学位申請論文

佐藤文隆

主論文

General Relativistic Instability of the Supermassive Stars *)

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Abstract

The instability criterion of a star without rotation is derived by the method of variation, in the post-Newtonian approximation, and proved to coincide with the results of Chandrasekhar's paper.

To apply this criterion to the supermassive stars, we have calculated the adiabatic exponent in the case of the electron pair creation, and then obtained the following results: the instability results from the general relativistic effect for the stars of $M > 3.5 \cdot 10^4 M_\odot$, and from the electron pair creation for $M < 3.5 \cdot 10^4 M_\odot$.

The application of our treatments are limited to $10^{9.8} M_\odot > M > 10^3 M_\odot$ from the following reasons: the upper limit results from that the star above this limit can not be in a quasi-static equilibrium but in a free fall contraction, and the lower limit results from our assumption on the structure taking a simple polytrope of index 3.

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§1 Introduction and Summary

In an attempt to understand the source of the energy emitted by the radio glaxies including the star-like objects, Hoyle and Fowler suggested the possibility that a mass of the order of $10^8 M_\odot$ has condensed into a super-massive star in the galactic nucleus.[1] Several attempts[2),3),4),*)] to derive the energy from this supermassive star have met many difficulties, but their model has proposed many theoretically interesting problem; electron pair creation in the stellar interior,[2] equilibrium configuration and stability of the star allowing general relativistic effect,[5),6)] and general relativistic collapse.[8] The purpose of this paper is to clarify the effects of the electron pair creation and the general relativity on the star, but a relation to the radio galaxies will not be discussed directly in this paper.

In the case of non-relativistic gravitational equilibrium, an instability of a star occurs when the adiabatic exponent γ is smaller than 4/3 as a result of the endothermic phase

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*) Recently Fowler proposed a new theory and has asserted their supermassive star to be a possible model of the radio galaxies.[7] According to him, a small amount of rotation of the star removes the general relativistic instability and the star with $M \simeq 10^8 M_\odot$ can evolve stably into the stage where the hydrogen burning commences.

transition*) of the constituent matter.[9] In the case of general relativistic equilibrium, on the contrary, this criterion is revised so that an instability occurs even when γ is greater than 4/3 as shown by Chandrasekhar and other authors.[10),11),12)] In the investigation of the supermassive star, we must take into consideration the both effects: (1) the value of γ becomes smaller than 4/3 because of the electron pair creation in the high temperature gaseous mass around $T \simeq 10^{9\,o}K$, and (2) the general relativistic effect becomes significant as supposed from that the Shwarzschild radius increases with mass and gets to the same order of the stellar radius. Then, we must make clear which effect between the above two is really operative as the cause of the instability.

In this paper, we investigate mainly the above problem, and then make clear the applicability of our treatments. The conclusions of this paper are summarized in Fig. 1 and 2. In these figures, the lines designated by the mass values show the relations between the central temperature T_c and the central rest-mass density ρ_{oc} of the stars with these masses, and the shaded regions indicate the instability region and the free fall region, in which the star can not be in a stable equilibrium state. As a model of the star, we have taken a polytrope with index N = 3 because, in the supermassive star,

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*) These phase transitions are caused accompanying with dissociation of hydrogen molecule, ionization of atoms, electron pair creation, dissociation of Fe nucleus and so on.

the configuration is considered to be in a wholly convective state and the radiation pressure is dominant over the gas pressure.

Fig. 1, Fig. 2

From Fig. 1, we can see that for the stars with mass $M > 3.5 \cdot 10^4 M_\odot$ the instability is caused by the general relativistic effect at temperatures lower than those at which $\gamma = 4/3$. This general relativistic effect can be treated in the post-Newtonian approximation. In fact, the relativistic parameter (i.e., Schwarzschild radius divided by the actual radius of the star) is very smaller than unity, i.e., we have $2GM/Rc^2 < 3.10^{-3}$ even at the outset of the instability.

We can also see that the stars with mass $M < 3.5 \cdot 10^4 M_\odot$ become unstable in the $\gamma < 4/3$ region. The cause of $\gamma < 4/3$ is due to the electron pair creation for $M < (50 \sim 100) M_\odot$ and the dissociation of Fe nucleus into alpha-particles and neutrons for the smaller mass stars. In these cases, the instability will really occur only when a fairly large region of the stellar interior becomes to be contained in the $\gamma < 4/3$ region, and the distance between the curve of the outset of instability and the distance between the curve of the outset of instability and the curve of $\gamma = 4/3$ becomes greater as the mass becomes smaller.

For the smaller mass star (for example, $M < 10^3 M_\odot$), the configuration at the outset point of instability can not be represented by the simple polytrope with $N = 3$, because the influence of neutrino losses and nuclear burning to the stellar structure may be important.

It can be shown that, for a star with mass smaller than the Chandrasekhar mass limit, T_c decreases steeply with the increase of ρ_0 after attaining a maximum temperature and at the center it becomes finally very low temperature and high density.[13] Chandrasekhar's remark[11] that the general relativistic instability is significant for the white dwarfs is concerned with these stars.

Figure 2 shows that T_c at the outset of instability decreases with increasing mass and becomes far below the critical temperature of hydrogen burning. In this case, the energy to retain the energy output determined by the opacity must be supplied from the gravitational energy. Then, if the cooling time by the energy output by photons t_{cp} is shorter than the free fall time of the star t_{ff}, the star can not be in an quasi-static equilibrium state and is in a free fall contraction state. This situation is the same as in the early stage of the formation of ordinary mass stars.[14]

The free fall region in Fig. 2 shows the region where $t_{cp} < t_{ff}$, and we notice from this figure that the stars with mass $M > 10^9 M_\odot$ is always restricted in the instability region or the free fall region and can not be a stable quasi-static star throughout the whole lives.

In the following sections, the above mentioned results will be derived by the treatments as follows: In §2, the criterion of instability is recalculated in the post-Newtonian approximation using the method of variation*). This treatment differs from that using the full set of Einstein's field equations, but gives the same results. In §3, the value of γ in

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the case of electron pair creation is calculated. In §4, using the results of §2 and §3, we represent the instability region in T_c- ρ_{oc} diagram. In §5, the relation between T_c and ρ_{oc} is obtained assuming the polytrope with N = 3, and so we can obtain the values of T_c and ρ_{oc} at the outset of instability. In §6, we consider about the ranges of applicable mass of our hydrostatic instability criterion and the assumption of a polytropic star.

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*) In the process of preparation of this paper, the present author was aware of that the essentially same calculation with that in §2 is carried out in the book by Harrison et al.[15]. In comparison with their calculation, ours is more simple and more instructive to compare with the Newtonian case.

§2. Hydrostatic instability in post-Newtonian approximation

In this section, we derive a criterion of instability of a spherically symmetric star without rotation, by the energy principle taking the variational method. The calculation is carried out in the post-Newtonian approximation for the sake of simplicity and later applications.

We consider a spherically symmetric system with motions, if any, only in the radial direction. There, we use the Schwarzschild metric such as[16]

$$ds^2 = c^2 e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{2.1}$$

where ν and λ are functions of r and t only. If the radial motions of a gaseous mass are adiabatic, all the components of the energy-momentum tensor in the outer region of the star are always zero, and then the outer solutions of λ and ν are independent of time. This statement which has been known as Birkoff's theorem[17)] implies also that the inertial mass measured by an external observer, M is independent of time, where M is given by

$$M^2 = -\int_{\text{(all space)}} T_o^{\,o}\, dV \tag{2.2}$$

and

$$-T_o^o = (\rho c^2 + (v/c)^2 P)/\sqrt{1-(v/c)^2}\,, \tag{2.3}$$

where $V = (4\pi/3)\, r^3$, v the velocity of matter referring to the coordinate of Eq. (2.1) and ρc^2 the energy density in the system locally moving with v.[18)]

We separate T_o^o into three terms; rest mass energy density of the particle whose number is conserved in the course of motion $\rho_o c^2$, interval energy density u and kinetic energy density of the mass motion. Then, Eq. (2.3) becomes as

$$-T_o^o = \rho_c c^2 + u + \frac{\rho}{2} v^2, \tag{2.4}$$

where we have assumed $(v/c)^2 \ll 1$.

Using the rest mass energy density, we define the proper rest mass M_o as

$$M_o = \int_o^R \rho_o\, e^{\lambda/2}\, dV, \tag{2.5}$$

where λ is given from the one of Einstein's field equation as[16)]

$$e^{-\lambda} = 1 - \frac{2G\,(M_r + \frac{1}{2}\int_0^r \rho (v/c)^2\, dV)}{r c^2} , \tag{2.6}$$

with

$$M_r = \int_0^r \rho \; dV. \tag{2.7}$$

Next, we make the post-Newtonian approximation in which the effects of general relativity are treated as first order correction to the Newtonian theory. In this approximation, we may assume for all r as

$$\left(\frac{v}{c}\right)^2 \ll \frac{u}{\rho_0 c^2} \sim \frac{GM_r}{rc^2} \ll 1, \tag{2.8}$$

since $u/\rho_0 c^2$ is of the order of GM_r / rc^2 as it will be verified later in Eq. (2.26), and since $(v/c)^2$ can be assumed infinitely small because we consider perturbational motions. Substituting Eq. (2.6) into Eq. (2.5), we can express the binding energy E (taken minus) in the post-Newtonian approximation as follows,

$$E \fallingdotseq (M - M_0)\, c^2 - T \tag{2.9}$$

$$= c^2 \int_0^{M_0} \left[\epsilon - \frac{r_s}{r} - \epsilon \frac{r_s}{r} - \frac{1}{2}\left(\frac{r_s}{r}\right)^2 \right] dM_{or} , \tag{2.10}$$

where $$T = \int_0^R \frac{1}{2}\rho \mathbf{v}^2 \, dV, \qquad (2.11)$$

$$M_{or} = \int_0^r \rho_0 e^{\lambda/2} dV, \qquad (2.12)$$

$$r_s = GM_r/c^2 \quad \text{and} \quad \epsilon = u/\rho_0 c^2 . \qquad (2.13)$$

We represent by δ the differences of the physical quantities between the hydrostatic equilibrium state and the perturbed state with radial mass motions. Taking the variation of each side of Eq. (2.9) and considering Birkoff's theorem, we have

$$\delta M c^2 = \delta E + \delta T = 0. \qquad (2.14)$$

If the perturbed states are realized by accompanying the mass motions. δ T must be positive and δ E is negative, that is the criterion of instability.[9)]

The variation of the binding energy, δ E, is calculated in Appendix A, where δ E is calculated in terms of the displacement of each proper rest mass element δr (M_{or}). The final results are expressed in terms of $g \equiv \delta V$ in place of δr up to the second order of g.

In these displacements, each proper mass element expands or contracts so quickly that the variation of the internal energy subjects to the adiabatic process which can be written as

$$\delta \epsilon = -P \, \delta v + (\gamma P/2v)(\delta v)^2 \qquad (2.15)$$

and $$\gamma = -(v/P)(\partial P/\partial v) \text{ adi}, \qquad (2.16)$$

where P is pressure and $v \equiv /(\rho_0 c^2)$.

As the coefficient of the first order term of g always vanishes as a result of an equilibrium condition of the unperturbed state, δE can be obtained up to the second order of g from Eq. (A.5) as follows,

$$\delta E = (\Gamma_0 - 4/3) \int_0^R (P/2)(g/V)^2 \, dV \tag{2.17}$$

where Γ_0 is given by Eq. (A.6). Therefore, the instability criterion is written as

$$\Gamma_0 < 4/3. \tag{2.18}$$

If the contraction or expansion is uniform, i.e. $g \propto V$, and if γ exceeds 4/3 only by a small amount of the order of ϵ, the expression of Eq. (A.6) can be simplified into Eqs. (A.7) and (A.8) as follows,

$$\Gamma_0 = \int_0^R (\gamma - a) P dV \Big/ \int_0^R P dV \tag{2.19}$$

and
$$a = \frac{\rho_0 c^2}{9P} \left[\left(\frac{r_s}{r}\right)^2 + \frac{6GPV}{c^4 r} + \frac{8 P r_s}{\rho_0 c^2 r} \right] . \tag{2.20}$$

The above result is the same as that of Chandrasekhar[10] and can be rewritten as follows,

$$\bar{\gamma} - 4/3 < C q, \tag{2.21}$$

where
$$\bar{\gamma} = \int_0^R \gamma P dV \Big/ \int_0^R P dV \tag{2.22}$$

$$q = P_c / \rho_{oc} c^2, \tag{2.23}$$

and

$$C = \int_0^R a\,PdV \,/\, \left(q \int_0^R PdV\right), \qquad (2.24)$$

P_c and ρ_{oc} being the central pressure and central rest-mass density respectively.

Considering a polytropic star with index N, Eq. (2.24) is written as

$$C = \frac{1}{9}\Big[\int_0^{\xi_1}\Big\{(N+1)^2\,\theta'^2\xi^2 + 2(N+1)\,\theta^{N+1}\,\xi^2$$

$$+ 8\,(N+1)(-\xi\,\theta')\,\theta\,\xi^2 d\xi\Big]\Big/\int_0^{\xi_1}\theta^{N+1}\,\xi^2\,d\xi \quad , \qquad (2.25)$$

where θ is the Lane-Emden function. In this case, we can also show that

$$\frac{P}{\rho_o c^2} \leq q = \frac{1}{(N+1)(-\xi_1\theta_1{}')}\,\frac{GM}{Rc^2} \quad , \qquad (2.26)$$

and the coefficient of GM/Rc^2 is a number of the order of unity.[19) As u is the same order as P, r_s/r for all r implies that $u/\rho_o c^2 \ll 1$ as assumed in Eq. (2.8), which will be satisfied not only for the polytropic star but also for the nonpolytropic stars.

§3. Electron pair creation and adiabatic exponent

In a highly contracted supermassive star, the temperature reaches high enough to create the electron pair. To know the gaseous properties at these temperatures, we need to clarify the values of degeneracy parameter ψ (i.e., Gibbs' free energy per electron divided by kT) and the ratio of the gas pressure to the total pressure, β . The constant ψ curves and the constant β curves whose characteristic properties are explained in Appendix B are shown in Fig. 3.

Fig. 3

Within the ranges of T_c and ρ_{oc} in the supermassive stars, about which we shall mention in §5, the electron gas is almost non-degenerate and the radiation pressure is dominant over the gas pressure, as can be seen by comparing Fig. 1 with Fig. 3.

The pressure and internal energy of gas are composed of those of electron, position and ions. In the non-degenerate case, these are given as[20),21)]

$$P_g = kT(n_- + n_+ + \mu_e n_o/\mu_I) \tag{3.1}$$

and

$$U_g = (\langle\varepsilon\rangle n_- - m_e c^2 n_o) + \langle\varepsilon\rangle n_+ + 3n_o\mu_e kT/(2\mu_I), \tag{3.2}$$

where n_- and n_+ are number densities of electron and position respectively which are given by

$$n_\mp = \frac{1}{\pi^2}\left(\frac{m_e c}{\hbar}\right)^3 \frac{K_2(x)}{x}\, e^{\pm\varphi} \tag{3.3}$$

with $x = m_e C^2/kT$; φ is chemical potential divided by kT; n_o is the non-created electron number defined by

$$n_0 = n_- - n_+ \quad , \tag{3.4}$$

μ_e and μ_I are molecular weights of electron and ion respectively; $\langle \varepsilon \rangle$ is the total average energy of an electron defined as

$$\langle \varepsilon \rangle = m_e c^2 \left(\frac{1}{x} + \frac{K_1(x)}{K_2(x)} \right) \longrightarrow \begin{cases} m_e c^2 + 3kT/2 & \text{for } x \gg 1 \\ 3kT & \text{for } x \ll 1, \end{cases} \tag{3.5}$$

and K_1 and K_2 are the modified Bessel functions of 1st and 2nd orders respectively.

For the later uses, we shall give the expressions of β and the rest mass density ρ_0 as follows,

$$\beta \equiv P_g / (P_g + P_r) \tag{3.6}$$

P_r being radiation pressure defined as

$$P_r = U_r/3 = \pi^2 (kT)^4 / (45 \hbar^3 c^3) \tag{3.7}$$

and $\rho_0 = m_H \mu_e n_0$, (3.8)

m_H being the proton mass.

Next, we calculate γ given in Eq. (2.16) taking account of the electron pair creation. In general, γ can be given as[22)]

$$\gamma = -\frac{V}{P} \frac{(P_\eta \epsilon_\xi - \epsilon_\eta P_\xi) + P(P_\eta V_\xi - P_\xi V_\eta)}{V_\eta \epsilon_\xi - V_\xi \epsilon_\eta} \tag{3.9}$$

where ξ and η are some thermodynamical parameters and P_η

represents a partial derivative of P with respect to η and so on. This expression is approximated assuming $\beta \ll 1$ as follows,

$$\gamma = \frac{4}{3} + \frac{h}{U_r} - \frac{4}{3}\frac{\partial h}{\partial U_r} - \left(\frac{n_0}{U_r} - \frac{4}{3}\frac{\partial n_0}{\partial U_r}\right)\left(\frac{\partial h}{\partial \varphi}\right) \Big/ \left(\frac{\partial n_0}{\partial \varphi}\right) \tag{3.10}$$

and $$h = U_g - 3P_g , \tag{3.11}$$

where we have taken U_r and φ in place of η and ζ .

Substituting Eqs. (3.1) $\sim$ (3.7) into Eq. (3.10), we have

$$\gamma = 4/3 + \beta f(x, n_0)/6 , \tag{3.12}$$

$$f(x, n_0) = \frac{2}{3(1+\mu_e/\mu_I)}\left[x^2\left(\frac{K_1}{K_2}\right)^2 A + (3x\frac{K_1}{K_2} - x^2)A^{-1} + \frac{3}{2}\frac{\mu_e}{\mu_I}\right] , \tag{3.13}$$

$$\beta = (4/\pi^2)(\hbar/m_e c)^3 n_0 x^3 (1+\mu_e/\mu_I), \tag{3.14}$$

and

$$A = n_0 \Big/ \sqrt{\{(m_e c/\hbar)^3 2K_2/(\pi^2 x)\}^2 + n_0^2} . \tag{3.15}$$

For $x \gg 1$, $f(x, n_0)$ tends to unity like

$$f \longrightarrow 1 - 5/\{2x(1+\mu_e/\mu_I)\} , \tag{3.16}$$

and γ becomes

$$\gamma = 4/3 + \beta/6 , \tag{3.17}$$

which coincides with the adiabatic exponent Γ_1 of a mixture composed of radiation and a gas with $\gamma = 5/3$ in the case of $\beta \ll 1$.[23)]

The relation between T and ρ_0 when $\gamma = 4/3$ can be obtained by solving the equation $f(x, n_0) = 0$. The ratio n_+/n_- at $\gamma = 4/3$ is not constant but decreases with decreasing β. For $x \gg 1$, the ratio decreases like

$$n_+/n_- \simeq 3(1+\mu_e/\mu_I)/(8x^2). \tag{3.18}$$

In table I, we show the density and the ratio n_+/n_- at $\gamma = 4/3$ for several values of temperature, taking $\mu_e = 2$ and $\mu_I = 56$.

Table I

In Fig. 3, the region where $\gamma < 4/3$ is represented by the shaded region in the T- ρ_0 diagram. This region has been obtained taking into account the following facts: (1) the upper boundary of the $\gamma < 4/3$ region is limited by the curve which approaches a horizontal line with $x = 1.75$ in the limit $n_0 \to 0$ or $\varphi \to 0$ (see Appendix C), and (2) the $\gamma = 4/3$ curve turns to leftward and has a maximum density as known from the calculation by Chiu.[24)] In the upper branch of this region, the effect of the endothermic nuclear reaction is superposed on the electron pair creation. In the case of the dissociation $Fe^{56} \to 13\, He^4 + 4n$, the $\gamma < 4/3$ region is drawn in Fig. 3, according to the calculations of Kaminishi.[25)]

§4. Instability region on T_c- ρ_{oc} diagram and binding energy

In Newtonian theory, an instability will occur within the $\gamma < 4/3$ region obtained in §3. But, if we take account of the general relativistic effect, the instability will occur even outside the $\gamma < 4/3$ region as supposed from Eq. (2.21). Then, the general relativistic instability region will be more extensive than the Newtonian instability region.

To obtain the instability region in the $T_c - \rho_{oc}$ diagram, we derive a relation between T_c and P_{oc} from Eqs. (2.21) and (3.12) assuming that (1) the value of C is taken as 2.63 assuming a polytrope with N = 3,[10] and (2) the average value $\bar{\gamma}$ is replaced by the central value γ_c. As will be mentioned in §5, the above second assumption will be verified for the stars with $M > 3.5 \cdot 10^4 M_\odot$, but not for the stars with $M < 3.5 \cdot 10^4 M_\odot$.

Under the assumptions, the instability marginal curve can be written from Eqs. (2.21) and (3.12) as

$$Cq = \beta f(x, n_o) / 6, \tag{4.1}$$

where the density and the temperature are the values at the center of the star but hereafter in this section we shall omit a subscript C.

To solve Eq. (4.1) graphically, we draw the curves of $f(x, n_o)$ and $6C\,q/\beta$ as a function of x^{-1} for the several fixed values of n_o as shown in Fig. 4, which shows that the $6Cq/\beta$ curves cross with the $f(x, n_o)$ curves when $f(x, n_o)$ is nearly equal to unity for $n_o < n_o^*$ and when $f(x, n_o)$ is nearly equal to zero for $n_o > n_o^*$, where n_o^* is defined as

$$n_o^* = 10^{-5} \frac{2}{\pi^2} \left(\frac{m_e c}{\hbar}\right)^3 = 3.53 \cdot 10^{25} \text{ cm}^{-3}. \tag{4.2}$$

That is, we can approximate the solution of Eq. (4.1) as follows

$$6Cq / \beta = 1 \qquad \text{for } \rho_o < \rho_o^* \tag{4.3}$$

and $f(x, \rho_0) = 0$ for $\rho_0 > \rho_0^*$, (4.4)

where $\rho_0^*/\mu_e = m_H \; n_0^* = 5.86 \cdot 10 \quad gcm^{-3}$. (4.5)

Equation (4.3) is also written as

$$kT/m_e c^2 = 2.38 \; 10^{-2} \{2(1+\mu_e/\mu_I)\rho_0^2/\mu_e\}^{1/7}. \qquad (4.6)$$

Fig. 4

In the vicinity of $\rho_0 = \rho_0^*$, the above approximation will not be so good and there will be a solution to connect Eq. (4.3) with Eq. (4.4) smoothly.

In this way, we have obtained the instability region as shown in Fig. 1. For $\rho_0 > \rho_0^*$, a real boundary of the instability region would be above the $\gamma = 4/3$ curve which has been taken as the boundary in our approximation (see the footnote in §5.).

For $\rho_0 < \rho_0^*$, the instability occurs on account of the relativistic effect, which can be treated in the post-Newtonian approximation. We can verify this fact as follows: as the instability marginal curve has its form like $T \propto \rho_0^{2/7}$ and q varies as $q \propto T^4 \; \rho_0^{-1}$ for $\beta \ll 1$, q changes like $q \propto \rho_0^{1/7}$ on the instability marginal curve. Then, q's value is limited to

$$q < 1.24 \cdot 10^{-3}, \qquad (4.6)$$

for the general relativistic region, i.e. $\rho_0 < \rho_0^*$.

In the case of a polytrope, the binding energy given by Eq. (2.10) can be rewritten as

$$E = - \int_0^R (3P - u)(1 + \frac{r_s}{r})\, dV + \frac{9}{2} qC \int_0^R PdV. \tag{4.7}$$

It has been known that the binding energy in consideration of the relativistic effect becomes to increase after passing through a minimum values and becomes a positive value as the homologous contraction of a star with a fixed proper rest mass proceeds.[5),6),26),27)] In our case, this is also true and E takes the minimum value when

$$\overline{\Gamma_4} + q(\partial\overline{\Gamma_4}/\partial q)_{homo.} = 4/3 + 3qC \tag{4.8}$$

and E vanishes when

$$\overline{\Gamma_4} = 4/3 + 3qC/2, \tag{4.9}$$

where $\overline{\Gamma_4}$ is the average value of Γ_4 defined by Fowler[7)] as

$$\Gamma_4 \equiv 1 + P/u \,. \tag{4.10}$$

When $\beta \ll 1$ and pair electrons are neglected, we notice that

$$\Gamma_4 = \gamma = \frac{4}{3} + \frac{1}{6}\beta \quad , \tag{4.11}$$

and then we have

$$(\partial \bar{\Gamma}_4/\partial q) \simeq (\partial \bar{\beta}/6\partial q) \simeq 0 \qquad (4.12)$$

because $\bar{\beta}$ is constant in the course of the homologous contraction under our assumptions.

In conclusion, the relativistic instability marginal curve given by Eq. (4.3) and the minimum binding energy curve given by Eq. (4.8) are found to be the same, and the binding energy zero curve given by Eq. (4.9) is always contained in the instability region. This implies that it is wrong to think the criterion E = 0 as that of the instability as it had been considered in Hoyle-Fowler's paper,[2] about which Fowler himself has mentioned in the later paper.[28]

§5. Central temperature and density of a polytropic star

In an equilibrium state of a polytropic star, the relation between the central pressure P_c and the central density is given as

$$P_c = \kappa G M_0^{2/3} \rho_{oc}^{4/3}, \qquad (5.1)$$

here κ is a nondimensional number of the order of unity and its value does not vary so large with polytropic index N.[13] In Table II, the values for N = 2 and 3 have been given.

Table II

If we take into consideration of the relativistic effect, varies with the relativistic parameter q. For the post-Newtonian case, this relation is given as follows,

$$\mathcal{K}(N,\ q) = \mathcal{K}(N,\ 0)\ (1 + Kq) \tag{5.2}$$

and

$$K = -(2/3)\left[N\int_0^{\xi_1}\Phi\,\theta^{N-1}\xi^2 d\xi + 3\theta_1'\xi_1^3(3-N)/(5-N)\right]/(-\xi_1^2\theta_1') + 4N/3, \tag{5.3}$$

where the function Φ is defined in such a may that Tooper's relativistic polytropic solution Θ [26] is approximated in the post-Newtonian case as

$$\Theta = \theta + q\Phi\ . \tag{5.4}$$

Using the table of Φ given by Chandrasekhar,[10] we obtain the values of K as shown in Table II. As our consideration is limited to the case $q < 10^{-3}$, we may neglect the relativistic effect on the equilibrium configuration. Then, hereafter, we do not distinguish M_0 from M.

When the radiation pressure is dominant, Eq. (5.1) can be approximated as

$$\frac{kT}{m_e c^2} = \left[\frac{45}{4}\left(\frac{3}{\pi^4}\right)^{1/3}\right]^{1/4}\left(\frac{\hbar}{m_e c}\right) n_0^{1/3}\left(\frac{M}{M(\mathcal{K})}\right)^{1/6}\left[1 - \left(1+\frac{\mu_e}{\mu_1}\right)\left(\frac{15}{4\pi^4}\right)^{1/4}\left(\frac{M(\mathcal{K})}{M}\right)^{1/2}\right], \tag{5.5}$$

here $M(\kappa)$ is defined as

$$M(\kappa) = \frac{\sqrt{3}\,\pi}{8\,\kappa^{3/2}}\left(\frac{\hbar c}{G}\right)^{3/2}\frac{1}{m_H^2 \mu_e^2} \tag{5.6}$$

and, in the case of the polytrope with N = 3, this $M(\kappa)$ is Chandrasekhar's limiting mass defined as

$$M_{ch} = \frac{5.75}{\mu_e^2}\ M_{\odot}. \tag{5.7}$$

Now, β in Eq. (3.6) is represented in terms of M as

$$\beta = 1.77\left(\frac{M_{ch}}{M}\right)^{\frac{1}{2}}, \tag{5.8}$$

and the assumption that $\beta \ll 1$ is verified for $M > 10^3 M_{ch}$ at least.

As the mass of the star increases, a convective region tends to grow in the stellar interior.[1] Therefore, we assume a sholly convective star and then the structure of the supermassive star is well represented by the polytrope with N = 3 because of $\beta \ll 1$.

The evolutionary paths for several values of M are shown in Fig. 1, from which we can see that an instability of the larger mass results from the general relativistic effect but for the smaller mass star it results from the pair electron creation. Then, we may introduce a critical mass M_c at which the instability mechanism changes from one another. Assuming ρ_0^* given by Eq. (4.5) as the critical density and $\mu_e = 2$, this mass is evaluated as

$$M_c = 3.5 \cdot 10^4 M_{\odot}, \tag{5.9}$$

This value varies with the parameters in the vicinity of this density as follows,

$$M_c \quad \propto \quad C^{-1} K^{-1.5} \mu_e^{-1} . \tag{5.10}$$

For the star with $M > M_c$, the physical quantities at the outset point of the instability are given as

$$\rho_{oc} = 1.18 \cdot 10^{17} (M / M_{ch})^{-7/2} \quad g\ cm^{-3}, \tag{5.11}$$

$$T_c = 1.30 \cdot 10^{13} (M / M_{ch})^{-1} \quad {}^oK, \tag{5.12}$$

$$R = 4.87 \cdot 10^{-5} (M / M_{ch})^{3/2} \ \mu_e^{\ -1} \quad R_\odot, \tag{5.13}$$

and $$q = 5.36 \cdot 10^{-1} (M / M_{ch})^{-\frac{1}{2}} \ \mu_e^{-1}, \tag{5.14}$$

where $R_\odot$ is the solar radius. For these stars, the replacement of $\bar{\gamma}$ by γ_c can be verified because of $f = 1$ and β = constant throughout the stellar interior.

For the star with $M < M_c$, the approximation to replace $\bar{\gamma}$ by γ_c is not always verified. In fact, the star becomes unstable only after a considerable part of the interior gets in the $\gamma < 4/3$ region. The degree how the central parts are contained in the $\gamma < 4/3$ region depends on the stellar mass and becomes larger with decreasing mass for the stars with mass below M_c *).

Tooper[29] has applied the general relativistic instability theory even for the star with $M < M_c$ and has obtained the relations corresponding to Eqs. (5.11) $\sim$ (5.14). However, it must be noticed that the instability for these stars results from the electron pair creation.

For the smaller mass star ($M < (50 \sim 100) \times M_{ch}$), it is supposed from Fig. 1 that an instability results from the dissociation of Fe nucleus. In the treatment of the structure of these stars, however, our assumption of the polytrope with $N = 3$ would not be verified.

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*) As an example to know this degree, we only mention the following fact: For $10^3\ M_{ch}$ star, the instability criterion given by Eq. (2.21) is written as

$$4/3 - (\overline{\gamma})_{r_1} > 3.2 \cdot 10^{-3} \quad \text{if} \quad \rho_{oc} = 2\rho_o(r_1)$$

$$\text{and} \quad 4/3 - (\overline{\gamma})_{r_1} > 2.4 \cdot 10^{-2} \quad \text{if} \quad \rho_{oc} = 5\rho_o(r_1),$$

where r_1 is the radius of the layer inside which $\gamma < 4/3$ and $(\overline{\gamma})_{r_1}$ means the averaging taken for the region $r \leq r_1$, i.e. the $\gamma \leq 4/3$ region. Referring the calculation of γ by Chiu[24], we may conclude that the star is unstable for $\rho_{oc} = 5\rho_o(r_1)$, but it needs more detailed calculation to decide whether the star is unstable for $\rho_{oc} = 2\rho_o(r_1)$.

§6 Applicable ranges of stellar mass in this paper

In this section, we consider the applicability of our theory; first about an upper limit and next about a lower limit.

As seen from Eqs. (5.11) and (5.12), ρ_{oc} and T_c decrease with increasing mass, e.g., for $M = 10^{10} M_{ch}$ we have

$$\rho_{oc} \simeq 10^{-18} \mathrm{g\ cm^{-3}} \text{ and } T_c \simeq 10^{3\circ}\mathrm{K}. \tag{6.1}$$

In these low temperature and low density, we must reconsider critically about the following two points, i.e., the equation of state and the energy balance between the energy output and the energy generation.

The first point is that the gaseous mass in these state may be in atomic or molecular state and the new instability region due to atomic ionization or molecular dissociation may appear. For example, a critical mass corresponding to M_c in the case of the electron pair creation can be obtained for the case of hydrogen atom ionization as

$$M \simeq 10^{9.5} M_{ch} . \tag{6.2}$$

However, as it will be seen in the followings, this critical mass has not a physical meaning.

To consider the second point, we must first inquire the condition of opaqueness. The star becomes transparent with increasing its mass and it becomes completely transparent when the mass is above a critical mass such as

$$M \simeq 10^{11} \sqrt{\sigma / 10^{25}}\, M_{ch} , \tag{6.3}$$

here σ being the cross section of photon scattering in cm^2. Then, the star above this critical mass is out of our consideration.

The above opaqueness condition, however, is not sufficient for a quasi-static equilibrium star but we must inquire the condition of energy balance that the energy output determined by the opacity in an equilibrium state must be supplied fully. Since nuclear energy generation can not be expected in these low temperature, the output ,ust be supplied by the gravitational energy accompanying the contraction of the star.

The energy output L is evaluated following Eddington's standard model[30] as follows,

$$L = \frac{4\pi Gc(1-\beta)\ M}{\kappa_e} \quad , \tag{6.4}$$

here κ_e is an opacity for electron scattering given as $\kappa_e = 0.19 \cdot \mu_e/2$. The internal energy U in the case of $\beta \ll 1$ is approximately given as

$$U = \frac{3}{2}\ \frac{GM^2}{R}\ . \tag{6.5}$$

Then, the cooling time by photon diffusion t_{cp} is given as

$$t_{cp} = \frac{U}{L} = 3.6 \cdot 10^{10} \left(\frac{M}{M_\odot}\right)^{\frac{1}{2}} \left(\frac{kT}{m_e c^2}\right) \left(\frac{2}{\mu_e}\right) \text{ sec }. \tag{6.6}$$

The cooling time by photon-neutrino emission $t_{c\nu}$ is given as[31]

$$t_{c\nu} = 10^{53.3}\ n_o^{-1}\ T_5^{-4} \text{ sec}, \tag{6.7}$$

and it is evident that the neutrino process can be neglected.

On the other hand, the time scale of the gravitational energy generation is considered to be limited by the free fall time t_{ff}, which is given as[14)]

$$t_{ff} = 6.5 \cdot 10^2 \quad \rho_{oc}^{-1/2} \ . \tag{6.8}$$

The relation between T_c and ρ_{oc} derived from the equation $t_{ff} = t_{cp}$ takes the form of $T_c \propto \rho_{oc}^{1/8}$ and is shown in Fig. 2. In the $t_{ff} > t_{cp}$ region, the energy output can not be supplied even by the free fall contraction and the star can not be in a quasi-static equilibrium. In these stages, it is meaningless to consider an stability of a static star and our instability criterion is only applicable to the mass range such as

$$M \quad < \quad 10^{8.7} \, M_{ch}, \tag{6.9}$$

which defines the upper limit of our investigation of the instability.

Next, we consider about the lower limit, which arises from the use of the polytropic solution for the equilibrium configuration. As the mass decreases, the temperature at the outset point of instability increases and the neutrino flux becomes to dominate the photon flux. For example, their comparison is given in the case of $10^2 M_{ch}$ star at $n_o = 10^{28}$ cm^{-3} as follows,[31)]

$$t_{cp} \simeq 10^{11}\text{sec} \gg t_c = 10^{3.96} x^{-1} \; e^{2x} \simeq 10^7\text{sec}. \tag{6.10}$$

For the stars, the stellar structure can not be represented by a simple polytrope but may have a more complex structure. Then, our treatment on the structure must be modified.

The condition that $t_{cp} < t_{c\nu}$ on the $\gamma = 4/3$ curve implies approximately that $M > 10^3 M_{ch}$, which defines the lower mass limit. In this paper, we have neglected the effects of nuclear reactions except the dissociation of Fe nucleus but these effects on the stellar structure would become large for the stars of this order of mass.[21)]

However, it is worthwhile to mention that our treatment to obtain the criterion of instability assuming the adiabatic process is well founded even for the stars with these masses, because the time scale of the electron pair creation given by[32)]

$$t_{pair} \simeq 10^{-15.2} x^{-3} e^{2x} \quad \text{sec} \tag{6.11}$$

and t_{ff} are much shorter than the neutrino cooling time. Then, we may only alter the relation between T_c and ρ_{oc}, the evaluation of C and the averaging of .

Acknowledgements

The author wishes to express his thanks to Professor C. Hayashi for his stimulating discussion.

Appendix A. Variation of binding energy

The variation of binding energy is calculated by taking the variations of the terms on the right hand side of Eq. (2.10), using Eq. (2.15) and the following relations such as

$$g \equiv \delta V = 4\pi r^2 \delta r + 4\pi r (\delta r)^2 , \tag{A.1}$$

$$\delta\left(\frac{1}{r}\right) = -\frac{1}{3r}\left(\frac{g}{V}\right) + \frac{2}{9}\frac{1}{r}\left(\frac{g}{V}\right)^2 , \tag{A.2}$$

and $$\delta v/v = \left(\frac{d \ln g}{d \ln V} - \frac{r_s}{3r}\right)\left(\frac{g}{V}\right) + \left(\frac{2}{9} - \frac{1}{3}\frac{d \ln g}{d \ln V}\right)\frac{r_s}{r}\left(\frac{g}{V}\right)^2 . \tag{A.3}$$

In the calculation, it must be also noticed that $\delta M_r/M_r$ is the order of q and we can rewrite this term using the relation derived from the Newtonian theory.

The first order variation E of the binding energy is obtained as

$$\delta E = \frac{1}{3}\int_0^R \left\{ r\frac{dP}{dr} + \frac{GM_r\rho}{r}\left(1 + \frac{P}{\rho_0 c^2} + \frac{2r_s}{r} + \frac{3PV}{M_r c^2}\right)\right\}\left(\frac{g}{V}\right) dV . \tag{A.4}$$

The curly bracket of the above integrand is found to vanish. This is nothing but the equation of hydrostatic equilibrium in the post-Newtonian qpproximation.

The second order variation ^2E is obtained as

$$\delta^2 E = \left(\Gamma_0 - \frac{4}{3}\right)\int_0^R \frac{1}{2}\left(\frac{g}{V}\right)^2 P \, dV \tag{A.5}$$

and $$\Gamma_0 = \left[\int_0^R \left\{ \gamma \left(\frac{d \ln g}{d \ln V} \right)^2 - a \right\} \left(\frac{g}{V} \right)^2 P dV - \int_0^R b\, dV \right] \Big/ \int_0^R \left(\frac{g}{V} \right) P dV \tag{A.6}$$

Where $$a = \frac{\rho_0 c^2}{9P} \left[\left(\frac{r_s}{r} \right)^2 + \frac{6GPV}{c^4 r} + 6 \gamma \left(\frac{r_s}{r} \right) \left(\frac{P}{\rho_0 c^2} \right) \left(\frac{d \ln g}{d \ln V} \right) \right] \tag{A.7}$$

and $$b = \frac{G\rho_0}{c^2 r} \left[\int_0^R \left\{ \gamma \left(\frac{d \ln g}{d \ln V} \right)^2 - \frac{4}{3} \right\} \left(\frac{g}{V} \right)^2 P dV + \frac{8}{3} \int_0^R g P\, d\left(\frac{g}{V} \right) \right]. \tag{A.8}$$

Appendix B. Electron, positron and radiation gases

(i) Constant ψ curve

In a chemical equilibrium among electron, positron and radiation, we have the relation such as

$$\mu_- + \mu_+ = \mu_\gamma = 0 \tag{B.1}$$

and then

$$\mu_- = -\mu_+ \equiv kT\varphi \tag{B.2}$$

here μ_-, μ_+ and μ_γ are the chemical potentials of electron, positron and radiation respectively.

Gibbs' free energies per an electron divided by kT, $\psi_\mp$, are defined as

$$\psi_\mp = (\mu_\mp - m_e c^2) / kT \tag{B.3}$$

and then $$\psi_- = - \psi_+ - 2x \equiv \psi . \tag{B.4}$$

Therefore, a trivial condition that

$$n_o = n_- - n_+ \geq 0 \qquad \text{(B.5)}$$

introduces a restriction on ψ such as

$$\psi \geq - x . \qquad \text{(B.6)}$$

The behavior of the constant ψ curve for $\psi < 0$ and that for $\psi \geq 0$ are different from each other as shown in Fig. 3: for $\psi \geq 0$, ρ_o increases monotonously with T, but for $\psi < 0$, ρ_o takes a maximum value and tends to zero as temperature approaches to $kT/m_e c^2 = -\psi^{-1}$

In the case of non-degeneracy, i.e. $-\psi \gg 1$, each physical quantities are expressed in the expansion formulae by Chandrasekhar and Eqs. (3.1)$\sim$(3.5) are the first terms of these expansion formulae. The ratio of the second terms to the first terms are the order of $e^{-\psi-2x}$ for electron and e^{ψ} for positron and the ratio of the positron density to the electron density, n_+/n_- , is the order of $e^{2\psi}$.

(ii) Constant β curve

In the case of no electron pair creation, the constant curve takes the form such as ρ_o increases monotonously with T, but when the electron pair are created, the behavior of this curve varies with the value of β .

For 7 /11, $kT/m_e C^2$ is bounded by the maximum value $-1/\psi_o$ and ρ_o tends to zero when $x^{-1} \to -1/\psi_o$, where ψ_o is defined by the following equation such as

$$\frac{1}{\beta} - 1 = \frac{\pi^4}{\sqrt{2}\,45}\left(-\frac{1}{\psi_o}\right)^{3/2} \Big/ 2G\left(\psi_o - \frac{1}{\psi_o}\right) , \qquad \text{(B.7)}$$

where $G(\psi, \alpha)$ is one of the relativistic Fermi-Dirac functions defined as[33])

$$G(\psi, \alpha) = \frac{2}{3}\int_0^\infty \frac{(\chi + \alpha\chi^2/2)^{3/2}}{e^{\chi-\psi}+1}\,d\chi \tag{B.8}$$

In the limiting case that $\psi_0 \rightarrow 0$, β tends to

$$\beta = 7/11\ , \tag{B.9}$$

which is derived from Eq. (B.7)

For $\beta > 7/11$. the curve is monotonous one as in the case of no electron pair creation.

In the above discussion, we have neglected the effects by the phase changes due to nuclear dissociation or pair creations of the other particles. A general behavior of the curve in consideration of these effects is that the curve turns to leftward in Fig. 3 if the mean molecular weight of electron increases, and vice versa. For example, the $\beta = 7/11$ curve behaves in the F_e-dissociation zone as shown in Fig. 3.[25])

Appendix C. The upper boundary of the $\gamma < 4/3$ region

In the limit $\rho_0 \rightarrow 0$ without T tending to zero, the gas pressure nay be composed mainly of the created pair electrons. In this limit, which implies $|\varphi| \rightarrow 0$, we can approximate Eq. (3.9) as

$$\gamma = \frac{4 + xK_1 / D}{3 + x^2K_2 / (xK_1 + 4D)} \tag{C.1}$$

with $$D = K_2 + \pi^4 x^{-2} / 90. \tag{C.2}$$

Putting $\gamma = 4/3$, we have the temperature of the upper boundary of the $\gamma < 4/3$ region as

$$x = 1.75. \tag{C.3}$$

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Table I. The temperature-density relations and the ratio n_+ / n_- when $\gamma = 4/3$.

x	$T(^{\circ}K)$		ρ_c/μ_e (gcm^{-3})		n_+/n_-	
5	1.19	10^9	2.31	10^4	1.80	10^{-2}
8	7.43	10^8	8.45	10^2	6.42	10^{-3}
10	5.94	10^8	9.54	10	4.35	10^{-3}
12	4.95	10^8	1.14	10	3.05	10^{-3}
15	3.96	10^8	4.37	10^{-1}	2.48	10^{-3}
20	2.97	10^8	2.99	10^{-3}	9.55	10^{-4}

Table II. κ and C for polytropes of N = 2 and 3. In post-Newtonian case, κ is given as

$$\kappa(N,\ q) = \kappa(N,\ 0)(1 + Kq).$$

N	$\kappa(N,\ 0)$	K	$C^{(0)}$
2	0.435	9.35	2.4968
3	0.364	8.85	2.6325

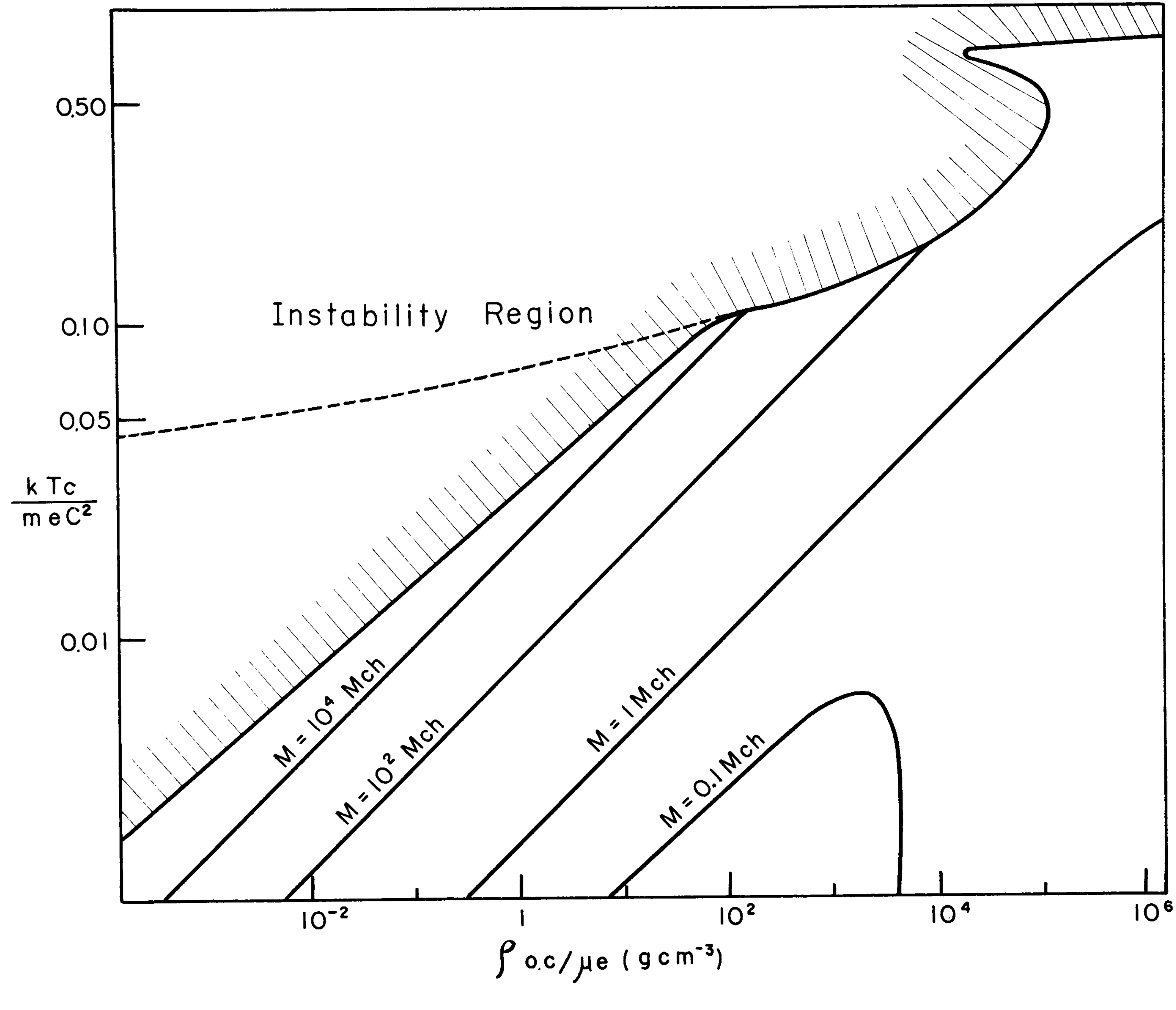
Figure Caption

Fig. 1. The relations of the central temperature and density of the stars with mass, 10^4 M_{ch}, 10^2 M_{ch} and 0.1 M_{ch}, where M_{ch} is the Chandrasekhar's limiting mass defined as $M_{ch} = 5.75/\mu_e^2\ M_\odot$. The shaded area represents the instability region in which the star can not be in a stable equilibrium.

Fig. 2. The free fall contraction region in which the star can not be in a quasi-static equilibrium and the instability region.

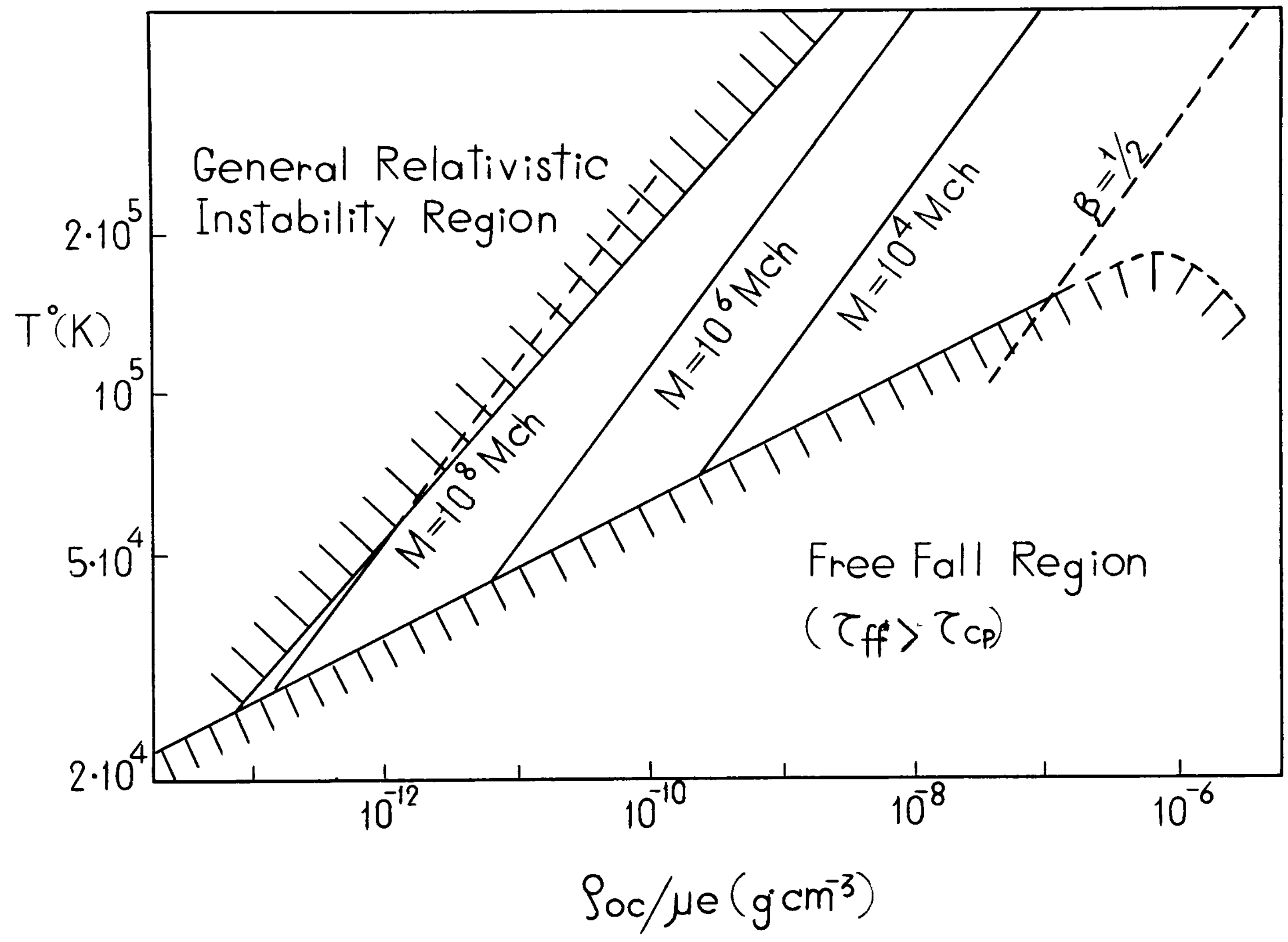
Fig. 3. The temperature-density diagram for the characteristics of electron, position and radiation gas. Gross features of the constant ψ curve, the constant β curve and the $\gamma < 4/3$ region are shown. The $\gamma < 4/3$ region is obtained refering the calculations in § 3 in this text, and the references 22) and 23).

Fig. 4. The solution of Eq. (4.1) for the densities $\rho_0 = (10^{-2}, 10^{-1}, \text{—}, 10^2)\ \rho_0^*$, where $\rho_0^*/\mu_e = 5.86\ \ 10\ \mathrm{g\ cm}^{-3}$.
The ordinate denotes the values of $f\ (\rho_0, x)$ and $6C\ q/\beta$, and the abscissa does the $kT/m_ec^2 = x^{-1}$.



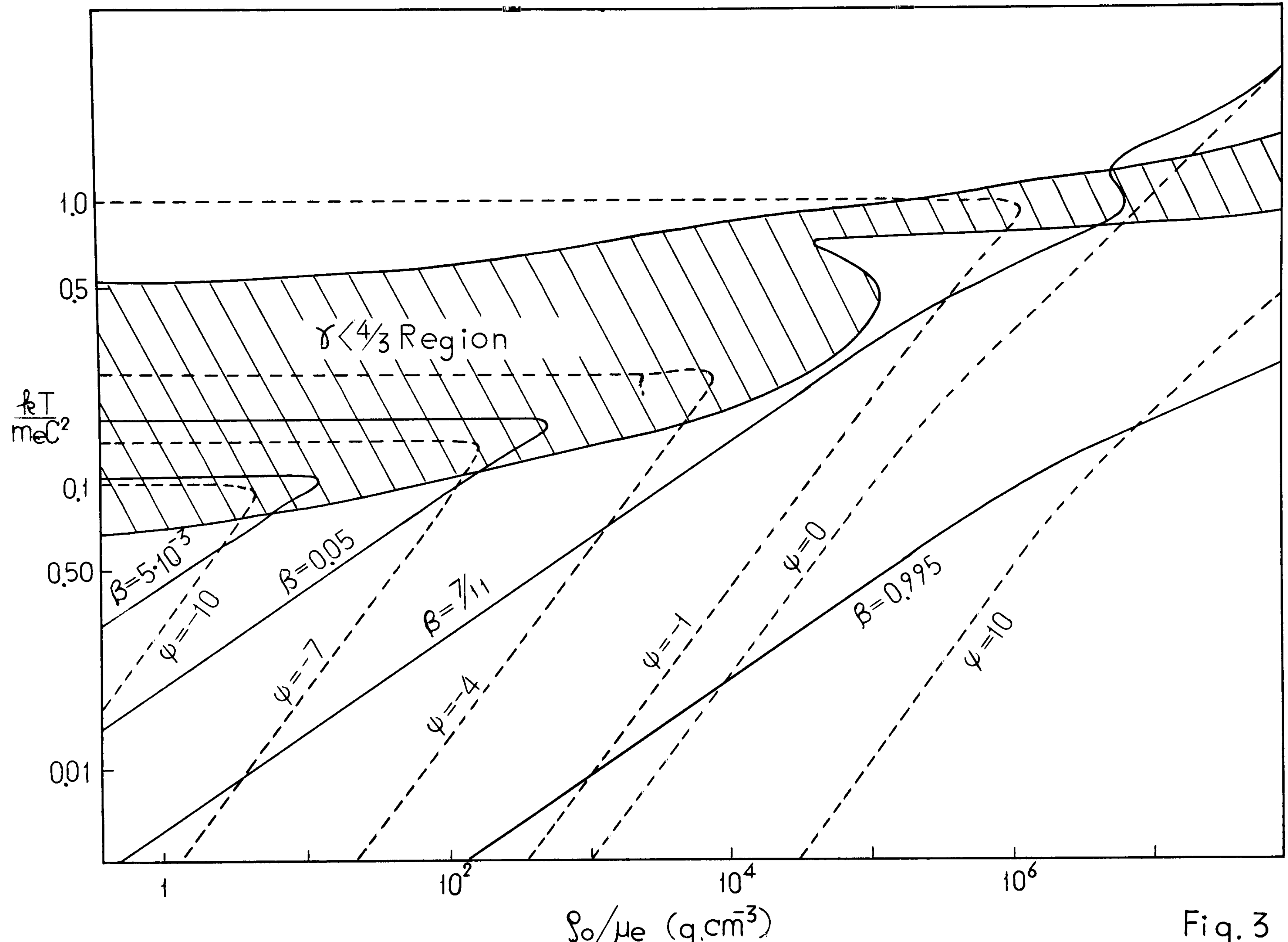
0.50
0.10
0.05
0.01
$\frac{kT_c}{m_e c^2}$
Instability Region
M = 10⁴ Mch
M = 10² Mch
M = 1 Mch
M = 0.1 Mch
10⁻²
1
10²
10⁴
10⁶
ρ o.c/μe (g cm⁻³)

Fig 1



General Relativistic
Instability Region
Free Fall Region
$(\tau_{ff} > \tau_{cp})$
$M=10^8 M_{ch}$
$M=10^6 M_{ch}$
$M=10^4 M_{ch}$
$\beta = 1/2$
$T°(K)$
$2\cdot10^5$
10^5
$5\cdot10^4$
$2\cdot10^4$
10^{-12}
10^{-10}
10^{-8}
10^{-6}
$\rho_{oc}/\mu_e\,(g\,cm^{-3})$

Fig. 2



$\gamma < 4/3$ Region
$\frac{kT}{m_e c^2}$
1.0
0.5
0.1
0.50
0.01
$\beta = 5\cdot 10^{-3}$
$\beta = 0.05$
$\beta = 7/11$
$\beta = 0.995$
$\varphi = -10$
$\varphi = -7$
$\varphi = -4$
$\varphi = -1$
$\varphi = 0$
$\varphi = 10$
1
10^2
10^4
10^6
ρ_0/μ_e (g.cm^{-3})

Fig. 3

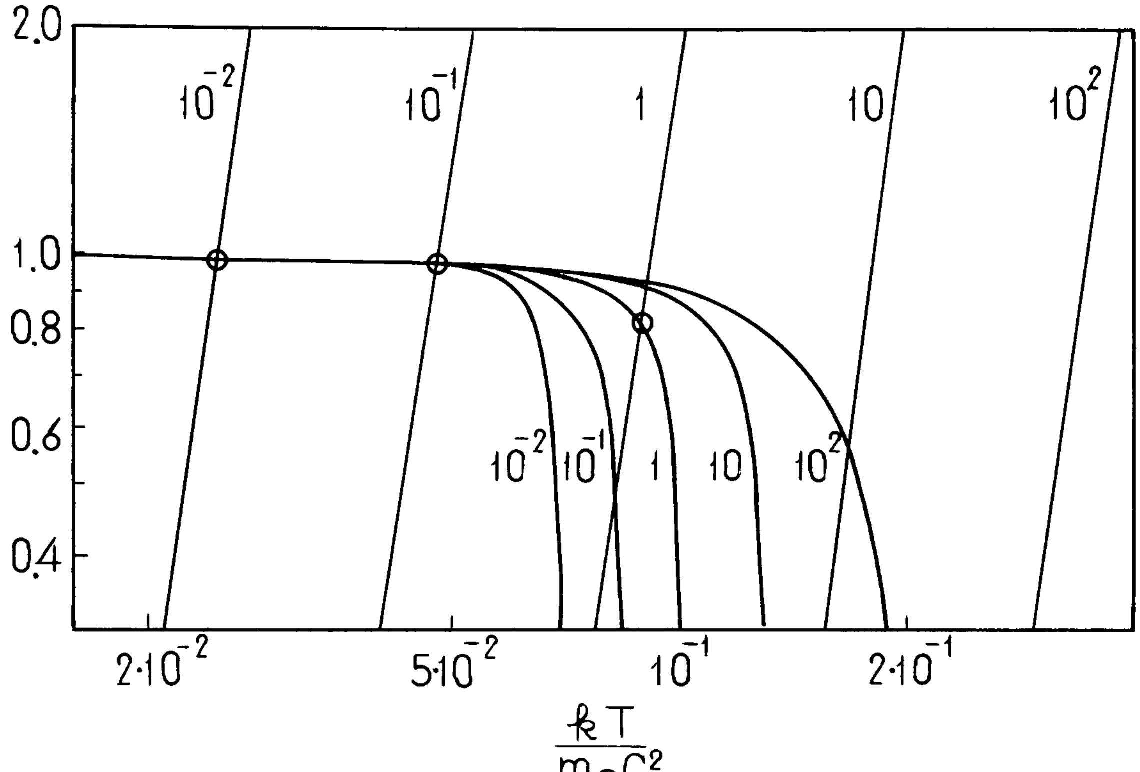

2.0
1.0
0.8
0.6
0.4
10^{-2}
10^{-1}
1
10
10^2
10^{-2}
10^{-1}
1
10
10^2
$2\cdot10^{-2}$
$5\cdot10^{-2}$
10^{-1}
$2\cdot10^{-1}$
$\frac{kT}{m_eC^2}$

Fig. 4