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主論文

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A Theorem on the Cohomology of Groups and Some Arithmetical Applications

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Introduction.

Various cohomology groups related to class field theory have been investigated by many authors. Especially there are important results on the Galois cohomology groups of ideles and idele classes of finite Galois extensions of algebraic number fields (see, for example [3]). The latter result was first obtained by J.Tate [9]. He also announced the corresponding result for the multiplicative group of the algebraic number field itself in [10], of which the proof was published later in [11], under a more general setting. Recently, we have investigated in [4] the Galois cohomology groups of the factor group of idele class group by its connected component of the unity. In [5], we have constructed an isomorphism between the Galois cohomology groups of the unit group of a local field and those of some Artin's splitting module.

In this paper, we shall prove the following theorem on the cohomology groups over finite groups snd show the known results cited above appear as its special cases.

Let G be a finite group. Suppose that we are given the following commutative diagram of G-modules with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_1 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & B_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Then we have the following theorem

Theorem (A). With the notation as above, we have

(III) Let A_3 and B_2 be cohomologically trivial G-modules. Then the long exact secuences derived from $0 \longrightarrow A_1 \longrightarrow B_{\cancel{2}\,1} \longrightarrow C_1 \longrightarrow 0$ and $0 \longrightarrow C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow 0$ are isomorphic.

We have the similar results for the following cases:

(I) C_2 and B_3 are cohomologically trivial.

(II) A_1 and C_2 are cohomologically trivial.

(IV) A_3 and C_1 are cohomologically trivial.

(V) A_2 and B_1 are cohomologically trivial.

(VI) A_3, B_3 and C_3 are cohomologically trivial.

(VII) A_2, B_2 and C_2 are cohomologically trivial (§1, Theorem 1.1).

In §1, we shall show the above theorem. In §2, we shall show the main result of [4] is considered to be a corollary of the case (III) of Theorem (A). In §3, we shall show several auxiliary lemmas on the cohomology groups of integral group rings. In §4, we shall construct a cohomologically triyial module $\widetilde{K}_p^X$ including K_p^X as a G-submodule, for every place p of k. Here K is a finite

Galois extension of an algebraic number field k with the group G, and K_p^X is a G-submodule of K_A^X consisting of all the ideles whose q-factors are 1 except when q divides p. In §5, we shall study a new treatment of the Galois cohomology groups of C_K/D_K using the result of §4, where C_K is the idele class group of K and D_K is its connected component of the unity. In §6, we shall show the announced result of [10] in a more general setting, but not quite general as that of [11].

Notation and Terminology

Let G be a finite group and A be a G-module. $H^r(G,A)$ always denotes the r-dimensional cohomology group and is often abbreviated to $H^r(A)$. For a cocycle x of $H^r(G,A)$, $\{x\}$ denotes the cohomology class containing x. Although, in §1, we use several symbols to express the connecting homomorphisms derived from short exact sequences, in other sections, we always denote the connecting homomorphism by δ_*. For a G-module A, ${}^G A$ denotes a G-submodule of A consisting of all G-invariant elements of A. Let B be a module and P be a condition on B. We denote by $\langle x \mid x$ is the element of B satisfying the condition $P \rangle$ the submodule of B generated by all the elements of B which satisfy the condition P.

§1. Suppose that we are given the two diagrams of modules

$$(1.1)\qquad \begin{tikzcd} M_1 \arrow[dr, "g_1"] & \\ & M_2 \arrow[dl, "g_2"] \\ M_3 \arrow[uu, "g_3"] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M_1' \arrow[dr, "g_1'"] & \\ & M_2' \arrow[dl, "g_2'"] \\ M_3' \arrow[uu, "g_3'"] & \end{tikzcd},$$

where g_i and g_i' are homomorphisms. For the sake of simplicity, we denote these diagrams by the symbols $\Delta(M_1, M_2, M_3)$ and $\Delta(M_1', M_2', M_3')$. Let h_i be the homomorphisms from M_i to M_i' $(1 \leqq i \leqq 3)$ which satisfy the conditions $g_1' \cdot h_1 = \pm h_2 \cdot g_1$, $g_2' \cdot h_2 = \pm h_3 \cdot g_2$, $g_3' \cdot h_3 = \pm h_1 \cdot g_3$. By abuse of language, we call the triplet of homomorphisms $h = (h_1, h_2, h_3)$ an anti-homomorphism from $\Delta(M_1, M_2, M_3)$ to $\Delta(M_1', M_2', M_3')$ when at least one of the diagrams is anticommutative. We call $h = (h_1, h_2, h_3)$ an anti-isomorphism in case every homomorphism h_i is an isomorphism. In case all the diagrams are commutative, we call the triplet $h = (h_1, h_2, h_3)$, as usual, a homomorphism from $\Delta(M_1, M_2, M_3)$ to $\Delta(M_1', M_2', M_3')$ and an isomorphism when every h_i is an isomorphism. If the triplet $h = (h_1, h_2, h_3)$ is either an anti-homomorphism or a homomorphism, that is, satisfies the conditions $g_1 \cdot h_1 = \pm h_2 \cdot g_1$, $g_2 \cdot h_2 = \pm h_3 \cdot g_2$, $g_3 \cdot h_3 = \pm h_1 \cdot g_3$, we call h an (a)-homomorphism and an (a)-isomorphism when every h_i is an isomorphism.

In the following, we shall prove a theorem on the cohomology of groups. Although one can generalize the result in a natural way, using the functors Tor or Ext, here we shall be concerned with only the case of the cohomology of groups. Let G be a finite group. We are given a commutative diagram of G-modules

(1.2)

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A_1 \arrow[r, "\varphi_1"] \arrow[d, "\alpha_1"] & A_2 \arrow[r, "\psi_1"] \arrow[d, "\alpha_2"] & A_3 \arrow[r] \arrow[d, "\alpha_3"] & 0 \\ 0 \arrow[r] & B_1 \arrow[r, "\varphi_2"] \arrow[d, "\beta_1"] & B_2 \arrow[r, "\psi_2"] \arrow[d, "\beta_2"] & B_3 \arrow[r] \arrow[d, "\beta_3"] & 0 \\ 0 \arrow[r] & C_1 \arrow[r, "\varphi_3"] \arrow[d] & C_2 \arrow[r, "\psi_3"] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

Here, all row and vertical sequences are exact. Let us define the graded modules X_i, Y_i, Z_i $(1 \leqq i \leqq 3)$ by

$$X_i = \sum_{r=-\infty}^{\infty} H^r(A_i),$$

$$Y_i = \sum_{r=-\infty}^{\infty} H^r(B_i),$$

$$Z_i = \sum_{r=-\infty}^{\infty} H^r(C_i) \quad (1 \leqq i \leqq 3).$$

Let α_i^r be the homomorphism from $H^r(G,A_i)$ to $H^r(G,B_i)$ induced from α_i. We denote the homomorphism $\prod_{r=-\infty}^{\infty} \alpha_i^r : X_i \longrightarrow Y_i$, by the same symbol α_i. The homomorphisms β_i, φ_i and ψ_i are defined in a similar way. Let us denote the connecting homomorphisms derived from (1.2) by

$$\gamma_i^r : H^r(C_i) \longrightarrow H^{r+1}(A_i) \quad (1 \leqq i \leqq 3),$$

$$\delta_1^r : H^r(A_3) \longrightarrow H^{r+1}(A_1),$$

$$\delta_2^r : H^r(B_3) \longrightarrow H^{r+1}(B_1),$$

$$\delta_3^r : H^r(C_3) \longrightarrow H^{r+1}(C_1) \quad (r \in Z).$$

We denote the homomorphisms $\prod_{r=-\infty}^{\infty} \gamma_i^r : Z_i \longrightarrow X_i$ by γ_*^i and $\prod_{r=-\infty}^{\infty} \delta_i^r$ by δ_*^i. Then, from (1.2), we have the following diagram

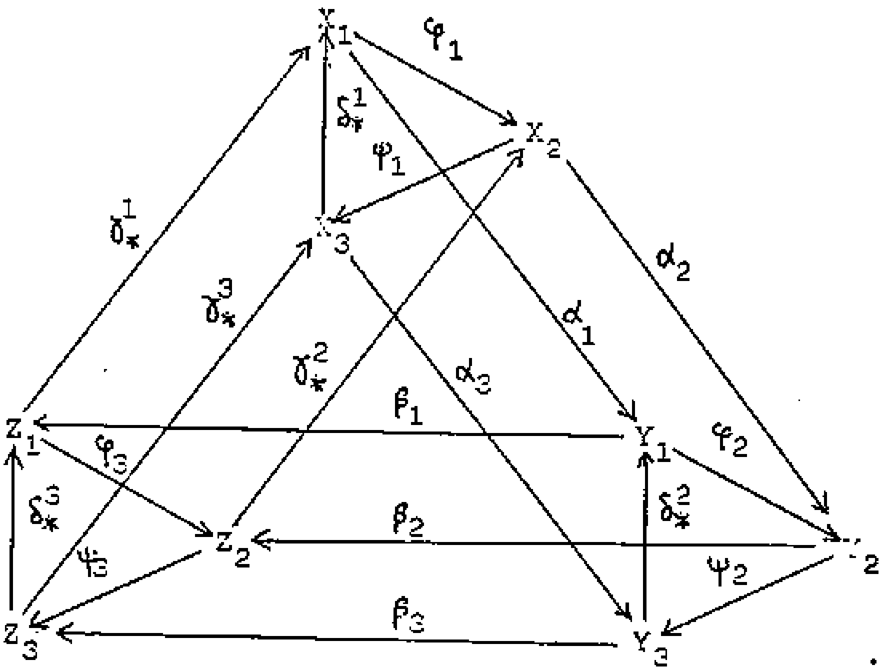
(1.3)
$$\begin{tikzcd}
& X_1 \arrow[rr, "\alpha_1"] \arrow[dd, "\varphi_1"] & & Y_1 \arrow[dd, "\varphi_2"] \\
X_3 \arrow[ur, "\delta_*^1"] \arrow[rr, "\alpha_3"] & & Y_3 \arrow[ur, "\delta_*^2"] & \\
& X_2 \arrow[ul, "\psi_1"] \arrow[rr, "\alpha_2"] & & Y_2 \arrow[ul, "\psi_2"]
\end{tikzcd}$$

Here the triplet $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a homomorphism. In the same way as α, one sees that $\beta = (\beta_1, \beta_2, \beta_3)$, $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ and $\psi = (\psi_1, \psi_2, \psi_3)$ are homomorphisms. On the other hand, one sees the following diagram is anticommutative (see for example, [2], Ch. III. §4)

(1.4)
$$\begin{tikzcd}
Z_3 \arrow[r, "\gamma_*^3"] \arrow[d, "\delta_*^3"] & X_3 \arrow[d, "\delta_*^3"] \\
Z_1 \arrow[r, "\gamma_*^1"] & X_1
\end{tikzcd}$$

Hence $\delta_* = (\delta_*^1, \delta_*^2, \delta_*^3)$ and $\gamma_* = (\gamma_*^1, \gamma_*^2, \gamma_*^3)$ are anti-homomorphism. Finally, we have the following diagram

(1.5)



In the following, we treat the case when two of the nine graded modules of (1.5) are zero module, especially the case when the two G-modules of (1.2) are cohomologically trivial. If the two of the nine G-modules of (1.2) are cohomologically trivial, there remain two short exact sequences contained in (1.2) such as non of the G-modules of the sequences are assumed to be cohomologically trivial. Then it is natural to expect the assumption implies some relation between the cohomology sequences derived from the remaining two short exact sequences. Certainly, if we suppose the G-modules C_1 and C_2 are cohomologically trivial, the diagram (1.5) coincides with the diagram (1.3) and the triplet $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is an isomorphism from $\Delta(X_1,X_2,X_3)$ to $\Delta(Y_1,Y_2,Y_3)$.

We restrict ourselves to the cases when the derived cohomology ~~groups~~ sequences of the remaining two sequences of (1.2) are (a)-isomorphic. One can easily show that there are only fifteen cases which satisfy the condition. The diagram (1.2) is symmetrical with respect to the diagonal line A_1, B_2, C_3. From the duality theorem of the cohomology groups, the diagram (1.2) can be considered to be also symmetrical with respect to the diagonal line A_3, B_2, C_1. Therefore, one sees that there are essentially following seven cases (I),...,(VII):

(I) C_2 and B_3 are cohomologically trivial. Hence $Z_2 = Y_3 = 0$.

(II) A_1 and C_2 are cohomologically trivial. Hence $X_1 = Z_2 = 0$.

(III) A_3 and B_2 are cohomologically trivial. Hence $X_3 = Y_2 = 0$.

(IV) A_3 and C_1 are cohomologically trivial. Hence $X_3 = Z_1 = 0$.

(V) A_2 and B_1 are cohomologically trivial. Hence $X_2 = Y_1 = 0$.

(VI) Any two of the three modules A_3, B_3 and C_3 are cohomologically trivial, that is, all the G-modules A_3, B_3 and C_3 are cohomologically trivial. Hence $X_3 = Y_3 = Z_3 = 0$.

(VII) Any two of the three modules A_2, B_2 and C_2 are cohomologically trivial, that is, all the G-modules A_2, B_2 and C_2 are cohomologically trivial. Hence $X_2 = Y_2 = Z_2 = 0$.

Theorem 1.1. *With the notation as above, we see that*

For the case (I), (X_1, X_2, X_3) *and* (X_1, Y_1, Z_1) *are anti-isomorphic,*

For the case (II), (X_3, Y_3, Z_3) *and* $\cancel{(Z_1, Z_2, Z_3)}$ (Y_1, Y_2, Y_3) *are anti-isomorphic,*

For the case (V), (Z_1, Z_2, Z_3) *and* (X_3, Y_3, Z_3) *are anti-isomorphic,*

For the case (VII), (X_1, Y_1, Z_1) *and* (X_3, Y_3, Z_3) *are anti-isomorphic,*

For the case (III), (X_1, Y_1, Z_1) *and* (Z_1, Z_2, Z_3) *are isomorphic,*

For the case (IV), (X_2, Y_2, Z_2) *and* (Y_1, Y_2, Y_3) *are isomorphic,*

For the case (VI), (X_1, Y_1, Z_1) *and* (X_2, Y_2, Z_2) *are isomorphic.*

Proof. Here, we shall prove the cases (II) and (III) which we shall use lator.

Case (III). It is sufficient to show the diagram (1.5) induces the following vertical isomorphisms u and u′ such as the following diagram is commutative

$$\begin{tikzcd}
\cdots \arrow[r] & H^r(C_1) \arrow[r, "\gamma_1^r"] \arrow[d, equal] \arrow[dr, phantom, "a"] & H^{r+1}(A_1) \arrow[r, "\alpha_1^{r+1}"] \arrow[d, "u", "\wr"'] \arrow[dr, phantom, "b"] & H^{r+1}(B_1) \arrow[r, "\beta_1^{r+1}"] \arrow[d, "u'", "\wr"'] \arrow[dr, phantom, "c"] & H^{r+1}(C_1) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & H^r(C_1) \arrow[r, "\varphi_3^r"] & H^r(C_2) \arrow[r, "\psi_3^r"] & H^r(C_3) \arrow[r, "\delta_3^{r+1}"] & H^{r+1}(C_1) \arrow[r] & \cdots \quad (r \in Z).
\end{tikzcd}$$

Construction of u. From the assumption, the homomorphisms $\varphi_1^{r+1} : H^{r+1}(A_1) \longrightarrow H^{r+1}(A_2)$ and $\gamma_2^r : H^r(C_2) \longrightarrow H^{r+1}(A_2)$ are bijective. The isomorphism $u : H^{r+1}(A_1) \longrightarrow H^r(C_2)$ is defined by $(\gamma_2^r)^{-1} \cdot \varphi_1^{r+1}$.

Then from the commutative diagram

$$\begin{tikzcd}
H^r(C_1) \arrow[r, "\gamma_1^r"] \arrow[d, "\varphi_3^r"] & H^{r+1}(A_1) \arrow[d, "\varphi_1^{r+1}", "\wr"'] \\
H^r(C_2) \arrow[r, "\gamma_2^r"] & H^{r+1}(A_2),
\end{tikzcd}$$

we see the diagram a is commutative.

Cnstruction of u′. From the assumption, the homomorphisms $\delta_2^r : H^r(B_3) \longrightarrow H^{r+1}(B_1)$ and $\beta_3^r : H^r(B_3) \longrightarrow H^r(C_3)$ are bijective. The isomorphism $u' : H^{r+1}(B_1) \longrightarrow H^r(C_3)$ is defined by putting $\beta_3^r \cdot (\delta_2^r)^{-1}$.

Then, from the commutative diagram

$$\begin{tikzcd}
H^r(B_3) \arrow[r, "\beta_3^r"] \arrow[d, "\wr\ \delta_2^r"'] & H^r(C_3) \arrow[d, "\delta_3^r"] \\
H^{r+1}(B_1) \arrow[r, "\beta_1^{r+1}"] & H^{r+1}(C_1),
\end{tikzcd}$$

we see the diagram ⓒ is commutative.

Now, we shall show the following diagram ⓑ is commutative

$$\begin{tikzcd}
H^{r+1}(A_1) \arrow[r, "\alpha_1^{r+1}"] \arrow[d, "\wr\ \varphi_1^{r+1}"'] & H^{r+1}(B_1) \\
H^{r+1}(A_2) \arrow[r, phantom, "ⓑ"] & H^r(B_3) \arrow[u, "\wr\ \delta_2^r"'] \arrow[d, "\wr\ \beta_3^r"'] \\
H^r(C_2) \arrow[u, "\wr\ \gamma_2^r"'] \arrow[r, "\psi_3^r"] & H^r(C_3).
\end{tikzcd}$$

Let a be any cocycle of $H^{r+1}(A_1)$. Then $\alpha_2(\varphi_1(a))$ is a cocycle of $H^{r+1}(B_2) = \{0\}$. Hence there exists a r-cochain b with values in B_2 such as $\delta(b) = \alpha_2(\varphi_1(a))$. Then $\delta(\beta_2(b)) = \beta_2(\delta(b)) = \beta_2(\alpha_2(\varphi_1(a))) = 0$, and so $\beta_2(b)$ is a r-cocycle with values in C_2. From the equation $\delta(b) = \alpha_2(\varphi_1(a))$, we see $\gamma_2^r\{(\beta_2(b))\} = \{\varphi_1(a)\}$ in $H^{r+1}(A_2)$. Therefore we have $u(\{a\}) = \{\beta_2(b)\}$, and so $\psi_3^r \cdot u(\{a\}) = \{\psi_3 \cdot \beta_2(b)\}$ in $H^r(C_3)$.

On the other hand, from the condition $\varphi_2(\alpha_1(a)) = \alpha_2(\varphi_1(a)) = \delta(b)$, we see $\delta_2^r\{\psi_2(b)\} = \{\alpha_1(a)\}$. So we have $u' \cdot \alpha_1^{r+1}(\{a\}) = \{\beta_3 \cdot \psi_2(b)\}$ in $H^r(C_3)$. Hence, by virtue of the fact $\beta_3 \cdot \psi_2 = \psi_3 \cdot \beta_2$, we have $\psi_3^r \cdot u = u' \cdot \alpha_1^{r+1}$, and so the diagram ⓑ is commutative.

Case (II). It is sufficient to show the diagram (1.5) induces the following isomorphisms v and v' suvh as the following diagram is either commutative or anticommutative

$$\begin{tikzcd}
\cdots \arrow[r] & H^r(B_3) \arrow[r, "\delta_2^r"] \arrow[d, equal] \arrow[dr, phantom, "ⓐ"] & H^{r+1}(B_1) \arrow[r, "\varphi_2^{r+1}"] \arrow[d, "v", "\wr"'] \arrow[dr, phantom, "ⓑ"] & H^{r+1}(B_2) \arrow[r, "\psi_2^{r+1}"] \arrow[d, "v'", "\wr"'] \arrow[dr, phantom, "ⓒ"] & H^{r+1}(B_3) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & H^r(B_3) \arrow[r, "\beta_3^r"] & H^r(C_3) \arrow[r, "\gamma_3^r"] & H^{r+1}(A_3) \arrow[r, "\alpha_3^{r+1}"] & H^{r+1}(B_3) \arrow[r] & \cdots \quad (r \in Z).
\end{tikzcd}$$

Construction of v. From the assumption, the homomorphisms $\beta_1^{r+1} : H^{r+1}(B_1) \longrightarrow H^{r+1}(C_1)$ and $\delta_3^r : H^r(C_3)$ are bijective. The isomorphism $v : H^{r+1}(B_1) \longrightarrow H^r(C_3)$ is defined by $(\delta_3^r)^{-1} \cdot \beta_1^{r+1}$.

$H^r(C_3) \to H^{r+1}(C_1)$

Then , from the commutative diagram

$$\begin{tikzcd}
H^r(B_3) \arrow[r, "\delta_2^r"] \arrow[d, "\beta_3^r"] & H^{r+1}(B_1) \arrow[d, "\beta_1^{r+1}", "\wr"'] \\
H^r(C_3) \arrow[r, "\delta_3^r", "\sim"'] & H^{r+1}(C_1),
\end{tikzcd}$$

we have the diagram ⓐ is commutative.

Construction of v'. From the assumption, the homomorphisms $\alpha_2^{r+1} : H^{r+1}(A_2) \longrightarrow H^{r+1}(B_2)$ and $\psi_1^{r+1} : H^{r+1}(A_2) \longrightarrow H^{r+1}(A_3)$ are bijective. The isomorphism $v' : H^{r+1}(B_2) \longrightarrow H^{r+1}(A_3)$ is defined by $\psi_1^{r+1} \cdot (\alpha_2^{r+1})^{-1}$.

Then, from the commutative diagram

$$\begin{tikzcd}
H^{r+1}(A_2) \arrow[r, "\psi_1^{r+1}", "\sim"'] \arrow[d, "\alpha_2^{r+1}", "\wr"'] & H^{r+1}(A_3) \arrow[d, "\alpha_3^{r+1}"] \\
H^{r+1}(B_2) \arrow[r, "\psi_2^{r+1}"] & H^{r+1}(B_3),
\end{tikzcd}$$

we see the diagram ⓒ is commutative.

Let us show the following diagram (5) is anticommutative

$$\begin{tikzcd}
H^{r+1}(B_1) \arrow[r, "\varphi_2^{r+1}"] \arrow[d, "\wr\ \beta_1^{r+1}"] & H^{r+1}(B_2) \\
H^{r+1}(C_1) \arrow[r, phantom, "(5)"] & H^{r+1}(A_2) \arrow[u, "\wr\ \alpha_2^{r+1}"'] \arrow[d, "\wr\ \psi_1^{r+1}"] \\
H^{r}(C_3) \arrow[u, "\wr\ \delta_3^{r}"'] \arrow[r, "\gamma_3^{r}"] & H^{r+1}(A_3) .
\end{tikzcd}$$

Let b be any cocycle of $H^{r+1}(B_1)$. Since $H^{r+1}(C_2) = 0$, there exists a r-cochain c with values in C_2 such as $\varphi_3(\beta_1(b)) = \delta(c)$. Hence $\delta(\psi_3(c)) = \psi_3(\varphi_3(\beta_1(b))) = 0$, and so $\psi_3(c)$ is a cocycle of $H^r(C_3)$. By the definition of the connecting homomorphism, we have $\delta_3^r(\{\psi_3(c)\}) = \{\beta_1(b)\}$. Hence we have $v(\{b\}) = \{\psi_3(c)\}$. Since β_2 is a surjective homomorphism, there exists a r-cochain $\bar{b}$ with values in B_2 such as $\beta_2(\bar{b}) = c$. Then we see $\varphi_2^r(\{b\}) = \{\varphi_2(b) - \delta(\bar{b})\}$ in $H^{r+1}(B_2)$ and $\beta_2(\varphi_2(b) - \delta(\bar{b})) = \varphi_3(\beta_1(b)) - \delta(c) = 0$. Since α_2 is an injective homomorphism, there exists a $(r+1)$-cocycle a of $H^{r+1}(A_2)$ satisfying $\alpha_2(a) = \varphi_2(b) - \delta(\bar{b})$. Then we have $v'.\varphi_2^{r+1}(\{b\}) = \{\psi_1(a)\}$. On the other hand, we see $\psi_3(c) = \psi_3(\beta_2(\bar{b})) = \beta_3(\psi_2(\bar{b}))$ and $\delta(\psi_2(\bar{b})) = \psi_2(\delta(\bar{b})) = \psi_2(\varphi_2(b) - \alpha_2(a)) = -\psi_2(\alpha_2(a)) = -\alpha_3(\psi_1(a))$. By the definition of the connecting homomorphism, we have $\gamma_3^r(\{\psi_3(c)\}) = -\{\psi_1(a)\}$. Hence, we have $v'.\varphi_2^{r+1} + \gamma_3^r.v = 0$, and so the diagram (5) is anticommutative. It is easy to show the other cases in the same way as above.

From this theorem, the following corollary is obvious.

Corollary 1.1. *For the case when one of the exponents of the graded modules* X_1, X_3, Z_1 *and* Z_3 *is at most* 2, *all the cuadrilateral diagrams contained in the diagram* (1.5) *are commutative. Hence, for all the cases* (I),...,(VII), *two triangular diagrams are isomorphic.*

We shall show another application of the above theorem. We assume G-modules A_2, A_3, B_2, B_3, C_2 and C_3 of the following diagram (1.6) are cohomologically trivial and all the row and vertical sequences are exact.

(1.6)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r, "\psi_1"] \arrow[d] & A_3 \arrow[r] \arrow[d] & A_4 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_1 \arrow[r] \arrow[d] & B_2 \arrow[r, "\psi_2"] \arrow[d] & B_3 \arrow[r] \arrow[d] & B_4 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_2 \arrow[r, "\psi_3"] \arrow[d] & C_3 \arrow[r] \arrow[d] & C_4 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

From this diagram, we get the following new commutative diagrams of G-modules with exact rows and columns

(1.7)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r, "\psi_1"] \arrow[d] & \mathrm{Im}\,\psi_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_1 \arrow[r] \arrow[d] & B_2 \arrow[r, "\psi_2"] \arrow[d] & \mathrm{Im}\,\psi_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_2 \arrow[r, "\psi_3"] \arrow[d] & \mathrm{Im}\,\psi_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$$(1.8)\qquad \begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Im}\psi_1 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & A_4 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Im}\psi_2 \arrow[r] \arrow[d] & B_3 \arrow[r] \arrow[d] & B_4 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Im}\psi_3 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & C_4 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd},$$

where $\operatorname{Im}\psi_i$ is a G-module consisting of all the elements $\psi_i(x)$, $x \in L_i$. Here $L_1 = A_2$, $L_2 = B_2$ and $L_3 = C_2$.

Let us denote the connecting homomorphisms of above diagrams by

$\gamma_i^r : H^r(C_i) \longrightarrow H^{r+1}(A_i) \quad (1 \leqq i \leqq 4)$,

$\bar{\gamma}^r : H^r(\operatorname{Im}\psi_3) \longrightarrow H^{r+1}(\operatorname{Im}\psi_1)$,

$\delta_i^r : H^r(\operatorname{Im}\psi_i) \longrightarrow H^{r+1}(M_i) \quad (1 \leqq i \leqq 3)$, where $M_1 = A_1$, $M_2 = B_1$ and $M_3 = C_1$.

$\bar{\delta}_i^r : H^r(N_i) \longrightarrow H^{r+1}(\operatorname{Im}\psi_i) \quad (1 \leqq i \leqq 3)$, where $N_1 = A_4$, $N_2 = B_4$ and $N_3 = C_4$.

Then , from Theorem 1.1 case (VII), we have the following diagram

$$(1.9)\qquad \begin{tikzcd}
\cdots \arrow[r] & H^r(A_4) \arrow[r] \arrow[d, "\wr\,\bar{\delta}_1^r"] \arrow[dr, phantom, "\bar{a}"] & H^r(B_4) \arrow[r] \arrow[d, "\wr\,\bar{\delta}_2^r"] \arrow[dr, phantom, "\bar{b}"] & H^r(C_4) \arrow[r, "\gamma_4^r"] \arrow[d, "\wr\,\bar{\delta}_3^r"] \arrow[dr, phantom, "\bar{c}"] & H^{r+1}(A_4) \arrow[r] \arrow[d, "\wr\,\bar{\delta}_1^{r+1}"] & \cdots \\
\cdots \arrow[r] & H^{r+1}(\operatorname{Im}\psi_1) \arrow[r] \arrow[d, "\wr\,\delta_1^{r+1}"] \arrow[dr, phantom, "a"] & H^{r+1}(\operatorname{Im}\psi_2) \arrow[r] \arrow[d, "\wr\,\delta_2^{r+1}"] \arrow[dr, phantom, "b"] & H^{r+1}(\operatorname{Im}\psi_3) \arrow[r, "\bar{\gamma}^{r+1}"] \arrow[d, "\wr\,\delta_3^{r+1}"] \arrow[dr, phantom, "c"] & H^{r+2}(\operatorname{Im}\psi_1) \arrow[r] \arrow[d, "\wr\,\delta_1^{r+2}"] & \cdots \\
\cdots \arrow[r] & H^{r+2}(A_1) \arrow[r] & H^{r+2}(B_1) \arrow[r] & H^{r+2}(C_1) \arrow[r, "\gamma_1^{r+2}"] & H^{r+3}(A_1) \arrow[r] & \cdots
\end{tikzcd} \quad (r \in Z)$$

In this diagram, the diagrams ⓐ, $\bar{ⓐ}$, ⓑ and $\bar{ⓑ}$ are commutative and ⓒ and $\bar{ⓒ}$ are anticommutative. Hence, we have

$$\gamma_1^{r+2}(\delta_3^{r+1}\circ\bar{\delta}_3^{r}) = (\gamma_1^{r+2}\circ\delta_3^{r+1})\,\bar{\delta}_3^{r} = (-\delta_1^{r+2}\circ\bar{\gamma}^{r+1})\,\bar{\delta}_3^{r}$$

$$= -\delta_1^{r+2}(\bar{\gamma}^{r+1}\circ\bar{\delta}_3^{r}) = \delta_1^{r+2}(\bar{\delta}_1^{r+1}\circ\gamma_4^{r}) = (\delta_1^{r+2}\circ\bar{\delta}_1^{r+1})\,\gamma_4^{r}.$$

Finally, we have the following commutative diagram

$$(1.10)\quad \begin{tikzcd} \cdots \arrow[r] & H^r(A_4) \arrow[r] \arrow[d, "\wr"] & H^r(B_4) \arrow[r] \arrow[d, "\wr"] & H^r(C_4) \arrow[r] \arrow[d, "\wr"] & H^r(A_4) \arrow[r] \arrow[d, "\wr"] & \cdots \\ \cdots \arrow[r] & H^{r+2}(A_1) \arrow[r] & H^{r+2}(B_1) \arrow[r] & H^{r+2}(C_1) \arrow[r] & H^{r+3}(A_1) \arrow[r] & \cdots \end{tikzcd} \quad (r\in Z).$$

Here the vertical arrows are the compositions of the connecting homomorphisms of (1.9). We define the graded modules X_i, Y_i and Z_i $(1\leqq i\leqq 4)$, in the same way as Theorem 1.1 and denote the homomorphisms corresponding to δ_i^r and $\bar{\delta}_i^r$ by δ_*^i and $\bar{\delta}_*^i$. Then we have

Corollary 1.2. *Suppose that the G-modules* A_2, A_3, B_2, B_3, C_2 *and* C_3 *of the diagram* (1.6) *are cohomologically trivial. Then the triplet of homomorphisms* $(\delta_*^1\circ\bar{\delta}_*^1,\ \delta_*^2\circ\bar{\delta}_*^2,\ \delta_*^3\circ\bar{\delta}_*^3)$ *is the isomorphism from* $\Delta(X_4,Y_4,Z_4)$ *to* $\Delta(X_1,Y_1,Z_1)$, *that is, the diagram* (1.10) *is commutative.*

§2. Let G be a finite group and A be a G-module. Let ξ be a 2-cohomology class of A and let $\bar{A}$ be Artin's splitting module of ξ. Then we have the following lemma which was proved by J.Tate ([9], Theorem 1).

Lemma 2.1. <u>With the notation as above, the following two conditions are equivalent:</u>

i) $H^1(N,A) = 0$ <u>and</u> $H^2(N,A)$ <u>is a cyclic group of the same order as</u> N, <u>generated by</u> $\rho_{G,N}\xi$, <u>for all subgroups</u> $N \subset G$. <u>Here</u> $\rho_{G,N}$ <u>is the restriction homomorphism from</u> G <u>to</u> N.

ii) $H^1(N,\bar{A}) = H^2(N,\bar{A}) = 0$ <u>for all subgroups</u> $N \subset G$.

Remark. It is well known that if $\bar{A}$ satisfies the condition ii) of this lemma, $\bar{A}$ is cohomologiclly trivial, that is, $H^r(N,\bar{A}) = 0$ for all subgroups $N \subset G$ and for all integers $r \in Z$ ([3], Ch.I, Th. 8.1).

In this section, we shall treat the G-module A and the cohomology class ξ satisfying the conditions i) and ii) of the above lemma. We assume we are given an exact sequence of G-modules:

$$(2.1) \qquad 0 \longrightarrow C \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0.$$

Let us denote the 2-cohomology class $\alpha_*(\xi) \in H^2(G,B)$ by η and Artin's splitting module of η by $\bar{B}$. Then we can define a G-homomorphism $\bar{\alpha}: \bar{A} \longrightarrow \bar{B}$ by putting $\bar{\alpha}(a + x) = \alpha(a) + x$, for every $a \in A$ and $x \in I[G]$. Here $I[G]$ is the augmentation ideal of the group ring $Z[G]$ generated by $d_\sigma = \sigma - 1$ $(\sigma \in G)$. Then it is easy to show $\ker \bar{\alpha} = C$. Hence we have the following exact sequence of G-modules

$$(2.2) \qquad 0 \longrightarrow C \longrightarrow \bar{A} \xrightarrow{\bar{\alpha}} \bar{B} \longrightarrow 0.$$

Combining (2.1) and (2.2), we have the following commutative diagram of G-modules

$$(2.3) \qquad \begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & C & = & C & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & A & \longrightarrow & \bar{A} & \longrightarrow & I[G] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\bar{\alpha}} & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & \bar{B} & \longrightarrow & I[G] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Since $\bar{A}$ is cohomologically trivial, we have the following theorem from (III) of Theorem 1.1.

Theorem 2.1. *Let* $A, \bar{A}, B, C$ *and* $I[G]$ *be G-modules in* (2.3). *Then the following diagram is commutative for every* $r \in Z$

$$\begin{array}{ccccccccc} \cdots \to & H^r(B) & \longrightarrow & H^r(\bar{B}) & \longrightarrow & H^r(I[G]) & \xrightarrow{\delta_*} & H^{r+1}(B) & \longrightarrow \cdots \\ & \| & & \| \wr \delta_* & & \| \wr \delta_* & & \| & \\ \cdots \to & H^r(B) & \xrightarrow{\delta_*} & H^{r+1}(C) & \longrightarrow & H^{r+1}(A) & \xrightarrow{\alpha_*} & H^{r+1}(B) & \longrightarrow \cdots, \end{array}$$

where δ_* *is the connecting homomorphisms and we have abbreviated* $H^q(G,X)$ *to* $H^q(X)$ *for a G-module* X.

Let us write the isomorphism $\delta_*: H^r(\bar{B}) \cong H^{r+1}(C)$ in a more explicit form. First, we fix a 2-cocycle u contained in ξ. Though the module $\bar{A}$ is detemined only up to G-isomorphisms, we can regard the module $\bar{A}$ as the splitting module of u. Since

$v = \alpha(u)$ is a 2-cocycle contained in η, the module $\bar{B}$ is similarly regarded as the splitting module of v. Therefore we can consider the mapping $d: G \longrightarrow I[G]$ satisfies the following equation in $\bar{A}$

$$\sigma d_\tau = d_{\sigma\tau} - d_\sigma + u[\sigma,\tau], \text{ for every } \sigma,\tau \in G,$$

where we set $d_1 = u[1,1]$. Then we see $H^1(G,I[G]) \cong Z/[G:1]Z$ is generated by the cohomology class $\{d\}$. For the sake of the following proposition, we replace the integer r by $p+1$. Let $N_p \subset H^p(G,Z)$ be the subgroup consisting of all the cocycles h satisfying the condition

$$(*) \qquad v \cup h = \delta g,$$

where g is a $(p+1)$-cochain with values in B and $\cup$ denotes the correspondence of cochains which induces the cup product (for details, see [3], Ch.I, §6.4). Let M_{p+1} be the subgroup of $H^{p+1}(G,\bar{B})$ consisting of all the cohomology classes $\{d \cup h - g\}$, where h and g satisfies the above condition $(*)$. It is easily verified that $M_{p+1} = H^{p+1}(G,\bar{B})$. So we obtain an explicit form of $\delta_*: H^{p+1}(G,\bar{B}) \cong H^{p+2}(G,C)$ by

$$\delta_*\{d \cup h - g\} = \{u \cup h - \delta(s \circ g)\},$$

where s is a cross section from B to A such as $\alpha \circ s = id_B$.

Here we shall show this explicit form implies the main theorem of our previous paper [5]. Let k be a local field and K be its Galois extension of finite degree. We denote the Galois group by G. Let us denote the unit group of K by U_K. Then we have the following exact sequence of G-modules

$$1 \longrightarrow U_K \longrightarrow K^X \xrightarrow{\alpha} Z \longrightarrow 1.$$

Here α is the normal exponential valuation with respect to K. Let $\xi_{K,k} = \{u\}$ be the canonical cohomology class for K/k. Let us denote $\alpha_*(\xi_{K,k}) = \{d \cdot u = v\}$ by $\eta_{K,k}$ and Artin's splitting module of $\eta_{K,k}$ by $\bar{Z}$. Then, in our previous paper [5], we have showed there exists an isomorphism $\nu_p : H^{p+2}(G, U_K) \cong H^{p+1}(G, \bar{Z})$ for every integer p.

Proposition 2.1. *For every integer $p \in Z$, we have an isomorphism*

$$\nu_p : H^{p+2}(G, U_K) \cong H^{p+1}(G, \bar{Z}),$$

such that the following diagram is commutative

$$\begin{array}{ccccccccc}
\cdots \to & H^{p+1}(Z) & \longrightarrow & H^{p+1}(\bar{Z}) & \longrightarrow & H^{p+1}(I[G]) & \longrightarrow & H^{p+2}(Z) & \to \cdots \\
& \| & & \downarrow \nu_p & & \wr\| & & \| & \\
\cdots \to & H^{p+1}(Z) & \longrightarrow & H^{p+2}(U_K) & \longrightarrow & H^{p+2}(K^X) & \longrightarrow & H^{p+2}(Z) & \to \cdots .
\end{array}$$

([5], Theorem).

Let us replace A, B and C of Theorem 2.1 by K^X, Z and U_K, respectively and other morphisms and symbols by corresponding ones. From the explicit form of δ_*, we have

$\delta_* : H^{p+1}(G, \bar{Z}) \cong H^{p+2}(G, U_K)$ and $\delta_*\{d \cup h - g\} = \{(u \cup h)/ \delta(s \cdot g)\}$,

where s is a cross section from Z to K^X and h is a p-cocycle p-cocycle of $H^p(G, Z)$ and g is a $(p+1)$-cochain with values in Z which satisfies the condition $(*)$. On the other hand, by the definition of ν_p, we can easily verify that

$$\nu_p\{(u \cup h)/ \delta(s \cdot g)\} = \{d \cup h - g\}.$$

Hence $\nu_p = (\delta_*)^{-1}$. Therefore Proposition 2.1 is obtained as a corollary of Theorem 2.1.

§3. Let G be a finite group and N be its subgroup of index n. We put $G = \bigcup_{i=1}^{n} \sigma_i N$ with $\sigma_1 = 1$ (the identity of G). Let us denote by $Z[G/N]$ the free Z-module generated by $\sigma_i N$ $(1 \leq i \leq n)$. Let ε_G be an onto G-homomorphism from $Z[G/N]$ to Z defined by putting $\varepsilon_G(\sigma_i N) = 1$ for every i. Then we have the following exact sequence of G-modules

$$0 \longrightarrow I[G/N] \longrightarrow Z[G/N] \xrightarrow{\varepsilon_G} Z \longrightarrow 0,$$

where $I[G/N]$ is the kernel of ε_G. Since G is a finite group, $Z[G/N] = \mathrm{Ind}_N^G Z$ is isomorphic to $\mathrm{Hom}_{Z[N]}(Z[G],Z) = \mathrm{Coind}_N^G Z$ ([1], Ch. III, (5.9) Prop.). Therefore we have

Lemma 3.1. _With the notation as above, we have the isomorphism_ $H^r(G,Z[G/N]) \cong H^r(N,Z)$ $(r \in Z)$.

Let us define a G-homomorphism ε_N from $Z[G]$ to $Z[G/N]$ by putting $\varepsilon_N(\sigma) = \sigma N$ (for every $\sigma \in G$). We denote the kernel of ε_N by $K[G,N]$. Then it is easily verified that $\varepsilon_N(I[G]) = I[G/N]$ and $K[G,N] \subset I[G]$. Therefore we have the following commutative diagram of G-modules with exact rows and columns

(3.1)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K[G,N] \arrow[r, equal] \arrow[d] & K[G,N] \arrow[d] & & \\
0 \arrow[r] & I[G] \arrow[r] \arrow[d] & Z[G] \arrow[r, "\varepsilon_G"] \arrow[d, "\varepsilon_N"] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I[G/N] \arrow[r] \arrow[d] & Z[G/N] \arrow[r] \arrow[d] & Z \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $Z[G]$ is cohomologically trivial, this diagram satisfies the conditions of case (III) of Theorem 1.1. Hence we have

Lemma 3.2. *The two cohomology sequences derived from* $0 \longrightarrow K[G,N] \longrightarrow I[G] \longrightarrow I[G/N] \longrightarrow 0$ *and* $0 \longrightarrow I[G/N] \longrightarrow Z[G/N] \longrightarrow Z \longrightarrow 0$ *are isomorphic, that is, the following diagram is commutative for every* $r \in Z$

$$\begin{array}{ccccccccc}
\cdots \to & H^r(G,K[G,N]) & \longrightarrow & H^r(G,I[G]) & \longrightarrow & H^r(G,I[G/N]) & \longrightarrow & H^{r+1}(G,K[G,N]) & \to \cdots \\
 & \wr\| & & \wr\| & & \| & & \wr\| & \\
\cdots \to & H^{r-1}(N,Z) & \xrightarrow{\tau^{N,G}} & H^{r-1}(G,Z) & \longrightarrow & H^r(G,I[G/N]) & \longrightarrow & H^r(N,Z) & \to \cdots
\end{array} ,$$

where $\tau^{N,G}$ *is the transfer homomorphism from* N *to* G.

Let us investigate the cohomology group of $Z[G/N]$ more precisely. Let H be another subgroup of G and let E be a set of representatives for the double cosets $H\sigma N$. Then as H-module

$$Z[G/N] = \sum_{\sigma \in E} Z[H\sigma N/N] \cong \sum_{\sigma \in E} Z[H/H \cap \sigma N \sigma^{-1}] \quad ([1], \text{Ch. III},$$

(5.6) Prop.). Hence, from Lemma 3.1, we have

Lemma 3.3. *Let* H *and* N *be subgroups of* G *and let* E *be a set of representatives for the double cosets* $H\sigma N$. *Then we have*

$$H^r(H,Z[G/N]) \cong \sum_{\sigma \in E} H^r(H \cap \sigma N \sigma^{-1}, Z) \quad (r \in Z).$$

From the cohomology sequences derived from $0 \longrightarrow K[G,N] \longrightarrow Z[G] \longrightarrow Z[G/N] \longrightarrow 0$, we have $H^{r+1}(H,K[G,N]) \cong H^r(H,Z[G/N])$ for every $r \in Z$. Therefore, from Lemma 3.3, we have

Corollary 3.1. *With the notation as above, we have*

$$H^{r+1}(H,K[G,N]) \cong H^r(H,Z[G/N]) \cong \sum_{\sigma \in E} H^r(H \cap \sigma N \sigma^{-1}, Z) \quad (r \in Z).$$

Now we consider the relation of the following two exact sequences

$$0 \longrightarrow I[N] \longrightarrow Z[N] \longrightarrow Z \longrightarrow 0,$$

$$0 \longrightarrow K[G,N] \longrightarrow Z[G] \longrightarrow Z[G/N] \longrightarrow 0.$$

By virtue of the fact that $Z[G]$ is N-projective, the functor Ind_N^G is an exact functor. So, from the upper exact sequence of above, we obtain an exact sequence of G-modules

$$0 \longrightarrow \mathrm{Ind}_N^G I[N] \longrightarrow \mathrm{Ind}_N^G Z[N] \longrightarrow \mathrm{Ind}_N^G Z \longrightarrow 0.$$

We can easily verify that $\mathrm{Ind}_N^G I[N] \cong K[G,N]$ and $\mathrm{Ind}_N^G Z[N] \cong Z[G]$. Hence we have

Lemma 3.4. <u>With the notation as above, the following two exact sequences of G-modules are isomorphic</u>

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ind}_N^G I[N] \arrow[r] \arrow[d, phantom, "\wr\|"] & \mathrm{Ind}_N^G Z[N] \arrow[r] \arrow[d, phantom, "\wr\|"] & \mathrm{Ind}_N^G Z \arrow[r] \arrow[d, phantom, "\wr\|"] & 0 \\
0 \arrow[r] & K[G,N] \arrow[r] & Z[G] \arrow[r] & Z[G/N] \arrow[r] & 0.
\end{tikzcd}$$

§4. Let k be either an algebraic number field of finite degree or an algebraic function field over a finite field, and K be its finite Galois extension with the group G. Let p be a place of k and P be one of its extensions to K. We denote the decomposition group of P by N. Let $K_p^{\times}$ be a G-submodule of $K_A^{\times}$ consisting of all the ideles whose Q-factors are 1 except when Q divides p. Let us fix a canonical class $\xi_{K,k}$ of K/k and denote Artin's splitting module of $\xi_{K,k}$ by $\bar{C}_K$. Here we shall construct a cohomologically trivial G-module $\widetilde{K_p^{\times}}$ such as the following diagram is commutative

$$(4.1)\qquad \begin{tikzcd} 0 \arrow[r] & K_p^{\times} \arrow[r] \arrow[d, "a_p"] & \widetilde{K_p^{\times}} \arrow[r] \arrow[d, "\tilde{a}_p"] & K[G,N] \arrow[r] \arrow[d, "\iota_N"] & 0 \\ 0 \arrow[r] & C_K \arrow[r] & \bar{C}_K \arrow[r] & I[G] \arrow[r] & 0, \end{tikzcd}$$

where a_p is a natural G-homomorphism and ι_N is also the natural embedding and $\tilde{a}_p$ is an into G-isomorphism. Let ι_P be the natural embedding from $K_P^{\times}$ (the multiplicative group of the P-completion of K) to $K_p^{\times}$. Then we have a commutative diagram

$$(4.2)\qquad \begin{tikzcd} H^2(N,K_P^{\times}) \arrow[r, tail, "\iota_P"] \arrow[dr, "\sim"'] & H^2(N,K_p^{\times}) \arrow[r, two heads, "a_p"] \arrow[d, "\tau^{N,G}"] & H^2(N,C_K) \\ & H^2(G,K_p^{\times}) & \end{tikzcd}$$

By virtue of the fact that the restriction homomorphism $\rho_{G,N}: H^2(G,C_K) \longrightarrow H^2(N,C_K)$ is an onto homomorphism, there exist a canonical class of K_P/k_p denoted by ξ_P such as

$a_p \circ l_p(\xi_p) = \rho_{G,N}(\xi_{K,k})$.

Hence we have a commutative diagram

$$
(4.3)\qquad
\begin{array}{ccccccccc}
1 & \longrightarrow & K_p^{\times} & \longrightarrow & W_{K_p,k_p} & \longrightarrow & N & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & C_K & \longrightarrow & W_{K,k} & \longrightarrow & G & \longrightarrow & 1,
\end{array}
$$

where $W_{K,k}$ and W_{K_p,k_p} are the Weil groups of $\xi_{K,k}$ and ξ_p, respectively. We put $G = \bigcup_{i=1}^{n} a_i N$, with $a_1 = 1$. We set $\bar{\sigma} = a_i$ when σ belongs to the coset $a_i N$. Then any element $\sigma \in G$ is uniquely written as the product $\bar{\sigma}\tilde{\sigma}$ ($\tilde{\sigma} \in N$). Therefore, if we consider the 2-cocycles of $\xi_{K,k}$ and ξ_p as the factor sets of the group extensions of (4.3), we can take the cocycles u and v which satisfy the following conditions.

$\xi_p = \{u\}$ such as $u[\sigma,\tau] = u_\sigma u_\tau u_{\sigma\tau}^{-1}$ $(\sigma,\tau \in N)$,

where $\sigma \longrightarrow u$ is a cross section from N to W_{K_p,k_p}, with $u_1 = 1$.

$\xi_{K,k} = \{v\}$ such as $v[\sigma,\tau] = v_\sigma v_\tau v_{\sigma\tau}^{-1}$ $(\sigma,\tau \in G)$,

where $\sigma \longrightarrow v_\sigma$ is the cross section from G to $W_{K,k}$ which satisfies $v_\sigma = v_{\bar{\sigma}} v_{\tilde{\sigma}} = v_{\bar{\sigma}} u_{\tilde{\sigma}}$, with $v_{\bar{1}} = 1$, where W_{K_p,k_p} is considered to be imbedded in $W_{K,k}$.

Let us define the G-module structure of $\widetilde{K_p^{\times}}$ using these cocycles u and v. We consider $K_p^{\times}$ is a G-submodule of $\widetilde{K_p^{\times}}$ and so, for the purpose of defining a G-module structure on $\widetilde{K_p^{\times}}$, it is sufficient to define the G-action on $K[G,N]$. Since $K[G,N]$ is, as Z-module

$\sum_{i=1,\ \rho\in N}^{n} Z(a_i\rho - a_i)$, we can define the G-module structure of $\widetilde{K_p^X}$ by

$$\sigma(a_i\rho - a_i) = \sigma a_i\rho - \sigma a_i + v[\sigma, a_i\rho]/v[\sigma, a_i] \quad (\sigma\in G,\ \rho\in N).$$

By the definition of v, we have

$$v[\sigma, a_i\rho]/v[\sigma, a_i]$$
$$=(v_{\sigma a_i} v_{a_i}^{-1} v_{\sigma}^{-1})(v_{\sigma} v_{a_i\rho} v_{\sigma a_i\rho}^{-1})$$
$$= v_{\overline{\sigma a_i}} v_{\widetilde{\sigma a_i}} v_{\rho} v_{\widetilde{\sigma a_i\rho}}^{-1} v_{\overline{\sigma a_i}}^{-1}$$
$$= \overline{\sigma a_i}\, u[\widetilde{\sigma a_i}, \rho] \in K_p^X.$$

Then the above definition is well defined and from the fact that $a_i\rho - a_i = d_{a_i\rho} - d_{a_i}$, the commutativity of the diagram (4.1) is obvious. Let us show the cohomological triviality of $\widetilde{K_p^X}$. Let $\overline{K_p^X}$ be Artin's splitting module of u. Then, as a Z-module, we see

$$\widetilde{K_p^X} = K[G,N] \oplus K_p^X = \sum_{i=1}^{n} a_i(\overline{K_p^X}),$$

where $a_i(\overline{K_p^X}) = a_i(K_p^X) \oplus a_i(I[N])$. Since the isotropy subgroup of K_p^X is N and $v[\sigma, a_i\rho]/v[\sigma, a_i] = \overline{\sigma a_i}\, u[\widetilde{\sigma a_i}, \rho]$ (for every $\sigma\in G$, $\rho\in N$), we see $\sigma(a_i(\overline{K_p^X})) = \overline{\sigma a_i}(\overline{K_p^X}) = \sigma a_i(\overline{K_p^X})$. Therefore, from a characterization theorem of the induced module (see, for example, [1], Chap. III. (5.3) Prop.), we see $\widetilde{K_p^X} \cong \mathrm{Ind}_N^G \overline{K_p^X}$. On the other hand, we see $K[G,N] \cong \mathrm{Ind}_N^G I[N]$ and $K_p^X \cong \mathrm{Ind}_N^G K_p^X$.

Summarizing these, we can easily show

Lemma 4.1. <u>With the notation as above, we have the following G-isomorphism of two exact sequences</u>

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ind}_N^G\, K_P^X & \longrightarrow & \mathrm{Ind}_N^G\, \overline{K_P^X} & \longrightarrow & \mathrm{Ind}_N^G\, I[N] & \longrightarrow & 0 \\ & & \wr\Vert & & \wr\Vert & & \wr\Vert & & \\ 0 & \longrightarrow & K_p^X & \longrightarrow & \widetilde{K_p^X} & \longrightarrow & K[G,N] & \longrightarrow & 0. \end{array}$$

From this lemma, we see $\widetilde{K_p^X}$ is cohomologically trivial.

Remark 1. In the above discussion, we have fixed cocycles u and v. But, as is well known, Artin's splitting module is uniquely defined by the cohomology class up to G-isomorphism. Hence, we can consider the module $\widetilde{K_p^X}$ is defined by the cohomology class ξ_p.

Remark 2. We can show the cohomological triviality of $\widetilde{K_p^X}$ in a more straight way. For every $H < G$, we have the following derived cohomology sequence of (4.1).

$$0 \longrightarrow H^1(H,\widetilde{K_p^X}) \longrightarrow H^1(H,K[G,N]) \longrightarrow H^2(H,K_p^X) \longrightarrow H^2(H,\widetilde{K_p^X})$$
$$\longrightarrow H^2(H,K[G,N]).$$

From Corollary 3.4, we have $H^2(H,K[G,N]) \cong H^1(H,Z[G,N]) = 0$. Therefore, to show the cohomological triviality of $\widetilde{K_p^X}$, it is necessary and sufficient to show the connecting homomorphism $\delta_*: H^1(H,K[G,N]) \to H^2(H,K_p^X)$ is an isomorphism. Let E be a set of representatives for the double cosets $H\sigma N$. Then, from Corollary 3.1 and $K_p^X \cong \mathrm{Ind}_N^G\, K_P^X$, we have the following commutative diagram

$$\begin{array}{ccc} \delta_*: H^1(H, K[G,N]) & \longrightarrow & H^2(H, K_p^{\times}) \\ \wr\!\!\| & & \wr\!\!\| \\ \sum_{\sigma\in\Xi} H^0(H\cap \sigma N\sigma^{-1}, Z) & \cong & \sum_{\sigma\in\Xi} H^2(H\cap\sigma N\sigma^{-1}, K^{\times}_{\sigma p}). \end{array}$$

Hence δ_* is an isomorphism, and so $\widetilde{K_p^{\times}}$ is cohomologically trivial.

§5. Let k be an algebraic number field of finite degree and K/k be a finite Galois extension with the group G. In the following, we assume the number of the real places of k which ramify in K is at most 1. Let us denote by C_K the idele class group of K and by D_K the connected component of the unity of C_K. We denote by N the decomposition group of the real place which ramifies in K. Let $\xi_{K,k}$ be the canonical cohomology class of $H^2(G, C_K)$. We denote by $\eta_{K,k}$ the image of $\xi_{K,k}$ by a natural homomorphism from C_K to C_K/D_K. Let us denote Artin's splitting modules of $\xi_{K,k}$ and $\eta_{K,k}$ by $\bar{C}_K$ and by $\overline{C_K/D_K}$, respectively. Then we have the following commutative diagram of G-modules with exact rows and columns

(5.1)
$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & D_K & = & D_K & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & C_K & \longrightarrow & \bar{C}_K & \longrightarrow & I[G] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & C_K/D_K & \longrightarrow & \overline{C_K/D_K} & \longrightarrow & I[G] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Since $\bar{C}_K$ is cohomologically trivial, we have the following lemma from Theorem 1.1 case (III).

Lemma 5.1. *For every* $r \in Z$, *the derived cohomology sequences of* $0 \longrightarrow D_K \longrightarrow C_K \longrightarrow C_K/D_K \longrightarrow 0$ *and* $0 \longrightarrow C_K/D_K \longrightarrow \overline{C_K/D_K} \longrightarrow I[G] \longrightarrow 0$ *are isomorphic, that is the following diagram is commutative*

$$\begin{tikzcd}
\cdots \arrow[r] & H^r(D_K) \arrow[r] \arrow[d, "\wr\,\delta_*"] & H^r(C_K) \arrow[r] \arrow[d, "\wr\,\delta_*"] & H^r(C_K/D_K) \arrow[r] \arrow[d, equal] & H^{r+1}(D_K) \arrow[r] \arrow[d, "\wr\,\delta_*"] & \cdots \\
\cdots \arrow[r] & H^{r-1}(\overline{C_K/D_K}) \arrow[r] & H^{r-1}(I[G]) \arrow[r] & H^r(C_K/D_K) \arrow[r] & H^r(\overline{C_K/D_K}) \arrow[r] & \cdots .
\end{tikzcd}$$

Now, we shall prove a general proposition concerning the extensions of groups as follows.

Proposition 5.1. *Let* G *be a finite group and* N *be its subgroup and* A *be a G-module. Let* $\rho_{G,N}$ *be the restriction homomorpism from* $H^2(G,A)$ *to* $H^2(N,A)$. *We fix a cohomology class* ξ *of* $H^2(G,A)$ *and denote* $\rho_{G,N}\xi$ *by* η. *Then the following six conditions are equivalent;*

1) $\eta = 0$ *in* $H^2(N,A)$.

2) *Let* $\bar{A}_\xi$ *and* $\bar{A}_\eta$ *be any of Artin's splitting modules corresponding to* ξ *and* η, *respectively. Then the extension* $0 \longrightarrow A \longrightarrow \bar{A}_\eta \longrightarrow I[N] \longrightarrow 0$ *is split as an extension of N-modules and there exists an injective N-homomorhism* κ *such that the following diagram is commutative*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & \bar{A}_\eta \arrow[r] \arrow[d, "\kappa"] & I[N] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & \bar{A}_\xi \arrow[r] & I[G] \arrow[r] & 0,
\end{tikzcd}$$

where the right arrow is a natural embedding.

3) Let $1 \longrightarrow A \longrightarrow \mathcal{G} \longrightarrow G \longrightarrow 1$ and $1 \longrightarrow A \longrightarrow \mathcal{H} \longrightarrow N \longrightarrow 1$ be any of the extensions of groups corresponding to ξ and η, respectively. Then the extension $1 \longrightarrow A \longrightarrow \mathcal{H} \longrightarrow N \longrightarrow 1$ is split and there exists an injective homomorphism λ such that the following diagram is commutative

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[d, equal] & \mathcal{H} \arrow[r] \arrow[d, "\lambda"] & N \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & A \arrow[r] & \mathcal{G} \arrow[r, "\pi"] & G \arrow[r] & 1, \end{tikzcd}$$

where the right arrow is the natural embedding.

4) There exists a cocycle u of ξ which satisfies the equation

$$u[\sigma, \tau\rho] = u[\sigma, \tau], \text{ for any } \sigma, \tau \in G \text{ and } \rho \in N;$$

5) For any of Artin's splitting modules $\bar{A}_\xi$ corresponding to ξ, there exists an injective G-homomorphism μ such that the following diagram is commutative

$$\begin{tikzcd} & K[G,N] \arrow[dl, "\mu"'] \arrow[d] \\ \bar{A}_\xi \arrow[r] & I[G] \end{tikzcd} \quad .$$

Here the vertical arrow is an natural embedding.

6) For any of Artin's splitting modules corresponding to ξ, there exists a G-module $\tilde{A}$ which is an extension of $I[G/N]$ with the kernel A and also exists a surjective G-homomorp-hism ν such that the following diagram is commutative

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & \bar{A}_\xi \arrow[r] \arrow[d, "\nu"] & I[G] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & \tilde{A} \arrow[r] & I[G/N] \arrow[r] & 0, \end{tikzcd}$$

where $I[G] \longrightarrow I[G/N]$ is the natural onto homomorphism.

Proof. We can consider the integral group ring $Z[G]$ to be a supplemented algebra with a Z-algebra homomorphism $\varepsilon : Z[G] \longrightarrow Z$. Then, from a well known relation of the extensions of groups, the 2-cohomology groups of groups and the extensions of augmentation ideals, it is easy to show the equivalence 1)$\Longleftrightarrow$2)$\Longleftrightarrow$3). (See, for example [2], Chap. XIV).

3)$\Longrightarrow$4). We can take a cross section u from N to $\mathfrak{N}$ such as

$$u_\sigma \cdot u_\tau = u_{\sigma\tau} \quad \text{for every} \quad \sigma, \tau \in N.$$

We set $G = \bigcup_{\alpha \in E} \alpha N$, where E is a set of representatives. For every $\alpha \in E$, we fix an element $u_\alpha \in \mathfrak{G}$ such as $\pi(u_\alpha)' = \alpha$. In the same way as §4, we set $\bar{\sigma} = \alpha$ when $\sigma N = \alpha N$ $(\alpha \in E)$, and denote $\bar{\sigma}^{-1} \cdot \sigma$ by $\tilde{\sigma}$. Since every element $\sigma \in G$ is uniquely written as the product $\bar{\sigma} \cdot \tilde{\sigma}$, we can define a cross section u from G to $\mathfrak{G}$ by putting $u_\sigma = u_{\bar{\sigma}} \cdot u_{\tilde{\sigma}}$.

Then, for every $\sigma, \tau \in G$ and $\rho \in N$, we have

$$\begin{aligned} u_{\tau\rho} \cdot u_{\sigma\tau\rho}^{-1} &= (u_{\bar{\tau}} \cdot u_{\tilde{\tau}\rho})(u_{\widetilde{\sigma\tau}\cdot\rho}^{-1} \cdot u_{\overline{\sigma\tau}}^{-1}) \\ &= u_{\bar{\tau}}(u_{\tilde{\tau}} \cdot u_\rho \cdot u_\rho^{-1} \cdot u_{\widetilde{\sigma\tau}}^{-1})\, u_{\overline{\sigma\tau}}^{-1} \\ &= u_{\bar{\tau}} \cdot u_{\tilde{\tau}} \cdot u_{\widetilde{\sigma\tau}}^{-1} \cdot u_{\overline{\sigma\tau}}^{-1} \\ &= u_\tau \cdot u_{\sigma\tau}^{-1}. \end{aligned}$$

Hence we have $u_\sigma \cdot u_{\tau\rho} \cdot u_{\sigma\tau\rho}^{-1} = u_\sigma \cdot u_\tau \cdot u_{\sigma\tau}^{-1}$. Therefore the factor set $\{u[\sigma, \tau] = u_\sigma \cdot u_\tau \cdot u_{\sigma\tau}^{-1} \mid \sigma, \tau \in G\}$ satisfies the condition 4).

4)$\Longrightarrow$5). Let $\bar{A}_u$ be Artin's splitting module of the cocycle u. Then the condition 4) is nothing but the condition in order that a module $K[G,N]$ may be a G-submodule of $\bar{A}_u$. Let $\bar{A}_{u'}$ be Artin's splitting module of any cocycle u' contained in ξ. Since $\bar{A}_u$ is equivalent to $\bar{A}_{u'}$, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & \bar{A}_u \arrow[r] \arrow[d, "\wr\mu'"] & I[G] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & \bar{A}_{u'} \arrow[r] & I[G] \arrow[r] & 0.
\end{tikzcd}$$

Then the restricted homomorphism $\mu = \mu'|K[G,N] : K[G,N] \longrightarrow \bar{A}_{u'}$ satisfies the required relation.

5)$\Longrightarrow$6). Since $\bar{A}_\xi / \mu(K[G,N])$ is a G-module, it is obvious that the sequence $0 \longrightarrow A \longrightarrow \bar{A}_\xi / \mu(K[G,N]) \longrightarrow I[G/N] \longrightarrow 0$ is exact as G-modules. So, if we denote $\bar{A}_\xi / \mu(K[G,N])$ by $\tilde{A}$, $\tilde{A}$ satisfies the condition 6).

6)$\Longrightarrow$1). From the exact sequence $0 \longrightarrow I[G/N] \longrightarrow Z[G/N] \longrightarrow Z \longrightarrow 0$, we see $H^1(G, I[G/N]) \cong Z/[G:N]Z$. Moreover, if we set $d_{\sigma N} = \sigma N - N$ in $I[G/N]$, we see $d[\sigma] = d_{\sigma N}$ $(\sigma \in G)$ is a cocycle contained in the generator of $H^1(G, I[G/N])$. From the commutative diagram of 6), we have the following commutative diagram of derived cohomology groups

$$\begin{tikzcd}
H^1(G, I[G]) \arrow[r] \arrow[d] & H^2(G,A) \\
H^1(G, I[G/N]) \arrow[ur] &
\end{tikzcd}$$

Hence, if we put $u[\sigma,\tau] = \sigma d_{\tau N} - d_{\sigma\tau N} + d_{\sigma N}$ (d_N ツメル is an element of A), we have $\xi = \{u\}$. Therefore, for $\sigma, \tau \in N$, we have

$u[\sigma,\tau] = \sigma d_N$. We define an 1-cochain β, with value A by $\beta[\sigma] = d_N$. Then we have $\sigma d_N = (\delta\beta)[\sigma,\tau]$ $(\sigma,\tau \in N)$. Hence $\rho_{G,N}\xi = \{\delta\beta\} = 0$, which completes the proof.

Remark. Let $E(I[G],A)$ be the set of all the equivalent classes of the extensions of G-modules of $I[G]$ and A. Then, as is well known, $E(I[G],A)$ is considered to be a commutative group with Baer multiplication and $E(I[G],A)$ is isomorphic to $H^2(G,A)$ in a natural way. In the above proposition, we have written "$\bar{A}_\xi$ ~~as~~ is any of Artin's splitting modules corresponding to ξ", which means $\bar{A}_\xi$ ~~as~~ is any of the G-modules belonging to the equivalent class which corresponds to $\xi \in H^2(G,A)$.

In the following discussion, one shall see Theorem 5.1 is trivial for the case when there is no real place of k which ramifies in K. Hence, in the following, we suppose there exists a real place p_0 of k which ramifies in K. Let us fix one of the extensions of p_0 to K, and denote its decomposition group by N. From Corollary 3 of our previous paper [4], we have $H^2(N,C_K/D_K) = 0$ and $H^2(G,C_K/D_K) = \langle \eta_{K,k}\rangle \cong Z/[G:N]Z$. Therefore $\eta_{K,k}$ satisfies the condition 1) of the above proposition. So, there exists an exact sequence of G-modules

$$(5.2)\qquad 0 \longrightarrow C_K/D_K \longrightarrow \widetilde{C_K/D_K} \longrightarrow I[G/N] \longrightarrow 0.$$

Now we shall show that $\widetilde{C_K/D_K}$ is cohomologically trivial. Let us denote the connected component of the unity of the idele group $K_A^\times$ by H_K. Since D_K is the closure of $\bar{H}_K = H_K K^\times/K^\times$ in C_K ([3], Ch. III., § 7.2., Lem. 2.), we have the following commutative diagram of G-modules with exact rows and columns

(5.3)

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & D_K/\bar{H}_K \arrow[d] & \\
0 \arrow[r] & H_K \arrow[r] \arrow[d] & C_K \arrow[r] \arrow[d, no head, shift left=0.5] \arrow[d, no head, shift right=0.5] & C_K/\bar{H}_K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D_K \arrow[r] \arrow[d] & C_K \arrow[r] & C_K/D_K \arrow[r] \arrow[d] & 0 \\
 & D_K/\bar{H}_K \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}$$

Since $D_K/\bar{H}_K$ is uniquely divisible, we have an isomorphism $H^2(G,C_K/\bar{H}_K) \cong H^2(G,C_K/D_K)$. We shall denote by $\widetilde{C_K/\bar{H}_K}$ the extension of $I[G/N]$ with the kernel $C_K/\bar{H}_K$ corresponding to $\eta_{K,k}$ by this isomorphism. Then, we have the following exact sequence of G-modules $0 \longrightarrow D_K/\bar{H}_K \longrightarrow \widetilde{C_K/\bar{H}_K} \longrightarrow \widetilde{C_K/D_K} \longrightarrow 0$. By virtue of the fact $D_K/\bar{H}_K$ is uniquely divisible, for the purpose of showing that $\widetilde{C_K/D_K}$ is cohomologically trivial, it is necessary and sufficient to show $\widetilde{C_K/\bar{H}_K}$ is cohomologically trivial.

H_K is written in the form $\prod_{p \neq p_0\text{, real}} (K_p^X)_+ \cdot \prod_{p\text{, imaginary}} K_p^X \cdot K_{p_0}^X$, where p runs all the infinite places of k except p_0, and $(K_p^X)_+$ denotes the subgroup of $K_p^X = \prod_{P|p} K_P^X$ consisting of non-negative elements of K_p^X. Let $\widetilde{H}_K$ be the G-module $\prod_{p \neq p_0\text{, real}} (K_p^X)_+ \cdot \prod_{p\text{, imaginary}} K_p^X \cdot \widetilde{K_{p_0}^X}$, where $\widetilde{K_{p_0}^X}$ is the extension of $K^{x}_{p_0}$ by $K[G,N]$ defined in §4. Then it is easy to show the following sequence of G-modules are exact

(5.4) $\quad 0 \longrightarrow \tilde{H}_K \longrightarrow \bar{C}_K \longrightarrow \widetilde{C_K/\bar{H}_K} \longrightarrow 0.$

We see $\tilde{H}_K$ is cohomologically trivial from Lemma 4.1, and $\bar{C}_K$ is also cohomologically trivial. Therefore $\widetilde{C_K/\bar{H}_K}$ is cohomologically trivial.

Lemma 5.2. <u>Let $\eta_{K,k}$ be a generator of $H^2(G,C_K/D_K) \cong Z/[G:N]Z$. Then there exists a cohomologically trivial G-module $\widetilde{C_K/D_K}$ such that the following diagram is exact as G-modules</u>

$$0 \longrightarrow C_K/D_K \longrightarrow \widetilde{C_K/D_K} \longrightarrow I[G/N] \longrightarrow 0.$$

From Proposition 5.1 6), there exists a commutative diagram of G-modules

(5.5)

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K[G,N] \arrow[r, equal] \arrow[d] & K[G,N] \arrow[d] & \\
0 \arrow[r] & C_K/D_K \arrow[r] \arrow[d, equal] & \overline{C_K/D_K} \arrow[r] \arrow[d] & I[G] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_K/D_K \arrow[r] & \widetilde{C_K/D_K} \arrow[r] \arrow[d] & I[G/N] \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since, for every $r \in Z$, $H^r(G,K[G,N]) \cong H^{r-1}(N,Z)$ is at most order 2, we have the following Lemma from Corollary 1.1

Lemma 5.3. <u>From the diagram</u> (5.5), <u>we see the derived cohomology sequences of</u> $0 \longrightarrow C_K/D_K \longrightarrow \overline{C_K/D_K} \longrightarrow I[G] \longrightarrow 0$ <u>and</u> $0 \longrightarrow K[G,N] \longrightarrow I[G] \longrightarrow I[G/N] \longrightarrow 0$ <u>are isomorphic, that is,</u>

the following diagram is commutative

$$\begin{tikzcd}
\cdots \arrow[r] & H^{r-1}(I[G/N]) \arrow[r] \arrow[d, "\wr"] & H^r(K[G,N]) \arrow[r] \arrow[d, "\wr"] & H^r(I[G]) \arrow[r] \arrow[d, equal] & H^r(I[G/N]) \arrow[r] \arrow[d, "\wr"] & \cdots \\
\cdots \arrow[r] & H^r(C_K/D_K) \arrow[r] & H^r(\overline{C_K/D_K}) \arrow[r] & H^r(I[G]) \arrow[r] & H^{r+1}(C_K/D_K) \arrow[r] & \cdots
\end{tikzcd}$$

Combining Lemma 3.2, Lemma 5.1 and Lemma 5.3, we have the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^{r-2}(N,Z) \arrow[r, "\tau^{N,G}"] \arrow[d, "\wr"] & H^{r-2}(G,Z) \arrow[r] \arrow[d, "\wr"] & H^{r-1}(G,I[G/N]) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & H^{r-1}(G,K[G,N]) \arrow[r] \arrow[d, "\wr"] & H^{r-1}(G,I[G]) \arrow[r] \arrow[d, equal] & H^{r-1}(G,I[G/N]) \arrow[r] \arrow[d, "\wr"] & \cdots \\
\cdots \arrow[r] & H^{r-1}(G,\overline{C_K/D_K}) \arrow[r] \arrow[d, "\wr"] & H^{r-1}(G,I[G]) \arrow[r] \arrow[d, "\wr"] & H^r(G,C_K/D_K) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & H^r(G,D_K) \arrow[r] & H^r(G,C_K) \arrow[r] & H^r(G,C_K/D_K) \arrow[r] & \cdots
\end{tikzcd} \quad (r \in Z),$$

where $\tau^{N,G}$ is the transfer homomorphism from N to G. Hence we have the following theorem

Theorem 5.1. *With the notation and assumption as above, we have the following commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & H^{r-2}(N,Z) \arrow[r, "\tau^{N,G}"] \arrow[d, "\wr"] & H^{r-2}(G,Z) \arrow[r] \arrow[d, "\wr"] & H^{r-1}(G,I[G/N]) \arrow[r] \arrow[d, "\wr"] & \cdots \\
\cdots \arrow[r] & H^r(G,D_K) \arrow[r] & H^r(G,C_K) \arrow[r] & H^r(G,C_K/D_K) \arrow[r] & \cdots
\end{tikzcd} \quad (r \in Z).$$

§6. First, we shall summarize the main results of [11].

Let k be an algebraic number field of finite degree, or an algebraic function field of one variable over a finite field. Let K/k be a finite Galois extension with the group G. S denotes a set of places of K satisfying the following conditions

(S1) S is stable under G.

(S2) S contains all archimedean places.

(S3) S contains all places ramified over $\cancel{k}$ R.

(S4) S is large enough so that every ideal classes of K contains an ideal with support in S.

There exist exact sequences of G-modules:

(A) $0 \longrightarrow E \xrightarrow{a'} J \xrightarrow{a} C \longrightarrow 0,$

(B) $0 \longrightarrow X \xrightarrow{b'} Y \xrightarrow{b} Z \longrightarrow 0,$

in which:

E is the group of S-units of K, that is, elements of K which are units at all places P not in S.

J is the group of S-ideles of K, that is, ideles whose P-component is an unit for each place P not in S.

C is the group of S-idele classes, which in view of condition (S4) is G-isomorphic to the group of all idele classes of K.

Z is the group of integers, G operating trivially.

Y is the free Z-module generated by the places P in S, an element $\sigma \in G$ operating by the rule

$$\sigma\Big(\sum_{P \in S} n_P\, P\Big) = \sum_{P \in S} n_P(\sigma P) = \sum_{P \in S} n_{(\sigma^{-1}P)}\, P.$$

X is the kernel of the natural map b which takes an element $y = \sum n_P\, P$ into its coefficient sum, $\sum n_P$.

In these statements, Tate proved the cohomology sequence derived from (A) is isomorphic to that derived from (B), after a dimension shift of two; that is, he has constructed a commutative diagram

(6.1)

$$\begin{tikzcd}
\cdots \arrow[r] & H^r(X) \arrow[r] \arrow[d, "\alpha_3^r"] & H^r(Y) \arrow[r] \arrow[d, "\alpha_2^r"] & H^r(Z) \arrow[r] \arrow[d, "\alpha_1^r"] & H^{r+1}(X) \arrow[r] \arrow[d, "\alpha_3^{r+1}"] & \cdots \\
\cdots \arrow[r] & H^{r+2}(E) \arrow[r] & H^{r+2}(J) \arrow[r] & H^{r+2}(C) \arrow[r] & H^{r+3}(E) \arrow[r] & \cdots ,
\end{tikzcd}$$

in which the vertical arrows α_i^r, for $i=1, 2, 3$ and $r \in Z$, are isomorphisms.

Here we shall prove the above result in somewhat restricted situation. We assume the set of places S satisfies the following additional ~~relation~~ condition

(S5) S is large enough so that $\langle G_P \mid G_P$ is the decomposition group of $P \in S \rangle = G$.

Under this assumption, any $\sigma \in G$ is written in a form

$$\sigma = \sigma_1 \cdots \sigma_m, \text{ where } \sigma_i \in G_{P_i} \quad (P_i \in S).$$

Let us denote $\tau_0 = 1$ and $\tau_i = \sigma_1 \cdots \sigma_i (1 \leqq i \leqq m)$. Then we see

$$\tau_i - \tau_{i-1} \in K[G, G_{P_i}] \quad (1 \leqq i \leqq m).$$

Hence we have

$$d_\sigma = \sigma - 1 \in \langle K[G, G_{P_i}] \mid 1 \leqq i \leqq m \rangle \subset I[G].$$

Therefore, from the fact that $\{d_\sigma\}$ $(\sigma \in G)$ is a Z-basis of $I[G]$, we obtain

(S5)$'$ $\langle K[G, G_P] \mid P \in S \rangle = I[G]$.

Conversely, we can easily show (S5)$' \Longrightarrow$ (S5). Hence the condition (S5) is equivalent to the condition (S5)$'$.

Remark. In case G is abelian, the condition (S5) is equivalent to that the homomorphism $H^0(G,J) \longrightarrow H^0(G,C)$ is surjective. From the fact $H^1(G,J) = 0$, we see the condition (S5) is satisfied, if and only if $H^1(G,E) = 0$.

Note that the conditions (S1),...,(S5) are automatically satisfied if S is the set of all places of K. Let S_0 be a set of places of k consisting of all the restrictions of $p \in S$. Then, from the condition (S1), S is considered to be the set of places of K consisting of all the extensions of $p \in S_0$. For a place p of k, we dente by $\widetilde{K_p^X}$ the G-module defined in §4, and by U_p the subgroup of K_p^X whose elements are unit units at all places lying over p. Let $\prod'_{p \in S_0} \widetilde{K_p^X}$ be the restricted product of $\{\widetilde{K_p^X}\}$ with respect to $\{U_p\}$.

Let $\{f_\lambda\}_{\lambda \in I}$ be the set of all mappings from S_0 to S such that, for every $p \in S_0$, $f_\lambda(p)$ is an extension of p to K. Since $\widetilde{K_p^X}$ is defined when one fixes an extension of p to K, one can define $\prod'_{p \in S_0} \widetilde{K_p^X} \times \prod_{p \notin S_0} U_p$ for every f_λ ($\lambda \in I$). From the condition (S5), there ~~are~~ exist finitely many $K[G, G_{P_i}]$ $(i=0,\dots,m)$ such that

$$\langle K[G, G_{P_i}] \mid 0 \leq i \leq m \rangle = I[G]. \tag{6.2}$$

Let $f_i = f_{\lambda_i}$ be the mapping of $\{f_\lambda\}$ which takes value P_i. J_i denotes the G-module $\prod'_{p \in S_0} \widetilde{K_p^X} \times \prod_{p \notin S_0} U_p$ corresponding to the mapping f_i. We denote the decomposition group of $f_i(p)$ by $G_i(p)$. Then it is obvious that J is a G-submodule of J_i and J_i/J is G-isomorphic to $\sum_{p \in S_0} K[G, G_i(p)]$. We also see that

$$Y \cong \sum_{p \in S_0} Z[G/G_i(p)] \cong \sum_{p \in S_0} Z[G]/K[G,G_i(p)]$$, where each $K[G,G_i(p)]$ is embedded in $Z[G]$ in a natural way. Then we have the following commutative diagram

(6.3)
$$\begin{array}{ccccccccccc}
0 & \longrightarrow & J & \xrightarrow{l_i''} & J_i & \xrightarrow{l_i'} & \sum_{p \in S_0} Z[G] & \xrightarrow{l_i} & Y & \longrightarrow & 0 \\
 & & \downarrow a & & \downarrow c_i & & \downarrow d_i & & \downarrow b & & \\
0 & \longrightarrow & C & \longrightarrow & \bar{C} & \longrightarrow & Z[G] & \longrightarrow & Z & \longrightarrow & 0,
\end{array}$$

in which row sequences are exact and the homomorphism a and b are surjective. The homomorphism d_i is also an onto homomorphism induced from the natural projections. Since J_i is, as an abelian group, the direct sum of J and $\sum_{p \in S_0} K[G,G_i(p)]$, c_i is a G-homomorphism defined by putting

$c_i(x) = a(x) = x \bmod E \in \bar{C}$, for any $x \in J$,

$c_i(y) = y \in \bar{C}$, for any $y \in K[G,G_i(p)]$.

From (6.2), we see $\prod_{i=0}^{n} c_i : \sum J_i \longrightarrow \bar{C}$ is a surjective G-homomorphism. Let us denote $\prod_{i=0}^{n} c_i$ by c. Then we have the following commutative diagram

(6.4)
$$\begin{array}{ccccccccccc}
0 & \longrightarrow & J & \xrightarrow{l''} & J_0 \oplus (\sum_{i=1}^{m} J_i) & \xrightarrow{l'} & (\sum_{p \in S_0} Z[G]) \oplus (\sum_{i=1}^{m} J_i) & \xrightarrow{l} & Y & \longrightarrow & 0 \\
 & & \downarrow a & & \downarrow c & & \downarrow d & & \downarrow b & & \\
0 & \longrightarrow & C & \longrightarrow & \bar{C} & \longrightarrow & Z[G] & \longrightarrow & Z & \longrightarrow & 0,
\end{array}$$

where $l'' = l_0$, $l' = l_0' \oplus \mathrm{id}$, $l = l_0 \oplus 0$ and $d = d_0 \oplus \sum_{i=1}^{m} d_i \cdot l_i$ (0 means the zero mapping). Since all the homomorphisms a, b, c, d are surjective, we have the following commutative diagram of G-modules with exact rows and columns

(6.5)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Xi \arrow[r] \arrow[d, "a'"] & \mathrm{Ker}\ c \arrow[r] \arrow[d] & \mathrm{Ker}\ d \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "b'"] & 0 \\
0 \arrow[r] & J \arrow[r] \arrow[d, "a"] & \bar{J} \arrow[r] \arrow[d, "c"] & \tilde{J} \arrow[r] \arrow[d, "d"] & Y \arrow[r] \arrow[d, "b"] & 0 \\
0 \arrow[r] & C \arrow[r] \arrow[d] & \bar{C} \arrow[r] \arrow[d] & Z[G] \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$
,

where we denote $J_0 \oplus (\sum_{i=1}^{m} J_i)$ by $\bar{J}$ and $(\sum_{p \in S_0} Z[G]) \oplus (\sum_{i=1}^{m} J_i)$ by $\tilde{J}$. From the assumption (S3) and the fact that $\widetilde{K_p^X}$ is cohomologically trivial, we see $\bar{J}$ and $\tilde{J}$ are cohomologically trivial. Hence we see G-modules Ker c and Ker d are also cohomologically trivial. Hence, from Corollaty 1.2, we have

Theorem 6.1. Let S be a set of places of K which satisfies the conditions (S1),...,(S5). Then the diagram (6.1) is commutative and the isomorphisms α_i^r $(r \in Z,\ 1 \leq i \leq 3)$ are obtained as the compositions of the connected homomorphisms δ_* derived from (6.5).

Let us generalize the above theorem to algebraic tori. Let M be a torsion free G-module. Then, from (6.5) we have the following commutative diagram of G-modules with exact rows and columns

(6.6)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & E \otimes M \arrow[r] \arrow[d] & (\mathrm{Ker}\ c) \otimes M \arrow[r] \arrow[d] & (\mathrm{Ker}\ d) \otimes M \arrow[r] \arrow[d] & X \otimes M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J \otimes M \arrow[r] \arrow[d] & \bar{J} \otimes M \arrow[r] \arrow[d] & \tilde{J} \otimes M \arrow[r] \arrow[d] & Y \otimes M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C \otimes M \arrow[r] \arrow[d] & \bar{C} \otimes M \arrow[r] \arrow[d] & Z[G] \otimes M \arrow[r] \arrow[d] & Z \otimes M \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

From [6] Theorem 2, G-modules $\bar{J} \otimes M$, $\tilde{J} \otimes M$, $\bar{C} \otimes M$ and $Z[G] \otimes M$ are G-modules of trivial cohomology. Therefore we have

Theorem 6.2. *Let* S *be the set of places of* K *satisfying* (S1),...,(S5). *Then, for any torsion free G-module* M, *the cohomology sequence derived from* $0 \to E \otimes M \to J \otimes M \to C \otimes M \to 0$ *is isomorphic to that derived from* $0 \to X \otimes M \to Y \otimes M \to M \to 0$; *that is, we have the following commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & H^r(G, X \otimes M) \arrow[r] \arrow[d, "\wr"] & H^r(G, Y \otimes M) \arrow[r] \arrow[d, "\wr"] & H^r(G, M) \arrow[r] \arrow[d, "\wr"] & \cdots \\
\cdots \arrow[r] & H^{r+2}(G, E \otimes M) \arrow[r] & H^{r+2}(G, J \otimes M) \arrow[r] & H^{r+2}(G, C \otimes M) \arrow[r] & \cdots
\end{tikzcd}$$,

where the vertical arrows are the isomorphisms induced from the connecting homomorphisms derived from (6.6).

Remark. The above theorems are not general as those of [11] and the way of approach does not really improves on that of [11], but is primitive and shows the essential relation of [9] and [11] .

In the rest, we shall refer to the cohomology of algebraic tori. For the sake of simplicity, we restrict ourselves to the case that

S is the set of all places of K. Let T be an algebraic torus defined over k which splits over K. From [7], there exists an isomorphism between the category of tori defined over k and split over K and the dual of the category of finitely generated Z-free G-modules. We denote by $\hat{T}$ the character module of T and by $M = \mathrm{Hom}(\hat{T}, Z)$ the integral dual module of T. Then Theorem 6.2 enables us to describe the Galois cohomology groups of the torus T in terms of the Z-free module M.

For example, we can describe the Tamagawa number of T of k by the cohomology of X, Y and M. Let T_k be the group of k-rational points of T and T_{A_k} be the adele group of T over k. The factor group T_{A_k}/T_k is called the adele class group of T over k and dented by $C_k(T)$. Since K is the splitting field of T, it is known that $T_K \cong M \otimes K^{X}$, $T_{A_K} \cong M \otimes K_A^{X}$ and $C_K(T) \cong M \otimes C_K$. In [8], T.Ono has defined the numbers $h(T)$ and $i(T)$ for a torus T

$$h(T) = [H^1(G,\hat{T})] = [H^{-1}(G,M)],$$

$$i(T) = [C_K(T)^G : C_k(T)] = [\mathrm{Ker}(H^1(G,T_K) \longrightarrow H^1(G,T_{A_K}))].$$

Let $\tau(T)$ be the Tamagawa number of T over k. Then one has the following fundamental formula ([8], Main theorem),

$$\tau(T)\, i(T) = h(T).$$

From Theorem 6.2, we have

$$i(T) = [\mathrm{Ker}(H^{-1}(G,X\otimes M) \longrightarrow H^{-1}(G,Y\otimes M))]$$

$$= [\mathrm{Coker}(H^{-2}(G,Y\otimes M) \longrightarrow H^{-2}(G,M))].$$

For every place p of k, we fix a extension of p to K and denote it by P. Since Y is G-isomorphic to $\sum_p Z[G/G_P]$ (p runs all places of k), we have

$$H^r(G, Y \otimes M) \cong \sum_p H^r(G_P, M) \quad (r \in Z).$$

Here G_P denotes the decomposition group of P. Hence we have

$$i(T) = \left[\mathrm{Coker}\left(\sum_p H^{-2}(G_P, M) \xrightarrow{\prod \tau_P} H^{-2}(G, M)\right)\right],$$

where τ_P is the transfer homomorphism from G_P to G. From the integral duality, we have

$$i(T) = \left[\mathrm{Ker}\left(H^2(G, \hat{T}) \xrightarrow{\prod \rho_P} \prod_p H^2(G_P, \hat{T})\right)\right],$$

where ρ_P is the restriction homomorphism from G to G_P. Therefore we have

$$\tau(T) = [H^1(G, \hat{T})] / [\mathrm{Ker} \prod \rho_P : H^2(G, \hat{T}) \longrightarrow \prod_p H^2(G_P, \hat{T})].$$

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