学位申請論文

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学位審査報告

氏名	上田勝
学位の種類	理学博士
学位記番号	理博第　号
学位授与の日付	昭和　年　月　日
学位授与の要件	学位規則第5条第1項該当
研究科・専攻	理学研究科数学専攻

（学位論文題目）

The decomposition of the spaces of cusp forms of half－integral weight and trace formula of Hecke operators

（重さ半整数の尖点形式の空間の分解とヘッケ作用素の跡公式）

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（論文内容の要旨）

Fuchs 群 $\Gamma_0(N)$ に関する， weight k, 指標 χ の尖点形式のなすベクトル空間を $S(k, N, \chi)$ で表す。上半面上の正則関数 $f=f(z)$ が $S(k, N, \chi)$ に属することを特徴づける保型性は

$$(1)\qquad f\left(\frac{az+b}{cz+d}\right)=\chi(\gamma)(cz+d)^{-k}f(z)$$

が任意の $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_0(N)$ について成立つことであった。ここで k は正整数であった。

N が 4 で割れるとき，(1)に於て， k を半整数 $k+\frac{1}{2}$ で置き換えた如き保型性を示す関数があり． half－integral weight の尖点形式と呼ばれ，その空間を $S(k+\frac{1}{2}, N, \chi)$ で表す。

尖点形式 f は保型性より，フーリエ展開， $f(z)=a_1\tau+a_2\tau^2+\cdots$， $\tau=e^{2\pi inz}$ をもつ， Hecke は，丁度 a_n を固有値としてもつような作用素 T (n) n=1，2，…を導入した。今日 Hecke 作用素と呼ばれるものであり、以後の保型形式論に中心的役割を果した。一方， half－integral weight の尖点形式に対しては， $T(n^2)$ (n= 1，2，…) だけしか定義できず，関数等式等について，意味のある結果は得られなかった。

1973 年頃，志村（Princeton 大学）は $S(k+\frac{1}{2}, N, \chi)$ の元 f に対し, $S(2k, N/2, \chi^2)$ の元 F_f を構成し，half－integral weight の尖点形式の理論が豊かな内容をもつことを示し，いくつかの重要な問題を提出した。以後，無限次表現論の発達とも重って，この分野は多くの研究者の関心を集めた。中でも上記の対応：$f\to F_f$ は志村対応と呼ばれ，その性質を解明することは中心問題の１つである。この問題に対し，現在有効と思われる唯一の方法は Seberg 跡公式により両空間への Hecke 作用素の跡を計算して比較することで，（未解決の部分を）集約するとこ

(i)　$S(k+\frac{1}{2}, N, \chi)$ 上での $T(n^2)$ の跡を求める。

(ii)　上記(2)を用いて志村対応 $f\to F_f$ の像を決定する。

の２つである。

(i)について，S, Niwa（Nagoya J. '77）が次の仮定

(2)　$x=1$，Ｎは３乗因子なし

の下で解を与えた。また(ii)については，Niwa（同上）が仮定

(3)　$ord_2(N)=2$（即ちＮは丁度４で割れる）

且つ　$N/4$ は平方因子なし

の下で解を与えた。更に W. Kohnen（Crelle. J. '82）は仮定

(4)　＋「$x^2=1$」

の下で，$S(K+\frac{1}{2}, N, x)$ の適当な部分空間（＝Kohnen space）を定義し，（Niwaの跡公式を用いて），Kohnen space上の跡を計算し，問題(ii)の解を得た。

さて，上田勝の申請論文は，

(i)に対し（仮定なしの）完全な解を与え，(ii)に対し，仮定「$x^2=1$」＋

(4')　$2\leq Nd_2(N)\leq 4$

の下で解を得ている。

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（論文審査の結果の要旨）

(I)　T（n^2）の跡公式について

跡公式の解析的部分は既に志村により（実際は，代数幾何的方法によって）与えられており，Niwa, 上田ともそれにより，具体的 Fuchs 群 Γ_0(N) に対し，残りの数論的物分の計算を遂行している。

Half－integral の場合に，その計算は，integral weight の場合と較べて，本質的に“局所化”できないという難しさがある。但し，この局所化とは，数論的意味で，各素数 p での p 進完備化の中だけでの考察では済まないということである。

Niwa の仮定(2)は，その困難さを軽減するためのもので，申請者はその困難をまともに乗り越えており，これだけでも十分評価できるものである。

(II)　志村対応の像の決定について

この問題に対する申請者の結果は論文要旨に記した通り最終的なものではない。また（詳しいことは避けるが），申請者の方法が最終結果に到る最良の途とは必ずしも考えられない。しかし，Niwa Kohnen の結果に対する，申請者の拡張は，「平方因子なし」という仮定をはずした点で，表現論的観点からも本質的に新しい部分を含んだ拡張であり，美しい最終結果の存在を示唆した点で貴重な貢献である。

よって本論文は理学博士の学位論文に価するものと認める。

なお，主論文及び参考論文の内容を中心として，関連分野について試問した結果，合格と認めた。

主論文

The decomposition of the spaces of cusp forms of half-integral weight and trace formula of Hecke operators.

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Introduction.

Let k be a positive integer and N a positive integer divisible by 4. For an even Dirichlet character χ modulo N, we denote by $S(k+1/2,N,\chi)$ the space of cusp forms with weight $k+1/2$, level N, and character χ. Suppose $k \geqq 2$. For a primitive form F of $S(2k,N/2,\chi^2)$. we define the subspace $S(k+1/2,N,\chi;F)$ by

$S(k+1/2,N,\chi;F)$

$$= \{S(k+1/2,N,\chi) \ni f \mid f|\tilde{T}(p^2) = \lambda_F(p)f \text{ for all prime number } p\nmid N\}$$

Here,we denote by $\tilde{T}(p^2)$ the Hecke operator of half-integral weight and $\{\lambda_F(p)\}$ is the system of eigenvalues associated with the primitive form F.

Then the following decomposition is well-known,

(1) $$S(k+1/2,N,\chi) = \bigoplus_F S(k+1/2,N,\chi;F),$$

where the sum is extended over primitive forms of $S(2k,N/2,\chi^2)$ (cf. [Sh 1] lemma 7.). Note that we can also obtain the similar decomposition for the case k=1 after slight modifications.

Let M be the odd part of N and H_M the $\mathbb{C}$-algebra generated by the double cosets $\Gamma_0(M)\begin{pmatrix}1 & 0\\0 & n\end{pmatrix}\Gamma_0(M)$ with natural number n such that $(n,2M) = 1$. By using the Hecke operator, we can define two representations of H_M; $R(2k)\colon H_M \to \mathrm{End}_{\mathbb{C}} S(2k,N/2)$ and $\tilde{R}(k+1/2,\chi)\colon H_M \to \mathrm{End}_{\mathbb{C}} S(k+1/2,N,\chi)$, (cf. §0 (b) and (c)). Then, from the decomposition (1), we can expect that there is some relation between traces of $R(2k)$ and traces of $\tilde{R}(k+1/2,\chi)$.

Our main purpose in this paper is to calculate the difference between these two traces for several cases. In [N] , S.Niwa already took up this problem for the case of the cubic free level $N = 4M$, $(M,2) = 1$ and trivial character χ_0. He calculated the trace of the Hecke operator $\tilde{T}(n^2)$ over $S(k+1/2,4M,\chi_0)$ for all natural number n with $(n,2M) = 1$ and compared them with the traces of the Hecke operator $T(n)$ over $S(2k,2M,\chi_0)$. Then, he proved that these two traces have a simple relation. For example, if M is squarefree, these two traces coincide.

We shall generalize these results in §1 and §3. In §1, we shall explicitly calculate the trace of the Hecke operator over $S(k+1/2,N,\chi)$ for $\chi^2 = 1$, and in §3, we shall prove the relation between these traces.

Next, suppose that $N = 4M$, $(M,2) = 1$ and $\chi^2 = 1$. Then, in [K] , W.Kohnen defined the canonical subspace $S(k+1/2,N,\chi)_K$ of $S(k+1/2,N,\chi)$ and the Hecke operator over that subspace, (cf.§0 (d)). Moreover, he calculated those traces when M is squarefree, and he proved that those traces coincide with the traces of the Hecke operators over $S(2k,M,\chi_0)$, where χ_0 is the trivial character.

We shall also generalize these results in §2 and §3. In §2, we shall explicitly calculate the traces for any odd integer M and discuss the relations between traces in §3. Moreover, in §4, we shall give the examples of the explicit decomposition of $S(k+1/2,N,\chi)$ and $S(k+1/2,N,\chi)_K$ for several cases.

The author wishes to express his hearty thanks to Professor H.Saito and Professor S.Niwa for their kind advices and warm encouragement.

§0. Preliminaries.

(a) General notations.

The letter k denotes a positive integer. If $z \in C$ and $x \in C$, we put $z^x = \exp(x\cdot\log(z))$ with $\log(z) = \log|z| + \sqrt{-1}\arg(z)$ and the argument determined by $-\pi < \arg(z) \leqq \pi$. For $z \in C$, we put $e(z) = \exp(2\pi\sqrt{-1}z)$.

Let H be the upper half plane. For a complex-valued function $f(z)$ on H and $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(R)$, we define functions $J(\alpha,z)$ and $f|[\alpha]_k(z)$ on H by $J(\alpha,z) = cz + d$ and $f|[\alpha]_k(z) = (\det \alpha)^{k/2}J(\alpha,z)^{-k}f(\alpha(z))$. Moreover, $z \in H$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in \Gamma_0(4)$, put $j(\gamma,z) = \left(\frac{-1}{d}\right)^{-1/2}\left(\frac{c}{d}\right)(cz + d)^{1/2}$.
For an odd natural number M, let H_M be the C-algebra generated by the double cosets $\Gamma_0(M)\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\Gamma_0(M)$ with all natural number n such that $(n,2M) = 1$. Then H_M have the C-basis consisting of the elements $\Gamma_0(M)\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\Gamma_0(M)$, where $a,d > 0$, $a|d$ and $(d,2M) = 1$.

For a natural number n, we denote by $\phi(n)$ the cardinality of $(Z/nZ)^\times$.

Put $h(-n)$ = the class number of proper ideals of the order with discriminant $-n$ in an imaginary quadratic number field $Q(\sqrt{-n})$ and $w(-n)$ = a half of the cardinality of the unit group of the above order. Then, for simplicity, we denote $h'(-n) = h(-n)/w(-n)$.

For a real number x, $[x]$ means the greatest integer m with $x \geqq m$. When $n = \prod_{q|n} q^{\nu}$ is the decomposition to the prime numbers q for an natural number n, we set

$$\alpha_u(x) = \prod_{q|n} \{(q^{\nu+1}-1) - \left(\frac{u}{q}\right)(q^{\nu}-1)\}/(q-1).$$

(b) Modular forms of integral weight.

Let N be a positive integer. By $S(2k,N)$ we denote the space of all holomorphic cusp forms of weight 2k with the trivial character on the group $\Gamma_0 = \Gamma_0(N)$.

Let $\alpha \in GL_2^+(R)$. If Γ_0 and $\alpha^{-1}\Gamma_0\alpha$ are commensurable, we define a linear operator $[\Gamma_0\alpha\Gamma_0]_{2k}$ on $S(2k,N)$ by

$f|[\Gamma_0\alpha\Gamma_0]_{2k} = (\det \alpha)^{k-1} \sum_{\alpha_i} f|[\alpha_i]_{2k}$, where α_i runs over a system of representatives for $\Gamma_0\backslash\Gamma_0\alpha\Gamma_0$.

Then, for the odd part of N, say M, we can define the representation

$R(2k)\colon H_M \longrightarrow \mathrm{End}_C(S(2k,N))$ by

$R(2k)(\Gamma_0(M)\xi\Gamma_0(M)) = [\Gamma_0(N)\xi\Gamma_0(N)]_{2k}$.

For a positive integer n with $(n,N) = 1$, we put

$T_{2k,N}(n) = \sum_{ad=n} [\Gamma_0\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\Gamma_0]_{2k}$, where the sum is extended over the pairs of natural numbers (a,d) such that $a,d > 0$, $a|d$, $ad = n$ and $(d,N) = 1$. We call $T_{2k,N}(n)$ the n-th Hecke operator over $S(2k,N)$.

(c) Modular forms of half-integral weight.

Let N be a positive integer divisible by 4. By M we understand the odd part of N. Let $\Gamma_0 = \Gamma_0(N)$ and χ an even Dirichlet character modulo N. We suppose that χ is quadratic, namely $\chi^2 = 1$. Then, there is a squarefree odd positive integer M_0 such that M_0 is the divisor of M and

$\chi = \left(\frac{M_0}{\ }\right)$ or $\left(\frac{2M_0}{\ }\right)$, (Kronecker symbol).

Let $G(k+1/2)$ be the group consisting of pairs $(\alpha,\phi(z))$, where $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(R)$ and $\phi(z)$ is a holomorphic function on H satisfying $\phi(z) = t(\det \alpha)^{-k/2-1/4} J(\alpha,z)^{k+1/2}$ with $t \in C$, $|t| = 1$. The group law is defined by $(\alpha,\phi(z))\cdot(\beta,\psi(z)) = (\alpha\beta,\phi(\beta z)\psi(z))$. For a complex-valued function f on H and $(\alpha,\phi(z)) \in G(k+1/2)$, we define a function $f|(\alpha,\phi(z))$ on H by $f|(\alpha,\phi(z))(z) = \phi(z)^{-1}f(\alpha z)$.

By $\Delta_0 = \Delta_0(N,\chi) = \Delta_0(N,\chi)_{k+1/2}$, we denote the subgroup of $G(k+1/2)$ consisting of all pairs $(\gamma,\phi(z))$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in \Gamma_0(N)$ and $\phi(z) = \chi(d)j(\gamma,z)^{2k+1}$. We denote by $G(k+1/2,N,\chi)$ (resp. $S(k+1/2,N,\chi)$) the space of integral (resp. cusp) forms of weight $k+1/2$ with character χ on the group $\Gamma_0(N)$, namely, the space of C-valued holomorphic function f on H which satisfies $f|\xi = f$ for all $\xi \in \Delta_0(N,\chi)$ and holomorphic (resp. holomorphic and vanish) at all cusps of $\Gamma_0(N)$. In particular, we write $S(k+1/2,N) = S(k+1/2,N,\chi)$ if χ is the trivial character.

Let $U(N,\chi)$ be the subspace of $S(3/2,N,\chi)$ generated by all the theta series of the following type:

$h_\psi(tz) = (1/2) \sum_{m\in Z} \psi(m)me(tm^2z)$, $z \in H$, where t is a positive integer and ψ is a primitive odd Dirichlet character modulo r which satisfy the following conditions: $4tr^2 | N$ and $\chi = \psi\left(\frac{-t}{}\right)$ as a character modulo N. By $V(N,\chi)$ we denote the orthogonal complement of $U(N,\chi)$ in $S(3/2,N,\chi)$ with respect to the Petersson inner product.

Let $\xi \in G(k+1/2)$. If $\Delta_0 = \Delta_0(N,\chi)$ and $\xi^{-1}\Delta_0\xi$ are commensurable, we define a linear operator $[\Delta_0\xi\Delta_0]_{k+1/2}$ on $\mathfrak{G}(k+\frac{1}{2},N,\chi)$ and $S(k+1/2,N,\chi)$ by $f|[\Delta_0\xi\Delta_0]_{k+1/2} = \sum_{\eta} f|\eta$, where η runs over a system of representatives for $\Delta_0 \backslash \Delta_0\xi\Delta_0$. Then we define a C-linear map $\tilde{R}(k+1/2,\chi) : H_M \to \mathrm{End}_C(S(k+1/2,N,\chi))$ by $\tilde{R}(k+1/2,\chi)(\Gamma_0(M)\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\Gamma_0(M))$

$$= a(ad)^{k-3/2}[\Delta_0(N,\chi)\left(\begin{pmatrix} a^2 & 0 \\ 0 & d^2 \end{pmatrix} , (d/a)^{k+1/2}\right)\Delta_0(N,\chi)]_{k+1/2} ,$$

where $a,d > 0$, $a|d$ and $(d,N) = 1$.

$\tilde{R}(k+1/2,\chi)$ is a representation of H_M (cf. [N] Introduction). In particular, for a natural number n with $(n,N) = 1$, we put

$\tilde{T}_{k+1/2,N,\chi}(n^2) = \tilde{R}(k+1/2,\chi)(\sum_{ad=n} \Gamma_0(M)\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\Gamma_0(M))$, where the sum

is extended over the pair (a,d) such that $a,d > 0$, $a|d$, $(d,N) = 1$ and $ad = n$. $\tilde{T}_{k+1/2,N,\chi}(n^2)$ is called the n-th Hecke operator over $S(k+1/2,N,\chi)$. Then, from [Sh 2] Theorem 1.7, it follows that $h_\psi(tz) \in U(N,\chi)$ is an eigenfunction of the Hecke operators $\tilde{T}_{3/2,N,\chi}(p^2)$ for the prime number $p\nmid N$. Since $\tilde{R}(k+1/2,\chi)(H_M)$ is generated by such operators, $U(N,\chi)$ is invariant under the action of the element of $\tilde{R}(k+1/2,\chi)(H_M)$. By using the self-adjointness of the Hecke operator with respect to the Petersson inner product, we see that $V(N,\chi)$ is also invariant under the action of the element of $\tilde{R}(k+1/2,\chi)(H_M)$ (cf. [Sh 1] lemma5).

$U(N,\chi)$ corresponds to the space of the Eisenstein series by means of Shimura correspondence and only the element of $V(N,\chi)$ corresponds to the cusp form (cf. [St]). Hence, when $k = 1$, namely weight 3/2, we shall be dealing with $V(N,\chi)$ in place of $S(3/2,N,\chi)$. When there is no confusion, for simplicity, we drop the subscripts $k+1/2,N,\chi$, etc.

(d) Kohnen subspace.

Suppose that $N = 4M$ and M is an odd natural number. Then $\chi = \left(\frac{M_0}{\ }\right)$ for some positive divisor M_0 of M (cf. §0,(c)). Put $\varepsilon = \left(\frac{-1}{M_0}\right)$. Then the Kohnen subspace $S(k+1/2,N,\chi)_K$ is defined as follows:

$$S(k+1/2,N,\chi)_K = \left\{ S(k+1/2,N,\chi) \ni f(z) = \sum_{n=1}^{\infty} a(n)e(nz);\ a(n) = 0 \text{ for } \varepsilon(-1)^k n \equiv 2,3 \pmod 4 \right\}.$$

Put $\xi = \xi_{k+1/2,\varepsilon} = \left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \varepsilon^{k+1/2} e((2k+1)/8)\right) \in G(k+1/2)$ and $Q = Q_{k+1/2,N,\chi} = [\Delta_0(N,\chi)\xi\Delta_0(N,\chi)]_{k+1/2}$. Then Q becomes the operator over $S(k+1/2,N,\chi)$. Moreover, from [K] Proposition 1, we know that the operator Q satisfies the quadratic equation $(Q-\alpha)(Q-\beta) = 0$, where $\alpha = (-1)^{[(k+1)/2]}\varepsilon 2\sqrt{2}$, $\beta = -\alpha/2$, and Q is hermitian and its α-eigenspace is $S(k+1/2,N,\chi)_K$.

From [K] §3 and §4, we know that $\tilde{R}(k+1/2,\chi)(H_M)$ preserve $S(k+1/2,N,\chi)_K$. Hence, we can define a C-linear map $\tilde{R}(k+1/2,\chi)_K : H_M \to \mathrm{End}_C(S(k+1/2,N,\chi)_K)$ by requiring that, for $\xi \in H_M$, $\tilde{R}(k+1/2,\chi)_K(\xi)$ is the restriction of the map $\tilde{R}(k+1/2,\chi)(\xi)$ to $S(k+1/2,N,\chi)_K$. For a natural number n with $(n,N) = 1$, we denote by $\tilde{T}_{k+1/2,N,\chi}(n^2)_K$ the restriction of the n-th Hecke operator $\tilde{T}_{k+1/2,N,\chi}(n^2)$ to $S(k+1/2,N,\chi)_K$.

(In particular, we write $S(k+1/2,N)_K = S(k+1/2,N,\chi)_K$ if χ is the trivial character.)

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From the definitions of $S(3/2,N,\chi)_K$ and $U(N,\chi)$, it is easily shown that $S(3/2,N,\chi)_K$ contains $U(N,\chi)$. Then we denote by $V(N,\chi)_K$ the orthogonal complement of $U(N,\chi)$ in $S(3/2,N,\chi)_K$ with respect to the Petersson inner product. $U(N,\chi)$ is invariant under the action of $\tilde{T}_{k+1/2,N,\chi}(n^2)_K$ with $(n,N) = 1$ (cf. §0,(c)). hence the Hecke operator $\tilde{T}_{k+1/2,N,\chi}(n^2)_K$ with $(n,N) = 1$ also preserve $V(N,\chi)_K$.

§1. The trace formula for the Hecke operator of half-integral weight.

Throughout this section, we shall use the same notations and assumptions as in §0 (a) and (c).

In the following calculations of the traces in §2 and §3, we use the trace formula of [Sh 3]. Now, we shall give the explanation of the formula. We take $\tau = (\alpha,h(z)) \in G(k+1/2)$ with $\alpha \in SL_2(R)$ which satisfies the following conditions:

(1.1) $\Gamma_0 = \Gamma_0(N)$ and $\alpha^{-1}\Gamma_0\alpha$ are commensurable.

(1.2) We define the proper lifting L by

$L(\gamma) = (\gamma,\chi(d)j(\gamma,z)^{2k+1}) \in G(k+1/2)$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0$.

Then, $L(\alpha\gamma\alpha^{-1}) = \tau L(\gamma)\tau^{-1}$ for all $\gamma \in \Gamma_0 \cap \alpha^{-1}\Gamma_0\alpha$.

For this τ, from [Sh 2] Proposition 1.1, we have the bijection: $\Gamma_0\alpha\Gamma_0 \ni \gamma_1\alpha\gamma_2 \longrightarrow L(\gamma_1)\tau L(\gamma_2) \in \Delta_0\tau\Delta_0$, where $\Delta_0 = \Delta_0(N,\chi)_{k+1/2}$ and $\gamma_1,\gamma_2 \in \Gamma_0$. Moreover, Δ_0 and $\tau\Delta_0\tau^{-1}$ are commensurable. For simplicity, we denote by $\beta^* = (\beta,h(\beta;z))$ the image of $\beta \in \Gamma_0\alpha\Gamma_0$ with respect to the above bijection.

Nextly, we put $\tau' = (\alpha^{-1}, h(\alpha^{-1}z)J(\alpha^{-1},z)^2)$. Then, τ' also satisfies the conditions (1.1) and (1.2) with respect to the proper lifting $L' : \Gamma_0 \ni \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow (\gamma, \chi(d) j(\gamma,z)^{3-2k})$. Hence, $\Delta_0' = \Delta_0(N,\chi)_{3/2-k}$ and $\tau'\Delta_0'\tau'^{-1}$ are also commensurable.

From [Sh 3] Theorem 4.5 and the assumption $\chi^2 = 1$, we have the following trace formula:

$$(1.3) \quad \mathrm{trace}([\Delta_0 \tau \Delta_0]_{k+1/2} | S(k+1/2,N,\chi)) - \mathrm{trace}([\Delta_0' \tau' \Delta_0']_{3/2-k} | G(3/2-k,N,\chi))$$

$$= \sum_{C \in \Phi(\Gamma_0 \alpha \Gamma_0 / \Gamma_0)} J(C) ,$$ where the meaning of the letters is as follows:

Let $\Phi(\Gamma_0 \alpha \Gamma_0)$ denote the subset of $\Gamma_0 \alpha \Gamma_0$ consisting of: all scalar elements of $\Gamma_0 \alpha \Gamma_0$, all elliptic elements of $\Gamma_0 \alpha \Gamma_0$, all hyperbolic elements of $\Gamma_0 \alpha \Gamma_0$ whose upper fixed points are cusps of Γ_0 (cf. [Sh 3] §3.6) and all parabolic elements of $\Gamma_0 \alpha \Gamma_0$ whose fixed points are cusps of Γ_0. We call two elements β and β' in $\Phi(\Gamma_0 \alpha \Gamma_0)$ equivalent if : $\gamma \beta \gamma^{-1} = \beta'$ for some $\gamma \in \Gamma_0$ when β and β' are scalars or elliptic or hyperbolic, $\gamma \beta' \gamma^{-1} \in Z_{\Gamma_0}(\beta)\beta$ for some $\gamma \in \Gamma_0$ when β and β' are parabolic, where $Z_{\Gamma_0}(\beta) = \{ \gamma \in \Gamma_0 \mid \gamma\beta = \beta\gamma \}$. We denote by $\Phi(\Gamma_0 \alpha \Gamma_0 / \Gamma_0)$ the set of all equivalence classes in $\Gamma_0 \alpha \Gamma_0$ with respect to the above equivalence relation. For each $C \in \Phi(\Gamma_0 \alpha \Gamma_0 / \Gamma_0)$, we pick any β from C. Then, a complex number $J(C)$ is given as follows:

If $\beta^* = (\pm 1, \eta)$, $J(C) = (1/8)(2k-1)\eta^{-1} |\Gamma_0(4) : \Gamma_0(N)|$.

If β is elliptic, $J(C) = (\sigma(\beta)\eta(1-\lambda^{-2}))^{-1}$. Here, the meaning of the letters is as follows: Let $z_0 \in H$ be the fixed point of β and $\alpha = \begin{pmatrix} \bar{z}_0 & z_0 \\ 1 & 1 \end{pmatrix}$, then $\alpha^{-1}\beta\alpha = \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & \lambda \end{pmatrix}$, $\eta = h(\beta; z_0)$ and $\sigma(\beta) = \#\{\gamma \in \Gamma_0 \mid \gamma z_0 = z_0\}$.

If β is hyperbolic, $J(C) = -(1/2)(\eta(1-\lambda^{-2}))^{-1}$. Here, the meaning of the letters is as follows: Let $z_0 \in Q \cup \{\infty\}$ be the upper fixed point of β. Take an element $\rho^* = (\rho, \phi) \in G(k+1/2)$ such that $\rho \in SL_2(R)$ and $\rho(\infty) = z_0$, then we denote $\rho^{*-1}\beta^*\rho^* = \left(\begin{pmatrix} \lambda^{-1} & x \\ 0 & \lambda \end{pmatrix}, \eta\right)$.

If β is parabolic, $J(C) = \begin{cases} \eta^{-1}e(\delta x)(1/2-\delta) & \text{if } \beta \in \Gamma_0 \\ \eta^{-1}e(\delta x)(1-e(x))^{-1} & \text{if } \beta \notin \Gamma_0. \end{cases}$

Here, the meaning of the letters is as follows: Let $z_0 \in Q \cup \{\infty\}$ be the fixed point of β and σ be an element of Γ_0 which generates $\{\gamma \in \Gamma_0 \mid \gamma z_0 = z_0\}/\{\pm 1\}$. Take an element $\rho^* = (\rho, \phi) \in G(k+1/2)$ such that $\rho \in SL_2(R)$, $\rho(\infty) = z_0$ and $\rho^{-1}\sigma\rho = \varepsilon_0 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ with $\varepsilon_0 = \pm 1$. Then $\rho^{*-1}L(\sigma)\rho^* = \left(\varepsilon_0 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, e(\delta)\right)$ with $0 < \delta \leqq 1$ and $\rho^{*-1}\beta^*\rho^* = \left(c\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \eta\right)$ with $c = \pm 1$ and $x \in R$.

Our purpose in this section is to prove the following Propositions.

Proposition 1.

Suppose $k \geqq 2$. The trace of the n-th Hecke operator $\tilde{T}_{k+1/2,N,\chi}(n^2)$ with $(n,N) = 1$ acting on $S(k+1/2,N,\chi)$ is given by the sum $T(p) + T(e) + T(h) + T'$, where the each term is given as follows:

For a prime number $p|M$, let $\nu = \nu_p = \mathrm{ord}_p N$ and $\mu = \mathrm{ord}_2 N$. By χ_p, we denote the p-part of the character χ for a prime number $p|N$ and by $f(\chi_p)$, the conductor of χ_p.

Put $\delta_0(\sqrt{n}) = \begin{cases} 1 & \text{if } n \text{ is square,} \\ 0 & \text{otherwise.} \end{cases}$

And $\delta_1 = \begin{cases} 1 & \text{if } \mu = 2, \\ 0 & \text{otherwise.} \end{cases}$

Let $n = n_0^2 n_1$, where n_0 is a positive integer and n_1 is a squarefree positive integer.

Then, for $n \equiv 1 \pmod 4$. we set

$$T(p) = -\delta_0(\sqrt{n})\, n^{k-1/2} \prod_{p|M} (p^{[\nu/2]} + p^{[(\nu-1)/2]})$$

$$\times \begin{cases} (1/2)(2^{[\mu/2]} + 2^{[(\mu-1)/2]}) & \text{if } \mu \geqq 5, \text{ or } \mu = 4 \text{ and } f(\chi_2)|4, \\ 2 & \text{if } \mu = 4 \text{ and } f(\chi_2) = 8, \\ 3/2 & \text{if } \mu = 3, \\ 1 & \text{if } \mu = 2, \end{cases}$$

$$+ \delta_1 \left((-1)^k/2\right)\left(\frac{-1}{M_0}\right)\chi(n)\, n^{k-1} \prod_{p|M} \left(p^{[\nu/2]} + \left(\frac{-n}{p}\right)^{\nu} p^{[(\nu-1)/2]}\right)$$

$$\times \sum_{0<a|n_0} h'(-4n/a^2).$$

For $n \equiv 3 \pmod 4$, we set

$$T(p) = (-1)^k \chi(n)\, n^{k-1} \prod_{p|M} \left(p^{[\nu/2]} + \left(\frac{p}{n}\right)^{\nu} p^{[(\nu-1)/2]}\right)$$

$$\times \sum_{0<a|n_0} h'(-n/a^2)$$

$$\times \begin{cases} 2^{[\mu/2]} + \left(\frac{2}{n}\right)^{\mu} 2^{[(\mu-1)/2]} & \text{if } \mu \geqq 5, \text{ or } \mu = 4 \text{ and } f(\chi_2)|4, \\ 2\left(1 + \left(\frac{2}{n}\right)\right) & \text{if } \mu = 4 \text{ and } f(\chi_2) = 8, \\ 3 & \text{if } \mu = 3 \text{ and } f(\chi_2)|4, \\ 3\left(\frac{2}{n}\right) & \text{if } \mu = 3 \text{ and } f(\chi_2) = 8, \\ (1/2)\left(5 - \left(\frac{2}{n}\right)\right) & \text{if } \mu = 2, \end{cases}$$

where the sum $\sum_{0<a|n_0}$ is extended over all positive divisors a of n_0.

Next, we define $T(e)$ by

$$T(e) = -\Sigma_1 (x^{2k-1} - y^{2k-1})(x-y)^{-1} h'(u)\alpha_u(t_0) \prod_{p|N} p^{-ord_p s} n_p(\theta_p),$$

where the sum Σ_1 is extended over the even integer s which satisfies $2\sqrt{n} > s > 0$ and the meaning of the letters is as follows: x and y are the solutions of $X^2 - sX + n = 0$ and $s^2 - 4n = t^2 u$, where t is a positive integer and u is a fundamental discriminant, namely the discriminant of an imaginary quadratic field. Put $t_0 = t \prod_{p|N} p^{-ord_p t}$ and $\theta = \theta_p = ord_p(st)$ for the prime divisor p of N.

The constant $n_p(\theta_p)$ is given by the following table:

Case (1). ($p|M$ and $p|s$).

$$n_p(\theta_p) = \chi_p(-n)p^\theta \times \begin{cases} p^{[\nu/2]} + \left(\frac{-n}{p}\right)^\nu p^{[(\nu-1)/2]} & \text{if } \theta \geqq [(\nu+1)/2] \\ (1 + \left(\frac{-n}{p}\right)) p^\theta & \text{if } \theta \leqq [(\nu-1)/2]. \end{cases}$$

Case (2). ($p|M$, $p\nmid s$ and $p|u$).

$$n_p(\theta_p) = \begin{cases} \{p^{\theta+1}(p^{[\nu/2]} + p^{[(\nu-1)/2]}) - p^\nu - p^{\nu-1}\}/(p-1) & \text{if } \theta \geqq [\nu/2], \\ 0 & \text{if } \theta < [\nu/2]. \end{cases}$$

Case (3). ($p|M$, $p\nmid s$ and $p\nmid u$).

$$n_p(\theta_p) = \begin{cases} (1 - \left(\frac{u}{p}\right)p^{-1})(p^{[\nu/2]} + p^{[(\nu-1)/2]})(p^{\theta+1} - p^{[\nu/2]+1})(p-1)^{-1} \\ \quad + p^{[\nu/2]}(p^{[\nu/2]} + \left(\frac{u}{p}\right)^\nu p^{[(\nu-1)/2]}) \quad \text{if } \theta \geqq [(\nu+1)/2], \end{cases}$$

$$\left\lfloor \; (1 + \left(\frac{u}{p}\right))p^{2\theta} \qquad\qquad \text{if } \theta \leqq [(\nu-1)/2]. \right.$$

Case (4). (p = 2 and μ = 2).

$$n_2(\theta_2) = \begin{cases} 2^{\theta+1} & \text{if } u \equiv 1 \pmod 8 \\ 3\times 2^{\theta} & \text{if } u \equiv 5 \pmod 8 \text{ and } s/2 \text{ is even,} \\ 3\times 2^{\theta+1} - 12 & \text{if } u \equiv 5 \pmod 8 \text{ and } s/2 \text{ is odd,} \\ 2^{\theta+2} - 6 & \text{if } u \equiv 0 \pmod 4 \text{ and } t \text{ is even,} \\ 2^{\theta} & \text{if } u \equiv 0 \pmod 4 \text{ and } t \text{ is odd.} \end{cases}$$

Case (5). (p = 2 and μ = 3).

$$n_2(\theta_2) = \begin{cases} 3\times 2^{\theta} & \text{if } u \equiv 1 \pmod 8, \\ 3\times 2^{\theta} & \text{if } u \equiv 5 \pmod 8 \text{ and } s/2 \text{ is even,} \\ 9\times 2^{\theta} - 24 & \text{if } u \equiv 5 \pmod 8 \text{ and } s/2 \text{ is odd,} \\ 3\times 2^{\theta+1} - 12 & \text{if } u \equiv 0 \pmod 4 \text{ and } t \text{ is even,} \\ 0 & \text{if } u \equiv 0 \pmod 4 \text{ and } t \text{ is odd.} \end{cases}$$

Case (6). (p = 2 and μ = 4).

$$n_2(\theta_2) = \begin{cases} 3\times 2^{\theta+1} & \text{if } u \equiv 1 \pmod 8 \text{ and } f(\chi_2)|4, \\ 2^{\theta+2} & \text{if } u \equiv 1 \pmod 8 \text{ and } f(\chi_2) = 8, \\ 6\times 2^{\theta} & \text{if } u \equiv 5 \pmod 8\text{, } s/2 \text{ is even and } f(\chi_2)|4, \\ 0 & \text{if } u \equiv 5 \pmod 8\text{. } s/2 \text{ is even and } f(\chi_2) = 8, \\ 9\times 2^{\theta+1} - 48 & \text{if } u \equiv 5 \pmod 8\text{. } s/2 \text{ is odd and } f(\chi_2)|4, \\ 3\times 2^{\theta+2} - 48 & \text{if } u \equiv 5 \pmod 8\text{. } s/2 \text{ is odd and } f(\chi_2) = 8, \end{cases}$$

$$\begin{cases} 3\times 2^{\theta+2} - 24 & \text{if } u \equiv 0 \pmod 4,\ t \text{ is even and } f(\chi_2) \mid 4, \\ 2^{\theta+3} - 24 & \text{if } u \equiv 0 \pmod 4,\ t \text{ is even and } f(\chi_2) = 8, \\ 0 & \text{if } u \equiv 0 \pmod 4 \text{ and } t \text{ is odd.} \end{cases}$$

Case (7). ($p = 2$ and $\mu = 2g + 1 \geqq 5$).

$$n_2(\theta_2) = \begin{cases} 2^{\theta+g+1} & \text{if } u \equiv 1 \pmod 8 \text{ and } \theta \geqq g+1, \\ 2^{2\theta+1} & \text{if } u \equiv 1 \pmod 8 \text{ and } \theta \leqq g, \\ 0 & \text{if } u \equiv 5 \pmod 8 \text{ and } s/2 \text{ is even}, \\ 3\times 2^{\theta+g+1} - 3\times 2^{2g+1} & \text{if } u \equiv 5 \pmod 8,\ s/2 \text{ is odd and } \theta \geqq g+1, \\ 0 & \text{if } u \equiv 5 \pmod 8,\ s/2 \text{ is odd and } \theta \leqq g, \\ 2^{\theta+g+2} - 3\times 2^{2g} & \text{if } u \equiv 0 \pmod 4,\ t \text{ is even and } \theta \geqq g, \\ 0 & \text{if } u \equiv 0 \pmod 4,\ t \text{ is even and } \theta \leqq g-1, \\ 0 & \text{if } u \equiv 0 \pmod 4 \text{ and } t \text{ is odd.} \end{cases}$$

Case (8) ($p = 2$ and $\mu = 2g \geqq 6$).

$$n_2(\theta_2) = \begin{cases} 3\times 2^{\theta+g-1} & \text{if } u \equiv 1 \pmod 8 \text{ and } \theta \geqq g, \\ 2^{2\theta+1} & \text{if } u \equiv 1 \pmod 8 \text{ and } \theta \leqq g-1, \\ 3\times 2^{\theta+g-1}\chi_2(5) & \text{if } u \equiv 5 \pmod 8,\ s/2 \text{ is even and } \theta \geqq g, \end{cases}$$

$$\begin{cases} 0 & \text{if } u \equiv 5 \pmod 8),\ s/2 \text{ is even and } \theta \leqq g-1, \\ 9\times 2^{\theta+g-1} - 3\times 2^{2g} & \text{if } u \equiv 5 \pmod 8),\ s/2 \text{ is odd and } \theta \geqq g, \\ 0 & \text{if } u \equiv 5 \pmod 8),\ s/2 \text{ is odd and } \theta \leqq g-1, \\ 3\times 2^{\theta+g} - 3\times 2^{2g-1} & \text{if } u \equiv 0 \pmod 4).\ t \text{ is even and } \theta \geqq g, \\ 0 & \text{if } u \equiv 0 \pmod 4),\ t \text{ is even and } \theta \leqq g-1, \\ 0 & \text{if } u \equiv 0 \pmod 4) \text{ and } t \text{ is odd.} \end{cases}$$

Next, we define $T(h)$ by

$T(h) = -\Sigma_2\ ((s-t)/2)^{2k-1} \prod_{p|N} m_p(\theta_p)$, where the sum Σ_2 is extended over the even integer s which satisfies $s > 2\sqrt{n}$ and $s^2 - 4n$ is square, and the meaning of the letters is as follows: Let t be the positive integer such that $s^2 - 4n = t^2$, and $\theta = \theta_p = \mathrm{ord}_p(st)$ for the prime divisor p of N. $m_p(\theta_p)$ is given by the following table:

Case (1). $(p|M)$

$$m_p(\theta_p) = \begin{cases} p^{[\nu/2]} + p^{[(\nu-1)/2]} & \text{if } \theta \geqq [(\nu+1)/2], \\ 2p^{\theta} & \text{if } \theta \leqq [(\nu-1)/2]. \end{cases}$$

Case (2). $(p = 2)$.

$$m_2(\theta_2) = \begin{cases} 2 & \text{if } \mu = 2, \\ 3 & \text{if } \mu = 3, \\ 6 & \text{if } \mu = 4 \text{ and } f(\chi_2)|4, \\ 4 & \text{if } \mu = 4 \text{ and } f(\chi_2) = 8, \end{cases}$$

$$\begin{cases} 2^{[\mu/2]} + 2^{[(\mu-1)/2]} & \text{if } \mu \geqq 5 \text{ and } \theta \geqq [(\mu+1)/2], \\ 2^{\theta+1} & \text{if } \mu \geqq 5 \text{ and } \theta \leqq [(\mu-1)/2]. \end{cases}$$

Finally, we define T' by

$$T' = \delta_0(\sqrt{n})\, n^{k-1} 2^{\mu-4} (2k-1) M \prod_{p|M} (p+1)/p.$$

Proposition 2. The trace of the n-th Hecke operator $\tilde{T}_{3/2,N,\chi}(n^2)$ with $(n,N) = 1$ acting on $V(N,\chi)$ is given by the sum $T(p) + T(e) + T(h) + T' + T''$, where the terms $T(p)$, $T(e), T(h)$ and T' are given by the same formulas as in Proposition 1 when we put $k = 1$. The term T'' is given by

$$T'' = \left(\Sigma_1\, a(\chi;\psi)\psi(n) - \Sigma_2\, b(\chi;\psi)\psi(n) \right) \prod_{p|n} (p^{\tau+1} - 1)/(p-1).$$

Here, the sum Σ_1 (resp. Σ_2) is extended over the primitive even (resp. odd) Dirichlet character ψ modulo $f(\psi)$ (which satisfies $4f(\psi)^2 | N$), and we denote by $a(\chi;\psi)$ (resp. $b(\chi;\psi)$) the number of the positive integer t which satisfies $4f(\psi)^2 t | N$ and $\chi = \psi\left(\frac{t}{\ }\right)$ (resp. $\chi = \psi\left(\frac{-t}{\ }\right)$) as a character modulo N, and $n = \prod_{p|n} p^{\tau}$.

Proofs of Proposition 1 and 2.

For simplicity, we put $\Gamma_0 = \Gamma_0(N)$ and $\Delta_0 = \Delta_0(N,\chi)_{k+1/2}$, and use the same notations as in the statement of Proposition 1 and 2. (Especially, we want to specify the contribution from the prime number 2. Therefore, we shall use the notations: For the prime number $p|N$, $\mathrm{ord}_p N = \tilde{\nu}_p = \tilde{\nu} = \begin{cases} \nu = \nu_p, & \text{if } p \text{ is odd} \\ \mu, & \text{if } p = 2. \end{cases}$)

From the definition, we have

$$\tilde{T}_{k+1/2,N,\chi}(n^2) = \sum_{0<a|n_0} a n^{k-3/2} \left[\Delta_0 \left(\begin{pmatrix} a^2 & 0 \\ 0 & (n/a)^2 \end{pmatrix}, (n/a^2)^{k+\frac{1}{2}} \right) \Delta_0 \right].$$

Since the scalar element $\left(\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}, 1 \right)$ of $G(k+1/2)$ trivially acts on $S(k+1/2,N,\chi)$, we have

$$[\Delta_0\left(\begin{pmatrix} a^2 & 0 \\ 0 & (n/a)^2 \end{pmatrix}, (n/a^2)^{k+1/2}\right)\Delta_0]$$
$$= [\Delta_0\left(\begin{pmatrix} (n/a^2)^{-1} & 0 \\ 0 & (n/a^2) \end{pmatrix}, (n/a^2)^{k+1/2}\right)\Delta_0]$$

as the operator on $S(k+1/2,N,\chi)$.

Therefore, firstly we shall apply the formula (1.3) to the operator $[\Delta_0\left(\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}, n^{k+1/2}\right)\Delta_0]$ and explicitly calculate that trace, and secondly we shall sum up them.

Let $\tilde{p}(n)$, $\tilde{e}(n)$, $\tilde{h}(n)$ and $\tilde{s}(n)$ be the contribution from the parabolic, elliptic, hyperbolic and scalar equivalence classes in $\Phi = \Phi(\Gamma_0 \begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix} \Gamma_0/\Gamma_0)$ respectively.

If $k \geqq 2$, $G(3/2-k,N,\chi) = \{0\}$. Hence, the contribution from $G(3/2-k,N,\chi)$ occurs only when $k = 1$. Then, we put

$$\tilde{d}(n) = \text{trace}\left([\Delta_0'\left(\begin{pmatrix} n & 0 \\ 0 & n^{-1} \end{pmatrix}, n^{-1/2}\right)\Delta_0']_{1/2}\middle| G(1/2,N,\chi)\right)$$
$$- \text{trace}\left([\Delta_0\left(\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}, n^{3/2}\right)\Delta_0]_{3/2}\middle| U(N,\chi)\right),$$

where $\Delta_0' = \Delta_0(N,\chi)_{1/2}$.

1. The part of $\tilde{s}(n)$.

Obviously, $\Gamma_0\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}\Gamma_0$ contains a scalar element if and only if $n = 1$. In that case, since $(\pm 1)^* = (\pm 1, 1)$, we have

$$\tilde{s}(n) = \begin{cases} 2^{\mu-4}(2k-1)\, M \prod_{p|M} (p+1)/p, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

The contribution to the trace of the n-th Hecke operator is the sum

$$\sum_{0<a|n_0} a\, n^{k-3/2}\, \tilde{s}(n/a^2) = \delta_0(\sqrt{n})n^{k-1}\, 2^{\mu-4}(2k-1)\, M \prod_{p|M} (p+1)/p.$$

This is the term T' in the statement of Proposition 1.

2. The part of $\mathfrak{d}(n)$.

Put $\beta_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$, $\tau_N = (\beta_N, N^{1/4}(-\sqrt{-1}z)^{1/2}) \in G(1/2)$

and $\Delta_0'' = \Delta_0(N,\chi\left(\frac{N}{\ }\right))_{1/2}$. Then, from [Sh 2] Proposition 1.4, we know that τ_N induces the isomorphism between $G(1/2,N,\chi)$ and $G(1/2,N,\chi\left(\frac{N}{\ }\right))$, and $\tau_N^{-1}\Delta_0''\tau_N = \Delta_0'$. Hence,

$$\tau_N^{-1}\Delta_0''\left(\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}, n^{1/2}\right)\Delta_0''\tau_N$$

$$= \Delta_0'\tau_N^{-1}\left(\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}, n^{1/2}\right)\tau_N\Delta_0' = \Delta_0'\left(\begin{pmatrix} n & 0 \\ 0 & n^{-1} \end{pmatrix}, n^{-1/2}\right)\Delta_0'.$$

If we write $\Delta_0''\left(\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}, n^{1/2}\right)\Delta_0'' = \bigcup_i \Delta_0''\xi_i$ (disjoint union), we have $\Delta_0'\left(\begin{pmatrix} n & 0 \\ 0 & n^{-1} \end{pmatrix}, n^{-1/2}\right)\Delta_0' = \bigcup_i \Delta_0'\tau_N^{-1}\xi_i\tau_N$. Therefore, as a operator on $G(1/2,N,\chi)$,

$$\tau_N^{-1}[\Delta_0''\left(\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}, n^{1/2}\right)\Delta_0'']_{1/2}\,\tau_N$$

$$= [\Delta_0'\left(\begin{pmatrix} n & 0 \\ 0 & n^{-1} \end{pmatrix}, n^{-1/2}\right)\Delta_0']_{1/2},$$

and $\operatorname{trace}([\Delta_0'\left(\begin{pmatrix} n & 0 \\ 0 & n^{-1} \end{pmatrix}, n^{-1/2}\right)\Delta_0']|G(1/2,N,\chi)) =$ $\operatorname{trace}([\Delta_0''\left(\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}, n^{1/2}\right)\Delta_0'']|G(1/2,N,\chi\left(\frac{N}{\ }\right)))$.

$G(1/2,N,\chi\left(\frac{N}{\ }\right))$ is spaned by the theta series of the following type: $\theta_\psi(tz) = (1/2)\sum_{m\in Z} \psi(m)e(tm^2z)$, $z \in H$, where t is a positive integer and ψ is a even primitive Dirichlet character modulo $f(\psi)$ which satisfy the following conditions: $4tf(\psi)^2|N$ and $\chi\left(\frac{N}{\ }\right) = \psi\left(\frac{t}{\ }\right)$ as a character modulo N (cf. [S-S]).

These theta series are linearly independent of each other.

From [Sh 2] Theorem 1.7. it follows that $\theta_\psi(tz)$ is the eigenfunction of the p-th Hecke operator $\tilde{T}_{1/2}(p^2) = p^{-3/2}\,[\Delta_0'' \left(\begin{pmatrix}1 & 0\\ 0 & p^2\end{pmatrix}, p^{1/2}\right)\Delta_0'']$ with the eigen value $\psi(p)(1+p^{-1})$. Let $n = \prod_{p|n} p^\tau$. Then, we have as a operator,

$$[\Delta_0'' \left(\begin{pmatrix}n^{-1} & 0\\ 0 & n\end{pmatrix}, n^{1/2}\right)\Delta_0''] = [\Delta_0'' \left(\begin{pmatrix}1 & 0\\ 0 & n^2\end{pmatrix}, n^{1/2}\right)\Delta_0'']$$

$$= \prod_{p|n} [\Delta_0'' \left(\begin{pmatrix}1 & 0\\ 0 & p^{2\tau}\end{pmatrix}, p^{\tau/2}\right)\Delta_0''] \text{ and}$$

$$[\Delta_0'' \left(\begin{pmatrix}1 & 0\\ 0 & p^{2\tau}\end{pmatrix}, p^{\tau/2}\right)\Delta_0'']$$

$$= [\Delta_0'' \left(\begin{pmatrix}1 & 0\\ 0 & p^2\end{pmatrix}, p^{1/2}\right)\Delta_0''] [\Delta_0'' \left(\begin{pmatrix}1 & 0\\ 0 & p^{2\tau-2}\end{pmatrix}, p^{(\tau-1)/2}\right)\Delta_0'']$$

$$- \begin{cases} p^2 + p & \text{for } \tau = 2, \\ p^2\,[\Delta_0'' \left(\begin{pmatrix}1 & 0\\ 0 & p^{2\tau-4}\end{pmatrix}, p^{(\tau-2)/2}\right)\Delta_0''] & \text{for } \tau \geqq 3, \end{cases}$$

(cf. [N 1] Introduction).

By using the induction, we see that $\theta_\psi(tz)$ has the eigen value $\psi(n)\sqrt{n}\prod_{p|n}(p^\tau + p^{\tau-1})$ with respect to the operator $[\Delta_0'' \left(\begin{pmatrix}n^{-1} & 0\\ 0 & n\end{pmatrix}, n^{1/2}\right)\Delta_0'']$.

If we take a positive integer t_1 such that $4f(\psi)^2 t_1 | N$ and $\chi\left(\frac{N}{\ }\right) = \psi\left(\frac{t_1}{\ }\right)$, $t_2 = N/(4f(\psi)^2 t_1)$ satisfies the conditions: $4f(\psi)^2 t_2 | N$ and $\chi = \psi\left(\frac{t_2}{\ }\right)$. Hence, when we fix the character ψ, the number of the positive integer t such that $4f(\psi)^2 t | N$ and $\chi\left(\frac{N}{\ }\right) = \psi\left(\frac{t}{\ }\right)$ equals to the constant $a(\chi;\psi)$ defined in the statement of Proposition 2

Therfore, trace $([\Delta_0'' \left(\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}, n^{1/2}\right) \Delta_0'']|G(1/2,N,\chi(\tfrac{N}{\cdot})))$

$= \Sigma_1\ a(\chi;\psi)\ \psi(n)\sqrt{n} \prod_{p|n} (p^\tau + p^{\tau-1})$, where Σ_1 is the same as in the statement of Proposition 2.

We can also discuss the case of $U(N,\chi)$ in a similar way and obtain the following results: When ψ and t run over the set described in §0 (c), the set of the theta series $\{ h_\psi(tz) \}$ gives the basis of $U(N,\chi)$ and $h_\psi(tz)$ has the eigen value $\psi(n)\sqrt{n} \prod_{p|n} (p^\tau + p^{\tau-1})$ with respect to the operator $[\Delta_0 \left(\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}, n^{3/2}\right) \Delta_0]$, where $n = \prod_{p|n} p^\tau$

Therefore, trace $([\Delta_0 \left(\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}, n^{3/2}\right) \Delta_0]|U(N,\chi))$

$= \Sigma_2\ b(\chi;\psi)\ \psi(n)\sqrt{n} \prod_{p|n} (p^\tau + p^{\tau-1})$, where Σ_2 and $b(\chi;\psi)$ are the same as in the statement of Proposition 2.

From the above, we have

$$\tilde{d}(n) = (\ \Sigma_1\ a(\chi;\psi)\ \psi(n) \quad - \quad \Sigma_2\ b(\chi;\psi)\ \psi(n)\ )\sqrt{n} \prod_{p|n} (p^\tau + p^{\tau-1}).$$

where $\tau = \mathrm{ord}_p(n)$ for the prime number $p|n$. The contribution to the trace of the n-th Hecke operator is calculated as follows:

$$\sum_{0<a|n_0} a\, n^{-1/2}\, \tilde{d}(n/a^2)$$

$$= \sum_{0<a|n_0} a\, n^{-1/2} (\ \Sigma_1 a(\chi;\psi)\psi(n/a^2) - \Sigma_2 b(\chi;\psi)\psi(n/a^2))$$

$$\times\sqrt{n/a^2} \prod_{p|(n/a^2)} (p^\tau + p^{\tau-1})$$

$$= (\Sigma_1 a(\chi;\psi)\psi(n) - \Sigma_2 b(\chi;\psi)\psi(n)) \sum_{0<a|n_0} \prod_{p|(n/a^2)} (p^\tau + p^{\tau-1})$$

$$= (\Sigma_1 a(\chi;\psi)\psi(n) - \Sigma_2 b(\chi;\psi)\psi(n)) \prod_{p|n} (p^{\tau+1} - 1)/(p - 1).$$

This is the term T'' in the statement of Proposition 2.

Before calculating the part of $\tilde{p}(n)$, $\tilde{e}(n)$ and $\tilde{h}(n)$, we give some remarks.

If an equivalence class $C \in \Phi$ is not scalar, we can choose an element $\beta = n^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ from C such that $(a,c) = 1$ and that $c \neq 0$ (cf. [N 1] Remark 1). Put $\beta' = n\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0 \begin{pmatrix} 1 & 0 \\ 0 & n^2 \end{pmatrix} \Gamma_0$, then there exist u, v and $w \in Z$ such that $\beta' = \sigma_1 \begin{pmatrix} 1 & 0 \\ 0 & n^2 \end{pmatrix} \sigma_2$ with $\sigma_1 = \begin{pmatrix} a & -v \\ c & u \end{pmatrix} \in \Gamma_0$ and $\sigma_2 = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \in \Gamma_0$. Therefore, from $c \equiv 0 \pmod N$.

$$\beta^* = L(\sigma_1)\left(\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}, n^{k+1/2}\right) L(\sigma_2) = (\sigma_1, \chi(u) j(\sigma_1, z)^{2k+1})$$

$$\times \left(\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}, n^{k+1/2}\right) (\sigma_2, j(\sigma_2, z)^{2k+1})$$

$$= \left(\beta, \chi(a)\left(\frac{-1}{a}\right)^{-k-1/2} \left(\frac{c}{a}\right) J(\beta, z)^{k+1/2}\right).$$

Now, suppose that β is parabolic or hyperbolic. Then, β has a fixed point κ which is the cusp of Γ_0. Since $c \neq 0$, we know $\kappa \neq \infty$. Let $\rho = \begin{pmatrix} \kappa & \kappa-1 \\ 1 & 1 \end{pmatrix} \in SL_2(R)$ and $\rho^* = (\rho, J(\rho, z)^{k+1/2}) \in G(k+1/2)$. Then, by calculating with attention to the signature of the branch, we have

$$\rho^{*-1}\beta^*\rho^* = (\rho^{-1}, J(\rho^{-1}, z)^{k+1/2})\left(\beta, \chi(a)\left(\frac{-1}{a}\right)^{-k-1/2}\left(\frac{c}{a}\right) J(\beta, z)^{k+1/2}\right)$$

$$\times (\rho, J(\rho, z)^{k+1/2})$$

$$= \left(\begin{pmatrix} \lambda^{-1} & y \\ 0 & \lambda \end{pmatrix}, \left(\frac{\mathrm{sgn}(\lambda)}{\mathrm{sgn}(c)}\right) \chi(a) \left(\frac{-1}{a}\right)^{-k-1/2} \left(\frac{c}{a}\right) \lambda^{k+1/2} \right), \text{ where}$$

$\lambda = (a - c\kappa)/n$, $y = (-a + d - c + 2c\kappa)/n$ and $\mathrm{sgn}(x) = 1, -1$ according to $x \geqq 0$, $x < 0$.

3. The part of $\tilde{p}(n)$.

Now, all the Γ_0-equivalence classes for the cusps of Γ_0 are represented by the number t^{-1}, where t runs over the following set S:

$$S = \left\{ \begin{array}{l} t = \zeta \prod_{p|N} p^e > 0,\ 0 \leqq e = e_p \leqq \mathrm{ord}_p(N) = \tilde{\nu} = \tilde{\nu}_p \text{ and } \zeta \\ \text{runs over a system of representatives, which are} \\ \text{prime to } N, \text{ for } (Z/\prod_{p|N} p^{\min(e,\tilde{\nu}-e)}Z)^{\times}. \end{array} \right\}$$

Let $A_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$. Then, the stabilizer of t^{-1} in $\Gamma_0/\{\pm 1\}$ is generated by $\sigma = \begin{pmatrix} 1-ut & u \\ -ut^2 & 1+ut \end{pmatrix} = A_t \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} A_t^{-1}$,

where u is the least natural number $ut^2 \equiv 0 \pmod N$, namely, $u = \prod_{p|N} p^{\max(\tilde{\nu}-2e,0)}$. For simplicity, we denote this product by $\prod'_{p|N} p^{\tilde{\nu}-2e}$.

Let us write out all the parabolic equivalence classes in $\Phi = \Phi(\Gamma_0 \begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix} \Gamma_0/\Gamma_0)$. Let β, β_1 be the parabolic elements in $\Phi(\Gamma_0 \begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix} \Gamma_0)$. Then, from the definition of the equivalence relation, it is easily seen that, if β and β_1 are equivalent, the fixed point of β must be Γ_0-equivalent to the fixed point of β_1. Now, suppose that the unique fixed point of β is t^{-1} with $t \in S$. Then, we have $A_t^{-1} \beta A_t =$

$\pm\, n^{-1}\begin{pmatrix} n & \tau \\ 0 & n \end{pmatrix}$ for some non-zero real number τ. Hence, $\beta' = n\beta$
$= \pm\, A_t \begin{pmatrix} n & \tau \\ 0 & n \end{pmatrix} A_t^{-1} = \pm \begin{pmatrix} n-t\tau & \tau \\ -t^2\tau & n+t\tau \end{pmatrix}$.

Since $\Gamma_0 \begin{pmatrix} 1 & 0 \\ 0 & n^2 \end{pmatrix} \Gamma_0 = \{M_2(Z) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \pmod N),\ (a,N) = 1,$

$(a,b,c,d) = 1,\ ad-bc = n^2\}$,

we see that $\beta' \in \Gamma_0 \begin{pmatrix} 1 & 0 \\ 0 & n^2 \end{pmatrix} \Gamma_0$ if and only if $t^2\tau \equiv 0 \pmod N$ and $(n,\tau) = 1$. In that case, τ becomes the multiple of u. Let $x = \tau/u$. For simplicity, we write

$$\beta'(t,x) = \begin{pmatrix} n-txu & xu \\ -t^2xu & n+txu \end{pmatrix}.$$

Suppose that such elements $\beta'(t,x_1)$ and $\beta'(t,x_2)$ are equivalent. Then, from the definition of the equivalence relation, there exist γ_1 and γ_2 in the stabilizer of t^{-1} in $\Gamma_0/\{\pm 1\}$ such that $\gamma_1^{-1}\gamma_2\, \beta'(t,x_1)\, \gamma_1 = \beta'(t,x_2)$. Since that stabilizer is generated by σ, we can write $\gamma_1 = \pm\sigma^a$ and $\gamma_2 = \pm\sigma^b$ with $a,\ b \in Z$. Hence, $\beta'(t,x_2) = A_t \begin{pmatrix} n & x_2u \\ 0 & n \end{pmatrix} A_t^{-1} = \gamma_1^{-1}\gamma_2\, \beta'(t,x_1)\, \gamma_1$
$= \pm\sigma^{b-a}\, \beta'(t, x_1)\sigma^a = \pm\, A_t \begin{pmatrix} n & x_1u + bnu \\ 0 & n \end{pmatrix} A_t^{-1}$. Therfore, we have $x_2 = x_1 + bn$.

From the above results, the system of representatives of all the parabolic equivalence classes in Φ is formed by the matrices $\beta(t,x) = n^{-1}\beta'(t,x)$, where $t = \zeta \prod_{p|N} p^e$ runs over the set S and x runs over the system of representatives for $(Z/nZ)^\times$ which satisfies the condition $x \neq 0$. Here, by the suitable choice of the representative, we may assume that $4|x$ and that $(\zeta,n) = 1$.

Now, we shall determine the number $J(\beta) = J(C)$ for the equivalence class C containing the matrix $\beta = \beta(t,x)$. Let $\rho = A_t \begin{pmatrix} \sqrt{u} & 0 \\ 0 & \sqrt{u}^{-1} \end{pmatrix} \in SL_2(R)$, then, we have $\rho^{-1}\sigma\rho = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Since, $\beta = 1 + (x/n)(\sigma - 1)$, we have $\rho^{-1}\beta\rho = \begin{pmatrix} 1 & x/n \\ 0 & 1 \end{pmatrix}$.

Next, we take the lift $\rho^{*} \in G(k+1/2)$ of ρ, then,we must determine the numbers δ and η such that $\rho^{*-1}L(\sigma)\rho^{*} = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, e(\delta)\right)$ with $0 < \delta \leqq 1$ and that $\rho^{*-1}\beta^{*}\rho^{*} = \left(\begin{pmatrix} 1 & x/n \\ 0 & 1 \end{pmatrix}, \eta\right)$. From elementary calculation, it is easily shown that δ and η are invariant when we replace ρ^{*} by $\left(\begin{pmatrix} t^{-1} & t^{-1} - 1 \\ 1 & 1 \end{pmatrix}, (z + 1)^{k+1/2}\right)$. Moreover, if we write $\beta = n^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, from the definitions of the letters, we have $c \neq 0$ and $(a,c) = 1$, and then, $t^{-1} = (a - d)/2c$ and $a - ct^{-1} = (a + d)/2 = n$. Therfore, by using the remark after the part of $\tilde{d}(n)$, we obtain $\eta = \chi(a)\left(\frac{-1}{a}\right)^{-k-1/2}\left(\frac{c}{a}\right) = \chi(n - txu)\left(\frac{-1}{n - txu}\right)^{-k-1/2}\left(\frac{-t^2xu}{n - txu}\right)$.

From $tu = \zeta \prod_{p|N} p^{\max(e,\tilde{\nu}-e)}$ and $4|x$, we have $txu \equiv 0 \pmod{8 \prod_{p|M} p}$ Hence, $\left(\frac{-1}{n - txu}\right) = \left(\frac{-1}{n}\right)$ and $\left(\frac{-t^2xu}{n - txu}\right) = \left(\frac{-1}{n}\right)\left(\frac{t^2xu}{n - txu}\right) = \left(\frac{-1}{n}\right)\left(\frac{xu}{n}\right)$. Moreover, from §0 (c), we have a square-free positive divisor M_0 of M such that $\chi = \left(\frac{M_0}{\quad}\right)$ or $\left(\frac{2M_0}{\quad}\right)$. Hence, we have $\chi(n - txu) = \chi(n)$. Thus, we obtain $\eta = \chi(n)\left(\frac{-1}{n}\right)^{-k+1/2}\left(\frac{xu}{n}\right)$.

In a similar way, we have

$$e(\delta) = \chi(1+ut)\left(\frac{-1}{1+ut}\right)^{-k-1/2}\left(\frac{-ut^2}{1+ut}\right) = \chi(1+ut)\left(\frac{-1}{1+ut}\right)^{-k+1/2}$$

$$\times\left(\frac{u}{1+ut}\right) = \chi(1+ut)\left(\frac{-1}{1+ut}\right)^{-k+1/2}\prod_{p|N}\left(\frac{p}{1+ut}\right)^{\max(\tilde{\nu}-2e,0)}$$

$$= \chi(1+ut)\left(\frac{-1}{1+ut}\right)^{-k+1/2}\prod_{p|N}{}'\left(\frac{p}{1+ut}\right)^{\tilde{\nu}},$$ where the meaning

of the symbol $\prod_{p|N}{}'$ is as follows: For any complex number $a(p)$,

we put $a'(p) = \begin{cases} a(p), & \text{if the prime number } p \text{ satisfies the condition } 2e < \tilde{\nu}, \\ 1, & \text{otherwise.} \end{cases}$

Then, we put $\prod_{p|N}{}' a(p) = \prod_{p|N} a'(p)$.

Now,we shall divide the cases.

(i) $\mu = \mathrm{ord}_2(N) \geqq 5$.

In this case, we have $\max(e_2, \mu-e_2) \geqq 3$. Hence,

$ut \equiv 0 \pmod{8 \prod_{p|M} p}$, and then $\chi(1+ut) = \left(\frac{-1}{1+ut}\right) = \left(\frac{p}{1+ut}\right)$

$= 1$ for all the prime divisor p of N. Therefore, we have

$e(\delta) = 1$ and $\delta = 1$.

(ii) $\mu = 4$.

In this case, we have $\max(e_2, \mu-e_2) \geqq 2$. Hence, we have

$ut \equiv 0 \pmod{4 \prod_{p|M} p}$ and $\left(\frac{-1}{1+ut}\right) = \left(\frac{p}{1+ut}\right) = 1$ for all the

odd prime divisor p of N. Moreover, by using $\mu = 4$, we have also $\prod_{p|N}{}' \left(\frac{p}{1+ut}\right)^{\tilde{\nu}} = 1$. If $f(\chi_2)|4$, $\chi(1+ut) = 1$ and if $f(\chi_2) = 8$, we have $\chi(1+ut) = \left(\frac{2}{1+ut}\right) = \begin{cases} 1, & \text{for } e_2 \neq 2, \\ -1, & \text{for } e_2 = 2. \end{cases}$

Thus, we obtain $\delta = \begin{cases} 1, & \text{if } f(\chi_2)|4, \text{ or } f(\chi_2) = 8 \text{ and } e_2 \neq 2, \\ 1/2, & \text{if } f(\chi_2) = 8 \text{ and } e_2 = 2. \end{cases}$

(iii) $\mu = 3$.

In a similar way as in the case (ii), we obtain the following results: If $f(\chi_2)|4$, $\delta = \begin{cases} 1, & \text{for } e_2 \neq 1, \\ 1/2, & \text{for } e_2 = 1. \end{cases}$

If $f(\chi_2) = 8$, $\delta = \begin{cases} 1, & \text{for } e_2 \neq 2, \\ 1/2, & \text{for } e_2 = 2. \end{cases}$

(iv) $\mu = 2$.

In this case, we have always $f(\chi_2)|4$. We write $\chi = \left(\frac{M_0}{\ }\right)$ as above. Then, we easily see that

$$e(\delta) = \begin{cases} 1, & \text{if } e_2 = 0,2, \\ \left(\frac{M_0}{1+ut}\right)(-1)^{-k+1/2} \prod_{p|M}{}' \left(\frac{p}{1+ut}\right)^{\tilde{\nu}}, & \text{if } e_2 = 1, \end{cases}$$

where the symbol $\prod_{p|M}{}'$ is defined by replacing N with M at the definition of the symbol $\prod_{p|N}{}'$.

For the case $e_2 = 1$, we have $1 + ut \equiv 3 \pmod 4$ and $ut \equiv 0 \pmod{\prod_{p|M} p}$. Hence,

$$1 = \left(\frac{1+ut}{M_0}\right) = \left(\frac{\left(\frac{-1}{M_0}\right) M_0}{1+ut}\right) = \left(\frac{\left(\frac{-1}{M_0}\right)}{1+ut}\right)\left(\frac{M_0}{1+ut}\right) = \left(\frac{-1}{M_0}\right)\left(\frac{M_0}{1+ut}\right).$$

And also, for the prime number $p|M$, $1 = \left(\frac{1+ut}{p}\right) =$

$\left(\frac{\left(\frac{-1}{p}\right)}{1+ut}\right)\left(\frac{p}{1+ut}\right) = \left(\frac{-1}{p}\right)\left(\frac{p}{1+ut}\right)$. Thus, we obtain

$$e(\delta) = (-1)^k\sqrt{-1}\left(\frac{-1}{M_0}\right)\prod_{p|M}{}'\left(\frac{-1}{p}\right)^{\nu} \quad \text{and}$$

$$\delta = (1/2) - (1/4)\left(\frac{-1}{2k+1}\right)\left(\frac{-1}{M_0}\right)\prod_{p|M}{}'\left(\frac{-1}{p}\right)^{\nu} \quad \text{for } e_2 = 1.$$

Now, we can determine $J(\beta)$. Obviously, Γ_0 contains $\beta = \beta(t,x)$ if and only if $n = 1$. In that case, we have $\eta = 1$. And also, from the assumption $4|x$, we have $e(\delta x) = 1$. Therefore, we obtain

$$J(\beta) = \begin{cases} (1/2) - \delta, & \text{if } n = 1. \\ \chi(n)\left(\frac{-1}{n}\right)^{k-1/2}\left(\frac{xu}{n}\right)e(\delta x/n)(1 - e(x/n))^{-1}, & \text{if } n > 1. \end{cases}$$

Suppose $n = 1$. We shall calculate $\tilde{p}(1)$ for the cases (i) - (iv) and in the following, for simplicity, we shall drop the subscript p from e_p and $\tilde{\nu}_p$, etc.

(i) $\mu \geqq 5$.

In this case, $\delta = 1$ and $J(\beta) = -1/2$. Hence,

$$\tilde{p}(1) = -(1/2) \times \#\{\beta(t,x) \mid t \in S\} = -(1/2) \times \#S$$

$$= -(1/2) \times \sum_{\substack{0 \leqq e_p \leqq \tilde{\nu}_p \\ p|N}} \prod_{p|N} \phi(p^{\min(e,\,\tilde{\nu}-e)})$$

$$= -(1/2) \times \prod_{p|N} \sum_{e=0}^{\tilde{\nu}} \phi(p^{\min(e,\,\tilde{\nu}-e)})$$

$$= -(1/2) \prod_{p|N} (p^{[\tilde{\nu}/2]} + p^{[(\tilde{\nu}-1)/2]}).$$

(ii) $\mu = 4$.

If $f(\chi_2)|4$, we have $\delta = 1$. Hence, in the same way as in the case (i), we have $\tilde{p}(1) = -(1/2) \prod_{p|N} (p^{[\tilde{\nu}/2]} + p^{[(\tilde{\nu}-1)/2]})$.

If $f(\chi_2) = 8$, we have $J(\beta) = \begin{cases} -1/2, & \text{for } e_2 \neq 2, \\ 0, & \text{for } e_2 = 2. \end{cases}$

Therefore, $\tilde{p}(1) = -(1/2) \times \#\{\beta(t,x) \mid S \ni t \text{ such that } e_2 \neq 2\}$

$$= -(1/2) \sum_{\substack{0 \leq e_p \leq \nu_p \\ p|M}} \prod_{p|M} \phi(p^{\min(e,\ \nu-e)})$$

$$\times \left(\sum_{e_2=0}^{4} \phi(2^{\min(e_2,\ 4-e_2)}) - \phi(2^{\min(2,\ 4-2)}) \right)$$

$$= -2 \prod_{p|M} \sum_{e=0}^{\nu} \phi(p^{\min(e,\ \nu-e)})$$

$$= -2 \prod_{p|M} (p^{[\nu/2]} + p^{[(\nu-1)/2]}).$$

(iii) $\mu = 3$.

In a similar way as in the case (ii), we obtain the following result: $\tilde{p}(1) = -(3/2) \prod_{p|M} (p^{[\nu/2]} + p^{[(\nu-1)/2]})$.

(iv) $\mu = 2$.

In this case, we have

$$J(\beta) = \begin{cases} -(1/2), & \text{if } e_2 \neq 1, \\ (1/4)\left(\frac{-1}{2k+1}\right)\left(\frac{-1}{M_0}\right) \prod'_{p|M} \left(\frac{-1}{p}\right)^{\nu}, & \text{if } e_2 = 1. \end{cases}$$

Hence, $\tilde{p}(1) = \tilde{p}_1 + \tilde{p}_2$ with

$\tilde{p}_1 = -(1/2) \times \#\{\beta(t,x) \mid S \ni t$ such that $e_2 \neq 1\}$ and

$$\tilde{p}_2 = (1/4)\left(\frac{-1}{2k+1}\right)\left(\frac{-1}{M_0}\right) \sum_{\substack{\beta(t,x) \\ e_1 = 1}} \prod_{p|M}{}' \left(\frac{-1}{p}\right)^{\nu}$$, where the sum

$\sum_{\substack{\beta(t,x) \\ e_1 = 1}}$ is extended over the matrices $\beta(t,x)$ such that $e_2 = 1$.

In a similar way as in the above cases, we have $\tilde{p}_1 =$

$$- \prod_{p|M} (p^{[\nu/2]} + p^{[(\nu-1)/2]})$$ and

$$\tilde{p}_2 = (1/4)\left(\frac{-1}{2k+1}\right)\left(\frac{-1}{M_0}\right) \phi(2^{\min(1,\ 2-1)})$$

$$\times \sum_{\substack{0 \leqq e_p \leqq \nu_p \\ p|M}} \prod_{p|M}{}' \left(\frac{-1}{p}\right)^{\nu} \phi(\prod_{p|M} p^{\min(e,\ \nu-e)})$$

$$= (1/4)\left(\frac{-1}{2k+1}\right)\left(\frac{-1}{M_0}\right) \prod_{p|M} \sum_{e=0}^{\nu} \phi(p^{\min(e,\ \nu-e)}) \times \begin{cases} 1, & \text{if } \nu \leqq 2e, \\ \left(\frac{-1}{p}\right)^{\nu}, & \text{if } \nu > 2e \end{cases}$$

$$= (1/4)\left(\frac{-1}{2k+1}\right)\left(\frac{-1}{M_0}\right) \prod_{p|M} (p^{[\nu/2]} + \left(\frac{-1}{p}\right)^{\nu} p^{[(\nu-1)/2]}) .$$

Next, we suppose $n > 1$.

In the following, for simplicity, we drop the subscript p and since $4|x$, we write $x = 4x_0$. Then,

$$J(\beta) = \chi(n)\left(\frac{-1}{n}\right)^{k-1/2} \left(\frac{4x_0u}{n}\right) e(4\delta x_0/n)(1 - e(4x_0/n))^{-1} .$$

We need the following lemma.

Lemma. We have the following equalities.

(i) $\sum_{a \in (Z/nZ)^{\times}} \left(\frac{4a}{n}\right) e(4a/n)(1 - e(4a/n))^{-1}$

$$= \begin{cases} -(1/2)\delta_0(\sqrt{n})\phi(n), & \text{if } n \equiv 1 \pmod 4. \\ \sqrt{-n}\; h'(-n), & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

(ii) $\sum_{a \in (Z/nZ)^{\times}} \left(\frac{4a}{n}\right) e(2a/n)(1 - e(4a/n))^{-1}$

$$= \begin{cases} 0, & \text{if } n \equiv 1 \pmod 4, \\ \left(\left(\frac{2}{n}\right) - 1\right)\sqrt{-n}\; h'(-n), & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

(iii) $\sum_{a \in (Z/nZ)^{\times}} \left(\frac{4a}{n}\right) e(a/n)(1 - e(4a/n))^{-1}$

$$= \begin{cases} (1/2)\sqrt{n}\; h'(-4n), & \text{if } n \equiv 1 \pmod 4, \\ (1/2)\left(1 - \left(\frac{2}{n}\right)\right)\sqrt{-n}\; h'(-n), & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Here, the above sum $\sum_{a \in (Z/nZ)^{\times}}$ is extended over a system of representatives for $(Z/nZ)^{\times}$. We shortly write $\sum_a$ for this sum.

Proof. Since $e(a/n)(1 - e(a/n))^{-1} = -(1/2) + (1/2)\sqrt{-1}\cot(\pi a/n)$, we have $\sum_a \left(\frac{4a}{n}\right) e(4a/n)(1 - e(4a/n))^{-1} = \sum_a \left(\frac{a}{n}\right) e(a/n)(1 - e(a/n))^{-1}$ $= -(1/2) \sum_a \left(\frac{a}{n}\right) + (1/2)\sqrt{-1} \sum_a \left(\frac{a}{n}\right) \cot(\pi a/n)$. Obviously, the first term is equal to $-(1/2)\delta_0(\sqrt{n})\phi(n)$. If $n \equiv 1 \pmod 4$, since $\cot(z)$ is an odd function, we have $\left(\frac{-a}{n}\right)\cot(-\pi a/n) = -\left(\frac{a}{n}\right)\cot(\pi a/n)$. Therefore, in this case, the second term is equal to zero. If $n \equiv 3 \pmod 4$, observing the expantion $\pi \cot(\pi z) = z^{-1} + \sum_{m=1}^{\infty} ((z - m)^{-1} + (z + m)^{-1})$, we have $\sum_a \left(\frac{a}{n}\right) \cot(\pi a/n) = (2n/\pi)\, L(1, \left(\frac{\ }{n}\right))$. Here, $L(s, \left(\frac{\ }{n}\right))$ is the

Dirichlet's L-function with respect to the character $\left(\frac{-}{n}\right)$ modulo n. From Dirichlet's class number formula, it follows that $\sum_a \left(\frac{a}{n}\right) \cot(\pi a/n) = 2\sqrt{n}\, h'(-n)$. Thus we obtain the equality (i).

For the equalities (ii) and (iii), we use the following identities: $e(2a/n)(1 - e(4a/n))^{-1} = (1/2)\{e(2a/n)\ (1 - e(2a/n))^{-1} + e(2a/n)\ (1 + e(2a/n))^{-1}\}$, $e(a/n)(1 - e(4a/n))^{-1}$
$= (1/4)\ \{\ e(a/n)(1 - e(a/n))^{-1} + e(a/n)(1 + e(a/n))^{-1} + e(a/n)(1 - \sqrt{-1}\ e(a/n))^{-1} + e(a/n)(1 + \sqrt{-1}\ e(a/n))^{-1}\}$
and $e(2a/n)(1 + e(2a/n))^{-1} = (1/2) - (1/2)\sqrt{-1}\ \cot(\pi(2a/n + 1/2))$, etc. By using these identities, we can also apply the similar procedure as in the proof of the equality (i) for the case of the equalities (ii) and (iii).

Now, we shall calculate $\tilde{p}(n)$ for the cases (i) - (iv).

(i) $\mu \geqq 5$.

In this case, $\delta = 1$. Hence, we have

$$\tilde{p}(n) = \chi(n)\left(\frac{-1}{n}\right)^{k-1/2} \sum_{x_0 \in (Z/nZ)^\times} \sum_{\substack{0 \leqq e_p \leqq \tilde{\nu}_p \\ p|N}} \sum_{\zeta} \left(\frac{4x_0 u}{n}\right) e(4x_0/n) \times (1 - e(4x_0/n))^{-1}$$

$$= \chi(n)\left(\frac{-1}{n}\right)^{k-1/2} \sum_{x_0 \in (Z/nZ)^\times} \left(\frac{4x_0}{n}\right) e(4x_0/n)(1 - e(4x_0/n))^{-1} \times \sum_{\substack{0 \leqq e_p \leqq \tilde{\nu}_p \\ p|N}} \sum_{\zeta} \left(\frac{u}{n}\right).$$

Here, we note that the matrices $\beta(t,x)$ depend only on x_0, ζ and e_p. Moreover, we have

$$\sum_{\substack{0 \leqq e_p \leqq \tilde{\nu}_p \\ p|N}} \sum_{\zeta} \left(\frac{u}{n}\right) = \sum_{\substack{0 \leqq e_p \leqq \tilde{\nu}_p \\ p|N}} \prod_{p|N}{}' \left(\frac{p}{n}\right)^{\tilde{\nu}} \prod_{p|N} \phi(p^{\min(e,\tilde{\nu}-e)})$$

$$= \prod_{p|N} \sum_{e=0}^{\tilde{\nu}} \phi(p^{\min(e,\tilde{\nu}-e)}) \times \begin{cases} 1, & \text{if } \tilde{\nu} \leqq 2e \\ \left(\frac{p}{n}\right)^{\tilde{\nu}}, & \text{if } \tilde{\nu} > 2e \end{cases}$$

$$= \prod_{p|N} \left(p^{[\tilde{\nu}/2]} + \left(\frac{p}{n}\right)^{\tilde{\nu}} p^{[(\tilde{\nu}-1)/2]}\right).$$

From this and lemma (i), it follows that $\tilde{p}(n) =$

$$\chi(n)\left(\frac{-1}{n}\right)^{k-1/2} \prod_{p|N} \left(p^{[\tilde{\nu}/2]} + \left(\frac{p}{n}\right)^{\tilde{\nu}} p^{[(\tilde{\nu}-1)/2]}\right)$$

$$\times \begin{cases} -(1/2)\delta_0(\sqrt{n})\phi(n), & \text{if } n \equiv 1 \pmod 4 \\ \sqrt{-n}\, h'(-n), & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

$$= \begin{cases} -(1/2)\delta_0(\sqrt{n})\phi(n) \prod_{p|N} \left(p^{[\tilde{\nu}/2]} + p^{[(\tilde{\nu}-1)/2]}\right), & \text{if } n \equiv 1 \pmod 4, \\ (-1)^k\sqrt{n}\, h'(-n)\chi(n) \prod_{p|N} \left(p^{[\tilde{\nu}/2]} + \left(\frac{p}{n}\right)^{\tilde{\nu}} p^{[(\tilde{\nu}-1)/2]}\right), & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Here, when $n \equiv 1 \pmod 4$, we used the identity: $\delta_0(\sqrt{n})\chi(n) = \delta_0(\sqrt{n})\left(\frac{p}{n}\right) = \delta_0(\sqrt{n})$. We note that these expressions also fit for the case $n = 1$.

(ii) $\mu = 4$.

If $f(\chi_2)|4$, we have also $\delta = 1$. Hence, we can calculate in the same way as in the case (i) and the result is also the same as in the case (i).

If $f(\chi_2) = 8$, $\delta = \begin{cases} 1, & \text{for } e_2 \neq 2 \\ 1/2, & \text{for } e_2 = 2. \end{cases}$

Hence, we have $\tilde{p}(n) = \tilde{p}_1 + \tilde{p}_2$ with

$$\tilde{p}_1 = \chi(n)\left(\frac{-1}{n}\right)^{k-1/2} \sum_{x_0 \in (Z/nZ)^\times} \sum_{\substack{0 \leq e_p \leq \tilde{\nu}_p \\ p|N,\ e_2 \neq 2}} \sum_{\zeta} \left(\frac{4x_0 u}{n}\right) \times e(4x_0/n)(1 - e(4x_0/n))^{-1}$$

and $$\tilde{p}_2 = \chi(n)\left(\frac{-1}{n}\right)^{k-1/2} \sum_{x_0 \in (Z/nZ)^\times} \sum_{\substack{0 \leq e_p \leq \tilde{\nu}_p \\ p|N,\ e_2 = 2}} \sum_{\zeta} \left(\frac{4x_0 u}{n}\right) \times e(2x_0/n)(1 - e(4x_0/n))^{-1} .$$

By using lemma (i) and (ii), we can calculate $\tilde{p}_1$ and $\tilde{p}_2$ as follows:

$$\tilde{p}_1 = \chi(n)\left(\frac{-1}{n}\right)^{k-1/2} \sum_{x_0 \in (Z/nZ)^\times} \left(\frac{4x_0}{n}\right) e(4x_0/n)(1 - e(4x_0/n))^{-1}$$

$$\times \sum_{\substack{0 \leq e_p \leq \tilde{\nu}_p \\ p|N,\ e_2 \neq 2}} \sum_{\zeta} \prod_{p|N} \left(\frac{p}{n}\right)^{\tilde{\nu}}$$

$$= \chi(n)\left(\frac{-1}{n}\right)^{k-1/2} \times \begin{Bmatrix} -(1/2)\delta_0(\sqrt{n})\phi(n), \text{ if } n \equiv 1 \pmod 4 \\ \sqrt{-n}\, h'(-n), \text{ if } n \equiv 3 \pmod 4 \end{Bmatrix}$$

$$\times \prod_{p|M} \sum_{e=0}^{\nu} \phi(p^{\min(e,\nu-e)}) \times \begin{Bmatrix} 1, \text{ if } \nu \leq 2e \\ \left(\frac{p}{n}\right)^{\nu}, \text{ if } \nu > 2e \end{Bmatrix}$$

$$\times \{\left(\frac{2}{n}\right)^4 \phi(2^{\min(0,4)}) + \left(\frac{2}{n}\right)^4 \phi(2^{\min(1,3)}) + \phi(2^{\min(3,1)}) + \phi(2^{\min(4.0)})\}$$

$$= \chi(n)\left(\frac{-1}{n}\right)^{k-1/2} \times \begin{Bmatrix} -(1/2)\delta_0(\sqrt{n})\phi(n), \text{ if } n \equiv 1 \pmod 4 \\ \sqrt{-n}\, h'(-n), \text{ if } n \equiv 3 \pmod 4 \end{Bmatrix}$$

$$\times 4 \prod_{p|M} (p^{[\nu/2]} + \left(\frac{p}{n}\right)^{\nu} p^{[(\nu-1)/2]})$$

$$= \begin{cases} -2\,\delta_0(\sqrt{n})\phi(n) \prod_{p|M} (p^{[\nu/2]} + p^{[(\nu-1)/2]}) , & \text{if } n \equiv 1 \pmod 4 \\ (-1)^k\, 4\, \chi(n)\sqrt{n}\, h'(-n) \prod_{p|M} \left(p^{[\nu/2]} + \left(\frac{p}{n}\right)^{\nu} p^{[(\nu-1)/2]}\right), & \\ \qquad \text{if } n \equiv 3 \pmod 4 \end{cases}$$

$$\tilde{P}_2 = \chi(n)\left(\frac{-1}{n}\right)^{k-1/2} \sum_{x_0 \in (Z/nZ)^\times} \left(\frac{4x_0}{n}\right) e(2x_0/n)(1 - e(4x_0/n))^{-1}$$

$$\times \sum_{\substack{0 \leqq e_p \leqq \tilde{\nu}_p \\ p|N,\ e_2 = 2}} \sum_{\zeta} \prod_{p|N} {}'\left(\frac{p}{n}\right)^{\tilde{\nu}}$$

$$= \chi(n)\left(\frac{-1}{n}\right)^{k-1/2} \times \left\{\begin{matrix} 0 , \text{ if } n \equiv 1 \pmod 4 \\ \left(\left(\frac{2}{n}\right) - 1\right)\sqrt{-n}\, h'(-n), \text{ if } n \equiv 3 \pmod 4 \end{matrix}\right\}$$

$$\times \prod_{p|M} \sum_{e=0}^{\nu} \phi(p^{\min(e,\nu-e)}) \times \left\{\begin{matrix} 1 , \text{ if } \nu \leqq 2e \\ \left(\frac{p}{n}\right)^{\nu} , \text{ if } \nu > 2e \end{matrix}\right\}$$

$$\times \phi(2^{\min(2,4-2)})$$

$$= \begin{cases} 0, \text{ if } n \equiv 1 \pmod 4 , & \\ 2\,(-1)^k \left(\left(\frac{2}{n}\right) - 1\right)\chi(n)\sqrt{n}\, h'(-n) \prod_{p|M} \left(p^{[\nu/2]} + \left(\frac{p}{n}\right)^{\nu} p^{[(\nu-1)/2]}\right), & \\ \qquad \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Therefore, we obtain

$$\tilde{p}(n) = \begin{cases} -2\delta_0(\sqrt{n})\phi(n) \prod_{p|M} (p^{[\nu/2]} + p^{[(\nu-1)/2]}), \text{ if } n \equiv 1 \pmod 4, & \\ 2\left(1 + \left(\frac{2}{n}\right)\right)(-1)^k\chi(n)\sqrt{n}\, h'(-n) \prod_{p|M} \left(p^{[\nu/2]} + \left(\frac{p}{n}\right)^{\nu} p^{[(\nu-1)/2]}\right), & \\ \qquad \text{if } n \equiv 3 \pmod 4. \end{cases}$$

We note that these expressions also fit for the case $n = 1$.

(iii) $\mu = 3$.

In this case, we can calculate in a similar way as in the case (ii). The results are as follows:

When $f(\chi_2)|4$, we have

$$\tilde{p}(n) = \begin{cases} -(3/2)\delta_0(\sqrt{n})\phi(n) \prod_{p|M} (p^{[\nu/2]} + p^{[(\nu-1)/2]}), \\ \qquad \text{if } n \equiv 1 \pmod 4, \\ 3(-1)^k \chi(n)\sqrt{n}\, h'(-n) \prod_{p|M} \left(p^{[\nu/2]} + \left(\frac{p}{n}\right)^{\nu} p^{[(\nu-1)/2]}\right), \\ \qquad \text{if } n \equiv 3 \pmod 4. \end{cases}$$

When $f(\chi_2) = 8$, we have

$$\tilde{p}\,(n) = \begin{cases} -(3/2)\delta_0(\sqrt{n})\phi(n) \prod_{p|M} (p^{[\nu/2]} + p^{[(\nu-1)/2]}), \\ \qquad \text{if } n \equiv 1 \pmod 4, \\ 3\left(\frac{2}{n}\right)(-1)^k \chi(n)\sqrt{n}\, h'(-n) \prod_{p|M} \left(p^{[\nu/2]} + \left(\frac{p}{n}\right)^{\nu} p^{[(\nu-1)/2]}\right), \\ \qquad \text{if } n \equiv 3 \pmod 4. \end{cases}$$

We note that these expressions also fit for the case $n = 1$.

(iv) $\mu = 2$.

In this case, we have

$$\delta = \begin{cases} 1, \text{ if } e_2 \neq 1, \\ (1/2) - (1/4)\left(\frac{-1}{2k+1}\right)\left(\frac{-1}{M_0}\right) \prod_{p|M}{}' \left(\frac{-1}{p}\right)^{\nu}, \text{ if } e_2 = 1. \end{cases}$$

Hence, we have $\tilde{p}(n) = \tilde{p}_1 + \tilde{p}_2$ with

$$\tilde{p}_1 = \chi(n)\left(\frac{-1}{n}\right)^{k-1/2} \sum_{x_0 \in (\mathbb{Z}/n\mathbb{Z})^\times} \sum_{\substack{0 \leq e_p \leq \tilde{\nu}_p \\ p|N, e_2 \neq 1}} \sum_{\zeta} \left(\frac{4x_0u}{n}\right) e(4x_0/n)(1 - e(4x_0/n))^{-1}$$

and $\tilde{p}_2 = \chi(n)\left(\frac{-1}{n}\right)^{k-1/2} \sum_{x_0 \in (Z/nZ)^\times} \sum_{\substack{0 \leq e_p \leq \tilde{\nu}_p \\ p|N,\ e_2 = 1}} \sum_{\zeta} \left(\frac{4x_0 u}{n}\right)$

$$\times\ e(2x_0/n - (x_0/n)\left(\frac{-1}{2k+1}\right)\left(\frac{-1}{M_0}\right)\prod_{p|M}{}'\left(\frac{-1}{p}\right)^{\nu})(1 - e(4x_0/n))^{-1}$$

We can calculate $\tilde{p}_1$ in a similar way as in the above cases and the result is as follows:

$$\tilde{p}_1 = \begin{cases} -\delta_0(\sqrt{n})\phi(n) \prod_{p|M} (p^{[\nu/2]} + p^{[(\nu-1)/2]}), \text{ if } n \equiv 1 \pmod 4, \\ 2(-1)^k \chi(n)\sqrt{n}\, h'(-n) \prod_{p|M} (p^{[\nu/2]} + \left(\frac{p}{n}\right)^{\nu} p^{[(\nu-1)/2]}), \\ \qquad \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Next, we put $(-1)^a = \left(\frac{-1}{2k+1}\right)\left(\frac{-1}{M_0}\right) \prod_{p|M}{}'\left(\frac{-1}{p}\right)^{\nu}$. Then,

$$\tilde{p}_2 = \chi(n)\left(\frac{-1}{n}\right)^{k-1/2} \sum_{\substack{0 \leq e_p \leq \tilde{\nu}_p \\ p|N,e_2 = 1}} \sum_{\zeta} \prod_{p|N}{}'\left(\frac{p}{n}\right)^{\tilde{\nu}}$$

$$\times \sum_{x_0 \in (Z/nZ)^\times} \left(\frac{4x_0}{n}\right) e((2 - (-1)^a)x_0/n)(1 - e(4x_0/n))^{-1}$$

$$= \chi(n)\left(\frac{-1}{n}\right)^{k-1/2} \sum_{\substack{0 \leq e_p \leq \nu_p \\ p|M}} \prod_{p|M}{}'\left(\frac{p}{n}\right)^{\nu} \prod_{p|M} \phi(p^{\min(e,\nu-e)})$$

$$\times \begin{cases} ((-1)^a/2)\sqrt{n}\, h'(-4n), \text{ if } n \equiv 1 \pmod 4, \\ ((1 - \left(\frac{2}{n}\right))/2)\sqrt{-n}\, h'(-n), \text{ if } n \equiv 3 \pmod 4. \end{cases}$$

Here, we used the identity:

$$e(3x_0/n)(1 - e(4x_0/n))^{-1} = - e(-x_0/n)(1 - e(-4x_0/n))^{-1}.$$

Therefore, when $n \equiv 1 \pmod 4$, we have

$$\tilde{p}_2 = (1/2)\left(\frac{-1}{2k+1}\right)\left(\frac{-1}{M_0}\right)\chi(n)\sqrt{n}\, h'(-4n)$$

$$\times \sum_{\substack{0 \leq e_p \leq \nu_p \\ p|M}} \prod_{p|M}{}' \left(\left(\frac{p}{n}\right)\left(\frac{-1}{p}\right)\right)^{\nu} \prod_{p|M} \phi(p^{\min(e,\nu-e)})$$

$$= ((-1)^k/2)\left(\frac{-1}{M_0}\right)\chi(n)\sqrt{n}\, h'(-4n) \prod_{p|M} \left(p^{[\nu/2]} + \left(\frac{-n}{p}\right)^{\nu} p^{[(\nu-1)/2]}\right).$$

When $n \equiv 3 \pmod 4$, we have

$$\tilde{p}_2 = ((1 - \left(\frac{2}{n}\right))/2)(-1)^k\chi(n)\sqrt{n}\, h'(-n)$$

$$\times \prod_{p|M} \left(p^{[\nu/2]} + \left(\frac{p}{n}\right)^{\nu} p^{[(\nu-1)/2]}\right).$$

Therefore, we obtain, when $n \equiv 1 \pmod 4$,

$$\tilde{p}(n) = -\delta_0(\sqrt{n})\phi(n) \prod_{p|M} \left(p^{[\nu/2]} + p^{[(\nu-1)/2]}\right)$$

$$+ ((-1)^k/2)\left(\frac{-1}{M_0}\right)\chi(n)\sqrt{n}\, h'(-4n) \prod_{p|M} \left(p^{[\nu/2]} + \left(\frac{-n}{p}\right)^{\nu} p^{[(\nu-1)/2]}\right),$$

and when $n \equiv 3 \pmod 4$.

$$\tilde{p}(n) = ((5 - \left(\frac{2}{n}\right))/2)(-1)^k\chi(n)\sqrt{n}\, h'(-n) \prod_{p|M} \left(p^{[\nu/2]} + \left(\frac{p}{n}\right)^{\nu} p^{[(\nu-1)/2]}\right).$$

We note that these expressions also fit for the case $n = 1$.

Finally, we must calculate the contribution to the trace of the n-th Hecke operator. But, that is easy calculation if we use the identity: $\sum_{0<a|n_0} a\,\phi(n/a^2) = n.$

4. The part of $\tilde{e}(n)$. (In this part, we follow the method in [N].)

Let C be an elliptic equivalence class in Φ. Take $\beta = n^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in C$ such that $(a,c) = 1$ and that $c \neq 0$.

Since N is divisible by 4, $\Gamma_0 = \Gamma_0(N)$ has no elliptic point. Therefore, by using the remark after the part of $\tilde{d}(n)$, we have $J(\beta) = (1/2)\chi(\beta)\zeta(\beta)^{-2k-1}(1 - \zeta(\beta)^{-4})^{-1}$ $[= J(C)]$ with

$\chi(\beta) = \chi(a)\left(\frac{c}{a}\right)$ and $\zeta(\beta) = \left(\frac{-1}{a}\right)^{-1/2} J(\beta, z_0)^{1/2} =$

$\left(\frac{-1}{a}\right)^{-1/2} (2\sqrt{n})^{-1} \left(\sqrt{2n + a + d} + \mathrm{sgn}(c)\sqrt{a + d - 2n} \right).$

Here, $z_0 \in H$ is the fixed point of β.

we put $w = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\Gamma_0^* = \Gamma_0 \cup \Gamma_0 w$ and $W(\beta) = |Z_{\Gamma_0^*}(\beta) : Z_{\Gamma_0}(\beta)|$, where $Z_{\Gamma_0^*}(\beta)$ (resp. $Z_{\Gamma_0}(\beta)$) is the centralizer of β in Γ_0^* (resp. Γ_0). We note that, in the elliptic case, the equivalence relation is the usual Γ_0-conjugacy relation and that any element of Γ_0^* acts on $\Gamma_0\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix}\Gamma_0$ by means of the inner automorphism. Then, it is easy to see $J(w\beta w) = \overline{J(\beta)}$ and, from the definition of $J(\beta)$, we have $J(-\beta) = J(\beta)$. Therefore, we have $\tilde{e}(n) = \Sigma_1 J(\beta) = \Sigma_1 J(w\beta w) = (1/2)\,\Sigma_1(J(\beta) + J(w\beta w)) = \Sigma_2(J(\beta) + J(w\beta w))W(\beta)^{-1},$

where β in the sum Σ_1 (resp. Σ_2) runs over all the representatives for the elliptic Γ_0 (resp. Γ_0^*) - conjugacy classes in Φ. Moreover, since $n\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0 \begin{pmatrix} 1 & 0 \\ 0 & n^2 \end{pmatrix} \Gamma_0$, we have $c \equiv 0 \pmod 4$ and $a \equiv d \equiv \pm 1 \pmod 4$. Hence, β is not Γ_0^*- conjugate to $-\beta$. Therefore, we have $\tilde{e}(n) = 2\, \Sigma_3 (J(\beta) + J(w\beta w)) W(\beta)^{-1}$, where β in the sum Σ_3 runs over all the representatives for the elliptic Γ_0^*- conjugacy classes in Φ which is congruent to $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ modulo 4.

Thus, we obtain $\tilde{e}(n) = \Sigma_3\ \chi(\beta)\Xi(\beta)W(\beta)^{-1}$, by using $|\zeta(\beta)| = 1$, where $\Xi(\beta) = (\zeta(\beta)^{-2k+1} - \zeta(\beta)^{2k-1})(\zeta(\beta)^2 - \zeta(\beta)^{-2})^{-1}$.

Now, we shall give all the representatives for the elliptic Γ_0^*- conjugacy classes in Φ which is congruent to $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ modulo 4, by using the method in [H] (or [D-M] chapter 6).

Let t be an integer such that $|t| < 2n$ and that $t \equiv 2 \pmod 4$. We write $t^2 - 4n^2 = m^2 u$, where m is a positive integer and u is a fundamental discriminant. Then, let f be a positive integer such that $f | m$ and that $(f,n) = 1$, and let ξ be a representative for $(Z / \prod_{p|N} p^{\tilde{\nu} + \rho} Z)$ which satisfies the conditions: $(\xi, Nn) = 1$, $\xi \equiv 1 \pmod 4$ and $F_t(\xi) \equiv 0 \pmod{Nf^2}$ with $\rho = \rho_p = \mathrm{ord}_p(f)$ and $F_t(X) = X^2 - tX + n^2$.

We put $S(\xi) = \left\{ \begin{array}{l} \text{the prime divisor } p \text{ of } N \text{ such that} \\ t^2 - 4n^2 \equiv 0 \pmod{p^{2\rho+1}} \text{ and that} \\ F_t(\xi) \equiv 0 \pmod{p^{\tilde{\nu}+2\rho+1}} \end{array} \right\}$

and let S be a subset of $S(\xi)$.

For these t, f, ξ and S, we define the matrix ϕ by

$\phi = \phi(t,f,\xi,S)$

$$= \begin{pmatrix} \xi & -f \prod_{p\in S} p^{-\tilde{\nu}} |F_t(\xi)f^{-2}|_p^{-1} \\ f^{-1}F_t(\xi) \prod_{p\in S} p^{\tilde{\nu}} |F_t(\xi)f^{-2}|_p & t - \xi \end{pmatrix}.$$

Moreover, we put $R = \begin{pmatrix} Z & Z \\ NZ & Z \end{pmatrix}$, $U_0 = \prod_q (R \otimes_Z Z_q)^\times \times GL_2^+(R)$,

where q runs over all prime numbers, $Q_A[\phi] = Q[\phi] \otimes_Q Q_A$ and $\Lambda = R \cap Q[\phi]$.

Then, we have the bijections:

$\Gamma_0 \backslash U_0 Q_A[\phi]^\times \cap GL_2(Q)/Q[\phi]^\times \simeq Q_A[\phi]^\times/((Q_A[\phi]^\times \cap U_0)Q[\phi]^\times)$

$\simeq$ the proper ideal class group of the order Λ

(cf. [H] or [D-M] chapter 6). Hence, we can choose a system $\{\delta\}$ of representatives for $\Gamma_0 \backslash U_0 Q_A[\phi]^\times \cap GL_2(Q)/Q[\phi]^\times$ as follows: For the double coset corresponding to the principal class by the above bijection, we take $\delta = 1$. In the other case, the corresponding proper ideal class contains a prime ideal P of Λ such that $(P, nN(t^2 - 4n^2)) = 1$ and that $\# \Lambda/P$ is a prime number p. Then, there are the elements $v_0 \in Q_A[\phi]^\times$ and $u_0 \in U_0$ such that $P = \Lambda v_0$ and that $u_0 v_0 \in GL_2(Q)$. If necessary, by multipling some element of Γ_0 from the left, we can see that $u_0 v_0 = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}$ with $0 \leqq j < p$. Then, we take $\delta = u_0 v_0$.

Now, when t, f, ξ, S and δ vary under the above conditions,

$\beta = n^{-1}\delta\phi\delta^{-1}$ forms a complete system of representatives for the elliptic Γ_0^*- conjugacy classes in Φ which is congruent to $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ modulo 4.

If we write the above $\beta = n^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $(a,c) = 1$ and $c > 0$ by using the above conditions and $F_t(\xi) > 0$. Hence, we have $\zeta(\beta) = (2\sqrt{n})^{-1}(\sqrt{t + 2n} + \sqrt{t - 2n})$. Therefore, $\zeta(\beta)$ and $\Xi(\beta)$ depend only on t, and so we write $\zeta(\beta) = \zeta_t$ and $\Xi(\beta) = \Xi_t$.

Next, since $Z_{\Gamma_0^*}(\beta) = \delta\Lambda^{\times}\delta^{-1}$, we have $W(\beta) = w((t^2-4n^2)f^{-2})$ and, by using the same method as in [N] p 196, we have

$\chi(n^{-1}\delta\phi\delta^{-1}) = \chi(\phi)\left(\frac{t + 2n}{p}\right)$ for $\delta = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}$.

Therefore, we obtain

$$\tilde{e}(n) = \sum_t \Xi_t \sum_f w((t^2-4n^2)f^{-2})^{-1} \sum_\xi \sum_S \chi(\phi) \times \sum_\delta \begin{cases} 1, & \text{if } \delta = 1, \\ \left(\frac{t + 2n}{p}\right), & \text{if } \delta = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}. \end{cases}$$

Since $\left(\frac{t + 2n}{}\right)$ can be considered as the genus character over the proper ideal class group of Λ (cf. [N] p 196-197), the last sum is equal to the class number $h((t^2-4n^2)f^{-2})$ or 0 according as $t + 2n$ is square or not.

Thus, we may assume that $t + 2n = s^2$ with $s > 0$, then, $t^2 - 4n^2 = s^2(s^2 - 4n)$ and $\zeta_t = (2\sqrt{n})^{-1}(s + \sqrt{s^2 - 4n})$. By using $|\zeta_t| = 1$, we have

$\Xi_t = -n^{-k+3/2}\, s^{-1}\, (x^{2k-1} - y^{2k-1})(x - y)^{-1}$, where x and y

are the solutions of $X^2 - sX + n = 0$.

Finally, we discuss $\sum_{\xi} \sum_{S} \chi(\phi) =$

$\sum_{\xi} \chi(\xi) \sum_{S} \left(\frac{f}{\xi}\right) \prod_{p \in S} \left(\frac{p^{\tilde{\nu}} |F_t(\xi)|_p}{\xi}\right)$. Since $(f, t - 2n, t + 2n)$

divides some power of 2, we have the following decomposition:

$f = 2^{\rho_2} f_1 f_2$, $(f_1 f_2, 2) = 1$, $f_1 > 0$, $f_2 > 0$, $f_1 | s$, $(f_1, s^2-4n) = 1$, $f_2^{\,2} | (s^2-4n)$ and $(f_2, s) = 1$. From $\xi \equiv 1 \pmod 4$ and the reciprocity law, $\left(\frac{f_i}{\xi}\right) = \left(\frac{\xi}{f_i}\right)$, $(i = 1,2)$. Put $t' = t/2$. Since $0 \equiv F_t(\xi) \equiv (\xi - t')^2 \pmod{f_1^{\,2} f_2^{\,2}}$. we have $\xi \equiv t' \pmod{f_1 f_2}$.

Hence, $\left(\frac{\xi}{f_1}\right) = \left(\frac{t'}{f_1}\right) = \left(\frac{2t}{f_1}\right) = \left(\frac{s^2-4n}{f_1}\right)$ and also $\left(\frac{\xi}{f_2}\right) = \left(\frac{t'}{f_2}\right)$

$\left(\frac{s^2}{f_2}\right) = 1$. Therefore, $\left(\frac{f}{\xi}\right) = \left(\frac{2}{\xi}\right)^{\rho_2} \left(\frac{s^2-4n}{f_1}\right)$.

Next, by using the same arguement in [N] p 198, we can prove that $\left(\frac{|F_t(\xi)|_p}{\xi}\right) = 1$ for $p \in S$ and that

$\left(\frac{p}{\xi}\right) = \left(\frac{\xi}{p}\right) = \left(\frac{2s^2-4n}{p}\right)$ for any odd prime $p \in S$. Therefore,

$$\sum_{\xi} \sum_{S} \chi(\phi) = \left(\frac{s^2-4n}{f_1}\right) \sum_{\xi} \chi(\xi) \left(\frac{2}{\xi}\right)^{\rho_2} \sum_{S} \prod_{p \in S} \left(\frac{p}{\xi}\right)^{\tilde{\nu}}$$

$$= \left(\frac{s^2-4n}{f_1}\right) \sum_{\xi} \chi(\xi) \left(\frac{2}{\xi}\right)^{\rho_2} \prod_{p \in S(\xi)} \left(1 + \left(\frac{p}{\xi}\right)^{\tilde{\nu}}\right)$$

$$= \left(\frac{s^2-4n}{f_1}\right) \sum_{\xi} \chi(\xi) \left(\frac{2}{\xi}\right)^{\rho_2} \prod_{p | N} \left(1 + \delta_p(\xi) \left(\frac{p}{\xi}\right)^{\tilde{\nu}}\right)$$

$$= \left(\frac{s^2-4n}{f_1}\right) \sum_{\xi} \chi_2(\xi) \left(\frac{2}{\xi}\right)^{\rho_2} \left(1 + \delta_2(\xi)\left(\frac{2}{\xi}\right)^{\mu}\right) \prod_{p | M} \chi_p(\xi) \left(1 + \delta_p(\xi)\left(\frac{2s^2-4n}{p}\right)^{\nu}\right).$$

Here, for the prime $p|N$ and the representative η of $(Z/p^{\tilde{\nu}+\rho}Z)$ such that $F_t(\eta) \equiv 0 \ (p^{\tilde{\nu}+2\rho})$, we define

$$\delta_p(\eta) = \begin{cases} 1, \text{ if } & s^2(s^2-4n) \equiv 0 \pmod{p^{2\rho+1}} \text{ and} \\ & F_t(\eta) \equiv 0 \pmod{p^{\tilde{\nu}+2\rho+1}}, \\ 0, \text{ otherwise.} \end{cases}$$

The quantity $\chi_p(\xi)(1 + \delta_p(\xi)\left(\frac{2s^2-4n}{p}\right)^{\nu})$ for the prime $p|M$ (resp. $\chi_2(\xi)\left(\frac{2}{\xi}\right)^{\rho_2}(1 + \delta_2(\xi)\left(\frac{2}{\xi}\right)^{\mu}))$ depends only on ξ modulo $p^{\nu+\rho}$ (resp. ξ modulo $2^{\mu+\rho}$). Therefore, we have

$$\sum_{\xi}\sum_{S}\chi(\phi) = \left(\frac{s^2-4n}{f_1}\right)\prod_{p|N} c_p(s,f). \quad \text{Here, we define}$$

$$c_p(s,f) = \begin{cases} \sum_{\eta}\chi_p(\eta)(1 + \delta_p(\eta)\left(\frac{2s^2-4n}{p}\right)^{\nu}), & p|M, \\ \sum_{\eta}\chi_2(\eta)\left(\frac{2}{\eta}\right)^{\rho}(1 + \delta_2(\eta)\left(\frac{2}{\eta}\right)^{\mu}), & p = 2, \end{cases}$$

where the above sums $\sum_{\eta}$ runs over all the representatives η of $(Z/p^{\tilde{\nu}+\rho}Z)$ such that $F_t(\eta) \equiv 0 \pmod{p^{\tilde{\nu}+2\rho}}$ and besides $\eta \equiv 1 \pmod 4$ in the case $p = 2$.

Combining all the above results, we obtain

$$\tilde{e}(n) = -n^{-k+3/2}\sum_{s} s^{-1}(x^{2k-1} - y^{2k-1})(x - y)^{-1}$$

$$\times \sum_{f} h'(s^2(s^2-4n)f^{-2})\left(\frac{s^2-4n}{f_1}\right)\prod_{p|N} c_p(s,f).$$

Here, the integer s runs over the even integer such that $0 < s < 2\sqrt{n}$, then we can write $s^2 - 4n = m_1{}^2 u$ with $m_1 > 0$. Put $\theta_p = \mathrm{ord}_p(sm_1)$ for the prime number $p|N$ and $s' = s \prod_{p|N} p^{-\mathrm{ord}_p(s)}$. Similarly, for f_1, f_2 and m_1, we define the number f_1', f_2' and m_1'. Moreover, we decompose $s' = r(s')u(s')$ and $m_1' = r(m_1')u(m_1')$ such that $(r(s'),s',m_1') = (r(m_1'),s',m_1') = 1$ and that $u(s')$ and $u(m_1')$ divide some power of (s',m_1').

Under these notation, we easily see that f_1' runs over the set $\{0 < f_1'|r(s')\ ,\ (f_1',u) = 1\}$ and that f_2' runs over the set $\{0 < f_2'|r(m_1')\}$. Hence, observing that $c_p(s,f)$ depends only on $\rho_p = \mathrm{ord}_p(f)$ if we fix s, we have

$$\sum_f h'(s^2(s^2-4n)f^{-2})\left(\frac{s^2 - 4n}{f_1}\right) \prod_{p|N} c_p(s,f)$$

$$= \sum_{\substack{0\leqq\rho_p\leqq\theta_p \\ p|N}} \ \sum_{\substack{0<f_1'|r(s') \\ (f_1',u)=1}} \ \sum_{0<f_2'|r(m_1')} h'\Big(\prod_{p|N} p^{2(\theta_p-\rho_p)}(s'm_1'/f_1'f_2')^2 u\Big)$$

$$\times\left(\frac{s^2-4n}{f_1}\right) \prod_{p|N} c_p(s,f)$$

$$= h'(u)\ u(s')\ u(m_1') \prod_{q|u(s')u(m_1')} \left(1 - \left(\frac{u}{q}\right) q^{-1}\right)$$

$$\times \sum_{\rho_2=0}^{\theta_2} 2^{\theta_2-\rho_2}\ c_2(s,f) \begin{cases} 1\ , \text{ if } \theta_2 = \rho_2 \\ 1 - \left(\frac{u}{2}\right) 2^{-1}\ , \text{ if } \theta_2 \neq \rho_2 \end{cases}$$

$$\times \prod_{p|M,\ p|s} \sum_{\rho_p=0}^{\theta_p} \left(\frac{u}{p}\right)^{\rho_p} p^{\theta_p-\rho_p}\ c_p(s,f) \begin{cases} 1, \text{ if } \theta_p = \rho_p \\ 1 - \left(\frac{u}{p}\right) p^{-1}, \text{ if } \theta_p \neq \rho_p \end{cases}$$

$$\times \prod_{p|M,p\nmid s} \sum_{\rho_p=0}^{\theta_p} p^{\theta_p-\rho_p} c_p(s,f) \begin{cases} 1, & \text{if } \theta_p = \rho_p \\ 1 - \left(\frac{u}{p}\right) p^{-1}, & \text{if } \theta_p \neq \rho_p \end{cases}$$

$$\times \sum_{\substack{0<f_1'|r(s') \\ (f_1',u)=1}} \left(\frac{u}{f_1'}\right)(r(s')/f_1') \prod_{q|(r(s')/f_1')} (1 - \left(\frac{u}{q}\right) q^{-1})$$

$$\times \sum_{0<f_2'|r(m_1')} (r(m_1')/f_2') \prod_{q|(r(m_1')/f_2')} (1 - \left(\frac{u}{q}\right) q^{-1}) \quad ,$$

where q denotes a prime number.

From elementary calculation, we see that the part of the sum $\sum_{\substack{0<f_1'|r(s') \\ (f_1',u)=1}}$ is equal to $r(s')$ and the part of the sum $\sum_{0<f_2'|r(m_1')}$ is equal to $\alpha_u(r(m_1'))$. Moreover, the part of the sum $\sum_{\rho_p=0}^{\theta_p}$ depends only on θ_p and so we denote this part by $n_p(\theta_p)$. Hence, observing that $q|u(s')u(m_1')$ if and only if $q|(s',m_1')$ and that $s^{-1}u(s')r(s') = \prod_{p|N} p^{-\operatorname{ord}_p(s)}$, we have

$$\tilde{e}(n) = -n^{-k+3/2} \sum_s (x^{2k-1} - y^{2k-1})(x-y)^{-1} h'(u)\, u(m_1')\alpha_u(r(m_1'))$$

$$\times \prod_{q|(s',m_1')} (1 - \left(\frac{u}{q}\right) q^{-1}) \prod_{p|N} p^{-\operatorname{ord}_p(s)} n_p(\theta_p).$$

Therefore, the contribution to the trace of the n-th Hecke operator is $\sum_{0<a|n_0} a\, n^{k-3/2}\, \tilde{e}(n/a^2) \quad =$

$$- \sum_{0<a|n_0} \sum_{s} a^{2k-2}(x^{2k-1} - y^{2k-1})(x - y)^{-1} h'(u)u(m_1')\alpha_u(r(m_1'))$$

$$\times \prod_{q|(s',m_1')} \left(1 - \left(\frac{u}{q}\right)q^{-1}\right) \prod_{p|N} p^{-\mathrm{ord}_p(s)} n_p(\theta_p).$$

Here, we understand that $s,x,y,$etc. are defined such as above, when we replace n with n/a^2.

Now, we put $\tilde{s} = as$, $\tilde{m}_1 = am_1$, $\tilde{x} = ax$, and $\tilde{y} = ay$. Then, we have that $\tilde{x}$ and $\tilde{y}$ are the solutions of $X^2 - \tilde{s}X + n = 0$ and that $\tilde{s}^2 - 4n = \tilde{m}_1^{\,2}u$. Since $(a,N) = 1$, $\mathrm{ord}_p(\tilde{s}\tilde{m}_1) = \theta_p$. Hence, $\sum_{0<a|n_0} a\, n^{k-3/2}\, \tilde{e}(n/a^2)$

$$= - \sum_{0<a|n_0} \sum_{\substack{\tilde{s}:\text{even} \\ 2\sqrt{n} > \tilde{s} > 0 \\ a|\tilde{s}}} (\tilde{x}^{2k-1} - \tilde{y}^{2k-1})(\tilde{x} - \tilde{y})^{-1} h'(u)$$

$$\times \prod_{p|N} p^{-\mathrm{ord}_p(\tilde{s})} n_p(\theta_p) \times u(m_1')\alpha_u(r(m_1'))$$

$$\times \prod_{q|(s',m_1')} \left(1 - \left(\frac{u}{q}\right) q^{-1}\right)$$

$$= - \sum_{\substack{\tilde{s}:\text{even} \\ 2\sqrt{n} > \tilde{s} > 0}} (\tilde{x}^{2k-1} - \tilde{y}^{2k-1})(\tilde{x} - \tilde{y})^{-1} h'(u) \prod_{p|N} p^{-\mathrm{ord}_p(\tilde{s})} n_p(\theta_p)$$

$$\times \sum_{\substack{0<a|n_0 \\ a|\tilde{s}}} u(m_1')\alpha_u(r(m_1')) \prod_{q|(s',m_1')} \left(1 - \left(\frac{u}{q}\right) q^{-1}\right).$$

We observe that an integer a divides the odd part of $(\tilde{s}, \tilde{m}_1)$ if and only if $a|n_0$ and $a|\tilde{s}$. Hence, from elementary calculation, we have $\sum_{\substack{0<a|n_0 \\ a|\tilde{s}}} u(m_1')\alpha_u(r(m_1')) \prod_{q|(s',m_1')} \left(1 - \left(\frac{u}{q}\right) q^{-1}\right)$

$= \alpha_u(\tilde{m}_1 \prod_{p|N} p^{-\mathrm{ord}_p(\tilde{m}_1)})$. Therefore, rewriting the notations:

$\tilde{s} \to s$, $\tilde{m}_1 \to t$, $\tilde{x} \to x$, $\tilde{y} \to y$. we obtain the assertion of Propositions. The determination of the constant $n_p(\theta_p)$ needs the elementary but very long calculation. So, we omit it.

5. The part of $\tilde{h}(n)$.

Let C be a hyperbolic equivalence class in Φ and take $\beta = n^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in C$ such that $(a,c) = 1$ and that $c \neq 0$. Let κ be the upper fixed point of β which is a cusp of Γ_0 and $\lambda(\beta) = (a - c\kappa)/n$. Since, $\kappa = (2c)^{-1}\{a - d - \mathrm{sgn}(a+d) \times \sqrt{(a+d)^2 - 4n^2}\}$, we have that $\mathrm{sgn}(\lambda(\beta)) = \mathrm{sgn}(a+d)$ and that $\lambda(\beta) = (2n)^{-1}\{a + d + \mathrm{sgn}(a+d)\sqrt{(a+d)^2 - 4n^2}\}$. Moreover, by using the remark after the part of $\tilde{d}(n)$, we have $J(\beta) =$

$$J(C) = -(1/2)\left(\frac{\mathrm{sgn}(a+d)}{\mathrm{sgn}(c)}\right) \chi(a)\left(\frac{-1}{a}\right)^{k+1/2}\left(\frac{c}{a}\right)\lambda(\beta)^{-k-1/2}(1 - \lambda(\beta)^{-2})^{-1}.$$

Hence, we have $J(-\beta) = J(\beta)$ and $J(w\beta w) = \left(\frac{-1}{a}\right)\mathrm{sgn}(a+d)J(\beta)$, where $w = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let Γ_0^* and $W(\beta)$ be the same as in the elliptic case. Then, we have $\tilde{h}(n) = \Sigma_1 J(\beta) = \Sigma_1 J(w\beta w) = (1/2)\Sigma_1(J(\beta) + J(w\beta w))$
$= (1/2)\Sigma_2(J(\beta) + J(w\beta w)) = (1/2)\Sigma_3(J(\beta) + J(-\beta) + J(w\beta w) + J(-w\beta w))$
$= \Sigma_3(J(\beta) + J(w\beta w)) = 2\Sigma_4(J(\beta) + J(w\beta w))W(\beta)^{-1} = 4\Sigma_4 J(\beta)W(\beta)^{-1}$,
where β in Σ_1 runs over all the representatives for the hyperbolic Γ_0- conjugacy classes in Φ, β in Σ_2 runs over those such that $\left(\frac{-1}{a}\right)\mathrm{sgn}(a+d) = 1$, β in Σ_3 runs over those such that $a + d > 0$ and that $a \equiv 1 \pmod 4$, and β in Σ_4 runs over those, such that $a + d > 0$ and that $a \equiv 1 \pmod 4$, for the hyperbolic Γ_0^*-conjugacy classes in Φ.

Thus, we obtain

$\tilde{h}(n) = -2\ \Sigma_4 \chi(\beta) W(\beta)^{-1} \lambda(\beta)^{-k-1/2}(1 - \lambda(\beta)^{-2})^{-1}$ with $\chi(\beta) = \chi(a)\left(\frac{c}{a}\right)$.

Let t be the integer such that $t \equiv 2 \pmod 4$ and that $t > 2n$ and that $t^2 - 4n^2$ is square. Then, we write $t^2 - 4n^2 = m^2$ with $m > 0$. For these t and m, let f, ξ, S and ϕ are the same as in the elliptic case. But, for simplifying the calculation, we assume the additional condition: $\xi \neq 1$.

Next, let $\Lambda = Q[\phi] \cap R$ with $R = \begin{pmatrix} Z & Z \\ NZ & Z \end{pmatrix}$. Then we know that $Q_A[\phi]^\times / ((\prod_p (\Lambda \otimes_Z Z_p)^\times \times R[\phi]^\times) Q[\phi]^\times)$ is isomorphic to the proper ideal class group of the order Λ. Then, for the principal ideal class, we set $\delta = 1$. In the other case, the proper ideal class contains the prime ideal P such that $\# \Lambda/P$ is a prime number p and that $(p, nN(t^2 - 4n^2)) = 1$. And so, there exist $v_0 \in Q_A[\phi]^\times$ and $u_0 \in U_0$ such that $\Lambda v_0 = P$ and that $GL_2(Q) \ni u_0 v_0 = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}$ with $0 \leqq j < p$. Then, we set $\delta = u_0 v_0$. Here, U_0 is the same as in the elliptic case.

When these t, f, ξ, S and $\{\delta\}$ vary under the above conditions, $\beta = n^{-1}\delta\phi\delta^{-1} = n^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ forms a complete system of all the representatives for the hyperbolic Γ_0^*-conjugacy classes in Φ, such that $a + d > 0$ and that $a \equiv 1 \pmod 4$.

Therefore, we have

$$\tilde{h}(n) = -2 \sum_{t} \sum_{f} \sum_{\xi} \sum_{S} \sum_{\delta} \chi(n^{-1}\delta\phi\delta^{-1})W(n^{-1}\delta\phi\delta^{-1})^{-1}\lambda(\phi)^{-k-1/2}$$

$$\times (1 - \lambda(\phi)^{-2})^{-1}.$$

Moreover, in the same way as in the elliptic case, we may assume that $t + 2n$ is square . Then we write $t + 2n = s^2$ with $s > 0$.

Hence, we have

$$\tilde{h}(n) = - \sum_{t} \sum_{f} \sum_{\xi} \sum_{S} \lambda(\phi)^{-k-1/2} (1 - \lambda(\phi)^{-2})^{-1} \phi(m/f) \chi(\phi)$$

$$= - n^{-k+3/2} \sum_{s} s^{-1} y^{2k-1}(x - y)^{-1} \sum_{f} \phi(s(s^2 - 4n)^{1/2}f^{-1})$$

$$\times \prod_{p|N} c_p(s,f) ,$$

where x and y are the solutions of $X^2 - sX + n = 0$ such that $x > y$, and the constant $c_p(s,f)$ are the same as in the elliptic case.

We can deduce the assertion of Propositions from the above formula $\tilde{h}(n)$, by using the same method as in the elliptic case.

§2. The trace formula for the Hecke operator over the Kohnen subspace.

Throughout this section, we shall use the same notations and assumptions as in §0 (a) and (d), and moreover, we suppose that n is the integer such that $(n,N) = 1$.

Let $Pr = Pr_{k+1/2,N,\chi} = (\alpha - \beta)^{-1}(Q_{k+1/2,N,\chi} - \beta)$ be the orthogonal projection from $S(k+1/2,N,\chi)$ onto $S(k+1/2,N,\chi)_K$. Then, we have $\mathrm{trace}(\tilde{T}_{k+1/2,N,\chi}(n^2)_K \mid S(k+1/2,N,\chi)_K)$

$$= \mathrm{trace}(\tilde{T}_{k+1/2,N,\chi}(n^2)\ Pr \mid S(k+1/2,N,\chi))$$

$$= (\sqrt{2}/6)(-1)^{[(k+1)/2]}\varepsilon\ \mathrm{trace}(\tilde{T}_{k+1/2,N,\chi}(n^2)\ Q)$$

$$+ (1/3)\ \mathrm{trace}(\tilde{T}_{k+1/2,N,\chi}(n^2)).$$

Write $\Delta_0 = \Delta_0(N,\chi)_{k+1/2}$ and $n = n_0^2 n_1$ with a positive integer n_0 and a positive square-free integer n_1. Then,

$$\tilde{T}_{k+1/2,N,\chi}(n^2)\ Q = \sum_{0<a|n_0} a\ n^{k-3/2}[\Delta_0\left(\begin{pmatrix}(n/a^2)^{-1} & 0\\ 0 & (n/a^2)\end{pmatrix},\ (n/a^2)^{k+1/2}\right)\Delta_0]\ Q.$$

Hence, from the results of §1, it is sufficient to calculate the trace of the operator $[\Delta_0\left(\begin{pmatrix}n^{-1} & 0\\ 0 & n\end{pmatrix},\ n^{k+1/2}\right)\Delta_0]\ Q$ for the calculation of $\mathrm{trace}(\tilde{T}_{k+1/2,N,\chi}(n^2)_K)$.

We can prove the following lemma by modufying the proof of lemma 1 in [K] ,§4.

Lemma. We have, as the elements of the abstract Hecke algebra,

$$\Delta_0(N,\chi)\left(\begin{pmatrix}1 & 0\\ 0 & n^2\end{pmatrix}, n^{k+1/2}\right)\Delta_0(N,\chi)\cdot \Delta_0(N,\chi)\xi_{k+1/2,\varepsilon}\Delta_0(N,\chi)$$

$$= \Delta_0(N,\chi)\left(\begin{pmatrix}4 & 1\\ 0 & 4n^2\end{pmatrix}, \varepsilon^{k+1/2}e((2k+1)/8)n^{k+1/2}\right)\Delta_0(N,\chi),$$

for any positive integer n with $(n,N) = 1$.

From this lemma, we have $[\Delta_0\left(\begin{pmatrix}n^{-1} & 0\\ 0 & n\end{pmatrix}, n^{k+1/2}\right)\Delta_0]\ Q$

$$= \varepsilon^{-k-1/2}e(-(2k+1)/8)[\Delta_0\left(\begin{pmatrix}n^{-1} & (4n)^{-1}\\ 0 & n\end{pmatrix}, n^{k+1/2}\right)\Delta_0].$$

For simplicity, we write $\Gamma_0 = \Gamma_0(N)$ and $\alpha = \begin{pmatrix}n^{-1} & (4n)^{-1}\\ 0 & n\end{pmatrix}$.

Let L be the same proper lifting as in §1. Then, it is easily shown $\tau = (\alpha, n^{k+1/2}) \in G(k+1/2)$ satisfies the conditions (1.1) and (1.2) with respect to L. Hence, we have the bijection: $\Gamma_0\alpha\Gamma_0 \ni \sigma_1\alpha\sigma_2 \to L(\sigma_1)\tau L(\sigma_2) \in \Delta_0\tau\Delta_0$, and denote by β^* the image of $\Gamma_0\alpha\Gamma_0 \ni \beta$.

Now, we shall calculate $\mathrm{trace}([\Delta_0\ (\alpha, n^{k+1/2})\ \Delta_0])$ by using Shimura's trace formula (1.3) in §1. Since, $\Gamma_0\alpha\Gamma_0$ has no scalar element, the contribution to the trace from the scalar elements is zero. Let $\tilde{p}_0(n)$, $\tilde{e}_0(n)$ and $\tilde{h}_0(n)$ be the contribution from the parabolic, elliptic and hyperbolic equivalence classes in $\Phi = \Phi(\Gamma_0\alpha\Gamma_0/\Gamma_0)$ respectively.

Moreover, when $k = 1$, we put

$$\tilde{d}_0(n) = \mathrm{trace}([\Delta_0'\tau'\Delta_0']_{1/2} \mid G(1/2,N,\chi))$$

$$- \mathrm{trace}([\Delta_0\tau\Delta_0]_{3/2} \mid U(N,\chi)),$$

where $\tau' = (\alpha^{-1}, n^{-1/2})$ and $\Delta_0' = \Delta_0(N,\chi)_{1/2}$

For calculating $\tilde{p}_0(n)$, $\tilde{e}_0(n)$ and $\tilde{h}_0(n)$, we give some remarks.

Remark 1. We have $\Gamma_0 \begin{pmatrix} 4 & 1 \\ 0 & 4n^2 \end{pmatrix} \Gamma_0$

$$= \left\{ M_2(Z) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \begin{array}{l} ad - bc = 16n^2, \ c \equiv 0 \pmod{16M}, \\ a \equiv d \equiv 0 \pmod 4, \ (a,M) = 1 \\ \text{and } (a,b,c,d) = 1. \end{array} \right\}$$

(cf. [K] , §4, lemma 2).

Remark 2. Let $\beta = (4n)^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0 \alpha \Gamma_0$. If $(b,d) = 1$, then we have

$$\beta^* = \left(\beta, \left(\frac{\operatorname{sgn}(d)}{-\operatorname{sgn}(c)}\right)\left(\frac{d}{b}\right)\left(\frac{-1}{b}\right)^{-k-1/2}\left(\frac{\varepsilon}{b}\right)\left(\frac{a}{M_0}\right)((cz+d)/4n)^{k+1/2}\right).$$

where M_0 is the square-free positive divisor of M such that $\chi = \left(\frac{M_0}{\ }\right)$, and $\operatorname{sgn}(x) = 1$, or -1, according as $x \geqq 0$, or $x < 0$. We can prove this assertion by slight modifying the proof of [K] , §4, lemma 3.

Remark 3. For an elliptic or hyperbolic matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(Z)$. Put $t = a + d$ and $f = (a - d, b, c)$. Then, t and f (and also the signature of c for elliptic A) are invariant under the $SL_2(Z)$-conjugation.

Moreover, every elliptic or hyperbolic Γ_0-conjugacy class in $\Gamma_0\alpha\Gamma_0$ contains an element $(4n)^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $d > 0$, $(b,d) = 1$ and $(b/f,\ (t^2 - 64n^2)/f^2) = 1$, where $t = a + d$ and $f = (a - d,\ b,\ c)$ (cf. [K] ,§4,lemma 4).

Remark 4. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(Z)$ be the elliptic or hyperbolic matrix such that $ad - bc = 16n^2$, $t = a + d \equiv 0 \pmod 4$ and $(a,b,c,d) = 1$ and that $f = (a - d,b,c)$ is an odd integer. For this A, we define the set D(A) by
$D(A) = \{SL_2(Z) \ni B \mid (4n)^{-1}B^{-1}AB \in \Gamma_0\alpha\Gamma_0\}$. Then, Γ_0 operates on D(A) by multiplication from the right and we have
$\# D(A)/\Gamma_0 = \prod_{p|M} \tilde{c}_p(t,f)$, where

$$\tilde{c}_p(t,f) = \begin{cases} p^{\nu-1}(p+1), & \text{if } \rho \geqq \nu, \\ p^{\rho}\times\#\{(Z/p^{\nu}Z) \ni x \mid x^2 - (t/4)x + n^2 \equiv 0 \pmod{p^{\nu+\rho}}\}, & \\ & \text{if } \rho < \nu, \end{cases}$$

and $\nu = \nu_p = \mathrm{ord}_p(N)$, $\rho = \rho_p = \mathrm{ord}_p(f)$.

The proof of this assertion will be given at Appendix.

1. The calculation of $\tilde{p}_0(n)$.

By using the remark 1 and the same arguement as in §1, we can write out all the parabolic equivalence classes in Φ. The result is as follows:

Put $\tilde{S} = \left\{ \begin{array}{l} t = 4\,\zeta \prod_{p|M} p^{e} > 0\ ;\ 0 \leqq e = e_p \leqq \mathrm{ord}_p(M) = \nu = \nu_p \\ \\ \text{and } \zeta \text{ runs over a system of representatives, such} \\ \text{that } (\zeta,N) = 1, \text{ for } (Z/\prod_{p|M} p^{\min(e,\nu-e)}Z)^{\times} \end{array} \right\}$.

For $t = 4\zeta \prod_{p|M} p^e \in \tilde{S}$, we write $u = \prod'_{p|M} p^{\nu-2e}$ and

$\tilde{\beta}(t,x) = (4n)^{-1}\begin{pmatrix} 4n - txu & xu \\ -t^2xu & 4n + txu \end{pmatrix}$, where the symbol $\prod'_{p|M}$

means the same as in §1. Then, the system of representatives of all the parabolic equivalence classes in Φ is formed by the matrices $\tilde{\beta}(t,x)$, where $t = 4\zeta \prod_{p|M} p^e$ runs over the set $\tilde{S}$ and x runs over the system of representatives for $(Z/4nZ)^{\times}$. Here, by the suitable choice of the representative, we may assume that $x > 0$.

Now, we shall determine the number $J(\beta) = J(C)$ for the equivalence class C containing the matrix $\beta = \tilde{\beta}(t,x)$. The stabilizer of t^{-1}, which is the fixed point of β, in $\Gamma_0/\{\pm 1\}$ is generated by $\sigma = \begin{pmatrix} 1 - ut & u \\ -ut^2 & 1 + ut \end{pmatrix}$.

Put $A_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ and $\rho = A_t \begin{pmatrix} \sqrt{u} & 0 \\ 0 & \sqrt{u}^{-1} \end{pmatrix}$. Then, we have

$\rho^{-1}\sigma\rho = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and, since $\beta = 1 + (x/4n)(\sigma - 1)$,

$\rho^{-1}\beta\rho = \begin{pmatrix} 1 & x/4n \\ 0 & 1 \end{pmatrix}$.

Let $\rho^* = (A_t,\ j(A_t,z)^{2k+1})\left(\begin{pmatrix} \sqrt{u} & 0 \\ 0 & \sqrt{u}^{-1} \end{pmatrix},\ u^{-k/2-1/4}\right)$, then,

since $t \equiv 0 \pmod 4$ and $tu \equiv 0 \ (\mathrm{mod} \prod_{p|M} p)$, we have

$$\rho^{*-1}L(\sigma)\rho^* = \rho^{*-1}(A_t,\ j(A_t,z)^{2k+1})\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix},\ 1\right)(A_t^{-1},\ j(A_t^{-1},z)^{2k+1})\rho^*$$

$= \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right)$. Moreover, we note that $(xu, 4n + txu) =$

$(xu, 4n) = 1$. Hence, by using the remark 2, we have

$$\beta^* = \left(\beta, \left(\frac{\varepsilon n}{xu}\right)\left(\frac{n}{M_0}\right)\left(\frac{-1}{xu}\right)^{-k-1/2} ((cz + d)/4n)^{k+1/2}\right) \quad \text{with}$$

$\beta = (4n)^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Here, we used the assumption $x > 0$.

Therefore, we have

$$\rho^{*-1}\beta^*\rho^* = \left(\begin{pmatrix} 1 & x/4n \\ 0 & 1 \end{pmatrix}, \left(\frac{\varepsilon n}{xu}\right)\left(\frac{n}{M_0}\right)\left(\frac{-1}{xu}\right)^{-k-1/2}\right).$$

Finally, since $\alpha \notin \Gamma_0$, we have always $\beta \notin \Gamma_0$. Thus, we obtain $J(\beta) = \left(\frac{\varepsilon n}{xu}\right)\left(\frac{n}{M_0}\right)\left(\frac{-1}{xu}\right)^{k+1/2} e(x/4n)(1 - e(x/4n))^{-1}$.

Since $e(x/4n)(1 - e(x/4n))^{-1} = -(1/2) + (\sqrt{-1}/2)\cot(\pi x/4n)$, we have $\tilde{p}_0(n) = \tilde{p}_1(n) + \tilde{p}_2(n)$ with

$$\tilde{p}_1(n) = -(1/2)\left(\frac{n}{M_0}\right) \sum_{\substack{0 \leq e_p \leq \nu_p \\ p \mid M}} \sum_{\zeta} \left(\frac{\varepsilon n}{u}\right) \sum_{x} \left(\frac{\varepsilon n}{x}\right)\left(\frac{-1}{xu}\right)^{k+1/2}$$

and $\tilde{p}_2(n) = (\sqrt{-1}/2)\left(\frac{n}{M_0}\right) \sum_{\substack{0 \leq e_p \leq \nu_p \\ p \mid M}} \sum_{\zeta} \left(\frac{\varepsilon n}{u}\right) \sum_{x} \left(\frac{\varepsilon n}{x}\right)\left(\frac{-1}{xu}\right)^{k+1/2} \cot(\pi x/4n)$.

Put $A_0 = \sum_{x} \left(\frac{\varepsilon n}{x}\right)\left(\frac{-1}{xu}\right)^{k+1/2}$ and $B_0 = \sum_{x} \left(\frac{\varepsilon n}{x}\right)\left(\frac{-1}{xu}\right)^{k+1/2} \cot(\pi x/4n)$,

where x in the sum $\sum_{x}$ runs over any system of all representatives for $(Z/4nZ)^{\times}$.

Then, we have $A_0 = (1/2) \sum_x \{ \left(\frac{\varepsilon n}{x}\right)\left(\frac{-1}{xu}\right)^{k+1/2} + \left(\frac{\varepsilon n}{-x}\right)\left(\frac{-1}{-xu}\right)^{k+1/2} \}$

$$= (1/2) \sum_x \left(\frac{\varepsilon n}{x}\right)\{ \left(\frac{-1}{xu}\right)^{k+1/2} + \varepsilon\left(\frac{-1}{-xu}\right)^{k+1/2} \}$$

$$= (1/2) \sum_x \left(\frac{\varepsilon n}{x}\right)\left(\frac{\varepsilon}{xu}\right) (1 + \varepsilon(-1)^k\sqrt{-1}) = ((1 + \varepsilon(-1)^k\sqrt{-1})/2)\left(\frac{\varepsilon}{u}\right) \sum_x \left(\frac{n}{x}\right)$$

$= (1 + \varepsilon(-1)^k\sqrt{-1})\left(\frac{\varepsilon}{u}\right) \delta_0(\sqrt{n})\phi(n)$, where $\delta_0(\sqrt{n})$ is the same as in §1. Similarly, by using Dirichlet's class number formula, we have

$$B_0 = (1/2) \sum_x \{ \left(\frac{\varepsilon n}{x}\right)\left(\frac{-1}{xu}\right)^{k+1/2} \cot(\pi x/4n) + \left(\frac{\varepsilon n}{-x}\right)\left(\frac{-1}{-xu}\right)^{k+1/2} \cot(-\pi x/4n)\}$$

$$= (1/2) \sum_x \left(\frac{\varepsilon n}{x}\right) \cot(\pi x/4n)\{ \left(\frac{-1}{xu}\right)^{k+1/2} - \varepsilon\left(\frac{-1}{-xu}\right)^{k+1/2} \}$$

$$= (1/2) \sum_x \left(\frac{\varepsilon n}{x}\right) \cot(\pi x/4n)\left(\frac{-\varepsilon}{xu}\right) (1 - \varepsilon(-1)^k\sqrt{-1})$$

$$= 2(1 - \varepsilon(-1)^k\sqrt{-1})\left(\frac{-\varepsilon}{u}\right)\sqrt{n}\, h'(-4n).$$

Hence, $\tilde{p}_1(n) = -(1/2)\left(\frac{n}{M_0}\right) \sum_{\substack{0 \leq e_p \leq \nu_p \\ p|M}} \sum_\zeta \left(\frac{\varepsilon n}{u}\right)(1 + \varepsilon(-1)^k\sqrt{-1})\left(\frac{\varepsilon}{u}\right)\delta_0(\sqrt{n})\phi(n)$

$$= -((1 + \varepsilon(-1)^k\sqrt{-1})/2)\delta_0(\sqrt{n})\phi(n) \sum_{\substack{0 \leq e_p \leq \nu_p \\ p|M}} \sum_\zeta 1$$

$$= -((1 + \varepsilon(-1)^k\sqrt{-1})/2)\delta_0(\sqrt{n})\phi(n) \prod_{p|M} (p^{[\nu/2]} + p^{[(\nu-1)/2]}),$$

and $\tilde{p}_2(n) = (\sqrt{-1}/2)\left(\frac{n}{M_0}\right) \sum_{\substack{0 \leq e_p \leq \nu_p \\ p|M}} \sum_\zeta \left(\frac{\varepsilon n}{u}\right)$

$$\times\ 2(1 - \varepsilon(-1)^k\sqrt{-1})\left(\frac{-\varepsilon}{u}\right)\sqrt{n}\, h'(-4n)$$

$$= (\sqrt{-1} + \varepsilon(-1)^k)\left(\frac{n}{M_0}\right)\sqrt{n}\, h'(-4n) \sum_{\substack{0 \leq e_p \leq \nu_p \\ p|M}} \sum_{\zeta} \prod_{p|M}{}' \left(\frac{-n}{p}\right)^{\nu}$$

$$= (\sqrt{-1} + \varepsilon(-1)^k)\left(\frac{n}{M_0}\right)\sqrt{n}\, h'(-4n) \prod_{p|M} \left(p^{[\nu/2]} + \left(\frac{-n}{p}\right)^{\nu} p^{[(\nu-1)/2]}\right).$$

Therefore, in the same way as in §1, we have

$$\sum_{0<a|n_0} a\, n^{k-3/2}\, \tilde{p}_0(n/a^2)$$

$$= -\,((1 + \varepsilon(-1)^k\sqrt{-1})/2)\ \delta_0(\sqrt{n})\ n^{k-1/2} \prod_{p|M} (p^{[\nu/2]} + p^{[(\nu-1)/2]})$$

$$+ (\sqrt{-1} + \varepsilon(-1)^k)\left(\frac{n}{M_0}\right) n^{k-1} \prod_{p|M} \left(p^{[\nu/2]} + \left(\frac{-n}{p}\right)^{\nu} p^{[(\nu-1)/2]}\right)$$

$$\times \sum_{0<a|n_0} h'(-4n/a^2).$$

2. The calculation of $\tilde{e}_0(n)$.

Let C be an elliptic equivalence class in Φ. By using remark 3, we can take $\beta = (4n)^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in C$ with $d \neq 0$, $(b,d) = 1$ and $(b/f,\ (t^2 - 64n^2)/f^2) = 1$, where $t = a + d$ and $f = (a - d,b,c)$. We note that $|t| < 8n$ because C is elliptic. From remark 2, we have $J(\beta) = J(C) = (1/2)\chi(\beta)\zeta(\beta)^{-2k-1}(1 - \zeta(\beta)^{-4})^{-1}$ with $\chi(\beta) = \left(\frac{a}{M_0}\right)\left(\frac{\varepsilon}{b}\right)\left(\frac{d}{b}\right)$ and $\zeta(\beta) = \left(\frac{-1}{b}\right)^{-1/2}((cz_0 + d)/4n)^{1/2}$ $= \left(\frac{-1}{b}\right)^{-1/2}(4\sqrt{n})^{-1}(\sqrt{t + 8n} + \mathrm{sgn}(c)\sqrt{t - 8n}\ )$. Here, $z_0 \in H$ is the fixed point of β.

Put $w = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Then, since $w\beta w = (4n)^{-1}\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$,

we have $\chi(w\beta w) = \varepsilon\chi(\beta)$, $\zeta(w\beta w) = -\sqrt{-1}\,\zeta(\beta)^{-1}$ and $J(w\beta w) = ((\varepsilon(-1)^k\sqrt{-1})/2)\chi(\beta)\zeta(\beta)^{2k+1}(1 - \zeta(\beta)^4)^{-1}$. Hence,

$$\tilde{e}_0(n) = \Sigma_1\, J(\beta) = \Sigma_2(J(\beta) + J(w\beta w)) = (1/2)\,\Sigma_2\,\chi(\beta) \times$$
$$(\zeta(\beta)^{-2k+1} - \varepsilon(-1)^k\sqrt{-1}\,\zeta(\beta)^{2k-1})(\zeta(\beta)^2 - \zeta(\beta)^{-2})^{-1},$$

where β in Σ_1 runs over all the representatives for the elliptic Γ_0-conjugacy classes in Φ and β in Σ_2 runs over those such that $c > 0$.

Put $\lambda(t) = (\sqrt{t + 8n} + \sqrt{t - 8n}\,)/2$ and $p(t) = (\lambda(t)^{-2k+1} - \overline{\lambda(t)}^{\,-2k+1})(\lambda(t) - \overline{\lambda(t)}\,)^{-1}$, then $\lambda(-t) = \sqrt{-1}\,\overline{\lambda(t)}$ and $p(-t) = (-1)^k(\lambda(t)^{-2k+1} + \overline{\lambda(t)}^{\,-2k+1})(\lambda(t) + \overline{\lambda(t)}\,)^{-1}$.
Then, for β in Σ_2, we have $\zeta(\beta) = \left(\frac{-1}{b}\right)^{-1/2}(2\sqrt{n})^{-1}\lambda(t)$.
Hence, from $|\zeta(\beta)| = 1$, we can rewrite $\tilde{e}_0(n)$ as follows:

$$\tilde{e}_0(n) = 2^{2k}n^{k+1/2}\ \Sigma_2\chi(\beta)\left(\frac{-1}{b}\right)^{k+1/2}(\lambda(t)^2 - \overline{\lambda(t)}^{\,2})^{-1}$$
$$\times\ (\lambda(t)^{-2k+1} - \varepsilon(-1)^k\sqrt{-1}\left(\frac{-1}{b}\right)\overline{\lambda(t)}^{\,-2k+1})$$
$$= 2^{2k-1}n^{k+1/2}\ \Sigma_2\chi(\beta)\ \{\left(\frac{-1}{b}\right)^{k+1/2}(1 + \varepsilon(-1)^k\sqrt{-1}\left(\frac{-1}{b}\right))\ p(t)(t + 8n)^{-1/2}$$
$$+\ (-1)^k\left(\frac{-1}{b}\right)^{k+1/2}(1 - \varepsilon(-1)^k\sqrt{-1}\left(\frac{-1}{b}\right))p(-t)(t - 8n)^{-1/2}\}$$
$$= 2^{2k-1}n^{k+1/2}\ \Sigma_2\chi(\beta)\ \{\left(\frac{\varepsilon}{b}\right)(1 + \varepsilon(-1)^k\sqrt{-1})p(t)(t + 8n)^{-1/2}$$
$$+\ (-1)^k\left(\frac{-\varepsilon}{b}\right)(1 - \varepsilon(-1)^k\sqrt{-1})p(-t)(t - 8n)^{-1/2}\}$$

Moreover, by using the correspondence: $\beta \to -w\beta w$, we have
$$\Sigma_2\chi(\beta)\left(\frac{-\varepsilon}{b}\right)p(-t)(t - 8n)^{-1/2} = \sqrt{-1}\,\varepsilon\ \Sigma_2\chi(\beta)\left(\frac{\varepsilon}{b}\right)p(t)(t + 8n)^{-1/2}.$$

Therefore, $\tilde{e}_0(n) = 2^{2k} n^{k+1/2}(1 + \varepsilon(-1)^k\sqrt{-1})$

$$\times \Sigma_2 \; \mathrm{sgn}(d)\left(\frac{a}{M_0}\right)\left(\frac{d}{b}\right) p(t)(t + 8n)^{-1/2}.$$

Now, we shall give a system of all the representatives for the elliptic Γ_0-conjugacy classes in Φ, such that $c > 0$.

Let t be an integer such that $|t| < 8n$ and that $t \equiv 0 \pmod 4$. Then, we write $t^2 - 64n^2 = m^2 u$ with a fundamental discriminant u and a positive integer m. Let f be a positive integer such that $f^2 \mid (t^2 - 64n^2)$ and that $(f,2n) = 1$. We put

$$B(t,f) = \left\{ M_2(Z) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{l} a + d = t, \; (a - d,b,c) = f, \\ (a,b,c,d) = 1, \; ad - bc = 16n^2, \\ c > 0. \end{array} \right\}$$

and, for $A \in B(t,f)$, we define the set $D(A)$ in the same way as in remark 4. Then, $SL_2(Z)$ operates on $B(t,f)$ by the inner automorphism in $GL_2(R)$. By $Z(A) = Z_{SL_2(Z)}(A)$, we denote the centralizer of A in $SL_2(Z)$. Then, $Z(A)$ operates on $D(A)$ by multiplication from the left.

Take a representative A of a $SL_2(Z)$-conjugacy class in $B(t,f)$, and, for such A, take a representative B of $Z(A)\backslash D(A)/\Gamma_0$. Then, when t, f, A and B vary under the above conditions, $(4n)^{-1}B^{-1}AB = (4n)^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ forms a complete system of all the representatives for the elliptic Γ_0-conjugacy classes in Φ, such that $c > 0$. Here, by the suitable choice of the representative, we may assume that $d > 0$, $(b,d) = 1$

and $(b/f, (t^2 - 64n^2)/f^2) = 1$ (cf. remark 3).

Thus, we have

$$\tilde{e}_0(n) = 2^{2k}n^{k+1/2}(1 + \varepsilon(-1)^k\sqrt{-1}) \sum_t p(t)(t + 8n)^{-1/2}$$

$$\times \sum_f \sum_A \sum_B \left(\frac{a}{M_0}\right)\left(\frac{d}{b}\right).$$

Since $(f, t + 8n, t - 8n) = (M_0, t + 8n, t - 8n) = 1$, we can decompose $f = f_1 f_2$ and $M_0 = M_1 M_2$ with $0 < f_1$, $0 < f_2$, $f_1^2 \mid (t + 8n)$, $f_2^2 \mid (t - 8n)$, $(f_1, t - 8n) = (f_2, t + 8n) = 1$, $0 < M_1$, $0 < M_2$, $M_1 \mid (t + 8n)$ and $(M_2, t + 8n) = 1$. In the same way as in [K 1] p 52, we have

$$\left(\frac{d}{b}\right) = \left(\frac{(t + 8n)/f_1^2}{b/f}\right)\left(\frac{t - 8n}{f_1}\right)\left(\frac{t + 8n}{f_2}\right) \text{ and } \left(\frac{a}{M_0}\right) = \left(\frac{a}{M_1}\right)\left(\frac{a}{M_2}\right)$$

$$= \left(\frac{t - 8n}{M_1}\right)\left(\frac{t + 8n}{M_2}\right).$$ Hence,

$$\tilde{e}_0(n) = 2^{2k}n^{k+1/2}(1 + \varepsilon(-1)^k\sqrt{-1}) \sum_t p(t)(t + 8n)^{-1/2}\left(\frac{t - 8n}{M_1}\right)\left(\frac{t + 8n}{M_2}\right)$$

$$\times \sum_f \left(\frac{t - 8n}{f_1}\right)\left(\frac{t + 8n}{f_2}\right) \sum_A \sum_B \left(\frac{(t + 8n)/f_1^2}{b/f}\right).$$

Therefore, from the same arguement in [K 1] p 53, we may assume that $t + 8n$ is a square integer, and then, by using remark 4, we have $\sum_A \sum_B \left(\frac{(t + 8n)/f_1^2}{b/f}\right) = h((t^2 - 64n^2)/f^2)\ \#(Z(A)\backslash D(A)/\Gamma_0)$

$$= h'((t^2 - 64n^2)/f^2) \prod_{p \mid M} \tilde{c}_p(t,f).$$

We can write $t + 8n = 4s^2$ with $s > 0$. Then, we have $t^2 - 64n^2 = 16s^2(s^2 - 4n)$, $\lambda(t) = s + (s^2 - 4n)^{1/2}$ and

$p(t) = -4^{-k}n^{-2k+1}(x^{2k-1} - y^{2k-1})(x - y)^{-1}$, where x and y are the solutions of $X^2 - sX + n = 0$.

Thus, we obtain

$$\tilde{e}_0(n) = -((1 + \varepsilon(-1)^k\sqrt{-1})/2)\, n^{-k+3/2}$$

$$\times \sum_s s^{-1}(x^{2k-1} - y^{2k-1})(x - y)^{-1}\left(\frac{s^2 - 4n}{M_1}\right)$$

$$\times \sum_{f_1, f_2}\left(\frac{s^2 - 4n}{f_1}\right) h'(16s^2(s^2 - 4n)(f_1f_2)^{-2}) \prod_{p|M} \tilde{c}_p(4s^2 - 8n, f_1f_2$$

where s runs over the integer such that $0 < s < 2\sqrt{n}$, f_1 runs over the set $\{ Z \ni f_1 > 0;\ (f_1, 2n) = 1,\ f_1 \mid s \}$ and f_2 runs over the set $\{ Z \ni f_2 > 0;\ (f_2, 2n) = 1,\ f_2^2 \mid (s^2 - 4n) \}$.

In the same way as in §1 (elliptic case), we rewrite the above formula as follows:

$$\tilde{e}_0(n) = -\ ((1 + \varepsilon(-1)^k\sqrt{-1})/2)\, n^{-k+3/2}$$

$$\sum_s (x^{2k-1} - y^{2k-1})(x - y)^{-1}\, h'(u)(4 - 2\left(\frac{u}{2}\right))$$

$$\times\, u(m_1')\alpha_u(r(m_1')) \prod_{q|(s', m_1')} (1 - \left(\frac{u}{q}\right)q^{-1})$$

$$\times\, 2^{\mathrm{ord}_2(m_1)} \prod_{p|M} p^{-\mathrm{ord}_p(s)}\, n_p(\theta_p),$$

where $s^2 - 4n = m_1^2u$ with $m_1 > 0$, x and y are the solutions of $X^2 - sX + n = 0$, $\theta_p = \mathrm{ord}_p(sm_1)$,

$$s' = s \prod_{p|N} p^{-\,\mathrm{ord}_p(s)} \quad \text{and} \quad m_1' = m_1 \prod_{p|N} p^{-\,\mathrm{ord}_p(m_1)} ,$$

$u(m_1')$ and $r(m_1')$ are the same in §1(elliptic case),

the constant $n_p(\theta_p)$ is given by the table, case (1) - (3), at the statement of the elliptic case of proposition 1.

Moreover, in the same way as in §1, we obtain

$$\sum_{0<a|n_0} a\, n^{k-3/2}\, \tilde{e}_0(n/a^2)$$

$$= -\,((1 + \varepsilon(-1)^k\sqrt{-1})/2) \sum_{0<\tilde{s}<2\sqrt{n}} (\tilde{x}^{\,2k-1} - \tilde{y}^{\,2k-1})(\tilde{x} - \tilde{y})^{-1}$$

$$\times\ h'(u)\ \left(4 - 2\left(\frac{u}{2}\right)\right)\ 2^{\mathrm{ord}_2(\tilde{m}_1)} \prod_{p|M} p^{-\,\mathrm{ord}_p(\tilde{s})}\ n_p(\theta_p)$$

$$\times\ \alpha_u(\tilde{m}_1 \prod_{p|N} p^{-\,\mathrm{ord}_p(\tilde{m}_1)}) ,$$

where $\tilde{s}$ runs over the integer such that $0 < \tilde{s} < 2\sqrt{n}$, $\tilde{x}$ and $\tilde{y}$ are the solutions of $X^2 - \tilde{s}X + n = 0$, $\tilde{s}^{\,2} - 4n = \tilde{m}_1^{\,2}u$ with a fundamental discriminant u and a positive integer $\tilde{m}_1$, $\theta_p = \mathrm{ord}_p(\widetilde{sm}_1)$.

3. The calculation of $\tilde{h}_0(n)$.

Let C be a hyperbolic equivalence class in Φ and $w = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ From the definition of $J(C)$, we have $J(C) = J(-C)$. Hence, $\tilde{h}_0(n) = \Sigma_1 J(C) = \Sigma_2(J(C) + J(-C)) = 2\,\Sigma_2 J(C) = \Sigma_2(J(C) + J(wCw))$, where C in Σ_1 runs over all the hyperbolic Γ_0-conjugacy classes in Φ and C in Σ_2 runs over those such that $a + d > 0$.

For C in Σ_2, we can take $\beta = (4n)^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in C$ with $d > 0$, $(b,d) = 1$ and $(b/f, (t^2 - 64n^2)/f^2) = 1$, where $t = a + d$ and $f = (a - d,b,c)$ (cf. remark 3). Then, from remark 2, we have

$$J(\beta) = J(C) = -\ (1/2)\left(\frac{d}{b}\right)\left(\frac{\varepsilon}{b}\right)\left(\frac{a}{M_0}\right)\left(\frac{-1}{b}\right)^{k+1/2}\ \lambda(\beta)^{-k-1/2}(1 - \lambda(\beta)^{-2})^{-1}$$

$$\text{with}\quad \lambda(\beta) = \left\{ \begin{array}{ll} (4n)^{-1}(a - cz_0), & \text{if } z_0 \neq \infty \\ (d/4n), & \text{if } z_0 = \infty \end{array} \right\},$$

where z_0 is the upper fixed point of β which is a cusp of Γ_0 (cf. Remark after the part of $\tilde{d}(n)$ in §1).

Hence, $J(\beta) + J(w\beta w)$ =

$$-\ (1/2)\left(\frac{d}{b}\right)\left(\frac{\varepsilon}{b}\right)\left(\frac{a}{M_0}\right)\lambda(\beta)^{-k-1/2}(1 - \lambda(\beta)^{-2})^{-1}\left(\left(\frac{-1}{b}\right)^{k+1/2} + \varepsilon\left(\frac{-1}{-b}\right)^{k+1/2}\right)$$

$$= -\ ((1 + \varepsilon(-1)^k\sqrt{-1})/2)\left(\frac{d}{b}\right)\left(\frac{a}{M_0}\right)\lambda(\beta)^{-k-1/2}(1 - \lambda(\beta)^{-2})^{-1}.$$

Let t be an integer such that $t \equiv 0 \pmod 4$ and that $t > 8n$ and that $t^2 - 64n^2$ is square. Then, we write $t^2 - 64n^2 = m^2$ with $m > 0$. Let f be a positive integer such that $f|m$ and that $(f,2n) = 1$. We put

$$B_1(t,f) = \left\{ M_2(Z) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{l} a + d = t,\ (a - d,b,c) = f, \\ (a,b,c,d) = 1,\ ad - bc = 16n^2 \end{array} \right\}$$

and, for $A \in B_1(t,f)$, let $D(A)$ and $Z(A)$ are the same as in the elliptic case. Then, $SL_2(Z)$ operates on $B_1(t,f)$ by the inner automorphism and the system of all the representatives of the $SL_2(Z)$-conjugacy classes in $B_1(t,f)$ is given by

$$X(t,f) = \left\{ M_2(Z) \ni \begin{pmatrix} \nu' & \tau \\ 0 & \nu \end{pmatrix} \;\middle|\; \begin{array}{l} \nu\nu' = 16n^2,\ 0 < \nu' < \nu, \\ 0 \leqq \tau < \nu \quad \nu',\ \nu + \nu' = t \\ f = (\nu - \nu', \tau) \end{array} \right\}$$

(cf. [K 1] p 55). For $A \in X(t,f)$, take a representative B of $Z(A)\backslash D(A)/\Gamma_0$. When t, f, $A = \begin{pmatrix} \nu' & \tau \\ 0 & \nu \end{pmatrix}$ and B vary

under the above conditions, the matrix $(4n)^{-1}B^{-1}AB = (4n)^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ forms a complete system of all the representatives for the hyperbolic Γ_0-conjugacy classes in Φ, such that $a + d > 0$. Here, by the suitable choice of the representative B, we may assume that $d > 0$, $(b,d) = 1$ and $(b/f, (t^2 - 64n^2)/f^2) = 1$ with $t = a + d$ and $f = (a - d, b, c)$ (cf. Remark 3).

From elementary calculation, we get $\lambda((4n)^{-1}B^{-1}\begin{pmatrix} \nu' & \tau \\ 0 & \nu \end{pmatrix}B)$ $= (\nu/4n) = (\nu'/4n)^{-1}$. Hence, $\tilde{h}_0(n) = -((1 + \varepsilon(-1)^k\sqrt{-1})/2)(4n)^{-k+3/2}$

$$\times \sum_t \sum_f \sum_A \sum_B \nu'^{\,k-1/2}(\nu - \nu')^{-1}\left(\frac{d}{b}\right)\left(\frac{a}{M_0}\right).$$

In the same way as in the elliptic case, we may assume that $t + 8n$ is square. Since $t^2 - 64n^2 = (t + 8n)(t - 8n)$ is also square, we can write $t + 8n = 4s^2$ and $t - 8n = 4r^2$ with $s > 0$ and $r > 0$. Since $\nu = (t + (t^2 - 64n^2)^{1/2})/2$ and $\nu' = (t - (t^2 - 64n^2)^{1/2})/2$, we have $\nu - \nu' = 4sr$ and $\nu'^{\,1/2} = ((t + 8n)^{1/2} - (t - 8n)^{1/2})/2 = s - r$.

Therefore, in the same way as in the elliptic case, we obtain $\tilde{h}_0(n) = -((1 + \varepsilon(-1)^k\sqrt{-1})/2)\, n^{-k+3/2}$

$$\sum_s ((s - r)/2)^{2k-1}(sr)^{-1} \sum_f \phi(4sr/f) \prod_{p|M} \tilde{c}_p(4s^2 - 8n, f),$$

where s runs over the integer such that $s > 2\sqrt{n}$ and that $s^2 - 4n$ is square, $r = (s^2 - 4n)^{1/2}$ and f runs over the positive divisor of $4sr$ such that $(f,2n) = 1$.

Finally, by applying the same argument as in §1 (elliptic case), we can deduce the following result from the above formula of $\tilde{h}_0(n)$:

$$\sum_{0<a|n_0} a\, n^{k-3/2}\, \tilde{h}_0(n/a^2)$$

$$= -\,(1 + \varepsilon(-1)^k\sqrt{-1}) \sum_{s} ((s - r)/2)^{2k-1} \prod_{p|M} m_p(\theta_p),$$

where s runs over the integer such that $s > 2\sqrt{n}$ and that $s^2 - 4n$ is square, $r = (s^2 - 4n)^{1/2}$, $\theta_p = \mathrm{ord}_p(sr)$ and the constant $m_p(\theta_p)$ is given by the table, case (1), at the statement of the hyperbolic case of proposition 1.

Remark. The integer s which satisfies the above conditions is always even. Hence, we may consider only the even integer s.

4. The calculation of $\tilde{d}_0(n)$.

We have $[\Delta_0{}'\tau'\Delta_0{}']_{1/2} = \left[\Delta_0{}'\left(\begin{pmatrix}4n^2 & -1\\ 0 & 4\end{pmatrix}, n^{-1/2}\right)\Delta_0{}'\right]_{1/2}$

$$= \left[\Delta_0{}'\left(\begin{pmatrix}4 & -1\\ 0 & 4\end{pmatrix}, 1\right)\Delta_0{}'\right]_{1/2}\left[\Delta_0{}'\left(\begin{pmatrix}n^2 & 0\\ 0 & 1\end{pmatrix}, n^{-1/2}\right)\Delta_0{}'\right]_{1/2},$$

as the operators on $G(1/2,N,\chi)$ (cf. Lemma in this section). Then, from the proof of [K 1] §2 proposition 1, and the fact that $G(1/2,N,\chi)$ is spaned by the theta series (cf. [S - S] and also §1), it follows that $G(1/2,N,\chi)$ is the eigen space of the operator $[\Delta_0' \left(\begin{pmatrix} 4 & -1 \\ 0 & 4 \end{pmatrix}, 1 \right) \Delta_0']_{1/2}$ with the eigen value $2(1 - \varepsilon\sqrt{-1})$. Similarly, from lemma and $U(N,\chi) \subsetneqq S(3/2,N,\chi)_K$, we have, as the operator on $U(N,\chi)$, $[\Delta_0 \tau \Delta_0]_{3/2} \mid U(N,\chi) = 2(1 - \varepsilon\sqrt{-1})[\Delta_0 \left(\begin{pmatrix} 1 & 0 \\ 0 & n^2 \end{pmatrix}, n^{3/2} \right) \Delta_0]_{3/2} \mid U(N,\chi)$.

Therefore, we get $\tilde{d}_0(n) = 2(1 - \varepsilon\sqrt{-1})\, \tilde{d}(n)$, where $\tilde{d}(n)$ is the same as in §1.

Thus, the calculation of trace $([\Delta_0(\alpha, n^{k+1/2})\Delta_0]_{k+1/2})$ has completed.

Now, we have

$$\begin{aligned} &\mathrm{trace}\, (\tilde{T}_{k+1/2,N,\chi}(n^2)_K \mid S(k+1/2,N,\chi)_K) \\ &= (\sqrt{2}/6)(-1)^{[(k+1)/2]}\varepsilon\ \mathrm{trace}(\sum_{0<a\mid n_0} a\, n^{k-3/2} [\Delta_0 \left(\begin{pmatrix} (n/a^2)^{-1} & (4n/a^2)^{-1} \\ 0 & n/a^2 \end{pmatrix}, \right. \\ &\left. \varepsilon((2k+1)/8)(n/a^2)^{k+1/2} \right) \Delta_0]) \quad + (1/3)\ \mathrm{trace}(\tilde{T}_{k+1/2,N,\chi}(n^2)) \\ &= ((1 - \varepsilon(-1)^k\sqrt{-1})/6) \sum_{0<a\mid n_0} a\, n^{k-3/2}\ \mathrm{trace}([\Delta_0 \left(\begin{pmatrix} (n/a^2)^{-1} & (4n/a^2)^{-1} \\ 0 & n/a^2 \end{pmatrix}, \right. \\ &\left. (n/a^2)^{k+1/2} \right) \Delta_0]) \quad + (1/3)\ \mathrm{trace}(\tilde{T}_{k+1/2,N,\chi}(n^2)). \end{aligned}$$

Hence, by combining the above calculations with the results of section 1, we obtain the following propositions.

Proposition 3. Suppose $k \geqq 2$. The trace of the n-th Hecke operator $\tilde{T}_{k+1/2,N,\chi}(n^2)_K$ with $(n,N) = 1$ acting on $S(k+1/2,N,\chi)_K$ is given by the sum $T(p) + T(e) + T(h) + T'$, where the each term is given as follows: For a prime number $p|M$, let $\nu = \nu_p = \mathrm{ord}_p(N)$. Let $\delta_0(\sqrt{n})$, n_0 and n_1 be the same as in the starement of proposition 1.

Then, for $n \equiv 1 \pmod 4$. we set

$$T(p) = -(1/2)\delta_0(\sqrt{n})\, n^{k-1/2} \prod_{p|M} (p^{[\nu/2]} + p^{[(\nu-1)/2]})$$

$$+ (\varepsilon(-1)^k/2)\left(\frac{n}{M_0}\right) n^{k-1} \prod_{p|M} \left(p^{[\nu/2]} + \left(\frac{-n}{p}\right)^{\nu} p^{[(\nu-1)/2]}\right)$$

$$\times \sum_{0<a|n_0} h'(-4n/a^2).$$

For $n \equiv 3 \pmod 4$. we set

$$T(p) = (\varepsilon(-1)^k/2)\left(3 - \left(\frac{2}{n}\right)\right)\left(\frac{n}{M_0}\right) n^{k-1} \prod_{p|M} \left(p^{[\nu/2]} + \left(\frac{-n}{p}\right)^{\nu} p^{[(\nu-1)/2]}\right)$$

$$\times \sum_{0<a|n_0} h'(-n/a^2).$$

Next, we define $T(e)$ by

$$T(e) = -\Sigma_1 (x^{2k-1} - y^{2k-1})(x-y)^{-1} h'(u)\alpha_u(t_1) \prod_{p|M} p^{-\mathrm{ord}_p(s)} n_p(\theta_p),$$

where the sum Σ_1 is extended over the integer s such that $0 < s < 2\sqrt{n}$, and the meaning of the letter is as follows: x and y are the solutions of $X^2 - sX + n = 0$, and $s^2 - 4n = t^2u$ with a positive integer t and a fundamental discriminant u. Put $t_1 = t \prod_{p|M} p^{-\mathrm{ord}_p(t)}$ and $\theta = \theta_p = \mathrm{ord}_p(st)$.

The constant $n_p(\theta_p)$ is given by the table (case (1) - (3)) at the statement of the elliptic case of proposition 1.

Next, we define $T(h)$ by

$T(h) = - \Sigma_2 \, ((s - t)/2)^{2k-1} \prod_{p|M} m_p(\theta_p)$, where the sum Σ_2 is extended over the even integer s which satisfies $s > 2\sqrt{n}$ and $s^2 - 4n$ is square, and the meaning of the letters is as follows: Put $t = (s^2 - 4n)^{1/2}$ and $\theta = \theta_p = \mathrm{ord}_p(st)$. The constant $m_p(\theta_p)$ is given by the table (case(1)) at the statement of the hyperbolic case of proposition 1.

Finally, we define T' by

$$T' = ((2k - 1)/12) \; \delta_0(\sqrt{n}) \; n^{k-1} \, M \prod_{p|M} (p + 1)/p.$$

Proposition 4. The trace of the n-th Hecke operator $\tilde{T}_{3/2,N,\chi}(n^2)_K$ with $(n,N) = 1$ acting on $V(N,\chi)_K$ is given by the sum $T(p) + T(e) + T(h) + T' + T''$, where the terms $T(p)$, $T(e)$, $T(h)$ and T' are given by the same formulas as in proposition 3 when we put $k = 1$. The term T'' is given by the same formula as in proposition 2.

§3. The relations.

From [H 1], the trace of the n-th Hecke operator $T_{2k,N}(n)$ with $(n,2N) = 1$ acting on $S(2k,N)$ is given by the sum

$T_0(p) + T_{00}(e) + T_0(e) + T_0(h) + T_0' + T_0''$

Here, the definition of the each term is as follows: For the prime number p, we write $\mathrm{ord}_p(N) = \tilde{\nu}_p = \tilde{\nu} = \begin{cases} \nu_p = \nu, & \text{if } p \text{ is odd,} \\ \mu, & \text{if } p = 2. \end{cases}$

Let $\delta_0(\sqrt{n})$ be the same as in §1 and let M be the odd part of N. We decompose $n = n_0{}^2 n_1$ with a positive integer n_0 and a square-free positive integer n_1

$$\text{Then,}\quad T_0(p) = -(1/2)\delta_0(\sqrt{n})\, n^{k-1/2} \prod_{p|N} (p^{[\tilde{\nu}/2]} + p^{[(\tilde{\nu}-1)/2]}).$$

For $n \equiv 1 \pmod 4$, we set

$$T_{00}(e) = \begin{cases} 0, & \text{if } \mu \geqq 2, \\ ((-1)^k/2)\, n^{k-1} \prod_{p|M} \left(1 + \left(\frac{-n}{p}\right)\right) \sum_{0<a|n_0} h'(-4n/a^2), & \text{if } \mu \leqq 1. \end{cases}$$

For $n \equiv 3 \pmod 4$, we set

$$T_{00}(e) = ((-1)^k/2)\, n^{k-1} \prod_{p|M} \left(1 + \left(\frac{-n}{p}\right)\right) \sum_{0<a|n_0} h'(-n/a^2) \times C_2$$

$$\text{with}\quad C_2 = \begin{cases} \left(3 - \left(\frac{2}{n}\right)\right) & \text{if } \mu = 0, \\ \left(5 - \left(\frac{2}{n}\right)\right) & \text{if } \mu = 1, \\ 6 & \text{if } \mu = 2, \end{cases}$$

$$4(1 + \left(\frac{2}{n}\right)) \quad \text{if} \quad \mu \geqq 3.$$

Next, we define $T_0(e)$ by

$$T_0(e) = -\Sigma_1 (x^{2k-1} - y^{2k-1})(x-y)^{-1}\, h'(u)\, \alpha_u(t_0) \prod_{p|N} n_{0,p}(\theta_{0,p}),$$

where the sum Σ_1 is extended over the integer s such that $2\sqrt{n} > s > 0$ and besides that s is even if $\mu \geqq 1$, and the meaning of the letters is as follows: x and y are the solutions of $X^2 - sX + n = 0$. $s^2 - 4n = t^2 u$ with a positive integer t and a fundamental discriminant u. Put

$$\theta_0 = \theta_{0,p} = \mathrm{ord}_p(t) \quad \text{and} \quad t_0 = t \prod_{p|N} p^{-\mathrm{ord}_p(t)}.$$

The constant $n_{0,p}(\theta_{0,p})$ is given by the following table:

Case (1) $(p|M$ and $p|u)$.

$$n_{0,p}(\theta_{0,p}) = \begin{cases} \{p^{\theta_0+1}(p^{[\nu/2]} + p^{[(\nu-1)/2]}) - p^{\nu} - p^{\nu-1}\}/(p-1), \\ \qquad \text{if} \quad \theta_0 \geqq [\nu/2], \\ 0, \text{ if } \quad \theta_0 < [\nu/2]. \end{cases}$$

Case (2) $(p|M$ and $p\nmid u)$.

$$n_{0,p}(\theta_{0,p}) = \begin{cases} (1 - \left(\frac{u}{p}\right)p^{-1})(p^{[\nu/2]} + p^{[(\nu-1)/2]})(p^{\theta_0+1} - p^{[\nu/2]+1})/(p-1) \\ \qquad + p^{[\nu/2]}(p^{[\nu/2]} + \left(\frac{u}{p}\right)^{\nu} p^{[(\nu-1)/2]}), \\ \qquad \text{if} \quad \theta_0 \geqq [(\nu+1)/2], \\ (1 + \left(\frac{u}{p}\right))p^{2\theta_0}, \text{ if } \quad \theta_0 \leqq [(\nu-1)/2]. \end{cases}$$

Case (3) (p = 2).

$$n_{0,2}(\theta_{0,2}) = \begin{cases} 2^{\theta_0}(2^{[\mu/2]} + 2^{[(\mu-1)/2]}), \\ \quad \text{if } u \equiv 1 \pmod 8 \text{ and } \theta_0 \geqq [(\mu+1)/2], \\ 2^{2\theta_0 + 1}, \text{ if } u \equiv 1 \pmod 8 \text{ and } \theta_0 \leqq [(\mu-1)/2], \\ 3\times 2^{\theta_0}(2^{[\mu/2]} + 2^{[(\mu-1)/2]}) - 3\times 2^{\mu}, \\ \quad \text{if } u \equiv 5 \pmod 8 \text{ and } \theta_0 \geqq [(\mu+1)/2], \\ 0, \text{ if } u \equiv 5 \pmod 8 \text{ and } \theta_0 \leqq [(\mu-1)/2], \\ 2^{\theta_0 + 1}(2^{[\mu/2]} + 2^{[(\mu-1)/2]}) - 3\times 2^{\mu-1}, \\ \quad \text{if } u \equiv 0 \pmod 4 \text{ and } \theta_0 \geqq [(\mu-1)/2], \\ 0, \text{ if } u \equiv 0 \pmod 4 \text{ and } \theta_0 < [(\mu-1)/2]. \end{cases}$$

Next, we define $T_0(h)$ by

$T_0(h) = -\Sigma_2((s-t)/2)^{2k-1} \prod_{p|N} m_{0,p}(\theta_{0,p})$, where the sum Σ_2 is extended over the even integer s such that $s > 2\sqrt{n}$ and that $s^2 - 4n$ is square, and put $t = (s^2 - 4n)^{1/2}$ and $\theta_0 = \theta_{0,p} = \mathrm{ord}_p(t)$ and, for the prime number $p|N$,

$$m_{0,p}(\theta_{0,p}) = \begin{cases} p^{[\tilde{\nu}/2]} + p^{[(\tilde{\nu}-1)/2]}, \text{ if } \theta_0 \geqq [(\tilde{\nu}+1)/2], \\ 2p^{\theta_0}, \text{ if } \theta_0 \leqq [(\tilde{\nu}-1)/2]. \end{cases}$$

Finally, we define T_0' and T_0'' by

$T_0' = ((2k-1)/12)\ \delta_0(\sqrt{n})\ n^{k-1}\ N \prod_{p|N} (p+1)/p,$

and $T_0'' = \delta(k) \prod_{p|n} (p^{\tau+1} - 1)/(p - 1)$ with $n = \prod_{p|n} p^{\tau}$ and

$$\delta(k) = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k \geqq 2. \end{cases}$$

Now, let N_0 be the positive divisor of N such that $(N_0, N/N_0) = 1$ and that $N_0 \neq 1$. Take any element $\gamma(N_0) \in SL_2(Z)$ which satisfies the conditions:

$$\gamma(N_0) \equiv \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \pmod{N_0}, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N/N_0}, \end{cases}$$ and put $$W(N_0) = \gamma(N_0)\begin{pmatrix} N_0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, it is well-known that $W(N_0)$ is the normalizer of $\Gamma_0 = \Gamma_0(N)$ and that $[W(N_0)]_{2k}$ induces the C-linear automorphism of order 2 on $S(2k,N)$.

In [Y], M.Yamauchi explicitly calculated the trace of the operator $[W(N_0)]_{2k}\, T_{2k,N}(n)$ with $(n,N) = 1$ acting on $S(2k,N)$. But his formula contains several errors in the hyperbolic and parabolic cases. Therefore, though we need only the trace formula of the operator $[W(N_0)]_{2k}\, T_{2k,N}(n)$ with $(n,2N) = 1$, we shall give the corrected version of Yamauchi's formula in all cases.

The trace of the operator $[W(N_0)]_{2k}\, T_{2k,N}(n)$ with $(n,N) = 1$ acting on $S(2k,N)$ is given by the sum $T_1(p) + T_{10}(e) + T_1(e) + T_1(h) + T_1''$. Here, the definition of the each term is as follows: Let $\tilde{\nu}$, ν, μ and $\delta_0(\sqrt{n})$ be the same as in the trace formula of $T_{2k,N}(n)$. We decompose $nN_0 = \alpha_0{}^2\alpha_1$ with a positive integer α_0 and a square-free positive integer α_1.

Then, $T_1(p) = -(1/2)\, \delta_2\, \delta_0(\sqrt{n})\, n^{k-1/2} \prod_{p|(N/N_0)} (p^{[\nu/2]} + p^{[(\nu-1)/2]})$,

with $\delta_2 = \begin{cases} 1, & \text{if } N_0 = 4, \\ 0, & \text{otherwise.} \end{cases}$

For $nN_0 \equiv 1 \pmod 4$, we set

$$T_{10}(e) = \begin{cases} 0, & \text{if } \operatorname{ord}_2(N/N_0) \geqq 2, \\ ((-1)^k/2)\, n^{k-1} \prod\limits_{\substack{p|(N/N_0) \\ p:\ \text{odd}}} \left(1 + \left(\dfrac{-nN_0}{p}\right)\right) & \\ \quad \times \sum\limits_{\substack{0<a|\alpha_0 \\ (a,N_0)=1}} h'(-4nN_0/a^2), & \text{if } \operatorname{ord}_2(N/N_0) \leqq 1. \end{cases}$$

For $nN_0 \equiv 3 \pmod 4$, we set

$$T_{10}(e) = ((-1)^k/2)\, n^{k-1} \prod_{\substack{p|(N/N_0) \\ p:\ \text{odd}}} \left(1 + \left(\frac{-nN_0}{p}\right)\right)$$

$$\times \sum_{\substack{0<a|\alpha_0 \\ (a,N_0)=1}} h'(-nN_0/a^2) \times C_2',$$

with $C_2' = \begin{cases} 3 - \left(\dfrac{2}{nN_0}\right) & \text{if } \operatorname{ord}_2(N/N_0) = 0, \\ 5 - \left(\dfrac{2}{nN_0}\right) & \text{if } \operatorname{ord}_2(N/N_0) = 1, \\ 6 & \text{if } \operatorname{ord}_2(N/N_0) = 2, \\ 4\left(1 + \left(\dfrac{2}{nN_0}\right)\right) & \text{if } \operatorname{ord}_2(N/N_0) \geqq 3. \end{cases}$

Next, we define $T_1(e)$ by

$$T_1(e) = -N_0^{1-k} \Sigma_1 (x^{2k-1} - y^{2k-1})(x - y)^{-1} h'(t_1^2 u) \alpha_u(t_0)$$

$$\times \prod_{p|(N/N_0)} n_{0,p}(\theta_{0,p}),$$ where the sum Σ_1 is

extended over the integer s such that $2\sqrt{nN_0} > s > 0$ and that $s \equiv 0 \pmod{N_0}$ and besides that s is even if N/N_0 is even. The meaning of the letters is as follows: x and y are the solutions of $X^2 - sX + nN_0 = 0$. $s^2 - 4nN_0 = t^2 u$ with a fundamental discriminant u and a positive integer t.

Put $\theta_0 = \theta_{0,p} = \mathrm{ord}_p(t)$, $t_0 = t \prod_{p|N} p^{-\mathrm{ord}_p(t)}$ and

$t_1 = \prod_{p|N_0} p^{\mathrm{ord}_p(t)}$. The constant $n_{0,p}(\theta_{0,p})$ is given by

the same table as in the case $T_0(e)$ of the trace formula of $T_{2k,N}(n)$

Next, we define $T_1(h)$ by

$$T_1(h) = -\delta_0(\sqrt{N_0}) \quad \Sigma_2 \ ((s - t)/2)^{2k-1} \phi(\sqrt{N_0}\, t_1)\, t_1^{-1}$$

$$\times \prod_{p|(N/N_0)} m_{0,p}(\theta_{0,p}),$$ where the sum Σ_2 is extended

over the integer s such that $s > 2\sqrt{n}$ and that $s \equiv 0 \pmod{\sqrt{N_0}}$ and that $s^2 - 4n$ is square, and put $t = (s^2 - 4n)^{1/2}$,

$\theta_0 = \theta_{0,p} = \mathrm{ord}_p(t)$ and $t_1 = \prod_{p|N_0} p^{\mathrm{ord}_p(t)}$ The constant

$m_{0,p}(\theta_{0,p})$ is given by the same table as in the case $T_0(h)$

of the trace formula of $T_{2k,N}(n)$.

Finally, we define T_1'' by

$$T_1'' = \delta(k) \prod_{p|n} (p^{\tau+1} - 1)/(p - 1) \quad \text{with} \quad n = \prod_{p|n} p^{\tau} \quad \text{and}$$

$$\delta(k) = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

From these trace formulas and the results of the previous sections, we can deduce the relations between traces.

Now, we fix the notations as follows: Let N be an integer such that $2 \leqq \mathrm{ord}_2(N) = \mu \leqq 4$, and let M be the odd part of N, namely, $N = 2^{\mu}M$. Let χ be an even Dirichlet character modulo N such that $\chi^2 = 1$ and besides that the conductor of χ is divisible by 8 if $\mu = 4$.

Theorem. Under the above notations and assumptions, we have the following relations (3.1) - (3.4):

(3.1) Suppose $k \geqq 2$, then, for a positive integer n with $(n,N) = 1$, we have

$$\mathrm{trace}(\tilde{T}_{k+1/2,N,\chi}(n^2) \mid S(k+1/2,N,\chi))$$

$$= \mathrm{trace}(T_{2k,N/2}(n) \mid S(2k,N/2))$$

$$+ \Sigma_1 \Lambda(n,L_0) \,\mathrm{trace}([W(L_0)]_{2k}\, T_{2k,2^{\mu-1}L_0L_1}(n) \mid S(2k,2^{\mu-1}L_0L_1)).$$

(3.2) Let k and n be the same as in (3.1) and suppose $N = 4M$, then we have

$$\mathrm{trace}(\tilde{T}_{k+1/2,N,\chi}(n^2)_K \mid S(k+1/2,N,\chi)_K)$$

$$= \mathrm{trace}(T_{2k,M}(n) \mid S(2k,M))$$

$$+ \Sigma_1\ \Lambda(n,L_0)\ \mathrm{trace}([W(L_0)]_{2k}\ T_{2k,L_0L_1}(n) \mid S(2k,L_0L_1)).$$

(3.3) Let n be the same as in (3.1), then we have

$$\mathrm{trace}(\tilde{T}_{3/2,N,\chi}(n^2) \mid V(N,\chi))$$

$$= \mathrm{trace}(T_{2,N/2}(n) \mid S(2,N/2))$$
$$+ \Sigma_1\ \Lambda(n,L_0)\ \mathrm{trace}([W(L_0)]_2\ T_{2,2^{\mu-1}L_0L_1}(n) \mid S(2,2^{\mu-1}L_0L_1)).$$

(3.4) Let n be the same as in (3.1) and suppose $N = 4M$, then we have

$$\mathrm{trace}(\tilde{T}_{3/2,N,\chi}(n^2)_K \mid V(N,\chi)_K)$$

$$= \mathrm{trace}(T_{2,M}(n) \mid S(2,M))$$

$$+ \Sigma_1\ \Lambda(n,L_0)\ \mathrm{trace}([W(L_0)]_2\ T_{2,L_0L_1}(n) \mid S(2,L_0L_1)).$$

Here, the sum Σ_1 is extended over all the integer L_0 such that $1 < L_0 | M$ and that L_0 is square, and

$L_1 = M \prod_{p|L_0} p^{-\mathrm{ord}_p(M)}$. The constant $\Lambda(n,L_0)$ is defined by

$$\Lambda(n,L_0) = \prod_{p|M} \lambda(p,n;(\mathrm{ord}_p(L_0))/2) \quad \text{with}$$

$$\lambda(p,n;a) = \begin{cases} 1, & \text{if } a = 0, \\ 1 + \left(\frac{-n}{p}\right), & \text{if } 1 \leqq a \leqq [(\nu-1)/2], \end{cases}$$

$\left\lfloor \chi_p(-n), \right.$ if ν is even and $a = \nu/2$,

where $\nu = \mathrm{ord}_p(N)$ and χ_p is the p-component of χ.

Proof. We can easily verify the following claims:

(1) When $n \equiv 1 \pmod 4$, the first term of $T(p)$ is equal to the contribution from the parts of $T_0(p)$ and $T_1(p)$ to the right-hand side of the above relations, and the second term of $T(p)$ is equal to the contribution from the parts of $T_{00}(e)$ and $T_{10}(e)$ to the right-hand side of the above relations.

(2) When $n \equiv 3 \pmod 4$, $T(p)$ is equal to the contribution from the parts of $T_{00}(e)$ and $T_{10}(e)$ to the right-hand side of the above relations.

(3) $T(e)$ (resp. $T(h)$) is equal to the contribution from the parts of $T_0(e)$ and $T_1(e)$ (resp. $T_0(h)$ and $T_1(h)$) to the right-hand side of the above relations.

(4) T' is equal to the contribution from the part of T_0' to the right-hand side of the above relations.

(5) When $k = 1$, T'' is equal to the contribution from the parts of T_0'' and T_1'' to the right-hand side of the above relations.

From these claimes, we can easily deduce the assertions of Theorem.

Corollary. Let the notations and the assumptions be the same as in the above Theorem. We suppose that M is square-free. Then, we have the following isomorphisms between the H_M-modules:

For $k \geqq 2$,
$$S(k+1/2,4M,\chi)_K \simeq S(2k,M),$$
$$S(k+1/2,4M,\chi) \simeq S(2k,2M),$$
$$S(k+1/2,8M,\chi) \simeq S(2k,4M) \text{ and}$$
$$S(k+1/2,16M,\chi) \simeq S(2k,8M),$$
(in the last isomorphism, we remark that the conductor of χ is divisible by 8 from our assumption).

For $k = 1$,
$$S(3/2,4M,\chi)_K = V(4M,\chi)_K \simeq S(2,M),$$
$$S(3/2,4M,\chi) = V(4M,\chi) \simeq S(2,2M),$$
$$S(3/2,8M,\chi) = V(8M,\chi) \simeq S(2,4M) \text{ and}$$
$$S(3/2,16M,\chi) = V(16M,\chi) \simeq S(2,8M),$$
(in the last isomorphism, we remark that the conductor of χ is divisible by 8 from our assumption).

§4. Applications.

By using Theorem in §3, we can give the decomposition of H_M-module $S(k+1/2,N,\chi)$ under the same assumptions as in Theorem. For simplicity, we shall discuss only the decomposition of $S(k+1/2,4p^m,\chi)_K$, where k and m are some integers with $k \geqq 1$ and $m \geqq 0$, and p is an odd prime number and χ is an even Dirichlet character (modulo $4p^m$), namely, $\chi = \left(\frac{1}{\ }\right)$ or $\left(\frac{p}{\ }\right)$.

Before the statement of the results, we must introduce some notations. Let δ_ψ be the twisting operator for the character $\psi = \left(\frac{\ }{p}\right)$ (cf. [S-Y]). By $S^0(2k,p^m)$, we denote the subspace of $S(2k,p^m)$ spaned by all newforms in $S(2k,p^m)$.

For $m \geqq 3$, we define (cf. [S-Y]):

$S_I(2k,p^m) = \{ S^0(2k,p^m) \ni f \mid f|[W]_{2k} = f,\ f|\delta_\psi[W]_{2k} = f|\delta_\psi\}$,

$S_{II}(2k,p^m) = \{ S^0(2k,p^m) \ni f \mid f|[W]_{2k} = f,\ f|\delta_\psi[W]_{2k} = - f|\delta_\psi\}$,

$S_{II_\psi}(2k,p^m) = \{ S^0(2k,p^m) \ni f \mid f|[W]_{2k} = - f,\ f|\delta_\psi[W]_{2k} = f|\delta_\psi\}$,

$S_{III}(2k,p^m) = \{ S^0(2k,p^m) \ni f \mid f|[W]_{2k} = - f,\ f|\delta_\psi[W]_{2k} = - f|\delta_\psi\}$,

where $W = W(p^m)$ (cf. §3).

For $m = 2$, let $S^n(2k,p^2)$ be the orthogonal complement of $S^0(2k,p)|\delta_\psi + S^0(2k,1)|\delta_\psi$ in $S^0(2k,p^2)$ with respect to the Petterson inner product. Then, we define (cf. [S-Y]):

$S_I(2k,p^2) = \{ S^n(2k,p^2) \ni f \mid f|[W]_{2k} = f,\ f|\delta_\psi[W]_{2k} = f|\delta_\psi\}$,

$S_{II}(2k,p^2) = \{ S^n(2k,p^2) \ni f \mid f|[W]_{2k} = f,\ f|\delta_\psi[W]_{2k} = - f|\delta_\psi\}$,

$S_{II_\psi}(2k,p^2) = \{ S^n(2k,p^2) \ni f \mid f|[W]_{2k} = - f,\ f|\delta_\psi[W]_{2k} = f|\delta_\psi\}$,

$S_{III}(2k,p^2) = \{ S^n(2k,p^2) \ni f \mid f|[W]_{2k} = - f,\ f|\delta_\psi[W]_{2k} = - f|\delta_\psi\}$,

where $W = W(p^2)$ (cf. §3).

Under these notations, we have the following decompositions as the H_{p^m} - modules.

Proposition 5. Suppose $k \geqq 2$. Then, we have the following decompositions as the H_{p^m} - modules:

(i) $(m = 0)$. $\quad S(k+1/2,4)_K \cong S^0(2k,1)$.

(ii) $(m = 1)$.

$$S(k+1/2,4p)_K \cong S(k+1/2,4p,\left(\tfrac{p}{}\right))_K$$

$$\cong S^0(2k,p) \oplus 2\, S^0(2k,1).$$

(iii) $(m = 2$ and $\chi = \left(\tfrac{1}{}\right))$.

$$S(k+1/2,4p^2)_K \cong 2\,(S_I(2k,p^2) \oplus S_{II}(2k,p^2))$$

$$\oplus\ (1 + \left(\tfrac{-1}{p}\right))(S^0(2k,p)|\delta_\psi \oplus S^0(2k,1)|\delta_\psi)$$

$$\oplus\ 2\, S^0(2k,p) \oplus 4\, S^0(2k,1).$$

(iv) $(m = 2$ and $\chi = \left(\tfrac{p}{}\right))$.

$$S(k+1/2,4p^2,\left(\tfrac{p}{}\right))_K$$

$$\cong (1 + \left(\tfrac{-1}{p}\right))(S_I(2k,p^2) \oplus S_{II_\psi}(2k,p^2))$$

$$\oplus\ (1 - \left(\tfrac{-1}{p}\right))(S_{II}(2k,p^2) \oplus S_{III}(2k,p^2))$$

$$\oplus\ S^0(2k,p)|\delta_\psi \oplus (1 + \left(\tfrac{-1}{p}\right))\ S^0(2k,1)|\delta_\psi$$

$$\oplus\ 3\, S^0(2k,p) \oplus 4\, S^0(2k,1).$$

(v) $(m = 2a + 3$ with $a \geqq 0)$.

$$S(k+1/2,4p^{2a+3})_K \cong S(k+1/2,4p^{2a+3},\left(\tfrac{p}{}\right))_K$$

$$\cong \sum_{b=1}^{a+1} (2a + 3 - 2b)\ S^0(2k,p^{2b+1})$$

$$\oplus \sum_{b=1}^{a+1} (3 + \left(\tfrac{-1}{p}\right))(a + 2 - b)\ S_I(2k,p^{2b})$$

$$\oplus \sum_{b=1}^{a+1} (3 - \left(\tfrac{-1}{p}\right))(a + 2 - b)\ S_{II}(2k,p^{2b})$$

$$\oplus \sum_{b=1}^{a+1} \left(1 + \left(\frac{-1}{p}\right)\right)(a + 2 - b)\ S_{II_\psi}(2k, p^{2b})$$

$$\oplus \sum_{b=1}^{a+1} \left(1 - \left(\frac{-1}{p}\right)\right)(a + 2 - b)\ S_{III}(2k, p^{2b})$$

$$\oplus \left(2 + \left(\frac{-1}{p}\right)\right)(a + 1)\ S^0(2k, p)\,|\,\delta_\psi$$

$$\oplus \left(1 + \left(\frac{-1}{p}\right)\right)(2a + 2)\ S^0(2k, 1)\,|\,\delta_\psi$$

$$\oplus (3a + 4)\ S^0(2k, p) \oplus (4a + 6)\ S^0(2k, 1).$$

(vi) ($m = 2a + 4$ with $a \geqq 0$ and $\chi = \left(\frac{1}{}\right)$).

$$S(k+1/2, 4p^{2a+4})_K$$

$$\cong 2\ (S_I(2k, p^{2a+4}) \oplus S_{II}(2k, p^{2a+4})) \oplus \sum_{b=1}^{a+1} (2a + 4 - 2b)\ S^0(2k, p^{2b+1})$$

$$\oplus \sum_{b=1}^{a+1} \left\{\left(3 + \left(\frac{-1}{p}\right)\right)(a + 2 - b) + 2\right\} S_I(2k, p^{2b})$$

$$\oplus \sum_{b=1}^{a+1} \left\{\left(3 - \left(\frac{-1}{p}\right)\right)(a + 2 - b) + 2\right\} S_{II}(2k, p^{2b})$$

$$\oplus \sum_{b=1}^{a+1} \left(1 + \left(\frac{-1}{p}\right)\right)(a + 2 - b)\ S_{II_\psi}(2k, p^{2b})$$

$$\oplus \sum_{b=1}^{a+1} \left(1 - \left(\frac{-1}{p}\right)\right)(a + 2 - b)\ S_{III}(2k, p^{2b})$$

$$\oplus \left\{\left(1 + \left(\frac{-1}{p}\right)\right)(a + 2) + a + 1\right\} S^0(2k, p)\,|\,\delta_\psi$$

$$\oplus \left(1 + \left(\frac{-1}{p}\right)\right)(2a + 3)\ S^0(2k, 1)\,|\,\delta_\psi$$

$$\oplus (3a + 5)\ S^0(2k, p) \oplus (4a + 8)\ S^0(2k, 1).$$

(vii) ($m = 2a + 4$ with $a \geqq 0$ and $\chi = \left(\frac{p}{}\right)$).

$$S(k+1/2, 4p^{2a+4}, \left(\frac{p}{}\right))_K$$

$$\cong \left(1 + \left(\frac{-1}{p}\right)\right)(S_I(2k,p^{2a+4}) \oplus S_{II_\psi}(2k,p^{2a+4}))$$

$$\oplus \left(1 - \left(\frac{-1}{p}\right)\right)(S_{II}(2k,p^{2a+4}) \oplus S_{III}(2k,p^{2a+4}))$$

$$\oplus \sum_{b=1}^{a+1} (2a + 4 - 2b)\ S^0(2k,p^{2b+1})$$

$$\oplus \sum_{b=1}^{a+1} \left\{\left(3 + \left(\frac{-1}{p}\right)\right)(a + 2 - b) + 1 + \left(\frac{-1}{p}\right)\right\} S_I(2k,p^{2b})$$

$$\oplus \sum_{b=1}^{a+1} \left\{\left(3 - \left(\frac{-1}{p}\right)\right)(a + 2 - b) + 1 - \left(\frac{-1}{p}\right)\right\} S_{II}(2k,p^{2b})$$

$$\oplus \sum_{b=1}^{a+1} \left(1 + \left(\frac{-1}{p}\right)\right)(a + 3 - b)\ S_{II_\psi}(2k,p^{2b})$$

$$\oplus \sum_{b=1}^{a+1} \left(1 - \left(\frac{-1}{p}\right)\right)(a + 3 - b)\ S_{III}(2k,p^{2b})$$

$$\oplus \left\{\left(1 + \left(\frac{-1}{p}\right)\right)(a + 1) + a + 2\right\} S^0(2k,p)|\delta_\psi$$

$$\oplus \left(1 + \left(\frac{-1}{p}\right)\right)(2a + 3)\ S^0(2k,1)|\delta_\psi$$

$$\oplus (3a + 6)\ S^0(2k,p) \oplus (4a + 8)\ S^0(2k,1).$$

Here, the symbols $\oplus$ and Σ mean the direct sum as the H_{p^m} - modules, and the coefficient in front of the H_{p^m} - module $S(2k, p^n)$ $(0 \leqq n \leqq m)$ is the multiplicity.

If we replace $S(k+1/2, 4p^m, \chi)_K$ by $V(4p^m, \chi)_K$ and put $k = 1$ at the right-hand side of the decompositions (i) - (vii), we have the decomposition of the H_{p^m} - module $V(4p^m, \chi)_K$.

Remark. Let m be a positive integer and let s be an integer such that $0 \leqq s \leqq m$. Then, for a positive integer n with $(n,p) = 1$, $T_{2k,p^s}(n)$ coinsides with $T_{2k,p^m}(n)$ as the operator on $S(2k,p^m)$. Therefore, we can naturally consider $S(2k,p^s)$ and $S^0(2k,p^s)$ as the H_{p^m} - modules.

Proof.

The decomposition (i) and (ii) are immediate consequences from Corollary in §3.

Hence, in the following, we assume that $m \geqq 1$ and that the letter n means any positive integer prime to $2p$.

Firstly. we note that, when m is odd, by using Theorem in §3, we have

$$\operatorname{trace}(\tilde{T}_{k+1/2,4p^m,\left(\frac{1}{\cdot}\right)}(n^2)_K \mid S(k+1/2,4p^m)_K)$$

$$= \operatorname{trace}(\tilde{T}_{k+1/2,4p^m,\left(\frac{p}{\cdot}\right)}(n^2)_K \mid S(k+1/2,4p^m,\left(\tfrac{p}{\cdot}\right))_K).$$

Since the operator $\tilde{T}_{k+1/2,4p^m,\chi}(n^2)_K$ is hermitian with respect to the Petterson inner product, we get, for any odd integer $m \geqq 1$, $S(k+1/2,4p^m)_K \cong S(k+1/2,4p^m,\left(\frac{p}{\cdot}\right))_K$ as the H_{p^m} - modules.

Next, from the definitions and lemma 5.1 of [S-Y], we can easily get the following identities:

$$\operatorname{trace}(T_{2k,p^m}(n) \mid S(2k,p^m))$$

$$= \sum_{a=0}^{m} (m + 1 - a) \operatorname{trace}(T_{2k,p^a}(n) \mid S^0(2k,p^a)),$$

and, for an integer $t \geqq 1$,

$$\mathrm{trace}([W(p^{2t})]_{2k}\, T_{2k,p^{2t}}(n) \mid S(2k,p^{2t}))$$

$$= \sum_{a=1}^{t} \mathrm{trace}(T_{2k,p^{2a}}(n) \mid S_{I}(2k,p^{2a}) \oplus S_{II}(2k,p^{2a}))$$

$$- \sum_{a=1}^{t} \mathrm{trace}(T_{2k,p^{2a}}(n) \mid S_{II_{\psi}}(2k,p^{2a}) \oplus S_{III}(2k,p^{2a}))$$

$$+ \left(\frac{-1}{p}\right) \mathrm{trace}(T_{2k,p^{2}}(n) \mid S^{0}(2k,p)|\delta_{\psi} \oplus S^{0}(2k,1)|\delta_{\psi})$$

$$+ \mathrm{trace}(T_{2k,1}(n) \mid S^{0}(2k,1)).$$

Moreover, from Proposition 1.1 (and section 5) of [S-Y], we have: $S_{I}(2k,p^{2a})|\delta_{\psi} = S_{I}(2k,p^{2a})$,

$$S_{II}(2k,p^{2a})|\delta_{\psi} = S_{II_{\psi}}(2k,p^{2a}),$$

$$S_{II_{\psi}}(2k,p^{2a})|\delta_{\psi} = S_{II}(2k,p^{2a}) \quad \text{and}$$

$$S_{III}(2k,p^{2a})|\delta_{\psi} = S_{III}(2k,p^{2a}) \quad \text{for an integer } a \geqq 1.$$

Now, for an integer $t \geqq 1$, we put

$$A_{2t} = \mathrm{trace}(\tilde{T}_{k+1/2,4p^{2t},\left(\frac{1}{\cdot}\right)}(n^{2})_{K} \mid S(k+1/2,4p^{2t})_{K})$$

$$- \mathrm{trace}(\tilde{T}_{k+1/2,4p^{2t-1},\chi}(n^{2})_{K} \mid S(k+1/2,4p^{2t-1},\chi)_{K}),$$

and $$B_{2t} = \mathrm{trace}(\tilde{T}_{k+1/2,4p^{2t},\left(\frac{p}{\cdot}\right)}(n^{2})_{K} \mid S(k+1/2,4p^{2t},\left(\tfrac{p}{\cdot}\right))_{K})$$

$$- \mathrm{trace}(\tilde{T}_{k+1/2,4p^{2t-1},\chi}(n^{2})_{K} \mid S(k+1/2,4p^{2t-1},\chi)_{K})$$

From Theorem in §3 and the above facts, we have

$$A_{2t} = \mathrm{trace}(T_{2k,p^{2t}}(n) \mid S(2k,p^{2t}))$$

$$+ \mathrm{trace}([W(p^{2t})]_{2k}\, T_{2k,p^{2t}}(n) \mid S(2k,p^{2t}))$$

$$- \mathrm{trace}(T_{2k,p^{2t-1}}(n) \mid S(2k,p^{2t-1}))$$

$$= 2 \sum_{a=1}^{t} \mathrm{trace}(T_{2k,p^{2a}}(n) \mid S_{I}(2k,p^{2a}) \oplus S_{II}(2k,p^{2a}))$$

$$+ \sum_{b=1}^{t} \mathrm{trace}(T_{2k,p^{2b-1}}(n) \mid S^{0}(2k,p^{2b-1}))$$

$$+ \left(1 + \left(\frac{-1}{p}\right)\right) \mathrm{trace}(T_{2k,p^{2}}(n) \mid S^{0}(2k,p)|\delta_{\psi} \oplus S^{0}(2k,1)|\delta_{\psi})$$

$$+ 2\ \mathrm{trace}(T_{2k,1}(n) \mid S^{0}(2k,1)),$$

and $$B_{2t} = \mathrm{trace}(T_{2k,p^{2t}}(n) \mid S(2k,p^{2t}))$$

$$+ \left(\frac{-n}{p}\right) \mathrm{trace}([W(p^{2t})]_{2k}\, T_{2k,p^{2t}}(n) \mid S(2k,p^{2t}))$$

$$- \mathrm{trace}(T_{2k,p^{2t-1}}(n) \mid S(2k,p^{2t-1}))$$

$$= \left(1 + \left(\frac{-1}{p}\right)\right) \sum_{a=1}^{t} \mathrm{trace}(T_{2k,p^{2a}}(n) \mid S_{I}(2k,p^{2a}) \oplus S_{II_{\psi}}(2k,p^{2a}))$$

$$+ \left(1 - \left(\frac{-1}{p}\right)\right) \sum_{a=1}^{t} \mathrm{trace}(T_{2k,p^{2a}}(n) \mid S_{II}(2k,p^{2a}) \oplus S_{III}(2k,p^{2a}))$$

$$+ \sum_{b=1}^{t} \mathrm{trace}(T_{2k,p^{2b-1}}(n) \mid S^{0}(2k,p^{2b-1}))$$

$$+ \mathrm{trace}(T_{2k,p^{2}}(n) \mid S^{0}(2k,p)|\delta_{\psi})$$

$$+ \left(1 + \left(\frac{-1}{p}\right)\right) \mathrm{trace}(T_{2k,p^{2}}(n) \mid S^{0}(2k,1)|\delta_{\psi})$$

$$+ \mathrm{trace}(T_{2k,p}(n) \mid S^{0}(2k,p))$$

$$+ 2\ \mathrm{trace}(T_{2k,1}(n) \mid S^{0}(2k,1)).$$

From this expression of A_2, we have

$$\mathrm{trace}(\tilde{T}_{k+1/2,4p^2,\left(\frac{1}{\cdot}\right)}(n^2)_K \mid S(k+1/2,4p^2)_K)$$

$$= 2 \quad \mathrm{trace}(T_{2k,p^2}(n) \mid S_I(2k,p^2) \oplus S_{II}(2k,p^2))$$

$$+ (1 + \left(\frac{-1}{p}\right)) \; \mathrm{trace}(T_{2k,p^2}(n) \mid S^0(2k,p)|\delta_\psi \oplus S^0(2k,1)|\delta_\psi))$$

$$+ 2 \quad \mathrm{trace}(T_{2k,p}(n) \mid S^0(2k,p))$$

$$+ 4 \quad \mathrm{trace}(T_{2k,1}(n) \mid S^0(2k,1)).$$

The decomposition (iii) follows from this equality of traces and we can prove the decomposition (iv) in a similar way.

When $m = 2a + 3$ with $a \geqq 0$, by using Theorem in §3, we have

$$\mathrm{trace}(\tilde{T}_{k+1/2,4p^{2a+3},\chi}(n^2)_K \mid S(k+1/2,4p^{2a+3},\chi)_K)$$

$$- \quad \mathrm{trace}(\tilde{T}_{k+1/2,4p^{2a+2},\left(\frac{1}{\cdot}\right)}(n^2)_K \mid S(k+1/2,4p^{2a+2})_K)$$

$$- \quad \mathrm{trace}(\tilde{T}_{k+1/2,4p^{2a+2},\left(\frac{p}{\cdot}\right)}(n^2)_K \mid S(k+1/2,4p^{2a+2},\left(\frac{p}{\cdot}\right))_K)$$

$$+ \quad \mathrm{trace}(\tilde{T}_{k+1/2,4p^{2a+1},\chi}(n^2)_K \mid S(k+1/2,4p^{2a+1},\chi)_K)$$

$$= \quad \mathrm{trace}(T_{2k,p^{2a+3}}(n) \mid S(2k,p^{2a+3}))$$

$$- 2 \quad \mathrm{trace}(T_{2k,p^{2a+2}}(n) \mid S(2k,p^{2a+2}))$$

$$+ \quad \mathrm{trace}(T_{2k,p^{2a+1}}(n) \mid S(2k,p^{2a+1}))$$

$$= \quad \mathrm{trace}(T_{2k,p^{2a+3}}(n) \mid S^0(2k,p^{2a+3})).$$

Hence, we get inductively

$$\mathrm{trace}(\tilde{T}_{k+1/2,4p^{2a+3},\chi}(n^2)_K \mid S(k+1/2,4p^{2a+3},\chi)_K)$$

$$= \mathrm{trace}(T_{2k,p^{2a+3}}(n) \mid S^0(2k,p^{2a+3})) + A_{2a+2} + B_{2a+2}$$

$$+ \mathrm{trace}(\tilde{T}_{k+1/2,4p^{2a+1},\chi}(n^2)_K \mid S(k+1/2,4p^{2a+1},\chi)_K)$$

$$= \sum_{b=0}^{a} \mathrm{trace}(T_{2k,p^{2b+3}}(n) \mid S^0(2k,p^{2b+3}))$$

$$+ \sum_{b=0}^{a} A_{2b+2} + \sum_{b=0}^{a} B_{2b+2}$$

$$+ \mathrm{trace}(\tilde{T}_{k+1/2,4p,\chi}(n^2)_K \mid S(k+1/2,4p,\chi)_K).$$

From this equality and the above expressions of A_{2t} and B_{2t}, we immediately obtain the decomposition (v).

Finally, by using the expressions of A_{2a+4} and B_{2a+4} with $a \geqq 0$, we can deduce the decompositions (vi) and (vii) from the decomposition (v).

In this Proposition 5, we note the following:
When m is zero, or an odd positive integer, the H_{p^m} - module $S^0(2k,p^m)$ occurs with the multiplicity one in the decomposition of the H_{p^m} - module $\begin{cases} S(k+1/2,4p^m,\chi)_K, & \text{if } k \geqq 2, \\ V(4p^m,\chi)_K, & \text{if } k = 1. \end{cases}$

Hence, the non-zero element f of $S(k+1/2,4p^m,\chi)_K$ or $V(4p^m,\chi)_K$, which corresponds to a primitive form F in $S^0(2k,p^m)$, also become the common eigen form with respect to the n-th Hecke operator for $n = 2$ and p. (cf. [K] Preliminaries (a) and §3, as in the definitions of the n-th Hecke operator for $n = 2$ and p.)

Let the Fourier expansion of f (resp. F) be as follows: $f = \sum_{n=1}^{\infty} a(n)\, e(nz)$ (resp. $F = \sum_{n=1}^{\infty} A(n)\, e(nz)$). If u is a fundamental discriminant with $\varepsilon(-1)^k u > 0$, then, their Fourier expansions are related as follows:

$$L(s - k + 1, \chi(\tfrac{u}{\cdot})) \sum_{n=1}^{\infty} a(|u|\, n^2)\, n^{-s}$$

$$= a(|u|) \sum_{n=1}^{\infty} A(n)\, n^{-s} \quad \text{(cf. [K] §5 Theorem 2).}$$

Therefore, all primitive forms in $S^0(2k,p^m)$ are constructed from some elements of $S(k+1/2,4p^m,\chi)_K$ or $V(4p^m,\chi)_K$ by means of Shimura (- Niwa - Kohnen) correspondence.

In fact, for more general situations, the similar results can be proved by using Theorem in §3. But, we omit the details.

Appendix.

Let A, $D(A)$ and Γ_0 be the same as in Remark 4 of §2. We shall calculate $n(A) = \#(D(A)/\Gamma_0)$.

For a representative x for $(Z/MZ)^\times$ and a prime divisor p of M, we define the sets $V(x)$, $V_2(x)$ and $V_p(x)$ as follows:

$$V(x) = \left\{ SL_2(Z) \ni B \;\middle|\; \begin{array}{l} B^{-1}AB \equiv \begin{pmatrix} 4x + 4M\nu & * \\ 0 & * \end{pmatrix} \pmod{16M} \\ \text{for some } \nu \in Z \end{array} \right\}.$$

$$V_2(x) = \left\{ SL_2(Z) \ni B \;\middle|\; \begin{array}{l} B^{-1}AB \equiv \begin{pmatrix} 4x + 4M\nu & * \\ 0 & * \end{pmatrix} \pmod{16} \\ \text{for some } \nu \in Z \end{array} \right\}.$$

$$V_p(x) = \left\{ SL_2(Z) \ni B \;\middle|\; B^{-1}AB \equiv \begin{pmatrix} 4x & * \\ 0 & * \end{pmatrix} \pmod{p^{\nu_p}} \right\}$$

with $\nu_p = \mathrm{ord}_p(M)$.

Then, $\Gamma_0 = \Gamma_0(4M)$, $\Gamma_0(4)$ and $\Gamma_0(p^{\nu_p})$ operate on the sets $V(x)$, $V_2(x)$ and $V_p(x)$ respectively by multiplication from the right, and $D(A) = \bigcup_{x \in (Z/MZ)^\times} V(x)$ (disjoint union).

Moreover, we can define the isomorphism ϕ from

$V_2(x)/\Gamma_0(4) \times \prod_{p|M} V_p(x)/\Gamma_0(p^{\nu_p})$ to $V(x)/\Gamma_0$ as follows:

Take any element $(B_2, (B_p)_{p|M})$ of $V_2(x) \times \prod_{p|M} v_p(x)$. Then, there exists an element B of $SL_2(Z)$ such that $B \equiv B_2 \pmod{16}$ and that $B \equiv B_p \pmod{p^{\nu_p}}$ for all prime divisors p of M. And we define $\phi((B_2\, \Gamma_0(4), (B_p\, \Gamma_0(p^{\nu_p}))_{p|M})) = B\, \Gamma_0$.

Now, from the discussion in [K] §4 Appendix (proof of lemma 5), we know $\#(V_2(x)/\Gamma_0(4)) = 1$.

Therefore, we obtain

$$n(A) = \sum_{x \in (Z/MZ)^\times} \#(V(x)/\Gamma_0)$$
$$= \sum_{x \in (Z/MZ)^\times} \prod_{p|M} \#(V_p(x)/\Gamma_0(p^{\nu_p}))$$
$$= \prod_{p|M} \left(\sum_{x \in (Z/p^{\nu_p}Z)^\times} \#(V_p(x)/\Gamma_0(p^{\nu_p})) \right) .$$

In order to calculate $\#(V_p(x)/\Gamma_0(p^{\nu_p}))$, we note the following general facts:

Let L be an positive integer. We denote by $C(L)$ the set consisting of all the elements of $(Z/LZ) \times (Z/LZ)$ whose order is exactly L. We define the equivalence relation $\sim$ of $C(L)$ as follows: For two elements $(\bar{c}_1, \bar{d}_1)$, $(\bar{c}_2, \bar{d}_2)$ of $C(L)$,

$(\bar{c}_1, \bar{d}_1) \sim (\bar{c}_2, \bar{d}_2)$ if and only if there exists $\bar{m} \in (Z/LZ)^\times$

such that $\bar{m}(\bar{c}_1, \bar{d}_1) = (\bar{c}_2, \bar{d}_2)$.

Then, we have the bijection $SL_2(Z)/\Gamma_0(L) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Gamma_0(L) \longrightarrow$ the equivalence class containing $(a \bmod L, b \bmod L) \in C(L)/\sim$.

Now, we shall calculate $\#(V_p(x)/\Gamma_0(p^{\nu_p}))$.

Let write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, for $B = \begin{pmatrix} u & w \\ v & z \end{pmatrix} \in SL_2(Z)$, the condition $B^{-1}AB \equiv \begin{pmatrix} 4x & * \\ 0 & * \end{pmatrix} \pmod{p^{\nu_p}}$ is equivalent to

$\begin{pmatrix} a-4x & b \\ c & d-4x \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \pmod{p^{\nu_p}}$. By using the elementary divisor theory, there exist U_1 and $U_2 \in SL_2(Z)$ such that

$\begin{pmatrix} a-4x & b \\ c & d-4x \end{pmatrix} = U_1 \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} U_2$. Here, $g_1 = (a-4x, b, c, d-4x)$

and $g_1 g_2 = ((a-4x)(d-4x) - bc) = ((4x)^2 - 4tx + 16n^2)$,

where $t = a + d$, $ad - bc = 16n^2$ from the assumptions.

Put $\alpha = \mathrm{ord}_p(g_1)$ and $\beta = \mathrm{ord}_p(g_2)$. Since p is odd,

$g_1 Z_p = (a - d, b, c, t - 8x) Z_p$ Hence, $\alpha = \min(\rho_p, \tau_{p,x})$,

where $\rho_p = \mathrm{ord}_p(f)$ with $f = (a - d, b, c)$, and $\tau_{p,x} = \mathrm{ord}_p(t - 8x)$.

Thus, we have

$$\#(V_p(x)/\Gamma_0(p^{\nu_p})) = \#\left\{ \begin{array}{l} C(p^{\nu_p})/\sim \ni \text{the equivalence class} \\ \text{containing } (u,v) \bmod p^{\nu_p} \text{ such that} \\ p^{\alpha}u \equiv p^{\beta}v \equiv 0 \pmod{p^{\nu_p}} \end{array} \right\}$$

$$= \begin{cases} p^{\nu_p} + p^{\nu_p - 1}, & \text{if } \alpha \geqq \nu_p, \\ p^{\alpha}, & \text{if } \beta \geqq \nu_p > \alpha, \\ 0, & \text{if } \nu_p > \beta. \end{cases}$$

Firstly, we assume $\rho_p \geqq \nu_p$. If $\tau_{p,x} < \nu_p$, we have

$\mathrm{ord}_p((t^2 - 64n^2)/4) \geqq 2\rho_p > 2\tau_{p,x} = \mathrm{ord}_p((4x - t/2)^2)$. Hence,

$\alpha + \beta = \mathrm{ord}_p((4x - t/2)^2 - (t^2 - 64n^2)/4) = 2\tau_{p,x} = \alpha + \tau_{p,x}$.

Therefore, if $\#(V_p(x)/\Gamma_0(p^{\nu_p})) \neq 0$, then we get $\tau_{p,x} \geqq \nu_p$, namely, $t \equiv 8x \pmod{p^{\nu_p}}$.

Thus, $$\sum_{x \in (Z/p^{\nu_p}Z)^{\times}} \#(V_p(x)/\Gamma_0(p^{\nu_p})) = p^{\nu_p} + p^{\nu_p - 1}.$$

Next, we assume that $\nu_p > \rho_p$ If $\tau_{p,x} < \rho_p$, we have

$\alpha = \tau_{p,x}$ and $\mathrm{ord}_p((4x - t/2)^2) = 2\,\tau_{p,x} < 2\,\rho_p \leqq \mathrm{ord}_p((t^2 - 64n^2)/4)$.

Hence, $\alpha + \beta = \mathrm{ord}_p((4x - t/2)^2 - (t^2 - 64n^2)/4) = 2\,\tau_{p,x}$

$= 2\,\alpha < \alpha + \nu_p$. Therefore, if $\#(V_p(x)/\Gamma_0(p^{\nu_p})) \neq 0$, we get

$\tau_{p,x} \geqq \rho_p$. Thus, we obtain

$$\sum_{x \in (Z/p^{\nu_p}Z)^\times} \#(V_p(x)/\Gamma_0(p^{\nu_p}))$$

$$= p^{\rho_p} \times \# \left\{ (Z/p^{\nu_p}Z) \ni x \;\middle|\; \begin{array}{l} t \equiv 8x \pmod{p^{\rho_p}} \text{ and} \\ x^2 - (t/4)x + n^2 \equiv 0 \pmod{p^{\nu_p+\rho_p}} \end{array} \right\}$$

Moreover, we can omit the assumption $t \equiv 8x \pmod{p^{\rho_p}}$ by using the assumption $\nu_p > \rho_p$ and the fact $\mathrm{ord}_p((t^2 - 64n^2)/4) \geqq 2\,\rho_p$

References.

[D-M] K.Doi and T. Miyake, Modular forms and number theory, Kinokuniya (1976) (in japanese).

[H] H. Hijikata, Explicite formula of the traces of Hecke operators for $\Gamma_0(N)$, J. Math. Soc. Japan, vol. 26, No. 1 (1974), p. 56 - 82.

[K] W. Kohnen, Newforms of half-integral weight, J. reine und angew. Math., 333 (1982), p. 32 - 72.

[N] S. Niwa, On Shimura's trace formula, Nagoya Math. J., vol. 66 (1977), p. 183 - 202.

[Sh 1] G. Shimura, The critical values of certain zeta functions associated with modular forms of half-integral weight, J. Math. Soc. Japan, vol. 33, No. 4 (1981), p. 649 - 672.

[Sh 2] ———, On modular forms of half integral weight, Ann. of Math., vol. 97 (1973), p. 440 - 481.

[Sh 3] ———, On the trace formula for Hecke operators, Acta Math., vol. 132 (1974), p. 245 - 281.

[S-S] J.-P. Serre and H. M. Stark, Modular forms of weight 1/2, Springer Lecture Notes in Math., vol. 627 (1977), p. 27 - 67.

[St] J. Sturm, Theta series of weight 3/2, J. of Number Theory, vol. 14 (1982). p. 353 - 361.

[S-Y] H. Saito and M. Yamauchi, Trace formula of certain Hecke operators for $\Gamma_0(q^\nu)$, Nagoya Math. J., vol. 76 (1979), p. 1 - 33.

[Y] M. Yamauchi, On the traces of Hecke operators for a normalizer of $\Gamma_0(N)$, J. Math. Kyoto Univ., vol. 13 (1973), p. 403 - 411.