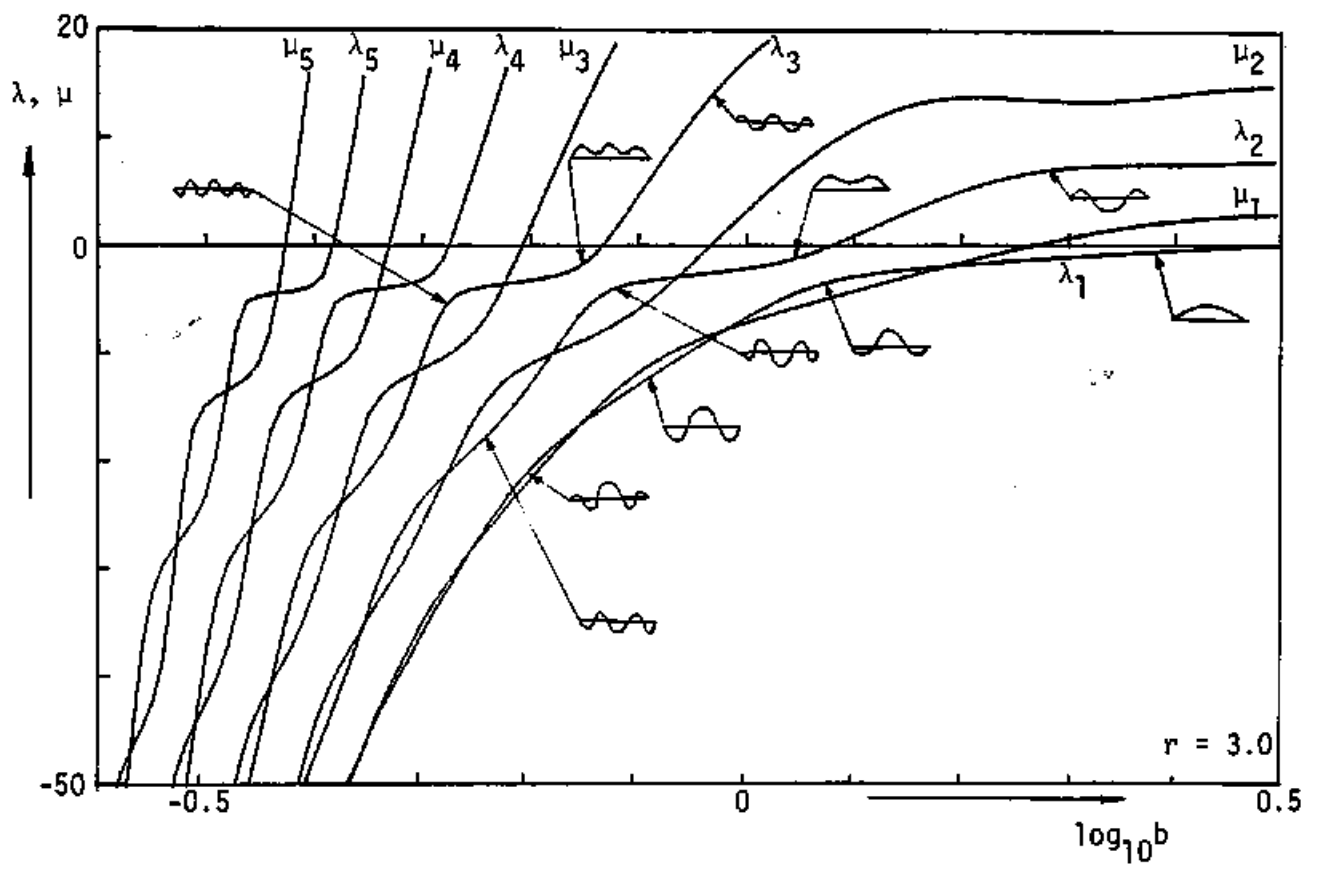
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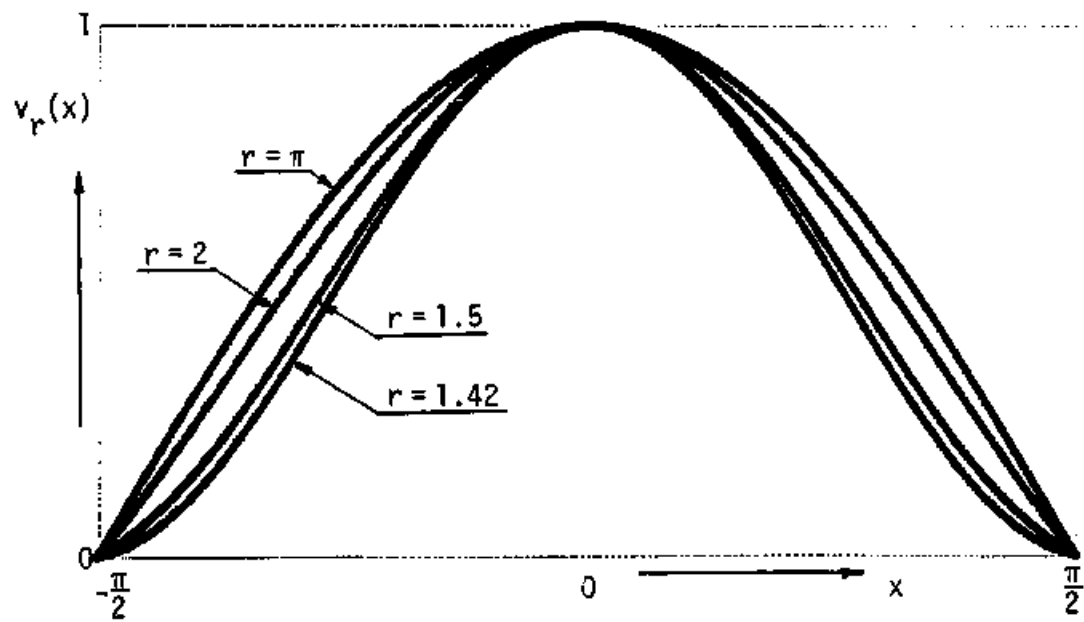
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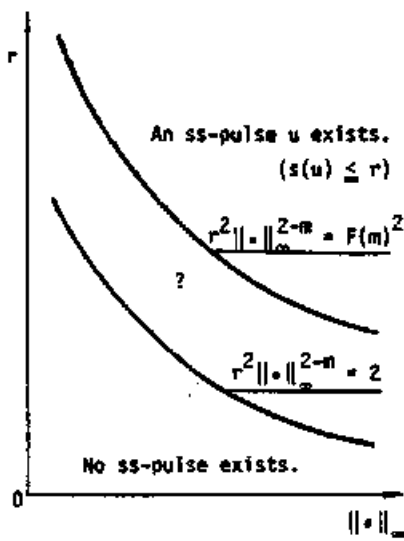
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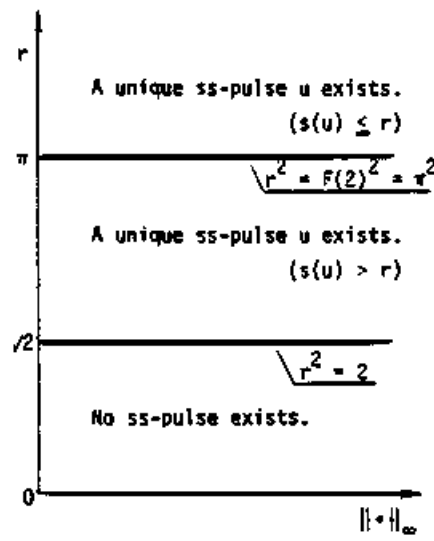
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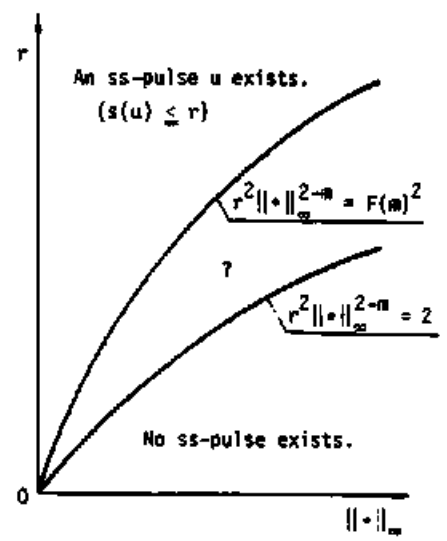
(Figure 5.6)



(1) $1 < m < 2$



(2) $m = 2$



(3) $m > 2$

(Figure 8.1)