学位申請論文

伊藤宏

Charge Transfer Model and
(2-cluster)→(2-cluster) Three-Body Scattering

by

Hiroshi T. Ito

Department of Mathematics
Kyoto University
Sakyo-ku, Kyoto 606, Japan

Abstract. We consider the scattering for a three-body system such that in the initial and final states a light particle is bound by one heavy particle and the other heavy particle moves free. We will show that when the masses of heavy particles go to infinity the limits of the scattering matrix and the total cross section at initial relative speed v_0 exist and that these limits can be expressed in terms of the quantities appearing in the scattering for the charge transfer model:

$$i\partial_t\psi(t)=[-(2m)^{-1}\Delta_x+V_{23}(x)+V_{13}(x-v_0\omega t-\eta)]\psi(t),$$

where ω is a unit vector, V_{j3} is the pair potential between the light particle and one of heavy particles, and η is a constant vector orthogonal to ω

§ 1 Introduction

We consider a three-body system consisting of two heavy particles (particles 1, 2) with the masses M_1, M_2 and a light particle (particle 3) with m. We set $\mu=(M_1,M_2)$ and write $\mu\gg 1$ $(\mu\to\infty)$ for M_1, $M_2 \gg 1$ $(M_1,M_2\to\infty)$ Let $r_j\in R^N$ $(j=1,2,3)$, $N\geqq 2$, be the position of particle j, and let V_{jk} be the pair potential between particle j and particle k. Then the three-body Hamiltonian is

(1 1) $$\widetilde{H}^{\mu}=-\sum_{j=1}^{2}(2M_j)^{-1}\Delta_{r_j}-(2m)^{-1}\Delta_{r_3}+V \qquad \text{in} \quad L^2(R^{3N}),$$

$$V=V(r_1,r_2,r_3)=V_{23}(r_3-r_2)+V_{13}(r_3-r_1)+V_{12}(r_2-r_1)$$

We assume the following throughout this paper:

(V) $V_{ij}(x)$ $(1\leqq i<j\leqq 3)$ is a smooth real-valued function on R^N, and there exists $\varepsilon_0>N+(3/2)$ such that

$$|\partial_x^{\gamma}\, V_{ij}(x)|\leqq C_\gamma(1+|x|)^{-\varepsilon_0}$$

for all multi-indices γ

Our main results are Theorems 1 1 and 1 3, which will be stated at the end of this section. For the proof of Theorem 1.1, we assume further

(V)' $V_{ij}(x)$ $(1\leqq i<j\leqq 3)$ satisfies (V) with

$\varepsilon_0>[(N-1)/2]+N+(3/2)$ ([ ] is Gauss' symbol.)

As usual, we remove the kinetic energy of the center of mass from $\widetilde{H}^{\mu}$ to get an operator H^{μ} in $L^2(R^{2N})$ A 2-cluster decomposition of the set $\{1,2,3\}$ is a partition of $\{1,2,3\}$ into two non-empty subsets, and in particular we use only the following 2-cluster decompositions:

$$a_1 := \{1,(2,3)\}, \qquad a_2 := \{2,(1,3)\}, \tag{1.2}$$

and we define $\mathbf{A} := \{a_1, a_2\}$

For each $a \in \mathbf{A}$, the Jacobi coordinates $\{x_a, y_a\}$ are defined by

$$\begin{aligned} x_a &:= r_3 - r_2, \qquad y_a := r_1 - \frac{M_2 r_2 + m r_3}{M_2 + m} \qquad \text{for} \quad a = a_1, \\ x_a &:= r_3 - r_1, \qquad y_a := \frac{M_1 r_1 + m r_3}{M_1 + m} - r_2 \qquad \text{for} \quad a = a_2. \end{aligned} \tag{1.3}$$

Let $m_a = m_a^\mu$ and $n_a = n_a^\mu$ $(a \in \mathbf{A})$ be the reduced masses defined by

$$\frac{1}{m_a} = \frac{1}{M_i} + \frac{1}{m}, \quad \frac{1}{n_a} = \frac{1}{M_j} + \frac{1}{M_i + m} \qquad \text{for} \quad a = \{j,(i,3)\} \tag{1.4}$$

Then H^μ is expressed as follows:

$$H^\mu = -\frac{1}{2m_a}\Delta_{x_a} - \frac{1}{2n_a}\Delta_{y_a} + V \qquad \text{in} \quad L^2(R^{2N}) \tag{1.5}$$

$\{x_{a_1}, y_{a_1}\}$ and $\{x_{a_2}, y_{a_2}\}$ are related as follows:

$$\begin{aligned} x_{a_1} &= \frac{m_{a_2}}{m} x_{a_2} + y_{a_2}, \qquad y_{a_1} = -\frac{m_{a_2}}{n_{a_1}} x_{a_2} + \frac{m_{a_1}}{m} y_{a_2}, \\ x_{a_2} &= \frac{m_{a_1}}{m} x_{a_1} - y_{a_1}, \qquad y_{a_2} = \frac{m_{a_1}}{n_{a_2}} x_{a_1} + \frac{m_{a_2}}{m} y_{a}, \end{aligned} \tag{1.6}$$

Under assumption (V), H^μ is self-adjoint in $\mathcal{H} := L^2(R^{2N})$ with domain $D(H^\mu) = H^2(R^{2N})$, the Sobolev space of order 2. For $a = \{i,(j,3)\} \in \mathbf{A}$ the 2-body Schrödinger operator h_a^μ is defined by

$$h_a^\mu := -(2m_a)^{-1}\Delta_{x_a} + V_{j3}(x_a). \tag{1 7}$$

which is self-adjoint in $L^2(R^N_{x_a})$ with domain $D(h_a^\mu) = H^2(R^N)$

Since $m_a \to m$ as $\mu \to \infty$, h_a^μ converges to a self-adjoint operator

(1.8) $$h_a^\infty := -(2m)^{-1}\Delta_{x_a} + V_{j3}(x_a)$$

in the norm resolvent sense as $\mu\to\infty$ Furthermore we note that $-z\Delta_{x_a} + V_{j3}(x_a)$, $z\in \mathbf{C}\setminus\{0\}$, is an analytic family of type (A) ([K], VII.2, [R-S]IV,XII.2) Let $k(a)$ be the number of negative eigenvalues (counting multiplicity) of h_a^∞. Under assumption (V), it is known that $k(a)$ is finite ([R-S]IV,XIII 3). We set

$$\mathbf{CH} := \{\alpha = (a,k);\ a\in \mathbf{A},\ 1\leqq k\leqq k(a),\ k\in \mathbf{N}\},$$

where $\mathbf{N} := \{1,2,\cdots\}$, and write $D(\alpha)=a$ for "channel" $\alpha=(a,k)$ $\in \mathbf{CH}$ For $\alpha=(a,k)\in \mathbf{CH}$ we denote by λ_α^∞ (<0) the k-th negative eigenvalue of h_a^∞ and by ϕ_α^∞ the eigenfunction of h_a^∞ with eigenvalue λ_α^∞ such that $\{\phi_\alpha^\infty\}$ $(\alpha\in \mathbf{CH},\ D(\alpha)=a)$ is an orthonormal system for each $a\in \mathbf{A}$ If $\mu\gg 1$, we can find negative eigenvalues λ_α^μ of $h_{D(\alpha)}^\mu$ and associated normalized eigenfunctions ϕ_α^μ $(h_{D(\alpha)}^\mu \phi_\alpha^\mu = \lambda_\alpha^\mu \phi_\alpha^\mu)$ for every $\alpha\in \mathbf{CH}$ such that (i) $\lambda_\alpha^\mu \to \lambda_\alpha^\infty$ as $\mu\to\infty$, (ii) $\phi_\alpha^\mu \to \phi_\alpha^\infty$ in $L^2(\mathbf{R}^N)$ as $\mu\to\infty$ and (iii) $\{\phi_\alpha^\mu\}$ $(\alpha\in \mathbf{CH},\ D(\alpha)=a)$ is an orthonormal system for each $a\in \mathbf{A}$ (See [K], II.1.4.)

For each $\mu\gg 1$ and each $\alpha\in \mathbf{CH}$, $D(\alpha)=a$, we define the channel embedding $P_\alpha^\mu \in \mathbf{B}(L^2(\mathbf{R}^N_{y_a}),\mathcal{H})$ and the channel Hamiltonian T_α^μ by

(1 9) $$(P_\alpha^\mu f)(x_a,y_a) = \phi_\alpha^\mu(x_a) f(y_a), \qquad T_\alpha^\mu := -(2n_a)^{-1}\Delta_{y_a} + \lambda_\alpha^\mu,$$

respectively Here we denote by $\mathbf{B}(X,Y)$ the space of all bounded linear operators from a Banach space X to a Banach space Y

Under assumption (V), the channel wave operators

$$(1.10)\qquad W^{\mu}_{\alpha\pm}:=\text{s-}\lim_{t\to\pm\infty} e^{itH^{\mu}}P^{\mu}_{\alpha}e^{-itT^{\mu}_{\alpha}}$$

exist in $\mathbf{B}(L^2(\mathbf{R}^N_{y_a}),\mathcal{H})$ (see [R-S]III, Theorem XI.35). For α, $\beta \in$ $\mathbf{C}\,\mathbf{H}$, the scattering operator for scattering $\alpha\to\beta$ is defined by

$$(1.11)\qquad S^{\mu}_{\beta\alpha}:=W^{\mu *}_{\beta +}W^{\mu}_{\alpha -}\ :L^2(\mathbf{R}^N_{y_{D(\alpha)}})\to L^2(\mathbf{R}^N_{y_{D(\beta)}})$$

Here A^* denotes the adjoint of the operator A

For each $\alpha\in\mathbf{C}\,\mathbf{H}$ we give the spectral representation of T^{μ}_{α}. We define maps $Z^{\mu}_{\alpha}(\lambda)$, $\lambda>\lambda^{\mu}_{\alpha}$, from $\mathcal{S}(\mathbf{R}^N)$ (the Schwartz space of rapidly decreasing functions) to $\Sigma:=L^2(S^{N-1})$ (S^{N-1} is the unit sphere in $\mathbf{R}^N$), by

$$(1\ 12)\qquad (Z^{\mu}_{\alpha}(\lambda)f)(\omega)=(2\pi)^{-N/2}n_a^{1/2}(2n_a(\lambda-\lambda^{\mu}_{\alpha}))^{(N-2)/4}\times$$
$$\times\int e^{-i(2n_a(\lambda-\lambda^{\mu}_{\alpha}))^{1/2}\omega\cdot y_a}f(y_a)dy_a,\quad a=D(\alpha),$$

where $\omega\in S^{N-1}$ It is known that $Z^{\mu}_{\alpha}(\lambda)$ can be extended to bounded operators from $L^2_s(\mathbf{R}^N_{y_a})$ to Σ for $s>1/2$, where $L^2_s(\mathbf{R}^N_y):=L^2(\mathbf{R}^N_y;\langle y\rangle^{2s}dy)$, $\langle y\rangle:=(1+|y|^2)^{1/2}$ (cf [G-M], Proposition 2.1) We define a map Z^{μ}_{α} from $L^2_s(\mathbf{R}^N_{y_a})$ to $L^2((\lambda^{\mu}_{\alpha},\infty);\Sigma)$ by

$$(1\ 13)\qquad (Z^{\mu}_{\alpha}f)(\lambda,*)=(Z^{\mu}_{\alpha}(\lambda)f)(*),\quad\text{for}\quad \lambda>\lambda^{\mu}_{\alpha}.$$

Then Z^{μ}_{α} can be extend to a unitary operator from $L^2(\mathbf{R}^N_{y_a})$ to $L^2((\lambda^{\mu}_{\alpha},\infty);\Sigma)$ and gives the spectral representation of T^{μ}_{α} .that is.

$$(1.14)\qquad (Z^{\mu}_{\alpha}T^{\mu}_{\alpha}f)(\lambda,*)=\lambda(Z^{\mu}_{\alpha}(\lambda)f)(*),\quad\text{for a.e. }\lambda>\lambda^{\mu}_{\alpha},$$

for $f\in D(T^{\mu}_{\alpha})$ We can see that $Z^{\mu}_{\beta}S^{\mu}_{\beta\alpha}Z^{\mu *}_{\alpha}$ is decomposable by a family of operators $\{S^{\mu}_{\beta\alpha}(\lambda)\}$ ([A-J-S] 15-3):

(1.15) $(Z_\beta^a S_{\beta\alpha}^a Z_\alpha^{a*} h)(\lambda) = S_{\beta\alpha}^a(\lambda) h(\lambda)$ in Σ for a.e. $\lambda \in (\lambda_{\beta\alpha}^a, \infty) \setminus \Lambda^a$,

for $h \in L^2((\lambda_{\beta\alpha}^a, \infty); \Sigma)$, where $\lambda_{\beta\alpha}^a := \max(\lambda_\alpha^a, \lambda_\beta^a)$ and $\Lambda^a = \{$the thresholds of $H^a\} \cup \sigma_p(H^a)$ ($\sigma_p(H^a)$ denotes the set of all eigenvalues of H^a) We will show that $S_{\beta\alpha}^a(\lambda)$ is a $\mathbf{B}(\Sigma)$-valued norm continuous function in $(\lambda_{\beta\alpha}^a, \infty) \setminus \Lambda^a$ ($\mathbf{B}(\Sigma) = \mathbf{B}(\Sigma, \Sigma)$) Furthermore, in Sect.2 we will show that

$$T_{\beta\alpha}^a(\lambda) := S_{\beta\alpha}^a(\lambda) - \delta_{\beta\alpha} \tag{1.16}$$

has an integral kernel $T_{\beta\alpha}^a(\lambda, \omega, \omega')$, which is continuous for $(\lambda, \omega, \omega') \in ((\lambda_{\beta\alpha}^a, \infty) \setminus \Lambda^a) \times S^{N-1} \times S^{N-1}$ Here $\delta_{\beta\alpha} = 1$ (resp. 0) if $\alpha = \beta$ (resp. $\alpha \neq \beta$) In particular, the total scattering cross section for scattering $\alpha \to \beta$ at relative energy λ and relative initial direction ω (see [A-J-S}, p.627),

$$\sigma_{\beta\alpha}^a(\lambda; \omega) := (2\pi)^{N-1} (2n_{D(\alpha)}(\lambda - \lambda_\alpha^a))^{(1-N)/2} \int_{S^{N-1}} |T_{\beta\alpha}^a(\lambda, \omega', \omega)|^2 d\omega' \tag{1.17}$$

is finite for all $\lambda \in (\lambda_{\beta\alpha}^a, \infty) \setminus \Lambda^a$ and $\omega \in S^{N-1}$

We next consider the following time-dependent Schrödinger equation for the charge transfer or impact parameter model:

$$i\partial_t \psi(t) = h_{\xi,\eta}(t) \psi(t) \qquad \text{in } L^2(\mathbf{R}^N).$$

$$h_{\xi,\eta}(t) = [-(2m)^{-1}\Delta_x + V_{23}(x) + V_{13}(x - \xi t - \eta)] \psi(t). \tag{1.18}$$

$$\xi \in \mathbf{R}^N \setminus \{0\}, \quad \eta \in \Pi_\xi := \{\eta \in \mathbf{R}^N;\ \xi \cdot \eta = 0\}$$

(See [Y], [Ha], [G}, [W]) The equation describes the motion of the light particle (particle 3) under the influence of interaction potential V_{13} and V_{23} due to two heavy particles 1 and 2; particle 2 is assumed to stay at the origin and particle 1 is assumed to move classically on the straight line $\xi t+\eta$.

Under assumption (V), (1.18) has a unique propagator

$$(1.19) \qquad U(t,s)=U(\xi,\eta;t,s), \qquad s,t\in \mathbf{R},$$

such that

(U-i) $U(t,s)$ is a unitary operator on $L^2(\mathbf{R}^N)$ and jointly strongly continuous in s and t.

(U-ii) $U(t,r)U(r,s)=U(t,s)$ for r, s, $t\in \mathbf{R}$

(U-iii) If $f\in H^2(\mathbf{R}^N)$, then $U(t,s)f\in H^2(\mathbf{R}^N)$ for s, $t\in \mathbf{R}$, and $U(t,s)f$ (which is strongly differentiable in s and t, respectively) satisfies

$$i\partial_t U(t,s)f=h_{\xi,\eta}(t)U(t,s)f, \quad i\partial_s U(t,s)f=-U(t,s)h_{\xi,\eta}(s)f$$

(see e.g. [R-S]III, Theorem X.71).

The purpose of this paper is to relate the scattering theory for equation (1.18) to that for the three-body system (1.5). We restrict ourselves to the (2-cluster)$\to$(2-cluster) scattering such that the initial and final channels belong to $\mathbf{C}\,\mathbf{H}$

For $\alpha\in \mathbf{C}\,\mathbf{H}$ we define a function $\psi_\alpha^\infty(x,t)=\psi_\alpha^\infty(\xi,\eta;x,t)$ by

$$\psi_\alpha^\infty(x,t):=e^{-i\lambda_\alpha^\infty t}\phi_\alpha^\infty(x) \qquad \text{for } D(\alpha)=a_1,$$

(1.20)

$$\psi_\alpha^\infty(x,t):=e^{im\xi\cdot x-i((m/2)|\xi|^2+\lambda_\alpha^\infty)t}\phi_\alpha^\infty(x-\xi t-\eta)$$

for $D(\alpha)=a_2$.

It is easy to see that $\psi_\alpha^\infty(x,t)$ satisfies

$$\begin{aligned} &i\partial_t \psi_\alpha^\infty(t)=h_{\alpha,1}^\infty \psi_\alpha^\infty(t) && \text{for } D(\alpha)=a_1, \\ (1.21) \quad & i\partial_t \psi_\alpha^\infty(t)=[-(2m)^{-1}\Delta_x+V_{13}(x-\xi t-\eta)]\psi_\alpha^\infty(t) && \text{for } D(\alpha)=a_2. \end{aligned}$$

Furthermore the strong limits

$$(1.22) \qquad \Omega_\alpha^\pm=\Omega_\alpha^\pm(\xi,\eta):=\mathrm{s}\text{-}\lim_{t\to\pm\infty} U(\xi,\eta;0,t)\psi_\alpha^\infty(t)$$

exist in $L^2(\mathbf{R}^N)$ for each $\alpha\in CH$ and

$$(1.23) \qquad (\Omega_\alpha^\pm,\Omega_\beta^\pm)=\delta_{\alpha\beta}$$

holds for α, $\beta\in CH$ ([Y],p.155), where (,) denotes the inner product in $L^2(\mathbf{R}^N)$ Let α, $\beta\in CH$ be, for example, such that $D(\alpha)=a_1$, $D(\beta)=a_2$. Then the quantity $|(\Omega_\alpha^-\ \Omega_\beta^+)|^2$ is the transition probability that particle 3 forming a bound state ϕ_α^∞ with particle 2 in the remote past will be captured by particle 1 (moving along the orbit $\xi t+\eta$) in ϕ_β^∞ in the far future.

Now we state the main results. For $\xi\in\mathbf{R}\setminus\{0\}$ and $\eta\in\Pi_\xi$, we define

$$(1.24) \qquad S_{\beta\alpha}^\infty(\xi,\eta):=e^{-i\int_{-\infty}^{\infty} V_{12}(-\xi t-\eta)dt}(\Omega_\alpha^-(\xi,\eta),\Omega_\beta^+(\xi,\eta))$$

<u>Theorem 1.1</u>. Let α, $\beta\in CH$, and assume (V) Then for $f\in C(S^{N-1})$, the continuous functions on S^{N-1}, and $v_0>0$, we have

$$(1.25) \qquad \lim_{\substack{\mu\to\infty \\ v\to v_0}} (S_{\beta\alpha}^\mu((1/2)n_{D(\alpha)}v^2+\lambda_\alpha^\mu)f)(\omega)=S_{\beta\alpha}^\infty(v_0\omega,0)f(\omega)$$

uniformly on S^{N-1}

Since $\|S^{\mu}_{\beta\alpha}(\lambda)\|_{B(\Sigma)} \leq 1$ for $\lambda > 0$ and $C(S^{N-1})$ is dense in Σ, we have

Corollary 1.2. Let $\alpha, \beta \in CH$, and assume (V)' and fix $v_0 > 0$. Then

$$(1.26)\quad \underset{\substack{\mu \to \infty \\ v \to v_0}}{\text{s-lim}}\, (S^{\mu}_{\beta\alpha}((1/2)n_{D(\alpha)}v^2 + \lambda^{\mu}_{\alpha})f)(*) = S^{\infty}_{\beta\alpha}(v_0 *, 0)f(*) \qquad \text{in } \Sigma$$

for any $f \in \Sigma$

Theorem 1.3. Let $\alpha, \beta \in CH$ and assume (V) and fix $v_0 > 0$. Then

$$(1.27)\quad \lim_{\substack{\mu \to \infty \\ v \to v_0}} \sigma^{\mu}_{\beta\alpha}((1/2)n_{D(\alpha)}v^2 + \lambda^{\mu}_{\alpha}; \omega) = \int_{\Pi_\omega} |S^{\infty}_{\beta\alpha}(v_0\omega, \eta) - \delta_{\beta\alpha}|^2 d\eta$$

uniformly for $\omega \in S^{N-1}$, where $d\eta$ is the Lebesgue measure on Π_ω.

Scattering theory for the charge transfer model has first been studied by Yajima [Y] He has proved asymptotic completeness for equation (1.18). His idea is to reduce the scattering theory for the time-dependent Hamiltonian to that for a time-independent Hamiltonian following Howland [Ho] and then to use the stationary method for three-body problem (cf.[G-M]). Hagedorn [Ha] has obtained similar results by a time-dependent approach. Recently Wüller [W] and Graf [G] have extended Yajima's results by using geometric methods of Enss [E]

Now we explain the organization of this paper. In the preliminary Sect.2 we shall give the exact form of the scattering matrix (Theorem 2.3) for scattering $\alpha \to \beta$, though Theorem 2.3 will be proved in Sect.8. We shall need certain uniform estimates for a family of self-adjoint operators which can be obtained by extending multiple commuator methods of Jensen, Mourre and Perry [J-M-P](see also [J]), that had been originated from Mourre's work [M] (see also [P-S-S], [F-H], [T], [A-B-G], [Yaf]) These resolvent estimates will be given in Sect.3 by an abstract setting. In Sect.4 we shall give a stationary expression for $(\Omega_{a}^{-}(\xi,\eta),\Omega_{b}^{+}(\xi,\eta))$ Our main theorem will be proved in Sect.5. Lemma 5.4 contains essential estimates in our proof. The proof of this lemma will be given in Sect.6 by using the abstract commutator estimates in Sect.3. Certain lemmas of Sect.5will be proved in Sect.7

§ 2 Preliminaries

For $k, s \in \mathbf{R}$ the weighted Sobolev space $H_s^k(\mathbf{R}^d)$ is defined by

$$(2.1)\quad H_s^k(\mathbf{R}^d) := \{ f \in \mathcal{S}'(\mathbf{R}^d);\ \|f\|_{k,s} := \|\langle\xi\rangle^s (1-\Delta)^{k/2} f\| < +\infty \},$$

where $\mathcal{S}'$ denotes the tempered distributions, Δ the d-dimensional Laplacian and $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$, $\xi \in \mathbf{R}^d$ Note that $\|f\|_{k,s}$ is equivalent to $\|(1-\Delta)^{k/2}\langle\xi\rangle^s f\|$ and $\sum_{|\gamma|\le k}\|\langle\xi\rangle^s D_\xi^\gamma f\|$ if $k \in \mathbf{N}\cup\{0\}$, where $|\gamma| = \gamma_1 + \cdots + \gamma_d$, $D_\xi^\gamma := D_{\xi_1}^{\gamma_1}\cdots D_{\xi_d}^{\gamma_d}$, $D_{\xi_j} := -i(\partial/\partial\xi_j)$ for multi-indices $\gamma = (\gamma_1, \cdots, \gamma_d)$. We write $H^k(\mathbf{R}^d) := H_0^k(\mathbf{R}^d)$ and $L_s^2(\mathbf{R}^d) := H_s^0(\mathbf{R}^d)$ Note that the Fourier transform on $\mathcal{S}'(\mathbf{R}^d)$ maps $H_s^k(\mathbf{R}^d)$ onto $H_k^s(\mathbf{R}^d)$ boundedly for all $k, s \in \mathbf{R}$

<u>Lemma 2.1</u>. Let $\alpha \in \mathbf{C}\,\mathbf{H}$ $(a = D(\alpha))$, and assume (V) Then ϕ_α^{+}, ϕ_α^{-}

$\in \mathcal{S}(R^N)$ and $\phi_\alpha^\mu \to \phi_\alpha^\infty$ in $\mathcal{S}(R^N)$ as $\mu \to \infty$

<u>Proof</u>. Let $a=(i,(j,3))$ By $(-\Delta+1)\phi_\alpha^\mu=(2m_a(\lambda_\alpha^\mu - V_{j3})+1)\phi_\alpha^\mu$, we have for any $k \in N$

$$(2.2) \qquad \phi_\alpha^\mu = [(-\Delta+1)^{-1}(2m_a(\lambda_\alpha^\mu - V_{j3})+1)]^k \phi_\alpha^\mu.$$

Similarly we have for any $k \in N$

$$(2.3) \qquad \phi_\alpha^\infty = [(-\Delta+1)^{-1}(2m(\lambda_\alpha^\infty - V_{j3})+1)]^k \phi_\alpha^\infty.$$

Since for any $\ell \in N \cup \{0\}$ and $f \in H^\ell(R^N)$

$$\underset{\mu\to\infty}{\text{s-lim}}\ (-\Delta+1)^{-1}(2m_a(\lambda_\alpha^\mu - V_{j3})+1)f = (-\Delta+1)^{-1}(2m(\lambda_\alpha^\infty - V_{j3})+1)f$$

in $H^{\ell+2}(R^N)$, we see that

$$(2.4) \qquad \phi_\alpha^\mu \to \phi_\alpha^\infty \quad \text{strongly in } H^k(R^N)$$

as $\mu\to\infty$ for any $k \in N$ The following estimates are easily verified:

$$(2.5) \qquad \sup_{\mu>1}\{\|[(-\Delta+1)^{-1}(2m_a(\lambda_\alpha^\mu - V_{j3})+1)]^k\|_{B(L^2_s, H^{2k}_s)} + \|[(-\Delta+1)^{-1}(2m(\lambda_\alpha^\infty - V_{j3})+1)]^k\|_{B(L^2_s, H^{2k}_s)}\} < \infty$$

for any $k \in N$ and $s \geqq 0$. Here $\sup_{\mu>1}\{\cdots\} := \sup_{M_1, M_2 > M_0}\{\cdots\}$ for some large M_0. We claim that for each $k \in N$ and $s \geqq 0$:

$$(2.6) \qquad \sup_{\mu>1}\{\|\phi_\alpha^\mu\|_{2k,s} + \|\phi_\alpha^\infty\|_{2k,s}\} < \infty$$

Indeed, (2.6) for $k=0$ follows from [Ag], p.52, and (2.6) for $k \geqq 1$ from (2.2), (2.3) and (2.5) Thus by the Schwarz inequality. (2.4) and (2.6), we obtain

$$\|\langle x\rangle^{s}(-\Delta+1)^{k}(\phi_{\alpha}^{\mu}-\phi_{\alpha}^{\infty})\|^{2}$$

$$\leqq\|(-\Delta+1)^{k}(\phi_{\alpha}^{\mu}-\phi_{\alpha}^{\infty})\|\|\langle x\rangle^{2s}(-\Delta+1)^{k}(\phi_{\alpha}^{\mu}-\phi_{\alpha}^{\infty})\|$$

$$\leqq\|\phi_{\alpha}^{\mu}-\phi_{\alpha}^{\infty}\|_{2k,0}(\|\phi_{\alpha}^{\mu}\|_{2k,2s}+\|\phi_{\alpha}^{\infty}\|_{2k,2s})\ \to\ 0$$

as $\mu\to\infty$ Since k and s are arbitrary, this and (2.6) imply the desired results. ∎

The following limiting absorption principle is important for a representation of $S_{\beta\alpha}^{\mu}(\lambda)$

<u>Lemma 2.2</u> ([M], [P-S-S]) Assume (V) and fix $\mu\gg 1$. Let J be any compact interval in $\mathbb{R}\setminus\Lambda^{\mu}$ and fix $s>½$ Then the norm limits

(2.7) $$(H^{\mu}-\lambda\pm i0)^{-1}:=\lim_{\varepsilon\downarrow 0}(H^{\mu}-\lambda\pm i\varepsilon)^{-1}$$

exist in $\mathbf{B}(L_{s}^{2}(\mathbb{R}^{2N}),L_{-s}^{2}(\mathbb{R}^{2N}))$ uniformly for $\lambda\in J$, and $\mathbf{B}(L_{s}^{2}(\mathbb{R}^{2N}).L_{-s}^{2}(\mathbb{R}^{2N}))$-valued functions $(H^{\mu}-\lambda\pm i0)^{-1}$ are Hölder continuous in $\lambda\in J$

<u>Remark</u>. Resolvent estimates for three-body Schrödinger operators have been studied by Mourre [M] for more general class of potentials including long range potentials, and Mourre's results have been extended by Perry, Sigal and Simon [P-S-S] (see also [F-H]) to many-body Schrödinger operators. Recently these results have been developed by Tamura[T] and Amrein, Berthier and Georgescu [A-B-G]

For $a=\{i,(j,3)\}\in \mathbf{A}$, we define the intercluster potential I_a by $I_a := V - V_{j3}(r_3 - r_j)$ For $\alpha \in \mathbf{CH}$ $(D(\alpha)=a)$, $P_a^{\mu*}$ is given by

$$(2.8) \qquad (P_\alpha^{\mu*}f)(y_a) = \int \overline{\phi_\alpha^\mu(x_a)} f(x_a, y_a) dx_a \qquad \text{(see (1.9))}$$

Thus, by Lemma 2.1, P_α^μ and $P_\alpha^{\mu*}$ can be regarded as operators in

$$\mathbf{B}(L_t^2(\mathbf{R}_{y_a}^N), L_s^2(\mathbf{R}_{x_a}^N)\otimes L_t^2(\mathbf{R}_{y_a}^N)), \mathbf{B}(L_s^2(\mathbf{R}_{x_a}^N)\otimes L_t^2(\mathbf{R}_{y_a}^N), L_t^2(\mathbf{R}_{y_a}^N)),$$

respectively, for any $s, t \in \mathbf{R}$ Thus, by (V), we can see that

$$P_\beta^{\mu*} I_a P_\alpha^\mu \in \mathbf{B}(L_{-s}^2(\mathbf{R}_{y_a}^N), L_s^2(\mathbf{R}_{y_b}^N)),$$

$$(2.9) \qquad I_a P_\alpha^\mu \in \mathbf{B}(L_{-s}^2(\mathbf{R}_{y_a}^N), L_{\varepsilon_0 - s}^2(\mathbf{R}^{2N})),$$

$$P_\beta^{\mu*} I_b \in \mathbf{B}(L_{s-\varepsilon_0}^2(\mathbf{R}^{2N}), L_s^2(\mathbf{R}_{y_b}^N))$$

for any α, $\beta \in \mathbf{CH}$ $(a=D(\alpha)$, $b=D(\beta))$ and s with $1/2 < s \leqq \varepsilon_0/2$. Furthermore, we note that $Z_\alpha^\mu(\lambda)^* \in \mathbf{B}(\Sigma, L_{-s}^2(\mathbf{R}_{y_a}^N))$, $s>1/2$ (see (1.12)). Now we give an expression of $S_{\beta\alpha}^\mu(\lambda)$ for each $\mu \gg 1$.

Theorem 2.3. Let α, $\beta \in \mathbf{CH}$, and asssume (V). Then

$$(2.10) \qquad \begin{aligned} &S_{\beta\alpha}^\mu(\lambda) \\ &= \delta_{\beta\alpha} + 2\pi i Z_\beta^\mu(\lambda) P_\beta^{\mu*}[-I_{D(\alpha)} + I_{D(\beta)}(H^\mu - \lambda - i0)^{-1} I_{D(\alpha)}] P_\alpha^\mu Z_\alpha^\mu(\lambda)^* \end{aligned}$$

for a.e. $\lambda \in (\lambda_{\beta\alpha}^\mu, \infty)\backslash\Lambda^\mu$ Furthermore, the R.H.S. of (2.10) is a $\mathbf{B}(\Sigma)$-valued norm continuous function of $\lambda \in (\lambda_{\beta\alpha}^\mu, \infty)\backslash\Lambda^\mu$

Remark. $S_{\beta\alpha}^\mu(\lambda)$ is well defined for *all* $\lambda \in (\lambda_{\beta\alpha}^\mu, \infty)\backslash\Lambda^\mu$ by (2.10).

Proof. We only prove the second half of the statement. The formula

(2.10) will be shown in Sect.8. Let $1/2 < s \leqq \varepsilon_0/2$. Then, it follows from (2.9) and Lemma 2.2 that

(2.11) $P_\beta^{\mu *}[-I_a + I_b(H^\mu - \lambda - i0)^{-1} I_a] P_\alpha^\mu \in \mathbf{B}(L^2_s(R^N_{y_a}), L^2_{s}(R^N_{y_b}))$

$(a = D(\alpha),\ b = D(\beta))$ Furthermore $Z_\gamma^\mu(\lambda)$, $\gamma \in \mathrm{CH}$, is a $\mathbf{B}(L^2_s(R^N), \Sigma)$-valued norm continuous function in λ (cf [G-M], Proposition (2.1)), which together with (2.11) and Lemma 2.2 implies the second half of the theorem. ∎

Proposition 2.4. Let $\alpha, \beta \in \mathrm{CH}$, and assume (V). Then $T^\mu_{\beta\alpha}(\lambda)$ (see (1.16)) has an integral kernel $T^\mu_{\beta\alpha}(\lambda, \omega, \omega')$ given by

$$T^\mu_{\beta\alpha}(\lambda, \omega, \omega') = i(2\pi)^{-N+1}(n_a n_b)^{1/2}(4n_a n_b(\lambda - \lambda^\mu_\alpha)(\lambda - \lambda^\mu_\beta))^{(N-2)/4} \times$$

$$\times([-I_a + I_b(H^\mu - \lambda - i0))^{-1} I_a]\phi^\mu_\alpha e^{i(2n_a(\lambda - \lambda^\mu_\alpha))^{1/2}\omega' \cdot y_a}, \tag{2.12}$$

$$\phi^\mu_\beta e^{i(2n_b(\lambda - \lambda^\mu_\beta))^{1/2}\omega \cdot y_b})_{L^2(R^{2N})}$$

$(a = D(\alpha),\ b = D(\beta))$

Furthermore, $T^\mu_{\beta\alpha}(\lambda, \omega, \omega')$ is continuous in $(\lambda, \omega, \omega') \in ((\lambda^\mu_{\beta\alpha}, \infty) \setminus \Lambda^\mu) \times S^{N-1} \times S^{N-1}$, and so $T^\mu_{\beta\alpha}(\lambda) \in \mathbf{B}(C(S^{N-1}))$ and is also a Hilbert-Schmidt operator on Σ

Proof Fix a real s with $N/2 < s \leqq \varepsilon_0/2$. Since the map $R^N \ni \xi \to \exp(i\xi \cdot *) \in L^2_s(R^N)$ is strongly continuous, the continuity of $T^\mu_{\beta\alpha}(\lambda, \omega, \omega')$ in $(\lambda, \omega, \omega')$ follows in the same way as Theorem 2.3. To finish the proof, we have only to show that $T^\mu_{\beta\alpha}(\lambda)$ is an integral operator with kernel $T^{\#}_{\beta\alpha}(\lambda,\omega,\omega')$ Let $\gamma=\alpha,\ \beta$. For each $h_\gamma\in$ $C(S^{N-1})$,

$$(Z^{\#*}_{\gamma}(\lambda)h_\gamma)(y)=(2\pi)^{-N/2}n_c{}^{1/2}(2n_c(\lambda-\lambda^{\#}_{\gamma}))^{(N-2)/4}\times$$

(2.13)

$$\times\int e^{i(2n_c(\lambda-\lambda^{\#}_{\gamma}))^{1/2}\omega\cdot y}h_\gamma(\omega)d\omega \quad (c=D(\gamma)),$$

which follows from (1.12). Therefore we have

$$(T^{\#}_{\beta\alpha}(\lambda)h_\alpha,h_\beta)$$

$$=\int\int_{S^{N-1}\times S^{N-1}}T^{\#}_{\beta\alpha}(\lambda,\omega,\omega')h_\alpha(\omega')\overline{h_\beta(\omega)}d\omega' d\omega,$$

where $T^{\#}_{\beta\alpha}(\lambda,\omega,\omega')$ is the R.H.S. of (2.12) Since $C(S^{N-1})$ is dense in Σ, this implies that $T^{\#}_{\beta\alpha}(\lambda)$ is an integral operator with kernel $T^{\#}_{\beta\alpha}(\lambda,\omega,\omega')$ This completes the proof. ∎

§ 3 Abstract resolvent estimates

This section is devoted to extending the abstract results developed in [J-M-P]

Let H be a self-adjoint operator in a Hilbert space $\mathbf{H}$ whose inner product and norm will be denoted by (f,g) and $\|f\|$. Then we define the scale of spaces $\mathbf{H}_{+2}$ and $\mathbf{H}_{-2}$ associated to the self-adjoint operator H as follows. $\mathbf{H}_{+2}$ is the domain $D(H)$ with the graph norm $\|f\|_{+2}=\|(H+i)f\|$ and $\mathbf{H}_{-2}$ is the dual of $\mathbf{H}_{+2}$ obtained via the inner product in $\mathbf{H}$

Let H, A be self-adjoint operators in $\mathbf{H}$, I a compact interval in $\mathbf{R}$, and $d \in \mathbf{N}$

Assumption 3.1.

(H-i) $D(A) \cap D(H)$ is a core for H

(H-ii) $e^{i\theta A}$ leaves $D(H)$ invariant, and for each $f \in D(H)$

$$\sup_{|\theta|<1} \| H e^{i\theta A} f \| < \infty$$

(H-iii) Let $H^{(0)} = H$ There are self-adjoint operators $iH^{(1)}, \cdots, i^d H^{(d)}$ satisfying the following :

$$D(i^j H^{(j)}) \supset D(H) \quad (j = 1, \cdots, d).$$

the form $i[i^{j-1} H^{(j-1)}, A]$, defined on $D(H) \cap D(A)$ is bounded from below and closable, and the self-adjoint operator associated with its closure is $i^j H^{(j)} (j = 1, \cdots, d)$

$(([B, C]u, v) := (Cu, B^* v) - (Bu, C^* v).)$

(H-iv) The form $[H^{(d)}, A]$ defined on $D(A) \cap D(H)$ extends to a bounded operator from $\mathbf{H}_{+2}$ to $\mathbf{H}_{-2}$, which is denoted by $[H^{(d)}, A]_0$.

(H-v) There exist $C_0 > 0$ and $\phi \in C_0^\infty(R)$ supported in a sufficiently small neighborhood of I and satisfying $0 \leqq \phi \leqq 1$, $\phi \equiv 1$ on I, such that

$$\phi(H) i H^{(1)} \phi(H) \geqq C_0 \phi(H)^2 \tag{3.1}$$

Let W be a bounded operator on $\mathbf{H}$, and A a self-adjoint operator in $\mathbf{H}$

Assumption 3.2. Let $W^{(0)} = W$ There are bounded operators $W^{(1)}, \cdots, W^{(d)}$ on $\mathbf{H}$ satisfying the following properties:

The form $[W^{(j-1)}, A]$, defined on $D(A)$, extends to the bounded operator $W^{(j)}$ $(j=1, \cdots, d)$

Theorem 3.3. Let H, A be self-adjoint operators in $\mathbb{H}$, I a compact interval in $\mathbf{R}$, and $d \in \mathbb{N}$ Furthermore, if $d \geqq 2$, let $W_1, \cdots, W_{d-1}$ be bounded operators on $\mathbb{H}$ Assume assumptions 3.1 and 3.2 with $W = W_1, \cdots, W_{d-1}$ Fix a real $s > d - ½$, and set

$$I_{\pm} = \{ z \in \mathbf{C} ; \operatorname{Re} z \in I, \ 0 < \pm \operatorname{Im} z < 1 \}$$

Define

$$D(z) = \langle A \rangle^{-s} R(z) W_1 R(z) \cdots W_{d-1} R(z) \langle A \rangle^{-s} \qquad \text{for } d \geqq 2,$$

$$D(z) = \langle A \rangle^{-s} R(z) \langle A \rangle^{-s} \qquad \text{for } d = 1,$$

for $z \in \mathbf{C} \setminus \mathbf{R}$, where $R(z) = (H - z)^{-1}$, $\langle A \rangle = (1 + A^2)^{1/2}$ Then there exists a constant K, such that:

(i) $\sup_{z \in I_{\pm}} \| D(z) \| \leqq K$.

(ii) $\| D(z) - D(z') \| \leqq K |z - z'|^{\delta_1}$ for $z, z' \in I_{\pm}$, where

(3.2) $$\delta_1 = \delta_1(s, d) = \cfrac{1}{1 + \cfrac{sd}{s - d + ½}}$$

(iii) For $\lambda \in I$ the norm limits

$$D(\lambda \pm i0) := \lim_{\delta \downarrow 0} D(\lambda \pm i\delta)$$

exist in $B(\mathbb{H})$, and $D(\lambda \pm i0)$ are Hölder continuous with exponent δ_1 in $\lambda \in I$ in the operator norm.

Moreover, if H, A, ($W_1, \cdots, W_{d-1}$ if $d \geqq 2$) depend on a parameter ν such that ϕ, C_0 can be taken independently of ν, and that

$$\|H^{(j)}R(i)\| \ (j=1,\cdots,d),\ \|R(i)[H^{(d)},A]_0R(i)\|, \tag{3.3}$$
$$(\|W_k^{(j)}\| \ (j=0,\cdots,d;\ k=1,\cdots,d-1) \text{ if } d\geqq 2)$$

remain bounded in ν, then K can be taken independently of ν

Theorem 3.3 gives an extension of Theorem 2.2 of [J-M-P], in which all W_k are the identity operator Note that assumption 3.1 implies the non-existence of the point spectrum of H in I ([M])

Furthermore under assumption 3.1 the absence of the singular continuous spectrum in I can be proved ([M],[P-S-S])

We prove Theorem 3.3 by the commutator method of [J-M-P] The following Lemma 3.4 plays an important role in our proof.

For small $|\varepsilon|>0$ the operator

$$Q_d(\varepsilon)=\sum_{j=1}^{d}\frac{\varepsilon^j}{j!}H^{(j)} \tag{3.4}$$

is H-bounded with H-bound <1, since each $H^{(j)}$ is H-bounded by (H-iii) and the closed graph theorem. Thus the operator $H+Q_d(\varepsilon)$ is a closed operator with $D(H+Q_d(\varepsilon))=D(H)$, and furthermore the resolvents of this operator have the following properties.

Lemma 3.4 ([J-M-P],Lemma 3.1). Let H, A be self-adjoint operators in $\mathbf{H}$, I a compact interval in $\mathbf{R}$ Assume assumption 3.1. Then there exists a positive constant ε_1 such that for $0<\pm\varepsilon<\varepsilon_1$, $z\in I_\pm$, the following results hold:

(i) There exists a bounded inverse $G_z(\varepsilon)$ of $H+Q_d(\varepsilon)-z$.

(ii) The following estimates hold for $G_z(\varepsilon)$:

(3.5) $$\|G_z(\varepsilon)\| \leqq C\cdot|\varepsilon|^{-1},$$

(3.6) $$\|(H+i)G_z(\varepsilon)\| \leqq C\cdot|\varepsilon|^{-1}, \quad \|G_z(\varepsilon)(H+i)\| \leqq C\,|\varepsilon|^{-1},$$

(3.7) $$\begin{aligned} &\|(H+i)G_z(\varepsilon)\langle A\rangle^{-1}\| \leqq C\,|\varepsilon|^{-1/2}, \\ &\|\langle A\rangle^{-1}G_z(\varepsilon)(H+i)\| \leqq C\,|\varepsilon|^{-1/2} \end{aligned}$$

(iii) The form $[A, G_z(\varepsilon)]$, defined on $D(A)$, extends to a bounded operator on $\mathbf{H}$, which is denoted by $[A, G_z(\varepsilon)]_0$. Furthermore $G_z(\varepsilon)$ maps $D(A)$ into $D(A)\cap D(H)$

(iv) For each $z\in I_+$(resp. I_-) ,

$G_z(\varepsilon) \in C^1((0,\varepsilon_1);\mathbf{B}(\mathbf{H}))$ (resp. $C^1((-\varepsilon_1,0);\mathbf{B}(\mathbf{H}))$, and

(3.8) $$\frac{d}{d\varepsilon}G_z(\varepsilon) = [G_z(\varepsilon), A]_0 + \frac{\varepsilon^d}{d!}G_z(\varepsilon)[H^{(d)}, A]_0 G_z(\varepsilon)$$

Moreover, if H, A depend on a parameter ν such that ϕ, C_0 can be taken independently of ν, and that

(3.9) $$\|H^{(j)}R(i)\| \ (j=1,\cdots,d), \ \|R(i)[H^{(d)}, A]_0 R(i)\|,$$

remain bounded in ν, then C can be taken independently of ν

See [J-M-P] for the proof of the first half of the lemma. The last half can be shown by carefully checking the estimates carried out in [J-M-P] (see also [M],[P-S-S]).

Moreover we need the following elementary lemma.

Lemma 3.5. Fix an integer $k\geqq 0$ and let $f_k(\varepsilon) = |\log\varepsilon|$ for $k=0$, $f_k(\varepsilon)=\varepsilon^{-k}$ for $k\in N$ Assume that a $\mathbf{B}(\mathbf{H})$-valued C^1-function $X(\varepsilon)$,

$\varepsilon \in (0, \varepsilon_1)$ $(\varepsilon_1 > 0)$, satisfies the following inequalities:

$$(3.10) \qquad \|\frac{d}{d\varepsilon} X(\varepsilon)\| \leq C_1 \, (\|X(\varepsilon)\|^p \, \varepsilon^{-q} + f_k(\varepsilon) + 1).$$

$$(3.11) \qquad \|X(\varepsilon)\| \leq C_2 \, \varepsilon^{-r},$$

where p, q, r, C_1, C_2 are constants satisfying $0 \leq p, q < 1$, $r \geq 0$, $C_1 > 0$, $C_2 > 0$. Then the following estimates hold:

$$(3.12) \qquad \|X(\varepsilon)\| \leq C \cdot \varepsilon^{-k+1} \qquad \text{when } k \geq 2,$$

$$(3.13) \qquad \|X(\varepsilon)\| \leq C \cdot |\log \varepsilon| \qquad \text{when } k = 1,$$

$$(3.14) \qquad \|X(\varepsilon)\| \leq C \qquad \text{when } k = 0,$$

where $C = C(C_1, C_2, \varepsilon_1, p, q, r) > 0$. Furthermore, when $k = 0$, the norm limit $X(0) := \lim_{\varepsilon \downarrow 0} X(\varepsilon)$ exists in $\mathbf{B}(\mathbf{H})$

<u>proof</u>. Putting (3.11) in (3.10), we have

$$(3.15) \qquad \|\frac{d}{d\varepsilon} X(\varepsilon)\| \leq C \cdot (\varepsilon^{-pr-q} + f_k(\varepsilon) + 1).$$

We first consider for $k \geq 1$. By integrating with respect to ε, we have (3.12), (3.13) when $pr + q \leq k$. When $pr + q > k$, we get $\|X(\varepsilon)\| \leq C \cdot \varepsilon^{-r_1}$, where $r_1 = pr + q - 1$ (Note that $r - r_1 = (1-p)r + (1-q) > 1-q > 0$.) Putting this into (3.10), we have the inequality replaced r by r_1 in (3.15) If $pr_1 + q \leq k$, we obtain (3.12), (3.13) If $pr_1 + q > k$, we get $\|X(\varepsilon)\| \leq C \cdot \varepsilon^{-r_2}$, where $r_2 = pr_1 + q - 1$. Continuing this process, we can find some $r_n (= pr_{n-1} + q - 1)$ with $pr_n + q \leq k$, $pr_{n-1} + q > k$, since $r_j - r_{j-1} \geq 1 - q > 0$. Thus we obtain (3.12), (3.13). When $k = 0$, (3.14) is obtained similarly and the existence of $X(0)$ follows from the integrability of the R.H.S. of (3.10).∎

<u>Proof of Theorem 3.3</u>. We have only to prove this theorem for $d \geq 2$

because the theorem for $d=1$ has been proved in [P-S-S] Moreover we give only the proof for $z\in I_+$.

(i) For multi-indices of nonnegative integers $\alpha=(\alpha_1,\cdots,\alpha_{d-1})$, $\beta=(\beta_1,\cdots,\beta_{d-1})$ we write $|\alpha|=\alpha_1+\cdots+\alpha_{d-1}$, and $\alpha\leqq\beta$ if and only if $\alpha_j\leqq\beta_j$ for all j. Let Γ_α be a family of all multiindices β with $\alpha\leqq\beta$, $|\beta|=|\alpha|+1$. Namely $\beta\in\Gamma_\alpha$ implies that $\alpha_j=\beta_j-1$ for some j and $\beta_i=\alpha_i$ for $i\neq j$. We set

$$F_z^\alpha(\varepsilon):=\langle A\rangle^{-s}G_z(\varepsilon)W_1^{(\alpha_1)}G_z(\varepsilon)W_2^{(\alpha_2)}G_z(\varepsilon)\cdots W_{d-1}^{(\alpha_{d-1})}G_z(\varepsilon)\langle A\rangle^{-s}$$

for $z\in I_+$, $\varepsilon>0$, $\alpha=(\alpha_1,\cdots,\alpha_{d-1})$ with $|\alpha|\leqq d$.

By Lemma 3.4 (iv), we have for $|\alpha|\leqq d-1$,

$$\begin{aligned}
&\frac{d}{d\varepsilon}F_z^\alpha(\varepsilon)\\
&\quad=\langle A\rangle^{-s}\Big(\frac{d}{d\varepsilon}G_z(\varepsilon)\Big)W_1^{(\alpha_1)}G_z(\varepsilon)\cdots W_{d-1}^{(\alpha_{d-1})}G_z(\varepsilon)\langle A\rangle^{-s}\\
&\qquad+\cdots+\langle A\rangle^{-s}G_z(\varepsilon)W_1^{(\alpha_1)}G_z(\varepsilon)\cdots W_{d-1}^{(\alpha_{d-1})}\Big(\frac{d}{d\varepsilon}G_z(\varepsilon)\Big)\langle A\rangle^{-s}\\
(3.16)\quad&\quad=\langle A\rangle^{-s}\Big\{[G_z(\varepsilon),A]_0W_1^{(\alpha_1)}G_z(\varepsilon)\cdots W_{d-1}^{(\alpha_{d-1})}G_z(\varepsilon)\\
&\qquad+\cdots+G_z(\varepsilon)W_1^{(\alpha_1)}G_z(\varepsilon)\cdots W_{d-1}^{(\alpha_{d-1})}[G_z(\varepsilon),A]_0\Big\}\langle A\rangle^{-s}\\
&+\frac{\varepsilon^d}{d!}\Big\{\langle A\rangle^{-s}G_z(\varepsilon)[H^{(d)},A]_0G_z(\varepsilon)W_1^{(\alpha_1)}G_z(\varepsilon)\cdots W_{d-1}^{(\alpha_{d-1})}G_z(\varepsilon)\langle A\rangle^{-s}\\
&+\cdots+\langle A\rangle^{-s}G_z(\varepsilon)W_1^{(\alpha_1)}G_z(\varepsilon)\cdots W_{d-1}^{(\alpha_{d-1})}G_z(\varepsilon)\times\\
&\qquad\qquad\times[H^{(d)},A]_0G_z(\varepsilon)\langle A\rangle^{-s}\Big\}
\end{aligned}$$

$= I_1(\varepsilon) + I_2(\varepsilon)$

First we estimate $I_2(\varepsilon)$ Since $s>1$ and

$(H+i)^{-1}[H^{(d)}, A]_0(H+i)^{-1}$, $W_j^{(\alpha_j)}$ $(j=1,\cdots,d-1)$ are bounded, by assumption 3.1 (iv) and assumption 3.2, we have

$$\|I_2(\varepsilon)\| \leqq C\cdot\varepsilon^{d}\cdot\varepsilon^{-1/2}\ \varepsilon^{-d+1}\ \varepsilon^{-1/2} \leqq C \tag{3.17}$$

by Lemma 3.4 (ii).

Next we estimate $I_1(\varepsilon)$ Noting that $G_*(\varepsilon)$ maps $D(A)$ into $D(A)$ and $W_j^{(\alpha_j)}$ maps $D(A)$ into $D(A)$, as follows from assumption 3.2 and Lemma 3.4 (iii), we have, by elementary computation,

$$I_1(\varepsilon) = [F_*^{\alpha}(\varepsilon), A] - \sum_{\beta\in\Gamma_\alpha} F_*^{\beta}(\varepsilon)$$

Since $\|\langle A\rangle^s\ F_*^{\alpha}(\varepsilon)\|$, $\|F_*^{\alpha}(\varepsilon)\langle A\rangle^s\| \leqq C\ \varepsilon^{-d+(1/2)}$ by Lemma 3.4 (ii) and

$$\|F_*^{\alpha}(\varepsilon)\langle A\rangle\| \leqq \|F_*^{\alpha}(\varepsilon)\|^{1-(1/s)}\|F_*^{\alpha}(\varepsilon)\langle A\rangle^s\|^{1/s},$$

$$\|\langle A\rangle F_*^{\alpha}(\varepsilon)\| \leqq \|F_*^{\alpha}(\varepsilon)\|^{1-(1/s)}\|\langle A\rangle^s F_*^{\alpha}(\varepsilon)\|^{1/s}$$

by interpolation, we have

$$\|[F_*^{\alpha}(\varepsilon), A]\| \leqq \|F_*^{\alpha}(\varepsilon)\langle A\rangle\| + \|\langle A\rangle F_*^{\alpha}(\varepsilon)\|$$

$$\leqq C\cdot\|F_*^{\alpha}(\varepsilon)\|^{1-(1/s)}\varepsilon^{(-d+(1/2))/s}$$

Thus we get

$$\|I_1(\varepsilon)\| \leqq C(\|F_*^{\alpha}(\varepsilon)\|^{1-(1/s)}\varepsilon^{(-d+(1/2))/s} + \sum_{\beta\in\Gamma_\alpha}\|F_*^{\beta}(\varepsilon)\|) \tag{3.18}$$

Therefore $F^{\alpha}_{*}(\varepsilon)$ satisfies

$$(3.19)\quad \left\|\frac{d}{d\varepsilon}F^{\alpha}_{*}(\varepsilon)\right\| \leqq C\left(\|F^{\alpha}_{*}(\varepsilon)\|^{1-(1/s)}\varepsilon^{(-d+(1/2))/s}+\sum_{\beta\in\Gamma_{\alpha}}\|F^{\beta}_{*}(\varepsilon)\|+1\right)$$

for all multi-indices α with $|\alpha|\leqq d-1$.

Furthermore, it follows from Lemma 3.4 (ii) that

$$(3.20)\qquad \|F^{\gamma}_{*}(\varepsilon)\|\leqq C\varepsilon^{-d+1}$$

for all multi-indices γ with $|\gamma|\leqq d$.

Let $|\alpha|=d-1$. Then we have by (3.19) and (3.20)

$$\left\|\frac{d}{d\varepsilon}F^{\alpha}_{*}(\varepsilon)\right\| \leqq C\left(\|F^{\alpha}_{*}(\varepsilon)\|^{1-(1/s)}\varepsilon^{(-d+(1/2))/s}+\varepsilon^{-d+1}+1\right)$$

Applying Lemma 3.5 to this, we have

$$(3.21)\qquad \|F^{\alpha}_{*}(\varepsilon)\|\leqq C\varepsilon^{-d+2}$$

Next let $|\alpha|=d-2$. Then $|\beta|=d-1$ for $\beta\in\Gamma_{\alpha}$.

Thus we obtain by (3.19)

$$\left\|\frac{d}{d\varepsilon}F^{\alpha}_{*}(\varepsilon)\right\| \leqq C\left(\|F^{\alpha}_{*}(\varepsilon)\|^{1-(1/s)}\varepsilon^{(-d+(1/2))/s}+\varepsilon^{-d+2}+1\right).$$

Applying Lemma 3.5, we have

$$\|F^{\alpha}_{*}(\varepsilon)\|\leqq C\varepsilon^{-d+3}$$

Continuing, we have for $|\alpha|=0$

$$(3.22)\quad \left\|\frac{d}{d\varepsilon}F^{\alpha}_{*}(\varepsilon)\right\| \leqq C\left(\|F^{\alpha}_{*}(\varepsilon)\|^{1-(1/s)}\varepsilon^{(-d+(1/2))/s}+|\log\varepsilon|+1\right).$$

Thus we have the following estimate, by Lemma 3.5, (3.23) $$\sup_{z\in I_+,\,0<\varepsilon<1}\|F_z(\varepsilon)\|\leqq K<\infty,$$

where $F_z(\varepsilon):=F_z^{\alpha}(\varepsilon)$ for $|\alpha|=0$.

Since $\lim_{\varepsilon\downarrow 0}\|Q_d(\varepsilon)R(z)\|=0$ for each $z\in \mathbf{C}\setminus\mathbf{R}$, $1+Q_d(\varepsilon)R(z)$ has a bounded inverse, and so

$$G_z(\varepsilon)=R(z)(1+Q_d(\varepsilon)R(z))^{-1}$$

holds for each $z\in \mathbf{C}\setminus\mathbf{R}$ when $\varepsilon>0$ is small. Therefore we get

$$\lim_{\varepsilon\downarrow 0} G_z(\varepsilon)=R(z)$$

for each $z\in \mathbf{C}\setminus\mathbf{R}$, and so we have by (3.23)

$$\sup_{z\in I_+}\|D(z)\|\leqq K.$$

(ii) For simplicity we write $n=(d-½)/s$. By (3.22), (3.23) we obtain

$$\left\|\frac{d}{d\varepsilon}F_z(\varepsilon)\right\|\leqq C(\varepsilon^{-n}+1).$$

Integrating this we have

(3.24) $$\|F_z(\varepsilon)-F_z(0)\|\leqq C\cdot\varepsilon^{1-n}$$

On the other hand $G_z(\varepsilon)$ is differetiable in $z\in I_+$ for each $\varepsilon>0$ by Lemm 3.4. We have the following estimate by Lemma 3.4(ii):

$$\left\|\frac{d}{dz}F_z(\varepsilon)\right\|\leqq\|\langle A\rangle^{-s}G_z(\varepsilon)^2W_1\cdots G_z(\varepsilon)\langle A\rangle^{-s}\|+$$

$$+\cdots+\|\langle A\rangle^{-s}G_z(\varepsilon)W_1\cdots W_{d-1}G_z(\varepsilon)^2\langle A\rangle^{-s}\|$$

$$\leq C\cdot \varepsilon^{-d},$$

which implies

$$\|F_z(\varepsilon)-F_{z'}(\varepsilon)\|\leq C\ \varepsilon^{-d}|z-z'| \tag{3.25}$$

for $z,\ z'\in I_+,\ \varepsilon>0$. Let $\varepsilon=|z-z'|^{\delta_2},\ \delta_2=(1-n)^{-1}\delta_1$ (see (3.2) for δ_1) Then by (3.24), (3.25) we have

$$\|F_z(0)-F_{z'}(0)\|$$

$$\leq\|F_z(0)-F_z(\varepsilon)\|+\|F_z(\varepsilon)-F_{z'}(\varepsilon)\|+\|F_{z'}(\varepsilon)-F_{z'}(0)\|$$

$$\leq C\cdot|z-z'|^{\delta_1}$$

Thus we have proved (ii). (iii) follows from (ii)

The proof of the last half can be obtained if one takes into consideration the last part of Lemma 3.4 and the proof carried out above. ∎

§ 4 The quantity $(\Omega_-^{\pm}(\xi,\eta),\Omega_+^{\pm}(\xi,\eta))$

We fix $\xi\in R^N\setminus\{0\}$ and $\eta\in\Pi_\xi$ and assume (V) throughout this section. We define an operator in $L^2(R^{N+1})$ $(R^{N+1}=R_x^N\times R_t)$:

$$K_{\xi,\eta}:=-\frac{1}{2m}\Delta_x-i\partial_t+V_{23}(x)+V_{13}(x-\xi t-\eta)+V_{12}(-\xi t-\eta) \tag{4.1}$$

Since $-\frac{1}{2m}\Delta_x-i\partial_t$ is a self-adjoint operator with domain

$D_0:=\{u\in L^2(R^{N+1});\ -\frac{1}{2m}\Delta_x u-i\partial_t u\in L^2(R^{N+1})\}$ and has a core

$\mathcal{S}(R^{N+1})$. $K_{\varepsilon,\eta}$ is a self-adjoint operator with domain D_0 and with core $\mathcal{S}(R^{N+1})$ by (V).

Lemma 4.1. Let J be a compact interval and $s>\frac{1}{2}$ Then the norm limits

$$(4.2)\qquad (K_{\varepsilon,\eta}-\lambda\pm i0)^{-1}:=\lim_{\varepsilon\downarrow 0}(K_{\varepsilon,\eta}-\lambda\pm i\varepsilon)^{-1}$$

exist in $\mathbf{B}(L^2(R^N)\otimes L^2_s(R), L^2(R^N)\otimes L^2_{-s}(R))$ uniformly for $\lambda\in J$

Proof. We shall apply Theorem 3.3. We set $\mathbf{H}=L^2(R^{N+1})$, $d=1$, $H=K_{\varepsilon,\eta}$, $A=t\times$, and $I=J$. Then assumption 3.1,(H-i) is satisfied because $\mathcal{S}(R^{N+1})$ is a common core for H and A (H-ii) is obvious. Since $i[H,A]=1$, (H-iii)~(H-v) follow. The conclusion follows from Theorem 3.3. ∎

It is easy to verify that

$$(4.3)\qquad U_1(t,s):=e^{-i\int_s^t V_{12}(-\xi\tau-\eta)d\tau}U(\xi,\eta;t,s),\quad (t,s\in R)$$

is the unique propagator of the following equation:

$$(4.4)\qquad i\partial_t\psi(t)=[h_{\varepsilon,\eta}(t)+V_{12}(-\xi t-\eta)]\psi(t)\ ,$$

where $U(\xi,\eta;t,s)$ and $h_{\varepsilon,\eta}(t)$ have been defined in (1.19), (1.18), respectively We identify $L^2(R^{N+1})$ with $L^2(R_t;L^2(R^N_x))$ and introduce a family of unitary operators $\widetilde{U}(\tau)$ $(\tau\in R)$ on $L^2(R^{N+1})$:

$$(4.5)\qquad (\widetilde{U}(\tau)f)(x,t)=(U_1(t,t-\tau)f(*,t-\tau))(x)$$

for $f\in L^2(R^{N+1})$

This family is a strongly continuous unitary group on $L^2(R^{N+1})$, and $\tilde{U}(\tau)f$ is strongly differentiable in τ for each $f \in \mathcal{S}(R^{N+1})$ and

$$\frac{d}{d\tau}\tilde{U}(\tau)f\Big|_{\tau=0} = -iK_{\xi,\eta}f.$$

Thus we have $\tilde{U}(\tau) = e^{-i\tau K_{\xi,\eta}}$, and so

$$(4.6) \qquad (K_{\xi,\eta} - i\varepsilon)^{-1} = i\int_0^\infty e^{-\varepsilon\tau}\,\tilde{U}(\tau)d\tau \qquad \text{for } \varepsilon > 0$$

(cf. [Ho] and [Y] for the above discussion)

For $a \in \mathbf{A}$ we define $W_a(x,t) = W_a(\xi,\eta;x,t)$ by

$$(4.7) \qquad \begin{aligned} W_{a_1}(x,t) &:= V_{13}(x-\xi t-\eta) + V_{12}(-\xi t-\eta), \\ W_{a_2}(x,t) &:= V_{23}(x) + V_{12}(-\xi t-\eta). \end{aligned}$$

Lemma 4.2. Let $\alpha,\ \beta \in \mathbf{CH}$ with $a = D(\alpha)$, $b = D(\beta)$, and assume (V) Then

$$(4.8) \qquad \begin{aligned} &e^{-i\int_{-\infty}^{\infty} V_{12}(-\xi\tau-\eta)d\tau}(\Omega_\alpha^-(\xi,\eta),\Omega_\beta^+(\xi,\eta))_{L^2(R^N)} - \delta_{\alpha\beta} \\ &= -i(W_a\psi_\alpha^-,\psi_\beta^\infty)_{L^2(R^{N+1})} + i((K_{\xi,\eta}-i0)^{-1}W_a\psi_\alpha^-,W_b\psi_\beta^-)_{L^2(R^{N+1})}, \end{aligned}$$

where $\psi_\gamma^\infty = \psi_\gamma^\infty(x,t)$ ($\gamma = \alpha,\ \beta$; see (1.20)) and $W_c = W_c(x,t)$ ($c \in \mathbf{A}$).

Remark. By (V) and Lemma 2.1, it is easy to see that $W_{D(\gamma)}\psi_\gamma^\mp \in L_s^2(R^{N+1})$ for some $s > \frac{1}{2}$ Therefore the second term in the R.H.S. of (4.8) is well-defined by Lemma 4.1.

Proof. By $V_{12}(-\xi t-\eta) \in L^1(R_t)$ and (4.3), we have

$$(4.9)\qquad \Omega^{\pm}_{1,\gamma} := e^{-i\int_{\pm\infty}^{0} V_{12}(-\xi\tau-\eta)d\tau}\,\Omega^{\pm}_{\gamma}(\xi,\eta) \qquad \text{(see (1.22))}$$

$$= s\text{-}\lim_{t\to\pm\infty} U_1(0,t)\psi^{\infty}_{\gamma}(*,t) \quad \text{in} \quad L^2(R^N)$$

and $(\Omega^{\pm}_{1,\alpha},\Omega^{\pm}_{1,\beta}) = \delta_{\beta\alpha}$ (see (1.23)). Thus the L.H.S. of (4.8) equals $(\Omega^{-}_{1,\alpha} - \Omega^{+}_{1,\alpha},\Omega^{+}_{1,\beta})$ Since for $\gamma \in CH$, $\psi^{\infty}_{\gamma}(*,t) \in H^2(R^N)$ for each $t \in R$, $U_1(0,t)\psi^{\infty}_{\gamma}(*,t)$ is continuously differentiable with respect to t in $L^2(R^N)$ and satisfies

$$(4.10)\qquad \partial_t U_1(0,t)\psi^{\infty}_{\gamma}(*,t) = iU_1(0,t)W_{D(\gamma)}\psi^{\infty}_{\gamma}(*,t)$$

Therefore we obtain

$$(\Omega^{-}_{1,\alpha} - \Omega^{+}_{1,\alpha},\Omega^{+}_{1,\beta})$$

$$= -\lim_{\tau\to\infty}(U_1(0,\tau)\psi^{\infty}_{\alpha}(*,\tau) - U_1(0,-\tau)\psi^{\infty}_{\alpha}(*,-\tau),\Omega^{+}_{1,\beta})$$

$$= -\lim_{\tau\to\infty}\int_{-\tau}^{\tau}\frac{d}{dt}(U_1(0,t)\psi^{\infty}_{\alpha}(*,t),\Omega^{+}_{1,\beta})dt$$

$$(4.11)\qquad = -i\int_{-\infty}^{\infty}(U_1(0,t)W_{\alpha}(*,t)\psi^{\infty}_{\alpha}(*,t),\Omega^{+}_{1,\beta})dt.$$

Here we note that the integral converges absolutely because the following estimates follow from (V) and Lemma 2.1:

$$(4.12)\qquad \|W_{D(\gamma)}(*,t)\psi^{\infty}_{\gamma}(*,t)\|_{L^2(R^N)} \le \text{const}\cdot(1+|t|)^{-\varepsilon_0}, \quad \gamma \in CH$$

By using (4.10) for $\gamma=\alpha$, we have in the same way as (4.11)

$$(4.13)\qquad (U_1(0,t)W_{\alpha}(*,t)\psi^{\infty}_{\alpha}(*,t),\Omega^{+}_{1,\beta})$$

$$= (U_1(0,t)W_{\alpha}(*,t)\psi^{\infty}_{\alpha}(*,t),U_1(0,t)\psi^{\infty}_{\beta}(*,t))$$

$$+\lim_{\tau\to\infty}\ (U_1(0,t)W_a(*,t)\psi_a^\infty(*,t),$$

$$[U_1(0,\tau)\psi_\beta^\infty(*,\tau)-U_1(0,t)\psi_\beta^\infty(*,t)])$$

$$=(W_a(*,t)\psi_a^\infty(*,t),\psi_\beta^\infty(*,t))$$

$$-i\int_t^\infty(U_1(0,t)W_a(*,t)\psi_a^\infty(*,t),U_1(0,s)W_b(*,s)\psi_\beta^\infty(*,s))ds$$

$$=(W_a(*,t)\psi_a^\infty(*,t),\psi_\beta^\infty(*,t))$$

$$-i\int_t^\infty(U_1(s,t)W_a(*,t)\psi_a^\infty(*,t),W_b(*,s)\psi_\beta^\infty(*,s))ds\quad(\text{by (U-ii)}),$$

where the integral converges absolutely by (4.12) for $\gamma=\beta$ Thus by (4.11) and (4.13) we obtain

$$(\Omega_{1,-}^{-}-\Omega_{1,\alpha}^{+},\Omega_{1,\beta}^{+})$$

$$=-i(W_a\psi_a^\infty,\psi_\beta^\infty)_{L^2(R^{N+1})}$$

$$-\int_{-\infty}^{\infty}dt\int_t^\infty(U_1(s,t)W_a(*,t)\psi_a^\infty(*,t),W_b(*,s)\psi_\beta^\infty(*,s))ds$$

The double integral absolutely converges, since the inner integral is $O(|t|^{-2\varepsilon_0+1})$ by (4.12). and this is calculated as

$$\int_{-\infty}^{\infty}ds\int_{-\infty}^{s}(U_1(s,t)W_a(*,t)\psi_a^\infty(*,t),W_b(*,s)\psi_\beta^\infty(*,s))dt$$

$$=\int_{-\infty}^{\infty}ds\int_0^\infty(U_1(s,s-t)W_a(*,s-t)\psi_a^\infty(*,s-t),W_b(*,s)\psi_\beta^\infty(*,s))dt$$

$$=\int_0^\infty(\widetilde{U}(t)W_a\psi_a^\infty,W_b\psi_\beta^\infty)_{L^2(R^{N+1})}dt\qquad(\text{see (4.5)})$$

$$= \lim_{\varepsilon \downarrow 0} \int_0^\infty e^{-\varepsilon t} (\widetilde{U}(t) W_a \psi_\alpha^\infty, W_b \psi_\beta^\infty)_{L^2(R^{N+1})} \, dt$$

$$= -i((K_{\varepsilon,\mu} - i0)^{-1} W_a \psi_\alpha^\infty, W_b \psi_\beta^\infty)_{L^2(R^{N+1})} ,$$

where we have used (4.6) and Lemma 4.1 in the last step. This completes the proof. ∎

§ 5 Proofs of the main theorems

In this section we will prove Theorems 1.1, and 1 3. To do so we prepare some lemmas and propositions. Throughout this section, we assume (V) and $\mu \gg 1$, and fix $v_0 > 0$ and a 2-body initial channel $\alpha \in \mathbf{CH}$ with $D(\alpha) = a$ and a 2-body final channel $\beta \in \mathbf{CH}$ with $D(\beta) = b$.

<u>5.1</u> The purpose of this subsection is to rewrite $T_{\beta\alpha}^{\mu}((1/2)n_a v^2 + \lambda_\alpha^\mu; \omega, \omega')$ in a form convenient for later purposes. We first note that when $\mu \to \infty$,

$$(5.1) \qquad m_a,\ m_b \to m;\ n_a,\ n_b \to \infty;\ n_a/n_b \to 1;\ \lambda_\gamma^\mu \to \lambda_\gamma^\infty$$

for $\gamma = \alpha, \beta$. Thus for any $v > 0$, there exists a unique $v' = v'(v, \mu) > 0$ such that

$$(5.2) \qquad \frac{1}{2} n_a v^2 + \lambda_\alpha^\mu = \frac{1}{2} n_b v'^2 + \lambda_\beta^\mu,$$

since $\mu \gg 1$. Clearly $v' \to v_0$ as $\mu \to \infty$, $v \to v_0$. Throughout this section we assume $|v - v_0| \ll 1$ in addition to $\mu \gg 1$. Therefore, using the coordinates (x_a, y_a), we have for $\lambda=(1/2)n_a v^2+\lambda_\alpha^\mu$

(5.3) $$T_{\beta\alpha}^{\mu}(\lambda;\omega,\omega')=C_1(v,\mu)\int e^{in_a v\omega'\cdot y_a}dy_a\int \phi_\alpha^{\mu}(x_a)\times$$

$$\overline{\times([-I_a+I_a(H^{\mu}-\lambda+i0)^{-1}I_b]\phi_\beta^{\mu}(x_b(\ ,\ ))e^{in_b v'\omega\cdot y_b(\ ,\ )})(x_a,y_a)dx_a}$$

(see (2.12) for $T_{\beta\alpha}^{\mu}(\lambda;\omega,\omega')$ and (1.6) for $x_b=x_b(x_a,y_a)$, $y_b=y_b(x_a,y_a)$), where

(5.4) $$C_1(v,\mu)=i(2\pi)^{-N+1}(vv')^{\frac{N-2}{2}}(n_a n_b)^{\frac{N-1}{2}}$$

For each $\omega\in S^{N-1}$, v and μ we define a self-adjoint operator $H^{\mu}(\omega,v)$ in $L^2(\mathbb{R}^{2N})$ by

(5.5) $$H^{\mu}(\omega,v):=e^{-in_a v\omega\ y_a}(H^{\mu}-\frac{1}{2}n_a v^2)e^{in_a v\omega\cdot y_a}$$
$$=H^{\mu}-iv\omega\ \nabla_{y_a}$$

with domain $D(H^{\mu}(\omega,v))=H^2(\mathbb{R}^{2N})$, and denote the resolvent of $H^{\mu}(\omega,v)$ by

(5.6) $$R^{\mu}(\omega,v;z)=(H^{\mu}(\omega,v)-z)^{-1}$$

By Lemma 2.2 and (5.5) we have the norm limits

$$R^{\mu}(\omega,v;\lambda_\alpha^{\mu}\pm i0):=\lim_{\varepsilon\downarrow 0}R^{\mu}(\omega,v;\lambda_\alpha^{\mu}\pm i\varepsilon)$$

in $\mathbf{B}(L_s^2(\mathbb{R}^{2N}),L_{-s}^2(\mathbb{R}^{2N}))$, $s>1/2$, and

$$R^{\mu}(\omega,v;\lambda_\alpha^{\mu}\pm i0)$$
$$=e^{-in_a v\omega\ y_a}(H^{\mu}-(1/2)n_a v^2+\lambda_\alpha^{\mu}\mp i0)^{-1}e^{in_a v\omega\ y_a}.$$

Thus $T_{\beta\alpha}^{\mu}((1/2)n_a v^2+\lambda_\alpha^{\mu};\omega,\omega')$ can be written as follows:

$$T^{\mu}_{\beta\alpha}((1/2)n_a v^2+\lambda^{\mu}_{\alpha};\omega,\omega')=C_1(v,\mu)\int e^{in_a v(\omega'-\omega)\cdot y_a}dy_a\int\phi^{\mu}_{\alpha}(x_a)\times \tag{5.7}$$
$$\times(\overline{[-I_a+I_a R^{\mu}(\omega,v;\lambda^{\mu}_{\alpha}-i0)I_b]\phi^{\mu}_{\beta}(x_b(\ ,\ ))\chi^{\mu}(\omega,v))(x_a,y_a)}dx_a,$$

where

$$\chi^{\mu}(\omega,v)=\chi^{\mu}(\omega,v;x_a,y_a):=e^{in_b v\omega\cdot y_b-in_a v\omega\cdot y_a}, \tag{5.8}$$

$$y_b=y_b(x_a,y_a)$$

We define

$$E^{\mu}(\omega,v;x_a,y_a):=(2\pi)^{N/2}(n_a v)^{1-N}C_1(v,\mu)\phi^{\mu}_{\alpha}(x_a)\times$$
$$\overline{([-I_a+I_a R^{\mu}(\omega,v;\lambda^{\mu}_{\alpha}-i0)I_b]\phi^{\mu}_{\beta}(x_b(\ ,\ ))\chi^{\mu}(\omega,v))(x_a,y_a)}, \tag{5.9}$$
$$G^{\mu}(\omega,v;y_a):=\int E^{\mu}(\omega,v;x_a,y_a)dx_a,$$

and we denote by $^{\vee}$ the inverse Fourier transform i e.

$$\check{g}(\xi)=(2\pi)^{-\frac{N}{2}}\int e^{i\xi\cdot y_a}g(y_a)dy_a. \tag{5.10}$$

Then we obtain

$$T^{\mu}_{\beta\alpha}((1/2)n_a v^2+\lambda^{\mu}_{\alpha};\omega,\omega')=(n_a v)^{N-1}\check{G}^{\mu}(\omega,v;n_a v(\omega'-\omega)) \tag{5.11}$$

for each ω, $\omega'\in S^{N-1}$, $v>0$ sufficiently near v_0 and $\mu\gg 1$.

5.2 This subsection is devoted to showing the existence of the limit of $G^{\mu}(\omega,v;y_a)$ as $\mu\to\infty$ and $v\to v_0$ in an appropriate topology We write $x=x_a, y=y_a$ for simplicity x_b, y_b are linear combinations of x, y with μ-dependent coefficients (see (1 6)) There are four cases of pairs of the initial and final 2-cluster decompositions a and b: Case j-k means the pair (a, b) with $a=a_j$.

$b=a_k$ $(1\leqq j,k\leqq 2)$ We want to express V, I_a, I_b and ϕ^{μ}_{β} as functions of x, y with parameter μ.

(5.12)

Case 1-1: $V=V^{\mu}(x,y)=V_{23}(x)+V_{13}(\frac{m_3}{m}x-y)+V_{12}(-\frac{m_3}{M_2}x-y)$,

$$I_a=I_b=I^{\mu}_a(x,y)=I^{\mu}_b(x,y)=V_{13}(\frac{m_3}{m}x-y)+V_{12}(-\frac{m_3}{M_2}x-y).$$

$$\phi^{\mu}_{\beta}(x_b)=\Phi^{\mu}_{\beta}(x,y)=\phi^{\mu}_{\beta}(x),$$

Case 1-2: $V=V^{\mu}$ and $I_a=I^{\mu}_a$ are the same as Case 1-1,

$$I_b=I^{\mu}_b(x,y)=V_{23}(x)+V_{12}(-\frac{m_3}{M_2}x-y).$$

$$\phi^{\mu}_{\beta}(x_b)=\Phi^{\mu}_{\beta}(x,y)=\phi^{\mu}_{\beta}(\frac{m_3}{m}x-y),$$

Cas 2-1: $V=V^{\mu}(x,y)=V_{23}(\frac{m_1}{m}x+y)+V_{13}(x)+V_{12}(\frac{m_1}{M_1}x-y)$,

$$I_a=I^{\mu}_a(x,y)=V_{23}(\frac{m_1}{m}x+y)+V_{12}(\frac{m_1}{M_1}x-y).$$

$$I_b=I^{\mu}_b(x,y)=V_{13}(x)+V_{12}(\frac{m_1}{M_1}x-y),$$

$$\phi^{\mu}_{\beta}(x_b)=\Phi^{\mu}_{\beta}(x,y)=\phi^{\mu}_{\beta}(\frac{m_1}{m}x+y),$$

Case 2-2: $V=V^{\mu}$ and $I_a=I^{\mu}_a$ are the same as Case 2-1,

$$I^{\mu}_b(x,y)=I^{\mu}_a(x,y),$$

$$\phi^{\mu}_{\beta}(x_b)=\Phi^{\mu}_{\beta}(x,y)=\phi^{\mu}_{\beta}(x)$$

We see that when $\mu\to\infty$, V^{μ}, I^{μ}_a, I^{μ}_b and Φ^{μ}_{β} have limits V^{∞}, I^{∞}_a, I^{∞}_b and Φ^{∞}_{β} pointwise on R^{2N} (see.(1.4),). These limits have the following forms:

(5.13)

Case 1-1: $V^{\infty}(x,y)=V_{23}(x)+V_{13}(x-y)+V_{12}(-y)$,

$$I_a^\infty(x,y) = I_b^\infty(x,y) = V_{13}(x-y) + V_{12}(-y),$$

$$\Phi_\beta^\infty(x,y) = \phi_\beta^\infty(x).$$

Case 1-2: V^∞ and I_a^∞ are the same as Case 1-1.

$$I_b^\infty(x,y) = V_{23}(x) + V_{12}(-y),$$

$$\Phi_\beta^\infty(x,y) = \phi_\beta^\infty(x-y).$$

Case 2-1: $V^\infty(x,y) = V_{23}(x+y) + V_{13}(x) + V_{12}(-y).$

$$I_a^\infty(x,y) = V_{23}(x+y) + V_{12}(-y), \quad I_b^\infty(x,y) = V_{13}(x) + V_{12}(-y),$$

$$\Phi_\beta^\infty(x,y) = \phi_\beta^\infty(x+y)$$

Case 2-2: V^∞ and I_a^∞ are the same as Case 2-1.

$$I_b^\infty(x,y) = I_a^\infty(x,y),$$

$$\Phi_\beta^\infty(x,y) = \phi_\beta^\infty(x)$$

Now we investigate $\chi^\mu(\omega, v)$ as $\mu \to \infty$, $v \to v_0$. $\chi^\mu(\omega, v)$ are given as follows:

$$\chi^\mu(\omega, v; x, y) = e^{in_a(v'(v,\mu)-v)\omega \ y} \qquad \text{(Case 1-1, 2-2)},$$

$$(5.14) \qquad = e^{im_a v'(v,\mu)\omega \cdot x + i\left(\frac{m_b n_b v'(v,\mu)}{m} - n_a v\right)\omega \cdot y} \qquad \text{(Case 1-2)},$$

$$= e^{-im_a v'(v,\mu)\omega \cdot x + i\left(\frac{m_b n_b v'(v,\mu)}{m} - n_a v\right)\omega \cdot y} \qquad \text{(Case 2-1)}.$$

(see (5.2) for $v' = v'(v,\mu)$)

<u>Lemma 5.1</u>. Define $\chi(\omega) = \chi(\omega; x, y)$, $\omega \in S^{N-1}$, by

$$\chi(\omega;x,y) = e^{i(\lambda_\alpha^\infty-\lambda_\beta^\infty)v_0^{-1}\omega\cdot y} \qquad \text{(Case 1-1, 2-2)}.$$

$$(5.15) \qquad = e^{imv_0\omega\cdot x - i(\frac{m}{2}v_0^2+\lambda_\beta^\infty-\lambda_\alpha^\infty)v_0^{-1}\omega\cdot y} \qquad \text{(Case 1-2)}.$$

$$= e^{-imv_0\omega\cdot x - i(\frac{m}{2}v_0^2+\lambda_\beta^\infty-\lambda_\alpha^\infty)v_0^{-1}\omega\cdot y} \qquad \text{(Case 2-1)}.$$

Then we have for any $\delta>0$ and any multi-index γ,

$$(5.16) \qquad \lim_{\substack{\mu\to\infty \\ v\to v_0}} \sup_{\omega\in S^{N-1}} \|\langle x;y\rangle^{-\delta} D^\gamma\, [\chi^\mu(\omega,v)-\chi(\omega)]\|_{L^\infty(R^{2N})} = 0,$$

where $D=(\partial_x,\partial_y)$ and $\langle x;y\rangle := (1+|x|^2+|y|^2)^{1/2}$

Proof. We only prove this in Case 2-1, because the others can be proved similarly We first note that $m_a v'(v,\mu)\to mv_0$ as $\mu\to\infty$, $v\to v_0$. By (5.2) we have

$$\frac{m_b n_b v'}{m} - n_a v = \frac{(\frac{m_b^2 n_b}{m^2} - n_a)n_a v^2 + \frac{2m_b^2 n_b}{m^2}(\lambda_\alpha^\mu - \lambda_\beta^\mu)}{n_b[\frac{m_b v'}{m} + \frac{n_a}{n_b}v]}$$

Since $\frac{m_b^2 n_b}{m^2} - n_a = -\frac{M_2 m}{M_2+m}$, we obtain

$$\frac{m_b n_b v'}{m} - n_a v \to -(\frac{1}{2}mv_0^2 + \lambda_\beta^\infty - \lambda_\alpha^\infty)\frac{1}{v_0} \qquad \text{as } \mu\to\infty,\ v\to v_0.$$

Therefore, in view of (5.14), (5.15), it follows that

$$\lim_{\substack{\mu\to\infty \\ v\to v_0}} \sup_{\omega\in S^{N-1}} \|D^\gamma[\chi^\mu(\omega,v)-\chi(\omega)]\|_{L^\infty(K)} = 0$$

for any compact set K in R^{2N} and multi-index γ. The lemma follows from this together with the estimates

$$|D^\gamma\chi^\mu(\omega,v;x,y)| + |D^\gamma\chi(\omega;x,y)| \le C_\gamma \quad \text{on } R^{2N},$$

where C_γ is independent of $\mu\gg 1$, $\omega\in S^{N-1}$ and v with $|v-v_0|\ll 1$. ∎

We note that there exists a constant $C>0$ independent of $\mu\gg 1$

such that

$$(5.17)\qquad C^{-1}\langle x;y\rangle \leqq \rho^{\mu}(x,y) \leqq C\langle x;y\rangle \qquad \text{on} \quad \mathbf{R}^{2N}$$

for $\rho^{\mu}(x,y)=\langle \frac{m_*}{m}x\pm y;x\rangle$, $\langle \pm\frac{m_*}{M_j}x-y;x\rangle$ $(j=1,2)$,

$$\langle \frac{m_*}{m}x-y;-\frac{m_*}{M_2}x-y\rangle,\quad \langle \frac{m_*}{m}x+y;\frac{m_*}{M_1}x-y\rangle,\quad \langle x;x\pm y\rangle$$

<u>Lemma 5.2</u>. Let $k\in \mathbf{N}\cup\{0\}$ and let s be a real with $s<\varepsilon_0-N$, and assume (V). Then

$$(5.18)\qquad \operatorname*{s-lim}_{\substack{\mu\to\infty\\ v\to v_0}} I_b^{\mu}\Phi_{\beta}^{\mu}\chi^{\mu}(\omega,v)=I_b^{\infty}\Phi_{\beta}^{\infty}\chi(\omega) \quad \text{in} \quad H_s^k(\mathbf{R}^{2N})$$

uniformly for $\omega\in S^{N-1}$

<u>Proof</u> We set $g^{\mu}:=I_b^{\mu}\Phi_{\beta}^{\mu}$, $g^{\infty}:=I_b^{\infty}\Phi_{\beta}^{\infty}$. Then, from (V), Lemma 2.1 and (5.17), the estimate

$$(5.19)\qquad |D^{\gamma}g^{\mu}(x,y)|+|D^{\gamma}g^{\infty}(x,y)|\leqq C_{\gamma}\langle x;y\rangle^{-\varepsilon_0}$$

follows for any γ, where C_{γ} is independent of $\mu\gg1$ (5.19) yields $g^{\mu}\chi^{\mu}(\omega,v)$, $g^{\infty}\chi(\omega)\in H_s^k(\mathbf{R}^{2N})$ because of $s<\varepsilon_0-N$. Fix $\delta>0$ with $\delta+s<\varepsilon_0-N$. Then we get

$$\begin{aligned}
(5.20)\qquad &\|g^{\mu}\chi^{\mu}(\omega,v)-g^{\infty}\chi(\omega)\|_{k,s}\\
&\leqq \|g^{\mu}[\chi^{\mu}(\omega,v)-\chi(\omega)]\|_{k,s}+\|[g^{\mu}-g^{\infty}]\chi(\omega)\|_{k,s}\\
&\leqq C\Big(\sum_{|\gamma|\leqq k}\|\langle x;y\rangle^{-\delta}D^{\gamma}[\chi^{\mu}(\omega,v)-\chi(\omega)]\|_{L^{\infty}}\|g^{\mu}\|_{k,s+\delta}\\
&\quad+\sum_{|\gamma|\leqq k}\|\langle x;y\rangle^{-\delta}D^{\gamma}\chi(\omega)\|_{L^{\infty}}\|g^{\mu}-g^{\infty}\|_{k,s+\delta}\Big),
\end{aligned}$$

where C is independent of $\mu\gg1$, $\omega\in S^{N-1}$ and v near v_0.

$\|g^\mu\|_{k,s+\epsilon}$ are uniformly bounded for $\mu \gg 1$ by (5.19) So by Lemma 5.1 the first term in the R.H.S. of (5.20) goes to zero uniformly for $\omega \in S^{N-1}$ as $\mu \to \infty$, $v \to v_0$. On the other hand, by (V) and Lemma 2.1 we have

$$D^\gamma g^\mu \to D^\gamma g^\infty \qquad \text{pointwise on} \quad \mathbf{R}^{2N}$$

as $\mu \to \infty$ for any γ Thus, using (5.19) and the Lebesgue dominated convergence theorem, we get $\lim_{\mu \to \infty} \|g^\mu - g^\infty\|_{k,s+\epsilon} = 0$. Therefore the second term in the R.H.S. of (5.20) goes to zero uniformly for $\omega \in S^{N-1}$ as $\mu \to \infty$ ∎

Lemma 5.3. Let $k \in \mathbf{N} \cup \{0\}$ and $s \geqq 0$. Then the map T defined by

$$T f(y) := \int \langle x \rangle^{-N} f(x,y)dx \qquad \text{for} \quad f \in H^k_s(\mathbf{R}^{2N})$$

is a bounded operator from $H^k_s(\mathbf{R}^{2N})$ to $H^k_s(\mathbf{R}^N)$. Furthermore

$$D^\gamma_y T f = T D^\gamma_y f$$

for any γ with $|\gamma| \leqq k$ and any $f \in H^k_s(\mathbf{R}^{2N})$

Proof. Apply the Schwarz inequality ∎

For each $\omega \in S^{N-1}$, $-(2m)^{-1}\Delta_x - iv_0\omega\ \nabla_y$ is a self-adjoint operator in $L^2(\mathbf{R}^{2N})$ with domain

$$D_\omega = \{f \in L^2(\mathbf{R}^{2N}); -(2m)^{-1}\Delta_x f - iv_0\omega\ \nabla_y f \in L^2(\mathbf{R}^{2N})\},$$

and $\mathcal{S}(\mathbf{R}^{2N})$ is a core of this self-adjoint operator Therfore

$$(5.21) \qquad H(\omega) := -(2m)^{-1}\Delta_x - iv_0\omega\ \nabla_y + V^\infty$$

is a self-adjoint operator with domain D_ω, and $\mathcal{S}(\mathbf{R}^{2N})$ is a core of $H(\omega)$ since V^∞ is a bounded real-valued function. We denote the resolvent of $H(\omega)$ by

$$(5.22) \qquad R(\omega;z)=(H(\omega)-z)^{-1}$$

The next lemma wili be proved in Sect.6 by using the abstract resolvent estimates of Sect.3.

Lemma 5.4. Assume (V) Let J be any compact interval in $\mathbf{R}$, fix $k\in \mathbf{N}\cup\{0\}$ and $s\in \mathbf{R}$ with $0\leqq k\leqq \varepsilon_0-2$ and $k+\frac{1}{2}<s$, and set $J_\pm:=\{z\in \mathbf{C};\mathrm{Re}\, z\in J,\ 0<\pm\mathrm{Im}\, z<1\}$ and $\mathbf{B}_{k,s}:=\mathbf{B}(H^k_s(\mathbf{R}^{2N}),H^k_{-s}(\mathbf{R}^{2N}))$ Then:

(i) There exist a large $M>0$ and a small $\delta_0>0$ such that

$$\sup_{\substack{M_1,M_2>M,\ |v-v_0|<\delta_0\\ \omega\in S^{N-1},\ z\in J_\pm}} \{\|R^\mu(\omega,v;z)\|_{\mathbf{B}_{k,s}}+\|R(\omega;z)\|_{\mathbf{B}_{k,s}}\}<\infty,$$

where $\mu=(M_1,M_2)$

(ii) The norm limits

$$R^\mu(\omega,v;\lambda\pm i0):=\lim_{\varepsilon\downarrow 0} R^\mu(\omega,v;\lambda\pm i\varepsilon),\quad R(\omega;\lambda\pm i0):=\lim_{\varepsilon\downarrow 0} R(\omega;\lambda\pm i\varepsilon)$$

exist in $\mathbf{B}_{k,s}$ uniformly for $\lambda\in J$, $\omega\in S^{N-1}$, $\mu=(M_1,M_2)$ and v with $M_1,M_2>M$, $|v-v_0|<\delta_0$.

(iii) For each $f\in H^k_s(\mathbf{R}^{2N})$ and $\lambda\in J$,

$$\operatorname*{s-lim}_{\substack{\mu\to\infty,\ v\to v_0\\ J\ni\lambda'\to\lambda}} R^\mu(\omega,v;\lambda'\pm i0)f=R(\omega;\lambda\pm i0)f$$

in $H^k_{-s}(\mathbf{R}^{2N})$ uniformly for $\omega\in S^{N-1}$

(iv) Let $f\in H^k_s(\mathbf{R}^{2N})$ and $\lambda\in J$. Then $R(\omega;\lambda\pm i0)f$ is an $H^k_{-s}(\mathbf{R}^{2N})$-valued strongly continuous function of $\omega\in S^{N-1}$

We define

$$E(\omega\,;x,y):=i(2\pi)^{\frac{2-N}{2}}v_0^{-1}\phi_a^{\infty}(x)\times$$

(5.23)

$$\times\left(\overline{[-I_a^{\infty}+I_a^{\infty}R(\omega\,;\lambda_a^{\infty}-i0)I_b^{\infty}]\Phi_a^{\infty}\chi(\omega)}\right)(x,y)$$

$$G(\omega\,;y):=\int E(\omega\,;x,y)dx$$

<u>Lemma 5.5</u>. (i) Assume (V) Then

(5.24) $$\sup_{\substack{\mu>1,\,|v-v_0|<1\\ \omega\in S^{N-1}}}\{\|G^{\mu}(\omega,v)\|_{1,1}+\|G(\omega)\|_{1,1}\}<\infty,$$

(5.25) $$\lim_{\substack{\mu\to\infty\\ v\to v_0}}\ \sup_{\omega\in S^{N-1}}\|G^{\mu}(\omega,v)-G(\omega)\|_{1,1}=0,$$

where $\|\ \|_{1,1}$ is the $H_1^1(R^N)$-norm and $G^{\mu}(\omega,v)=G^{\mu}(\omega,v;y)$, etc. (see (5.9)), and

$$\sup_{\substack{\mu>1,\,|v-v_0|<1\\ \omega\in S^{N-1}}}\{\cdots\}:=\sup_{\substack{M_1,M_2>M,\ |v-v_0|<\delta\\ \omega\in S^{N-1}}}\{\cdots\}$$

for some $M>0$ and $\delta>0$.

(ii) If we replace (V) by (V)' and $\|\ \|_{1,1}$ by $\|\ \|_{[(N-1)/2]+1,(N+1)/2}$, then (i) still holds ([ ] is Gauss' symbol)

<u>Proof</u>. (i) Here we denote by $\|\ \|_{k,s}$ the $H_s^k(R^{2N})$-norm.

By (5.9), (5.23) and Lemma 5.3 we can see that it suffices to prove

(5.26) $$\sup_{\substack{\mu>1,\,|v-v_0|<1\\ \omega\in S^{N-1}}}\{\|\langle x\rangle^N E^{\mu}(\omega,v)\|_{1,1}+\|\langle x\rangle^N E(\omega)\|_{1,1}\}<\infty,$$

(5.27) $$\lim_{\substack{\mu\to\infty\\ v\to v_0}}\ \sup_{\omega\in S^{N-1}}\|\langle x\rangle^N[E^{\mu}(\omega,v)-E(\omega)]\|_{1,1}=0$$

We first note that

$$(5.28)\qquad (vv')^{\frac{N-2}{2}}(n_a n_b)^{\frac{N-1}{2}}(vn_a)^{1-N}\to v_0^{-1}\quad \text{as}\quad \mu\to\infty,\ v\to v_0.$$

By Lemmas 2.1 and 5.2, we have

$$\sup_{\substack{\mu>1,\,|v-v_0|<1\\ \omega\in S^{N-1}}}\{\|\langle x\rangle^N\phi_\alpha^\mu I_a^\mu\Phi_\beta^\mu\chi^\mu(\omega,v)\|_{1,1}$$

$$(5.29)\qquad +\|\langle x\rangle^N\phi_\alpha^\infty I_a^\infty\Phi_\beta^\infty\chi(\omega)\|_{1,1}\}<\infty,$$

$$\lim_{\substack{\mu\to\infty\\ v\to v_0}}\ \sup_{\omega\in S^{N-1}}\|\langle x\rangle^N[\phi_\alpha^\mu I_a^\mu\Phi_\beta^\mu\chi^\mu(\omega,v)-\phi_\alpha^\infty I_a^\infty\Phi_\beta^\infty\chi(\omega)]\|_{1,1}=0.$$

Set

$$(5.30)\qquad f^\mu(\omega,v):=R^\mu(\omega,v;\lambda_\alpha^\mu-i0)I_b^\mu\Phi_\beta^\mu\chi^\mu(\omega,v).$$

$$(5.31)\qquad f(\omega):=R(\omega;\lambda_\alpha^\infty-i0)I_b^\infty\Phi_\beta^\infty\chi(\omega)$$

Then (5.26) and (5.27) can be reduced to the following estimates:

$$(5.32)\quad \sup_{\substack{\mu>1,\,|v-v_0|<1\\ \omega\in S^{N-1}}}\{\|\langle x\rangle^N\phi_\alpha^\mu I_a^\mu\overline{f^\mu(\omega,v)}\|_{1,1}+\|\langle x\rangle^N\phi_\alpha^\infty I_a^\infty\overline{f(\omega)}\|_{1,1}\}<\infty,$$

$$(5.33)\quad \lim_{\substack{\mu\to\infty\\ v\to v_0}}\ \sup_{\omega\in S^{N-1}}\|\langle x\rangle^N[\phi_\alpha^\mu I_a^\mu\overline{f^\mu(\omega,v)}-\phi_\alpha^\infty I_a^\infty\overline{f(\omega)}]\|_{1,1}=0$$

Fix s with $3/2<s<\varepsilon_0-N$. Then, by Lemmas 5.2 and 5.4, we have

$$(5.34)\qquad \sup_{\substack{\mu>1,\,|v-v_0|<1\\ \omega\in S^{N-1}}}\{\|f^\mu(\omega,v)\|_{1,-s}+\|f(\omega)\|_{1,-s}\}<\infty$$

By (5.34) and the following estimates

(5.35)

$$\|\langle x;y\rangle^{1+s}D^\gamma(\langle x\rangle^N\phi_\alpha^\infty I_a^\infty)\|_{L^\infty}<\infty,\qquad \sup_{\mu>1}\|\langle x;y\rangle^{1+s}D^\gamma(\langle x\rangle^N\phi_\alpha^\mu I_a^\mu)\|_{L^\infty}<\infty$$

for $|\gamma|\leq 1$, we obtain (5.32)

Let θ, $\omega \in S^{N-1}$ We have

$$\|f^{\mu}(\omega, v)-f(\omega)\|_{1,-s}$$

(5.36)
$$\leq \|R^{\mu}(\omega, v; \lambda_{\alpha}^{\mu}-i0)\|_{B_{1,s}} \|g^{\mu}\chi^{\mu}(\omega, v)-g^{\infty}\chi(\omega)\|_{1,s}$$

$$+\|[R^{\mu}(\omega, v; \lambda_{\alpha}^{\mu}-i0)-R(\omega; \lambda_{\alpha}^{\infty}-i0)]g^{\infty}\chi(\theta)\|_{1,-s}$$

$$+\|R^{\mu}(\omega, v; \lambda_{\alpha}^{\mu}-i0)-R(\omega; \lambda_{\alpha}^{\infty}-i0)\|_{B_{1,s}} \|g^{\infty}(\chi(\omega)-\chi(\theta))\|_{1,s},$$

where $\mathbf{B}_{k,s}$ is as in Lemma 5.4, and g^{μ}, g^{∞} are the same as in the proof of Lemma 5.2. By Lemmas 5.2 and 5.4(iii), the first two terms go to zero uniformly for $\omega \in S^{N-1}$ as $\mu\to\infty$, $v\to v_0$ for fixed θ The operator norminthe third term is uniformly bounded for $\omega \in S^{N-1}$, $\mu \gg 1$ and v with $|v-v_0|\ll 1$ (Lemma 5.4 (i)) and $g^{\infty}\chi(\omega)$ is an H_2^1-valued strongly continuous function of ω Thus, we see that for any $\varepsilon>0$ and $\theta \in S^{N-1}$ there exist $M>0$, $\delta>0$ and a neighborhood of θ, $U(\theta)$, such that

$$\|f^{\mu}(\omega, v)-f(\omega)\|_{1,-s}<\varepsilon \qquad (\mu=(M_1, M_2))$$

if $\omega \in U(\theta)$, M_1, $M_2>M$, $|v-v_0|<\delta$ Therefore, by using the finite covering argument, we see that

(5.37)
$$\lim_{\mu\to\infty,\, v\to v_0} \sup_{\omega \in S^{N-1}} \|f^{\mu}(\omega, v)-f(\omega)\|_{1,-s}=0.$$

To prove (5.33) we write

$$\langle x\rangle^{N}[\phi_{\alpha}^{\mu} I_{\alpha}^{\mu}\overline{f^{\mu}(\omega, v)}-\phi_{\alpha}^{\infty} I_{\alpha}^{\infty}\overline{f(\omega)}]$$

(5.38)
$$=\langle x\rangle^{N}\phi_{\alpha}^{\mu} I_{\alpha}^{\mu}[\overline{f^{\mu}(\omega, v)}-\overline{f(\omega)}]+\langle x\rangle^{N}[\phi_{\alpha}^{\mu} I_{\alpha}^{\mu}-\phi_{\alpha}^{\infty} I_{\alpha}^{\infty}]\overline{f(\omega)}$$

By (5.35) and (5.37), the first term tends to zero in $H_1^1(R^{2N})$

uniformly for ω as $\mu \to \infty$ and $v \to v_0$. We next prove that the second term of (5.38) tends to zero uniformly for ω as $\mu \to \infty$ First we claim that $f(\omega)$ is $H^1_{-s}(R^{2N})$-valued strongly continuous in ω To see this we write for θ, $\omega \in S^{N-1}$

$$(5.39)\quad f(\omega)-f(\theta)=(R(\omega;\lambda^\infty_a-i0)-R(\theta;\lambda^\infty_a-i0))I^\infty_b\Phi^\infty_\beta\chi(\theta)$$

$$+R(\omega;\lambda^\infty_a-i0)I^\infty_b\Phi^\infty_\beta(\chi(\omega)-\chi(\theta)).$$

The first term goes to zero in $H^1_{-s}(R^{2N})$ as $\omega \to \theta$ by Lemma 5.4 (iv) Similarly for the second term by Lemmas 5.1 and 5.4(i) This proves the continuity of $f(\omega)$. Now we will prove

$$(5.40)\qquad \lim_{\mu\to\infty}\sup_{\omega\in S^{N-1}}\|F^\mu f(\omega)\|_{1,1}=0,\qquad F^\mu:=\langle x\rangle^N[\phi^\mu_a I^\mu_a-\phi^\infty_a I^\infty_a]$$

We have for ω, $\theta \in S^{N-1}$

$$\|F^\mu f(\omega)\|_{1,1}\leq\|F^\mu f(\theta)\|_{1,1}+C\sum_{|\gamma|\leq 1}\|\langle x;y\rangle^{1+s}D^\gamma F^\mu\|_{L^\infty}\|f(\omega)-f(\theta)\|_{1,-s}.$$

By Lemma 2.1, (V) and $f(\theta)\in H^1_{-s}(R^{2N})$, the first term tends to zero in $H^1_1(R^{2N})$ as $\mu \to \infty$ for fixed $\theta \in S^{N-1}$ In view of the continuity of $f(\omega)$ and the boundedness $\sum_{|\gamma|\leq 1}\|\langle x;y\rangle^{1+s}D^\gamma F^\mu\|_{L^\infty}$ (see (5.35)), the argument similar to that in the proof of (5.37) yields (5.40). and hence (5.33)

(ii) Fix s_1 with $[(N-1)/2]+1+(1/2)<s_1<\varepsilon_0-N$. If we replace s, $H^1_1(R^{2N})$ and $H^1_{-s}(R^{2N})$ by s_1, $H^M_L(R^{2N})$ and $H^M_{-s_1}(R^{2N})$, respectively ($M=[(N-1)/2]+1$, $L=(N+1)/2$) in the proof of (i), we obtain the desired results in the same way as above.∎

5.3 We fix $\omega \in S^{N-1}$ in this subsection. We first introduce a a family of operators $\{L_\omega(\eta)\}$ $(\eta\in\Pi_\omega)$ in $L^2(R^{N+1})$, $R^{N+1}=$

$\mathbf{R}^N_x \times \mathbf{R}^1_t$:

$$(5.41)\qquad \begin{aligned} L_\omega(\eta) &:= -(2m)^{-1}\Delta_x - iv_0\partial_t + V^\omega(x, t\omega+\eta), \\ D(L_\omega(\eta)) &:= \{f \in L^2;\ -(2m)^{-1}\Delta_x f - iv_0\partial_t f \in L^2\} \end{aligned}$$

By (V), $L_\omega(\eta)$ is self-adjoint for each $\eta \in \Pi_\omega$, and $\mathcal{S}(R^{N+1})$ is a core of $L_\omega(\eta)$ Furthermore the norm limits

$$(5.42)\qquad (L_\omega(\eta)-\lambda\pm i0)^{-1} := \lim_{\varepsilon\downarrow 0} (L_\omega(\eta)-\lambda\pm i\varepsilon)^{-1}$$

exist in $\mathbf{B}(L^2(\mathbf{R}^N)\otimes L^2_s(\mathbf{R}), L^2(\mathbf{R}^N)\otimes L^2_{-s}(\mathbf{R}))$ uniformly for λ in any compact set in $\mathbf{R}$, where $s > \frac{1}{2}$ (see Lemma 4.1)

By the correspondence $f(x,y) \to f(x, t\omega+\eta)$ $(t \in \mathbf{R},\ \eta \in \Pi_\omega)$ we have $L^2(\mathbf{R}^{2N}) = \int^\oplus_{\Pi_\omega} L^2(\mathbf{R}^{N+1})d\eta$ (see [R-S]IV.XIII.16 for constant fiber direct integrals) Then, by (5.5) and (5.41), we have

$$(5.43)\qquad H(\omega) = \int^\oplus_{\Pi_\omega} L_\omega(\eta)d\eta, \qquad (H(\omega)-z)^{-1} = \int^\oplus_{\Pi_\omega} (L_\omega(\eta)-z)^{-1}d\eta$$

for $z \in \mathbf{C}\setminus\mathbf{R}$ Thus, for each $f \in L^2(\mathbf{R}^N \times \mathbf{R}^N)$ and $z \in \mathbf{C}\setminus\mathbf{R}$, we have

$$(5.44)\qquad (R(\omega;z)f)(*, *\omega+\eta) = (L_\omega(\eta)-z)^{-1} f(*, *\omega+\eta)$$

in $L^2(\mathbf{R}^N_x \times \mathbf{R}_t)$ for a.e. $\eta \in \Pi_\omega$. Here we note that $(L_\omega(\eta)-z)^{-1}$ operates on the variable x, t (the first $*$ stands for x and the second $*$ for t) The limiting absorption principle for $H(\omega)$ (Lemma 5.4) and $L_\omega(\eta)$ ((5.42)) together with (5.44) yields the following lemma.

Lemma 5.6. Assume (V), and fix $\lambda \in \mathbf{R}$ and $f \in L^2_s(\mathbf{R}^{2N})$ $(s > \frac{1}{2})$ Then

(5.45) $(R(\omega;\lambda-i0)f)(*.*\omega+\eta)=(L_\omega(\eta)-\lambda+i0)^{-1}f(*.*\omega+\eta)$

in $L^2_s(R^{N+1})$ for a.e. $\eta\in\Pi_\omega$.

Proof. Set

$$B_\varepsilon(\eta):=\|\langle *.*\omega+\eta\rangle^{-s}[(R(\omega;\lambda-i\varepsilon)f)(*.*\omega+\eta)$$

$$-(R(\omega;\lambda-i0)f)(*.*\omega+\eta)]\|^2_{L^2(R^{N+1})}$$

for each $\varepsilon>0$ and $\eta\in\Pi_\omega$. Then, by Fubini's theorem, $B_\varepsilon(\eta)$ is well defined for a.e. $\eta\in\Pi_\omega$ and

$$\|B_\varepsilon(\eta)\|_{L^1(\Pi_\omega)}=\|[R(\omega;\lambda-i\varepsilon)-R(\omega;\lambda-i0)]f\|^2_{L^2_s(R^{2N})}.$$

Since the R.H.S. goes to zero as $\varepsilon\downarrow 0$ by Lemma 5.4, we can choose a sequence $\varepsilon_1>\varepsilon_2>\cdots\to 0$ and a null subset e_0 of Π_ω such that as $j\to\infty$, $B_{\varepsilon_j}(\eta)\to 0$ for every $\eta\in\Pi_\omega\backslash e_0$. This implies that

(5.46) $\|(R(\omega;\lambda-i\varepsilon_j)f)(*.*\omega+\eta)-(R(\omega;\lambda-i0)f)(*.*\omega+\eta)\|^2_{L^2_s(R^{N+1})}\to 0$

as $j\to\infty$ for every $\eta\in\Pi_\omega\backslash e_0$. On the other hand, there exists a null subset e_1 of Π_ω such that for every $\eta\in\Pi_\omega\backslash e_1$ and all j,

$$f(*.*\omega+\eta)\in L^2_s(R^{N+1}).$$

(5.47)

$$(R(\omega;\lambda-i\varepsilon_j)f)(*.*\omega+\eta)=(L_\omega(\eta)-\lambda+\varepsilon_j)^{-1}f(*.*\omega+\eta)$$

in $L^2_s(R^{N+1})$. Thus, by (5.42) and (5.47), we have

(5.48) $\operatorname*{s-lim}_{j\to\infty}(R(\omega;\lambda-i\varepsilon_j)f)(*.*\omega+\eta)=(L_\omega(\eta)-\lambda+i0)^{-1}f(*.*\omega+\eta)$

in $L^2_s(R^{N+1})$ for every $\eta\in\Pi_\omega\backslash e_1$. By (5.46) and (5.48) we get the desired result.∎

In Lemma 5.8 we shall prove that (5.45) holds for all $\eta \in \Pi_\omega$ under the stronger assumption (V)' if we require a regularity of f. To this end we review the trace theorem.

Trace Theorem (see e.g. the proof of Theorem IX.38 of [R-S]II). Let $p, q \in \mathbf{N}$, and let σ be a real with $\sigma > p/2$. Then there exists a constant C such that

$$\| f(z,*) \|_{L^2(\mathbf{R}^q)} \leq C \| f \|_{H^\sigma(\mathbf{R}^p \times \mathbf{R}^q)}$$

for all $f \in \mathcal{S}(\mathbf{R}^p \times \mathbf{R}^q)$ and $z \in \mathbf{R}^p$ In particular, the trace $T_z f := f(z,*) \in L^2(\mathbf{R}^q)$ is well-defined for all $f \in H^\sigma(\mathbf{R}^p \times \mathbf{R}^q)$ and $z \in \mathbf{R}^p$ Furthermore T_z is a $\mathbf{B}(H^\sigma(\mathbf{R}^p \times \mathbf{R}^q), L^2(\mathbf{R}^q))$-valued norm continuous function of $z \in \mathbf{R}^p$

For each $\eta \in \Pi_\omega$ we define a map γ_η from $\mathcal{S}(\mathbf{R}^{2N})$ to $\mathcal{S}(\mathbf{R}^{N+1})$ by $(\gamma_\eta f)(x,t) := f(x, t\omega + \eta)$ $\gamma_\eta f$ is the restriction of f on a plane of codimension N-1 in $\mathbf{R}^{2N}$ The trace theorem guarantees that γ_η can be uniquely extended to a bounded operator from $H^k(\mathbf{R}^{2N})$ to $L^2(\mathbf{R}^{N+1})$ for any $k > (N-1)/2$, and that γ_η is a $\mathbf{B}(H^k(\mathbf{R}^{2N}), L^2(\mathbf{R}^{N+1}))$-valued norm continuous function of $\eta \in \Pi_\omega$. Furthermore we have

Lemma 5.7 Fix $k > (N-1)/2$ and $s \in \mathbf{R}$ Then γ_η can be uniquely extended to a bounded operator from $H^k_s(\mathbf{R}^{2N})$ to $L^2_s(\mathbf{R}^{N+1})$, and γ_η is a $\mathbf{B}(H^k_s(\mathbf{R}^{2N}), L^2_s(\mathbf{R}^{N+1}))$-valued norm continuous function of $\eta \in \Pi_\omega$.

Proof The relation $\gamma_\eta \langle x;y\rangle^{-s} = \langle x; t\omega+\eta\rangle^{-s}\gamma_\eta$ and the trace theorem yield $\gamma_\eta \in \mathbf{B}(H^k_s(R^{2N}), L^2_s(R^{N+1}))$ The continuity is easily verified.∎

Lemma 5.8. Assume (V)', and fix $\lambda\in R$, $s>[(N-1)/2]+(3/2)$ and $f\in H_s^{[\frac{N-1}{2}]+1}(R^{2N})$ Then

$$(5.49)\qquad \gamma_\eta R(\omega;\lambda-i0)f=(L_\omega(\eta)-\lambda+i0)^{-1}\gamma_\eta f$$

in $L^2{}_s(R^{N+1})$ for all $\eta\in\Pi_\omega$.

Proof $R(\omega;\lambda-i\varepsilon)$ maps $H_s^{[\frac{N-1}{2}]+1}(R^{2N})$ into $H_{-s}^{[\frac{N-1}{2}]+1}(R^{2N})$ for each $\varepsilon>0$, and we have

$$(5.50)\qquad \gamma_\eta R(\omega;\lambda-i\varepsilon)f=(L_\omega(\eta)-\lambda+i\varepsilon)^{-1}\gamma_\eta f$$

in $L^2{}_s(R^{N+1})$ for a.e.$\eta\in\Pi_\omega$ by (5.44) and Lemma 5.7 On the other hand, it is easy to see that $(L_\omega(\eta)-\lambda+i\varepsilon)^{-1}$ is strongly continuous in $\eta\in\Pi_\omega$ as a $\mathbf{B}(L^2(R^{N+1}))$-valued function. Thus both sides of (5.50) are strongly continuous in $\eta\in\Pi_\omega$ in $L^2{}_s(R^{N+1})$ by Lemma 5.7. and so (5.50) holds for all $\eta\in\Pi_\omega$. We fix $\eta\in\Pi_\omega$, and let $\varepsilon\downarrow 0$ in both sides. Then Lemmas 5.4 and 5.7 yield

$$\gamma_\eta R(\omega;\lambda-i\varepsilon)f \longrightarrow \gamma_\eta R(\omega;\lambda-i0)f \qquad \text{in } L^2{}_s(R^{N+1})$$

Since $\gamma_\eta f\in L^2_s(R^{N+1})$, we get

$$(L_\omega(\eta)-\lambda+i\varepsilon)^{-1}\gamma_\eta f \longrightarrow (L_\omega(\eta)-\lambda+i0)^{-1}\gamma_\eta f \qquad \text{in } L^2{}_s(R^{N+1})$$

by (5.42) This completes the proof. ∎

Lemma 5.9. We assume (V) Then we have

$$(5.51) \qquad (2\pi)^{\frac{N-2}{2}} \int G(\omega\,;t\omega+\eta)dt = S^{\infty}_{\beta\alpha}(v_0\omega,\eta) - \delta_{\alpha\beta}$$

for a.e. $\eta \in \Pi_\omega$. (See (1.24) for $S^{\infty}_{\beta\alpha}(v_0\omega,\eta)$)

<u>Proof.</u> By (5.23), we have

$$(5.52) \qquad (2\pi)^{\frac{N-2}{2}} \int G(\omega\,;t\omega+\eta)dt = (2\pi)^{\frac{N-2}{2}} \iint E(\omega\,;x,t\omega+\eta)dxdt$$

for a.e. $\eta \in \Pi_\omega$. Using Lemma 5.6 and (5.23), in the R.H.S. of (5.52), we get

$$(5.53) \qquad (2\pi)^{\frac{N-2}{2}} \iint E(\omega\,;x,t\omega+\eta)dxdt$$

$$= iv_0^{-1}\{-(\phi^{\infty}_{\alpha}(x)I^{\infty}_{a}(x,t\omega+\eta),\Phi^{\infty}_{\beta}(x,t\omega+\eta)\chi(\omega\,;x,t\omega+\eta))$$

$$+(\phi^{\infty}_{\alpha}(x)I^{\infty}_{a}(x,t\omega+\eta),((L_\omega(\eta)-\lambda^{\infty}_{\alpha}+i0)^{-1}I^{\infty}_{b}(*,*\omega+\eta)\Phi^{\infty}_{\beta}(*,*\omega+\eta)\times$$

$$\times\chi(\omega\,;*,*\omega+\eta))(x,t))\}$$

for a.e. $\eta \in \Pi_\omega$. By scaling $t \to v_0 t$, $L_\omega(\eta)$ turns into

$$\widetilde{L_\omega}(\eta) := -(2m)^{-1}\Delta_x - i\partial_t + V_{23}(x) + V_{13}(x-v_0t\omega-\eta) + V_{12}(-v_0t\omega-\eta)$$

(for Case 1-1, 1-2)

$$:= -(2m)^{-1}\Delta_x - i\partial_t + V_{23}(x+v_0t\omega+\eta) + V_{13}(x) + V_{12}(-v_0t\omega-\eta)$$

(for Case 2-1, 2-2)

Thus this scaling yields

$$(5.54) \qquad \text{the R.H.S. of (5.53)} =$$

$$-i(\phi^{\infty}_{\alpha}(x)I^{\infty}_{a}(x,v_0t\omega+\eta),\Phi^{\infty}_{\beta}(x,v_0t\omega+\eta)\chi(\omega\,;x,v_0t\omega+\eta))$$

$$+i(((\widetilde{L_\omega}(\eta)-\lambda^{\infty}_{\alpha}-i0)^{-1}I^{\infty}_{a}(*,v_0*\omega+\eta)\phi^{\infty}_{\alpha}(*))(x,t),I^{\infty}_{b}(x,v_0t\omega+\eta)\times$$

$$\times\Phi_\beta^\infty(x, v_0 t\omega+\eta)\chi(\omega; x, v_0 t\omega+\eta))$$

To finish our proof, in view of (5.53), (5.52), (4.8) and (1.24), we have only to show that (5.54) is equal to the R.H.S. of (4.8) for $\xi = v_0\omega$ and each $\eta \in \Pi_\omega$.

<u>Case 1-1</u>. We can obtain the desired result by observing that

$$(5.55) \qquad I_c^\infty(x, v_0 t\omega+\eta) = W_c(v_0\omega, \eta; x, t) \quad \text{for} \quad c=a,\ b \ (\text{see } (4.7)),$$

$$(5.56) \qquad \chi(\omega; x, v_0 t\omega+\eta) = e^{i(\lambda_\alpha^\infty-\lambda_\beta^\infty)t}, \quad \Phi_\beta^\infty(x, v_0 t\omega+\eta) = \phi_\beta^\infty(x).$$

$$(5.57) \qquad e^{-i\lambda_\alpha^\infty t}(\widetilde{L}_\omega(\eta)-\lambda_\alpha^\infty-i0)^{-1} = (K_{v_0\omega,\eta}-i0)^{-1}e^{-i\lambda_\alpha^\infty t}$$

<u>Case 1-2</u>. (5.55) and (5.57) hold also in this case. Instead of (5.56), we have only to note that

$$(5.58) \qquad \begin{aligned} &\chi(\omega; x, v_0 t\omega+\eta) = e^{imv_0\omega x - i(\frac{m}{2}v_0^2+\lambda_\beta^\infty-\lambda_\alpha^\infty)t}, \\ &\Phi_\beta^\infty(x, v_0 t\omega+\eta) = \phi_\beta^\infty(x-v_0 t\omega-\eta) \end{aligned}$$

<u>Case 2-1</u> We use new variables $(X,s) = (x+v_0 t\omega+\eta, t)$ in this case and the next case. In terms of the new variables we can write

$$(5.59) \qquad I_c^\infty(x, v_0 t\omega+\eta) = W_c(v_0\omega, \eta; X, s) \qquad \text{for} \quad c=a,\ b,$$

$$(5.60) \qquad \chi(\omega; x, v_0 t\omega+\eta) = e^{-imv_0\omega\ X+i(\frac{m}{2}v_0^2+\lambda_\alpha^\infty-\lambda_\beta^\infty)s},$$

$$(5.61) \qquad \phi_\alpha^\infty(x) = \phi_\alpha^\infty(X-v_0 s\omega-\eta),$$

(5.62) $$\Phi^{\infty}_{\beta}(x, v_0 t\omega+\eta) = \phi^{\infty}_{\beta}(X).$$

$$(\widetilde{L_\omega}(\eta) - \lambda^{\infty}_{\alpha} - i0)^{-1}$$

$$= (-(2m)^{-1}\Delta_X - i\partial_s - iv_0\omega\ \nabla_X + V_{23}(X) + V_{13}(X - v_0 s\omega - \eta)$$

(5.63) $$+ V_{12}(-v_0 s\omega - \eta) - \lambda^{\infty}_{\alpha} - i0)^{-1}$$

$$= U\,(-(2m)^{-1}\Delta_X - i\partial_s + V_{23}(X) + V_{13}(X - v_0 s\omega - \eta) +$$

$$+ V_{12}(-v_0 s\omega - \eta) - i0)^{-1} U^*,$$

where $U = e^{-imv_0\omega X + i(\frac{m}{2}v_0^2 + \lambda^{\infty}_{\alpha})s}$, a unitary multiplication operator Noting that the Jacobian for $(x, t) \to (X, s)$ is one, we can compute the R.H.S. of (5.54) to obtain the desired result.

Case 2-2. We use the same variables as above. (5.59), (5.61), (5.63) hold also. Instead of (5.60), (5.62), we have only to note that

(5.64) $\chi(\omega; x, v_0 t\omega+\eta) = e^{i(\lambda^{\infty}_{\alpha} - \lambda^{\infty}_{\beta})s}$, $\Phi^{\infty}_{\beta}(x, v_0 t\omega+\eta) = \phi^{\infty}_{\beta}(X - v_0 s\omega - \eta)$

We have shown that (5.54) equals the R.H.S. of (4.8) for $\xi = v_0\omega$ and $\eta \in \Pi_\omega$, and have finished the proof of Lemma 5.9. ∎

Lemma 5.5 (ii) shows $G(\omega) \in H^{[\frac{N-1}{2}]+1}_{(N+1)/2}(R^N)$ if we assume (V)' Thus $G(\omega; t\omega+\eta) \in L^2_{(N+1)/2}(R_t) \subset L^1(R_t)$ is well-defined for each $\eta \in \Pi_\omega$, in view of the trace theorem. At the end of this subsection we prove the following.

Lemma 5.10. We assume (V)' Then (5.51) holds for all $\eta\in\Pi_\omega$.

Proof. By the trace theorem the L.H.S. of (5.51) is continuous in $\eta\in\Pi_\omega$, and by Lemma 5.8 and the smoothness of V_{ij}, ϕ_α^m, ϕ_β^m the R.H.S. of (5.53)(= the R.H.S. of (5.51)) is continuous in $\eta\in\Pi_\omega$. Hence, (5.51) holds for all $\eta\in\Pi_\omega$.■

5.4 We assume (V)' and give the proof of Theorem 1.1 in this subsection. The following lemma will be proved in Sect.7

Lemma 5.11. Let $k>\frac{N}{2}$, $s>(N-1)/2$, and $h\in C(S^{N-1})$. Then for any

$\varepsilon>0$ there exists $R_0=R_0(k,s,\varepsilon,h,N)>0$ such that

$$(5.65)\qquad \left| R^{N-1}\int_{S^{N-1}} F(R(\omega'-\omega))h(\omega')d\omega' - h(\omega)\int_{\Pi_\omega} F(\eta)d\eta \right|$$

$$\leq \varepsilon\|F\|_{k,s}$$

for all $R\geq R_0$, $\omega\in S^{N-1}$ and $F\in H_s^k(R^N)$

Remark. $F(R(*-\omega))\in L^2(S^{N-1})$ and $F(\eta)\in L_s^2(\Pi_\omega)\subset L^1(\Pi_\omega)$ are well-defined by the trace theorem.

Proof of Theorem 1.1. We fix $f\in C(S^{N-1})$ and set $M=[(N-1)/2]+1$, $L=(N+1)/2$. Then by Lemma 5.5 we have $\check{G}^\mu(\omega,v)$, $\check{G}(\omega)\in H_M^L(R^N)$, where $\check{G}^\mu(\omega,v;\xi)$, $\check{G}(\omega;\xi)$ are the inverse Fourier transform of $G^\mu(\omega,v;y)$, $G(\omega;y)$, respectively Thus applying Lemma 5.11 with $k=L$, $s=M$, $h=f$, $F=\check{G}^\mu(\omega,v)$, and using (5.11) (see (1.16)), $\lim_{\mu\to\infty,\,v\to v_0} vn_\mu\to\infty$, we obtain

$$|(T^{\mu}_{\beta\kappa}((1/2)n_{\kappa}v^2+\lambda^{\mu}_{\kappa})f)(\omega)-(\int_{\Pi_\omega}\check{G}(\omega;\eta)d\eta)f(\omega)|$$

$$\leq |(vn_{\kappa})^{N-1}\int_{S^{N-1}}\check{G}^{\mu}(\omega,v;vn_{\kappa}(\omega'-\omega))f(\omega')d\omega'-(\int_{\Pi_\omega}\check{G}^{\mu}(\omega,v;\eta)d\eta)f(\omega)|$$

(5.66)

$$+\|\check{G}^{\mu}(\omega,v;\eta)-\check{G}(\omega;\eta)\|_{L^1(\Pi_\omega)}|f(\omega)|$$

$$\leq \theta\|G^{\mu}(\omega,v)\|_{M,L}+C\|G^{\mu}(\omega,v)-G(\omega)\|_{M,L}\|f\|_{L^\infty(\Pi_\omega)},$$

where $\theta=\theta(\mu,f,v)$, which is independent of $\omega\in S^{N-1}$, satisfies $\theta\to 0$ as $\mu\to\infty$ and $v\to v_0$, and C is independent of ω, v and $\mu\gg 1$. In the last step we have used the following estimate, which follows from the Schwarz inequality and Lemma 5.7,

$$(5.67)\qquad \|\check{u}(\eta)\|_{L^1(\Pi_\omega)}\leq \text{const.}\|\check{u}(\eta)\|_{L^2_M(\Pi_\omega)}$$
$$\leq \text{const.}\|u\|_{H^M_L(R^N)}$$

for $u\in H^M_L(R^N)$ Thus, by Lemma 5.5 and (5.66), we get

$$(5.68)\qquad \lim_{\mu\to\infty,\, v\to v_0}(T^{\mu}_{\beta\kappa}((1/2)v^2+\lambda^{\mu}_{\kappa})f)(\omega)=(\int_{\Pi_\omega}\check{G}(\omega;\eta)d\eta)f(\omega)$$

uniformly on S^{N-1} Now direct calculation yields

$$(5.69)\qquad \int_{\Pi_\omega}\check{u}(\eta)d\eta=(2\pi)^{\frac{N-2}{2}}\int_{R}u(t\omega)dt$$

for each $\omega\in S^{N-1}$ and $u\in\mathcal{S}(R^N)$ By the Schwarz inequality and the trace theorem we have

$$(5.70)\qquad \|u(*\omega)\|_{L^1(R)}\leq \text{const.}\|u(*\omega)\|_{L^2_L(R)}\leq \text{const.}\|u\|_{H^M_L(R^N)}$$

for any $u\in H^M_L(R^N)$, regarding $u(*\omega)$ as the trace of u to the one dimensional subspace $\{t\omega;t\in R\}$ in R^N This together with (5.67) implies that (5.69) holds for any $u \in H^{m}_{L}(\mathbb{R}^N)$ In particular we have

$$\int_{\Pi_\omega} \check{G}(\omega;\eta)d\eta = (2\pi)^{\frac{N-2}{2}} \int_{\mathbb{R}} G(\omega;t\omega)dt \tag{5.71}$$

$$= S^{m}_{\beta\alpha}(v_0\omega, 0) - \delta_{\alpha\beta}$$

for all $\omega \in S^{N-1}$ by Lemma 5.9. Theorem 1.1 follows from (5.68) and (5.71) if we recall the definition of $T^{\mu}_{\beta\alpha}(\lambda)$.∎

5.5 We assume (V) and prove Theorem 1 3 in this subsection. The next lemma will be proved in Sect.7

Lemma 5.12. Let $0 < s < k - \frac{1}{2}$ Then for any $\varepsilon > 0$ there exists $R_0 = R_0(\varepsilon, k, s, N) \geq 1$ such that

$$\left| R^{N-1} \int_{S^{N-1}} |F(R(\omega' - \omega))|^2 d\omega' - \int_{\Pi_\omega} |F(\eta)|^2 d\eta \right| \leq \varepsilon \|F\|^2_{k,s}$$

for all $R \geq R_0$, $\omega \in S^{N-1}$ and $F \in H^k_s(\mathbb{R}^N)$

We define J by $(Jf)(\xi) = \overline{f(\xi)}$ (the complex conjugation) and $\tilde{P}^{\mu}_{\gamma}$, $\tilde{W}^{\mu}_{\gamma\pm}$ by $(\tilde{P}^{\mu}_{\gamma}f)(x_c, y_c) := \overline{\phi}^{\mu}_{\gamma}(x_c) f(y_c)$,

$$\tilde{W}^{\mu}_{\gamma\pm} := \underset{t\to\pm\infty}{\text{s-lim}}\ e^{itH^\mu} \tilde{P}^{\mu}_{\gamma} e^{-itT^{\mu}_{\gamma}} \tag{5.72}$$

for $\gamma = \alpha, \beta$, $c = D(\gamma)$ Since

$$J\tilde{P}^{\mu}_{\gamma}J = P^{\mu}_{\gamma},\quad e^{-itT^{\mu}_{\gamma}} = J e^{itT^{\mu}_{\gamma}} J,\quad e^{-itH^{\mu}} = J e^{itH^{\mu}} J \quad \text{for } \gamma = \alpha, \beta,$$

we have, by (1.10), $W^{\mu}_{\gamma\pm} = J\tilde{W}^{\mu}_{\gamma\mp}J$ and so

$$S^{\mu}_{\beta\alpha} = J\tilde{S}^{\mu*}_{\alpha\beta} J, \tag{5.73}$$

where $\tilde{S}^{a}_{\alpha\beta} := \tilde{W}^{a*}_{\alpha+} \tilde{W}^{a}_{\beta-}$. We can see that $Z^{a}_{\alpha} \tilde{S}^{a}_{\alpha\beta} Z^{a*}_{\beta}$ is decomposable in terms of a family of operators $\{\tilde{S}^{a}_{\alpha\beta}(\lambda)\}$ ([A-J-S], 15-3):

(5.74) $$Z^{a}_{\alpha} \tilde{S}^{a}_{\alpha\beta} Z^{a*}_{\beta} = \{\tilde{S}^{a}_{\alpha\beta}(\lambda)\} \qquad \text{(see (1.15))}.$$

For $\gamma = \alpha, \beta$ we note that $JZ^{a*}_{\gamma}(\lambda) = Z^{a*}_{\gamma}(\lambda)\hat{J}$, where $\hat{J}$ is defined by $(\hat{J}g)(\omega) = \overline{g(-\omega)}$ for $g \in \Sigma = L^2(S^{N-1})$ Then we have by (5.73), (5.74)

(5.75) $$S^{a}_{\beta\alpha}(\lambda) = \hat{J}\tilde{S}^{a}_{\alpha\beta}(\lambda)^{*}\hat{J} \quad \text{on} \quad \Sigma, \qquad \lambda \in (\lambda^{a}_{\beta\alpha}, \infty)\backslash\Lambda^{a},$$

where $\lambda^{a}_{\beta\alpha} := \max(\lambda^{a}_{\alpha}, \lambda^{a}_{\beta})$. $\Lambda^{a} := \{\text{the thresholds of } H^{a}\} \cup \sigma_{p}(H^{a})$ Since both sides are norm continuous by Theorem 2.3, (5.75) holds for all $\lambda \in (\lambda^{a}_{\beta\alpha}, \infty)\backslash\Lambda^{a}$ Let $\tilde{T}^{a}_{\alpha\beta}(\lambda, \omega, \omega')$ be the integral kernel of $\tilde{S}^{a}_{\alpha\beta}(\lambda) - \delta_{\alpha\beta}$. Then, by (5.75), we get

(5.76) $$T^{a}_{\beta\alpha}(\lambda, \omega', \omega) = \tilde{T}^{a}_{\alpha\beta}(\lambda, -\omega, -\omega') \qquad \text{(cf. (1 16))}$$

This equality holds for all $(\lambda, \omega, \omega') \in ((\lambda^{a}_{\beta\alpha}, \infty)\backslash\Lambda^{a}) \times S^{N-1} \times S^{N-1}$, because both sides are continuous in all variables (Proposition 2.4)

The arguments up to subsection 5.3 are valid even if we replace (the initial channel) α and (the final channel) β by β and α, respectively. and then relace ϕ^{a}_{γ} and ϕ^{-}_{γ} by $\overline{\phi^{a}_{\gamma}}$ and $\overline{\phi^{-}_{\gamma}}$, respectively for $\gamma = \alpha, \beta$. In the definition of $G^{a}(\omega, v; y_{a})$ and $G(\omega; y)$ (see (5.9), (5.23)), we replace α and β by β and α, respectively(the initial speed v is replaced by v' (see (5.2)), and replace ϕ^{a}_{γ} and ϕ^{-}_{γ} by $\overline{\phi^{a}_{\gamma}}$ and $\overline{\phi^{-}_{\gamma}}$, respectively Denote the resulting function by $p^{a}(\omega, v; y_{b})$ and $p^{-}(\omega; y)$. Then we have

(5.77) $$\begin{aligned} &\tilde{T}^{a}_{\alpha\beta}(\lambda, \omega, \omega') = (n_{b}v')^{N-1}\hat{p}^{a}(\omega, v; n_{b}v'(\omega' - \omega)), \\ &\lambda = (1/2)n_{a}v^{2} + \lambda^{a}_{\alpha} = (1/2)n_{b}v'^{2} + \lambda^{a}_{\beta} \end{aligned}$$

in the same way as (5.11) Thus, in virture of (5.76), the total cross section is represented as (see (1.17))

$$(5.78)\qquad \sigma^{\mu}_{\beta\alpha}((1/2)n_a v^2+\lambda^{\mu}_{\alpha};\omega)$$

$$= (2\pi)^{N-1}(n_a v)^{1-N}(n_b v')^{2N-2}\int_{S^{N-1}} |\check{p}^{\mu}(-\omega, v; n_b v'(\omega'+\omega))|^2 d\omega'$$

We have

$$|\sigma^{\mu}_{\beta\alpha}((1/2)n_a v^2+\lambda^{\mu}_{\alpha};\omega)-(2\pi)^{N-1}\|\check{p}^{\infty}(-\omega;*)\|^2_{L^2(\Pi_\omega)}|$$

$$(5.79)\qquad \leqq |\sigma^{\mu}_{\beta\alpha}((1/2)n_a v^2+\lambda^{\mu}_{\alpha};\omega)-(2\pi)^{N-1}\|\check{p}^{\mu}(-\omega, v;*)\|^2_{L^2(\Pi_\omega)}|$$

$$+(2\pi)^{N-1}(\|\check{p}^{\infty}(-\omega;*)\|_{L^2(\Pi_\omega)}+\|\check{p}^{\mu}(-\omega, v;*)\|_{L^2(\Pi_\omega)})\times$$

$$\times|\|\check{p}^{\infty}(-\omega;*)\|_{L^2(\Pi_\omega)}-\|\check{p}^{\mu}(-\omega, v;*)\|_{L^2(\Pi_\omega)}|$$

Under assumption (V), Lemma 5.5(i) holds even if $G^{\mu}(\omega, v;y_a)$ and $G(\omega;y)$ are replaced by $p^{\mu}(\omega, v;y_b)$ and $p^{\infty}(\omega;y)$, respectively Thus, by using the trace theorem, Lemm 5.12 and (5.78), we obtain

$$(5.80)\qquad \lim_{\mu\to\infty,\, v\to v_0} \sigma^{\mu}_{\beta\alpha}((1/2)n_a v^2+\lambda^{\mu}_{\alpha};\omega)=(2\pi)^{N-1}\|\check{p}^{\infty}(-\omega;*)\|^2_{L^2(\Pi_\omega)}$$

uniformly for $\omega\in S^{N-1}$ Since

$$\check{p}^{\infty}(-\omega;\xi)=(2\pi)^{-(N-1)/2}\int_{\Pi_\omega} e^{i\xi\cdot\eta}\Big((2\pi)^{-1/2}\int_R p^{\infty}(-\omega;t\omega+\eta)dt\Big)d\eta$$

for $\xi\in\Pi_\omega$, the R.H.S. of (5.80) equals

$$(2\pi)^{N-2}\int_{\Pi_\omega}|\int_R p^{\infty}(-\omega;t\omega+\eta)dt|^2 d\eta$$

by Parseval's equality.

We define $\widetilde{\psi}^{\infty}_{\gamma}(\xi,\eta;x,t)$ by replacing ϕ^{∞}_{γ} by $\overline{\phi}^{\infty}_{\gamma}$ in the definition of $\psi^{\infty}_{\gamma}(\xi,\eta;x,t)$ (see (1.20)) and define

$$(5.81)\qquad \widetilde{\Omega}_\gamma^{\pm}(\xi,\eta):=\operatorname*{s-lim}_{t\to\pm\infty} U(\xi,\eta;0,t)\widetilde{\psi}_\gamma^{\mp}(\xi,\eta;*,t) \quad \text{in } L^2(R^N)$$

for $\gamma=\alpha,\ \beta$. Then, in the same way as Lemma 5.9, we have

$$(5.82)\qquad (2\pi)^{(N-2)/2}\int_R p^{-}(-\omega;t\omega+\eta)\,dt$$

$$= e^{-i\int_{-\infty}^{\infty}V_{12}(v_0t\omega-\eta)dt}(\widetilde{\Omega}_\beta^{-}(-v_0\omega,\eta),\widetilde{\Omega}_\alpha^{+}(-v_0\omega,\eta))-\delta_{\beta\alpha},$$

for a.e. $\eta\in\Pi_\omega$. Thus, by(1.24), (5.80), we have only to prove the following lemma to finish the proof of Theorem 1.3.

Lemma 5.13. For $\gamma=\alpha,\ \beta$ and $\xi\in R^N\backslash\{0\}$, $\eta\in\Pi_\xi$, we have

$$(5.83)\qquad \widetilde{\Omega}_\gamma^{\pm}(-\xi,\eta)=J\Omega_\gamma^{\mp}(\xi,\eta)$$

Proof. Recall that $U(\xi,\eta;t,s)$ is the propagator of $h_{\xi,\eta}(t)$ (see (1.18)). $Q(t,s):=JU(-\xi,\eta;-t,-s)J$, $(s,t\in R)$ obviously satisfies (U-i) and (U-ii) of Sect.1. Moreover we have

$$\begin{aligned} i\partial_t Q(t,s) &= Jh_{-\xi,\eta}(-t)U(-\xi,\eta;-t,-s)J \\ &= h_{\xi,\eta}(t)Q(t,s), \end{aligned}$$

where we have used $h_{\xi,\eta}(t)=h_{-\xi,\eta}(-t)$ in the last step.
Thus we see that $Q(t,s)=U(\xi,\eta;t,s)$ for all $s,t\in R$ by the uniqueness of propagator Since $J\widetilde{\psi}_\gamma^{\mp}(-\xi,\eta;x,t)=\psi_\gamma^{\mp}(\xi,\eta;x,-t)$ for $\gamma=\alpha,\ \beta$, it follows that

$$\begin{aligned} \widetilde{\Omega}_\gamma^{\pm}(-\xi,\eta) &= \operatorname*{s-lim}_{t\to\pm\infty} U(-\xi,\eta;0,t)\widetilde{\psi}_\gamma^{\mp}(-\xi,\eta;*,t) \\ &= \operatorname*{s-lim}_{t\to\pm\infty} JU(\xi,\eta;0,-t)J\widetilde{\psi}_\gamma^{\mp}(-\xi,\eta;*,t) \\ &= J\Omega_\gamma^{\mp}(\xi,\eta) \end{aligned}$$ ∎

§ 6 Proof of Lemma 5 4

We shall prove Lemma 5.4, under assumption (V), by applying the abstract theorem obtained in Sect.3. We fix $s \in \mathbf{R}$ and an integer $k \geq 0$ with $0 \leq k \leq \varepsilon_0 - 2$, $k + \frac{1}{2} < s$, and a compact interval $J = [e_1, e_2]$ in $\mathbf{R}$ throughout this section. We may assume $a = a_1 = (1, (2,3))$, since the other case can be treated similarly

Let

$$(6.1) \qquad A_0 := (1/2i)(x \cdot \nabla_x + \nabla_x \, x + y \, \nabla_y + \nabla_y \, y)$$

be the generator of dilations on R^{2N}, which is self-adjoint in $\mathcal{H} = L^2(\mathrm{R}^{2N})$ with $\mathcal{S} = \mathcal{S}(\mathrm{R}^{2N})$ as a core. For a triplet $\theta = (\omega, \mu, v)$ with $\omega \in S^{N-1}$, $\mu \gg 1$ and $|v - v_0| \ll 1$ for fixed $v_0 > 0$, we define a operator

$$(6.2) \qquad A_\theta := n_a^{-1} A_0 + v\omega \cdot y = n_a^{-1} e^{-in_a v\omega \, y} A_0 e^{in_a v\omega \, y},$$

and for $\omega \in S^{N-1}$ we define

$$(6.3) \qquad A_\omega^\infty := v_0 \omega \cdot y$$

We also write

$$(6.4) \qquad H_\theta = H^a(\omega, v) = H^a - iv\omega \, \nabla_y \qquad \text{(see (5.5))}$$

for a triplet $\theta = (\omega, \mu, v)$. A_θ, A_ω^∞ and H_θ are self-adjoint operators in $\mathcal{H}$ with $\mathcal{S}$ as a core. A direct calculation yields

$$(6.5) \qquad i[H_\theta, A_\theta] = \frac{2}{n_a} H_\theta - \frac{i}{n_a}[A_0, V^a] - \frac{2}{n_a} V^a + v^2,$$

$$(6.6) \quad i[A_0, V^a] = V_{23}^{(1)}(x) + V_{13}^{(1)}\left(\frac{m_a}{m}x - y\right) + V_{12}^{(1)}\left(-\frac{m_a}{M_2}x - y\right)$$

on $\mathcal{D}$, where $V_{jk}^{(\ell)}(x)=(x\cdot\nabla_x)^\ell V_{jk}(x)$ (see (5 12)) Thus the R.H.S. of (6.5) can be extended to a bounded operator from $H^2(\mathbf{R}^{2N})$ to $\mathcal{H}$, and the commutator $i[H_\theta, A_\theta]$ defines a self-adjoint operator $iH_\theta^{(1)}$ in $\mathcal{H}$ If $\ell\leq\varepsilon_0$, the ℓ-th commutator

$$
(6.7)\qquad \begin{aligned} & i^\ell[A_0,[\cdots,[A_0,V^\mu]\cdots]] \\ &= V_{23}^{(\ell)}(x)+V_{13}^{(\ell)}(\frac{m_2}{m}x-y)+V_{12}^{(\ell)}(-\frac{m_3}{M_2}x-y) \end{aligned}
$$

is bounded by (V). Therefore, by using (6.5) we see that the ℓ-th commutator $i^\ell[[\cdots[H_\theta, A_\theta],\cdots,], A_\theta]$ on $\mathcal{D}$ can be uniquely extended to a self-adjoint operator $i^\ell H_\theta^{(\ell)}$ in $\mathcal{H}$ for $\ell\leq\varepsilon_0$.

Let ϕ be a C^∞-function on $\mathbf{R}$ such that $0\leq\phi\leq1$, $\phi\equiv1$ on J and $\operatorname{supp}\phi\subset[e_1-1, e_2+1]$ (supp = support)

<u>Lemma 6.1</u> Let d be a positive integer with $d\leq k+1$.

(i) There exist $M>0$ and $\delta_0>0$ such that assumption 3.1 ($\mathbf{H}=\mathcal{H}$, $H=H_\theta$, $A=A_\theta$, $I=J$) is satisfied for all triplets $\theta=(\omega,\mu,v)$ with $\omega\in S^{N-1}$, $\mu=(M_1,M_2)$, M_1, $M_2>M$, $|v-v_0|<\delta_0$, where we can take $C_0=(\frac{1}{2})v_0^2$ in (H-v) Furthermore, $\|i^\ell H_\theta^{(\ell)}R_\theta(i)\|$ ($\ell=1,\cdots,d+1$) is uniformly bounded for $\theta=(\omega,\mu,v)$, where we write

$$
(6.8)\qquad R_\theta(z)=(H_\theta-z)^{-1}
$$

(ii) For $\omega\in S^{N-1}$ assumption 3.1 ($\mathbf{H}=\mathcal{H}$, $H=H(\omega)$, $A=A_\omega^\infty$, $I=J$) is satisfied, where we can take $C_0=v_0^2$ in (H-v) Furthermore, $i[H(\omega), A_\omega^\infty]=v_0^2$ and $i^\ell[[\cdots[H(\omega), A_\omega^\infty],\cdots], A_\omega^\infty]=0$ if $2\leq\ell$.

<u>Proof.</u>(i) (H-i) is obvious because $\mathcal{D}$ is a common core for H_θ and A_θ. (H-ii) follows easily from

(6.9) $e^{itA_\theta} = e^{-in_* v\omega\cdot y} e^{itA_0/n_*} e^{in_* v\omega\cdot y}$

We can verify (H-iii),(H-iv) by using the arguments before this lemma and the fact that $\mathscr{S}$ is a common core for H_θ and A_θ. Since $n_* \to \infty$ as $\mu \to \infty$, we obtain by (6.5)

$$\phi(H_\theta) i H_\theta^{(1)} \phi(H_\theta)$$

$$(6.10) \quad \geq \left(\frac{2}{n_*}(e_1-1) - \frac{1}{n_*}\|[A_0, V^\mu]\| - \frac{2}{n_*}\|V^\mu\| + v^2\right)\phi(H_\theta)^2$$

$$\geq (1/2) v_0^2 \phi(H_\theta)^2$$

for all $\theta = (\omega, \mu, v)$ with $\omega \in S^{N-1}$, $\mu \gg 1$ and $|v - v_0| \ll 1$. This implies (H-v) with $C_0 = (1/2) v_0^2$. The uniform boundedness of

(6.11) $\|i^\ell H_\theta^{(\ell)} R_\theta(i)\| = \|((2/n_*)^\ell H_\theta + \text{bounded operators}) R_\theta(i)\|$

$(\ell = 1, \cdots, d+1)$ follows from (6.5), (6.7) and (V).

(ii) $\mathscr{S}$ is a common core for $H(\omega)$ and A_ω^∞, and (H-i) is satisfied. Noting that

$$D(H(\omega)) = \{f \in L^2(R^{2N}); (-(2m)^{-1}\Delta_x - iv_0\omega\cdot\nabla_y) f \in L^2(R^{2N})\},$$

we can easily see that (H-ii) holds. (H-iii)~(H-v) follow from $i[H(\omega), A_\omega^\infty] = v_0^2$ on $\mathscr{S}$. This completes the proof. ∎

We set $\Xi := \{\theta = (\omega, \mu, v);\ M_1, M_2 > M,\ |v - v_0| < \delta_0,\ \omega \in S^{N-1}$, where $\mu = (M_1, M_2)\}$. Here M and δ_0 are as in Lemma 6.1.

We denote any of $-i\partial_{x_j}$ or $-i\partial_{y_j}$ $(j = 1, \cdots, N)$ by D. For any $f \in H^1(R^{2N})$ and $z \in C \setminus R$,

$$(6.12)\qquad e^{-itD}R(\omega;z)f=(e^{-itD}H(\omega)e^{itD}-z)^{-1}e^{-itD}f.$$

Thus $e^{-itD}R(\omega;z)f$ is strongly differentiable in $t\in\mathbf{R}$ and

$$(6.13)\qquad \frac{d}{dt}e^{-itD}R(\omega;z)\Big|_{t=0}=-i\{R(\omega;z)D-R(\omega;z)(DV^{\omega})R(\omega;z)\}f,$$

which implies $R(\omega;z)$ leaves $H^1(\mathbf{R}^{2N})$ invariant and

$$(6.14)\quad DR(\omega;z)=R(\omega;z)D-R(\omega;z)(DV^{\omega})R(\omega;z)\qquad \text{on}\quad H^1(\mathbf{R}^{2N})$$

for each $z\in\mathbf{C}\setminus\mathbf{R}$ By using (6.14) and (V), we see that

$$(6.15)\qquad \sup_{\omega\in S^{N-1},z\in K}\|R(\omega;z)\|_{B(H^{\ell})}<\infty$$

for any compact set K in $\mathbf{C}\setminus\mathbf{R}$ and $\ell\geqq 0$, where $H^{\ell}=H^{\ell}(\mathbf{R}^{2N})$ In the same way as above for any compact set K in $\mathbf{C}\setminus\mathbf{R}$ and $\ell\geqq 0$ we have

$$(6.16)\qquad \sup_{\theta\in\Xi,z\in K}\|R_{\theta}(z)\|_{B(H^{\ell})}<\infty$$

Furthermore $R_{\theta}(z)$ $(z\in\mathbf{C}\setminus\mathbf{R})$ leaves $\mathcal{S}$ invariant (e.g. Proposition 1.3 of [P]). and so by (6.1) we have

$$(6.17)\qquad i[R_{\theta}(i),A_{\theta}]=-iR_{\theta}(i)[H_{\theta},A_{\theta}]R_{\theta}(i)$$
$$=-\frac{2}{n_*}R_{\theta}(i)+R_{\theta}(i)\{\frac{i}{n_*}[A_0,V^{\theta}]+\frac{2}{n_*}(V^{\theta}-i)-v^2\}R_{\theta}(i)$$

on $\mathcal{S}$

<u>Lemma 6.2</u>. Let ℓ be a nonnegative integer Then

$$\sup_{\theta\in\Xi}\ n_*^{-1}\|R_{\theta}(i)\|_{B(H^{\ell},H^{\ell+1})}<\infty$$

<u>Proof</u>. By (6.16), (V) and the resolvent equation

$$R_\theta(i)=(H_\theta-V^{\mu}-i)^{-1}-(H_\theta-V^{\mu}-i)^{-1}V^{\mu}R_\theta(i),$$

it suffices to prove

(6.18) $$\sup_{\theta\in\Xi} n_*^{-1}\|(H_\theta-V^{\mu}-i)^{-1}\|_{B(H^\ell,H^{\ell+1})}<\infty$$

Thus by (6.4) and the Fourier transform it suffices to show that

(6.19) $$\sup_{\substack{(\omega,\mu,v)\in\Xi\\ \xi,\eta\in\mathbb{R}^N}} \frac{1}{n_*}\cdot\frac{|\xi|+|\eta|+1}{|(2m_*)^{-1}|\xi|^2+(2n_*)^{-1}|\eta|^2+v\omega\cdot\eta|+1}<\infty$$

Taking account of the inequality $2ab\leqq a^2+b^2$ for real a, b, we have

$$|\xi|+|\eta|+1\leqq|(1/2m_*)|\xi|^2+(1/2n_*)|\eta|^2+v\omega\cdot\eta|+n_*(v^2+1)+1+(m_*/2)$$

Therefore (6.19) follows. ∎

<u>Proof of (i), (ii) of Lemma 5.4</u>.

(I) First we give the proof for $R(\omega;z)$

When $k=0$, we fix $s>\frac{1}{2}$ and $d=1$ By Lemma 6.1(ii) and Theorem 3.3, we have

(6.20) $$\sup_{z\in J_\pm,\,\omega\in S^{N-1}}\|\langle\omega\ y\rangle^{-s}R(\omega;z)\langle\omega\ y\rangle^{-s}\|<\infty$$

and the norm limits

(6.21) $$\lim_{\varepsilon\downarrow 0}\langle\omega\cdot y\rangle^{-s}R(\omega;\lambda\pm i\varepsilon)\langle\omega\cdot y\rangle^{-s}$$

exist in $\mathcal{H}$ uniformly for $\omega\in S^{N-1}$ and $\lambda\in J$. From this together with $\langle\omega\cdot y\rangle^{s}\langle x;y\rangle^{-s}\leqq 1$, $\langle x;y\rangle:=(1+|x|^2+|y|^2)^{1/2}$, the desired results follow.

When $k\geqq 1$, we have only to prove

(6.22) $$\sup_{z\in J_\pm,\,\omega\in S^{N-1}}\|\langle x;y\rangle^{-s}D^{\gamma}R(\omega;z)\langle D\rangle^{-k}\langle x;y\rangle^{-s}\|<\infty,$$

$$(6.23)\qquad \lim_{\varepsilon,\varepsilon'\downarrow 0} \|\langle x;y\rangle^{-s}D^{\gamma}[R(\omega;\lambda\pm i\varepsilon)-R(\omega;\lambda\pm i\varepsilon')]\langle D\rangle^{-k}\langle x;y\rangle^{-s}\|=0$$

$$(\ \langle D\rangle:=(-\Delta_x-\Delta_y+1)^{1/2}\ )$$

uniformly for $\lambda\in J$, $\omega\in S^{N-1}$ for $s>k+½$ and $|\gamma|\leqq k$. By using (6.14) repeatedly and by taking account of $\langle x;y\rangle^{s}D^{\gamma}\langle D\rangle^{-k}\langle x;y\rangle^{-s}\in \mathbf{B}(\mathcal{H})$ for $|\gamma|\leqq k$, in order to prove (6.22) and (6.23), it turns out to be sufficient to show

$$(6.24)\qquad \sup_{z\in J_{\pm},\,\omega\in S^{N-1}} \|\langle\omega\ y\rangle^{-s}R(\omega;z)V^{\infty}_{\gamma_1}R(\omega;z)\cdots$$

$$\cdots V^{\infty}_{\gamma_\ell}R(\omega;z)\langle\omega\ y\rangle^{-s}\|<\infty.$$

$$(6.25)\qquad \lim_{\varepsilon,\varepsilon'\downarrow 0}\|\langle\omega\cdot y\rangle^{-s}\{R(\omega;\lambda\pm i\varepsilon)V^{\infty}_{\gamma_1}R(\omega;\lambda\pm i\varepsilon)\cdots V^{\infty}_{\gamma_\ell}R(\omega;\lambda\pm i\varepsilon)$$

$$-R(\omega;\lambda\pm i\varepsilon')V^{\infty}_{\gamma_1}R(\omega;\lambda\pm i\varepsilon')\cdots V^{\infty}_{\gamma_\ell}R(\omega;\pm i\varepsilon')\}\langle\omega\cdot y\rangle^{-s}\|=0\ ,$$

uniformly for $\lambda\in J,\omega\in S^{N-1}$, where $V^{\infty}_{\gamma_j}=D^{\gamma_j}V^{\infty}$ and $1\leqq\ell\leqq k$, $|\gamma_j|\leqq k$ $(j=1,\cdots,\ell)$ Since $[V^{\infty}_{\gamma_j},A^{\infty}_{\omega}]=0$ for $j=1,\cdots,\ell$, taking account of Lemma 6.1, we can apply Theorem 3.3 with $\mathbf{H}=\mathcal{H}$, $H=H(\omega)$, $A=A^{\infty}_{\omega}$, $I=J$, $d=\ell+1$ and $W_j=V^{\infty}_{\gamma_j}$ for $j=1,\cdots,\ell$ to conclude (6.24) and (6.25) This completes the proof of (i), (ii) for $R(\omega;z)$

(II) We next prove (i),(ii) for $R_\theta(z)=R^{\#}(\omega,v;z)$.

When $k=0$, we fix s with $½<s<1$ and shall show that

$$(6.26)\qquad \sup_{z\in J_{\pm},\ \theta\in\Xi}\|\langle x;y\rangle^{-s}R_\theta(z)\langle x;y\rangle^{-s}\|<\infty,$$

$$(6.27)\qquad \lim_{\varepsilon,\varepsilon'\downarrow 0}\ \sup_{\lambda\in J,\,\theta\in\Xi} \|\langle x;y\rangle^{-s}[R_\theta(\lambda\pm i\varepsilon)-R_\theta(\lambda\pm i\varepsilon')]\langle x;y\rangle^{-s}\|=0.$$

By the resolvent equation we get

$$(6.28)\qquad R_\theta(z)=R_\theta(i)+(z-i)R_\theta(i)^2+(z-i)^2R_\theta(i)R_\theta(z)R_\theta(i)$$

for $z\in \mathbf{C}\setminus\mathbf{R}$ Thus, to obtain (6.26), (6.27) it suffices to show that

$$(6.29)\qquad \sup_{z\in J_\pm,\,\theta\in\Xi} \|\langle x;y\rangle^{-s}R_\theta(i)R_\theta(z)R_\theta(i)\langle x;y\rangle^{-s}\|<\infty,$$

$$(6.30)\qquad \lim_{\varepsilon,\varepsilon'\downarrow 0}\ \sup_{\lambda\in J,\,\theta\in\Xi} \|\langle x;y\rangle^{-s}R_\theta(i)\times$$

$$\times[R_\theta(\lambda\pm i\varepsilon)-R_\theta(\lambda\pm i\varepsilon')]R_\theta(i)\langle x;y\rangle^{-s}\|=0.$$

By Lemma 6.1(i) and Theorem 3.3 with $d=1$, we have

$$(6.31)\qquad \sup_{z\in J_\pm,\,\theta\in\Xi} \|\langle A_\theta\rangle^{-s}R_\theta(z)\langle A_\theta\rangle^{-s}\|<\infty,$$

$$(6.32)\qquad \lim_{\varepsilon,\varepsilon'\downarrow 0,}\ \sup_{\lambda\in J,\,\theta\in\Xi} \|\langle A_\theta\rangle^{-s}[R_\theta(\lambda\pm i\varepsilon)-R_\theta(\lambda\pm i\varepsilon')]\langle A_\theta\rangle^{-s}\|=0.$$

We have $A_\theta R_\theta(i)=R_\theta(i)A_\theta+[A_\theta,R_\theta(i)]$ on $\mathcal{S}$ $[A_\theta,R_\theta(i)]$ is uniformly bounded for $\theta\in\Xi$ by (6.6) and (6.17) Since

$$(6.33)\qquad A_\theta=-i(n_*)^{-1}\nabla_x\cdot x-i(n_*)^{-1}\nabla_y\cdot y+v\omega\cdot y+(N/in_*).$$

we have $\sup_{\theta\in\Xi}\|R_\theta(i)A_\theta\langle x;y\rangle^{-1}\|<\infty$ by Lemma 6.2, and so we get $\sup_{\theta\in\Xi}\|A_\theta R_\theta(i)\langle x;y\rangle^{-1}\|<\infty$ By using interpolation this yields

$$(6.34)\qquad \sup_{\theta\in\Xi}\|\langle A_\theta\rangle^{s}R_\theta(i)\langle x;y\rangle^{-s}\|<\infty$$

for $0\leqq s\leqq 1$. Thus, (6.29) and (6.30) follows from (6.34) together with (6.31) and (6.32), and so (6.26) and (6.27) are obtained.

When $k\geqq 1$, we fix a real s with $k+\frac{1}{2}<s<k+1$. In the same way as (I), it suffices to prove

$$(6.35)\qquad \sup_{z\in J_\pm,\,\theta\in\Xi}\|\langle x;y\rangle^{-s}R_\theta(z)V^\#_{\gamma_1}R_\theta(z)\cdots V^\#_{\gamma_\ell}R_\theta(z)\langle x;y\rangle^{-s}\|<\infty,$$

$$(6.36)\quad \lim_{\varepsilon,\varepsilon'\downarrow 0}\ \sup_{\lambda\in J,\,\theta\in\Xi}\|\langle x;y\rangle^{-s}\{R_\theta(\lambda\pm i\varepsilon)V^\#_{\gamma_1}R_\theta(\lambda\pm i\varepsilon)\cdots V^\#_{\gamma_\ell}R_\theta(\lambda\pm i\varepsilon)$$

$$-R_\theta(\lambda\pm i\varepsilon')V^\#_{\gamma_1}R_\theta(\lambda\pm i\varepsilon')\cdots V^\#_{\gamma_\ell}R_\theta(\lambda\pm i\varepsilon')\}\langle x;y\rangle^{-s}\|=0\ ,$$

where $V^\#_\gamma=D^\gamma V^\#$ and $1\leqq\ell\leqq k$, $|\gamma|\leqq k$.

Using (6.28) repeatedly, we get

$$(6.37)\qquad R_\theta(z)=\{\sum_{\ell_1,\ell_2;\text{finite}}(z-i)^{\ell_1}R_\theta(i)^{\ell_2}\}+$$

$$+(z-i)^{2(k+1)}R_\theta(i)^{k+1}R_\theta(z)R_\theta(i)^{k+1}$$

for $z\in\mathbf{C}\setminus\mathbf{R}$ Thus, by substituting (6.37) in (6.35) and (6.36), we finally see that the proof of (6.35), (6.36) can be reduced to that of the following

$$(6.38)\qquad \sup_{z\in J_\pm,\,\theta\in\Xi}\|\langle A_\theta\rangle^{-s}R_\theta(z)U_2R_\theta(z)\cdots U_mR_\theta(z)\langle A_\theta\rangle^{-s}\|<\infty,$$

$$(6.39)\qquad \lim_{\varepsilon,\varepsilon'\downarrow 0}\ \sup_{\lambda\in J,\,\theta\in\Xi}\|\langle A_\theta\rangle^{-s}\{R_\theta(\lambda\pm i\varepsilon)U_2R_\theta(\lambda\pm i\varepsilon)\cdots U_mR_\theta(\lambda\pm i\varepsilon)$$

$$-R_\theta(\lambda'\pm i\varepsilon)U_2R_\theta(\lambda'\pm i\varepsilon)\cdots U_mR_\theta(\lambda'\pm i\varepsilon)\}\langle A_\theta\rangle^{-s}\|=0,$$

$$(6.40)\qquad \sup_{\theta\in\Xi}\ \{\|\langle x;y\rangle^{-s}U_1\langle A_\theta\rangle^{s}\|+\|\langle A_\theta\rangle^{s}U_{m+1}\langle x;y\rangle^{-s}\|\}<\infty$$

for $2\leqq m\leqq k+1$, where each U_j is the form

$$(6.41)\qquad U_j=R_\theta(i)Q_1R_\theta(i)\cdots Q_hR_\theta(i)\qquad (h\geqq k)$$

with $Q_n = 1$ or $V_\gamma^\#$ $(|\gamma| \leqq k)$.

We first prove (6.38) and (6.39) by applying Theorem 3.3. It follows from (6.17) and $k \leqq \varepsilon_0 - 2$ that q-th commutators $(0 \leqq q \leqq k+2)$ $[\cdots[U_j, A_\theta], \cdots, A_\theta]$ on $\mathcal{S}$ can be extended to bounded operators $U_j^{(q)}$ on $\mathcal{H}$, and their operator norms are uniformly bounded for $\theta \in \Xi$. Thus each U_j $(j=1, \cdots, m+1)$ satisfies Assumption 3.2 with $A = A_\theta$ and $d \leqq k+1$, and so (6.38) and (6.39) follow from Lemma 6.1 (i) and Theorem 3.3 with $d \leqq k+1$. Next we shall prove (6.40). We have

$$(6.42) \qquad U_1 A_\theta^{k+1} = A_\theta^{k+1} U_1 + [U_1, A_\theta^{k+1}] \quad \text{on } \mathcal{S}.$$

A_θ^j $(j=0, \cdots, k+1)$ has the form

$$(6.43) \qquad A_\theta^j = \sum_{|\gamma_1+\gamma_2|, |\gamma_3| \leqq j} C_{\gamma_1\gamma_2\gamma_3} x^{\gamma_1} y^{\gamma_2} \left(\frac{1}{n_a} D\right)^{\gamma_3},$$

where $C_{\gamma_1\gamma_2\gamma_3}$ are constants uniformly bounded for $\theta \in \Xi$. Since U_1 containes at least $(k+1)$ $R_\theta(i)$, we obtain

$$(6.44) \qquad \sup_{\theta \in \Xi} \left\| \left(\frac{1}{n_a} D\right)^\gamma U_1 \right\| < \infty \quad \text{for } |\gamma| \leqq k+1$$

by Lemma 6.2 and (V), and so we have

$$(6.45) \qquad \sup_{\theta \in \Xi} \| \langle x; y \rangle^{-k-1} A_\theta^{k+1} U_1 \| < \infty$$

The commutator $[U_1, A_\theta^{k+1}]$ has the form

$$[U_1, A_\theta^{k+1}] = \sum_{j=0}^{k} C_j A_\theta^j U_1^{(k+1-j)},$$

where C_j are constants independent of $\theta \in \Xi$. We note that (6.17), (V) and Lemma 6.2 yield

$$(6.46) \qquad \sup_{\theta \in \Xi} \left\| \left(\frac{1}{n_a} D\right)^\gamma U_1^{(k+1-j)} \right\| < \infty$$

for $|\gamma| \leqq k+1$ and for $j=0, \cdots, k$, and so we obtain

$$(6.47) \qquad \sup_{\theta\in\Xi} \|\langle x;y\rangle^{-k-1}[U_1, A_\theta^{k+1}]\| < \infty$$

It follows from (6.45),(6.47) that

$$(6.48) \qquad \sup_{\theta\in\Xi} \|\langle x;y\rangle^{-k-1}U_1\langle A_\theta\rangle^{k+1}\| < \infty.$$

Similarly, we have

$$(6.49) \qquad \sup_{\theta\in\Xi} \|\langle A_\theta\rangle^{k+1}U_{m+1}\langle x;y\rangle^{-k-1}\| < \infty$$

Therefore (6.40) follows by interpolation. This completes the proof of (i),(ii) of Lemma 5.4. ∎

It remains to prove (iii),(iv) of Lemma 5.4.

Lemma 6.3. Let ℓ be a nonnegative integer and $f\in H^\ell$ Then for each $\delta>0$,

$$(6.50) \qquad \operatorname*{s-lim}_{\lambda'\to\lambda,\,\mu\to\infty,\,v\to v_0} R^\mu(\omega, v;\lambda'+i\delta)f = R(\omega;\lambda+i\delta)f \qquad \text{in} \quad H^\ell$$

uniformly for $\omega\in S^{N-1}$

Proof. By (6.15) and (6.16) we may assume $f\in\mathcal{S}$ We have

$$(6.51) \qquad [R^\mu(\omega, v;\lambda'+i\delta) - R(\omega;\lambda+i\delta)]f = -R^\mu(\omega, v;\lambda'+i\delta)\times$$
$$\times\ [(\frac{1}{2m}-\frac{1}{2m_*})\Delta_x - \frac{1}{2n_*}\Delta_y - i(v-v_0)\omega\nabla_y - \lambda' + \lambda]R(\omega;\lambda+i\delta)f$$
$$-R^\mu(\omega, v;\lambda'+i\delta)[V^\mu - V^\infty]R(\omega;\lambda+i\delta)f.$$

Taking account of (6.15),(6.16), we see that the first term in the R.H.S. goes to zero in H^ℓ uniformly for $\omega\in S^{N-1}$ as $\mu\to\infty$, $v\to v_0$, $\lambda'\to\lambda$. It is obvious that

$$(6.52) \qquad \lim_{\mu\to\infty} \|[V^\mu - V^\infty]R(\omega;\lambda+i\delta)f\|_{\ell,0} = 0$$

for each $\omega \in S^{N-1}$ Since $R(\omega;\lambda+i\delta)f$ is H^a-valued strongly continuous function of $\omega \in S^{N-1}$, we can see that (6.52) hold uniformly for ω by the finite covering argument (see (5.37)) Thus, by (6.15), the second term in the R.H.S. of (6.51) goes to zero in H^a uniformly for $\omega \in S^{N-1}$ as $\mu \to \infty$, $v \to v_0$, $\lambda' \to \lambda$. This proves the lemma.∎

Proof of (iii) of Lemma 5.4. Fix $f \in H^k_s$ for $s > k + \frac{1}{2}$ and a sufficiently small $\varepsilon > 0$. By (ii) of Lemma 5 4 already shown, we can take a $\delta > 0$ such that

$$(6.53) \qquad \sup_{\substack{\lambda, \lambda' \in J \\ \theta = (\omega, \mu, v) \in \Xi}} \{\|[R_\theta(\lambda'+i0) - R_\theta(\lambda'+i\delta)]f\|_{H^k_{-s}} + \|[R(\omega;\lambda+i\delta) - R(\omega;\lambda+i0)]f\|_{H^k_{-s}}\} < \varepsilon$$

This together with Lemma 6.3 gives the desired result. ∎

Proof of (iv) of Lemma 5.4. By (i) of Lemma 5.4 already shown, we may assume $f \in \mathcal{S}$ For any $\xi \in S^{N-1}$, we have

$$[R(\xi;\lambda\pm i0) - R(\omega;\lambda\pm i0)]f$$

$$(6.54) \quad = [R(\xi;\lambda\pm i0) - R(\xi;\lambda\pm i\varepsilon)]f + [R(\omega;\lambda\pm i\varepsilon) - R(\omega;\lambda\pm i0)]f + [R(\xi;\lambda\pm i\varepsilon) - R(\omega;\lambda\pm i\varepsilon)]f.$$

By using the resolvent equation,

$$(6.55) \qquad R(\xi;z) - R(\omega;z) = R(\omega;z)\, i v_0(\xi - \omega)\ \nabla_x R(\omega;z)$$

for $z \in \mathbf{C} \setminus \mathbf{R}$, we can easily show that the last term goes to zero as

$\omega \to \xi$ in $H^{\pm}_{s}(\mathbf{R}^{2N})$ for each $\varepsilon > 0$. By Lemma 5.4 (ii), the others go to zero uniformly for ξ, $\omega \in S^{N-1}$ as $\varepsilon \to 0$. This completes the proof ∎

§ 7 Proof of Lemma 5.11 and Lemma 5.12

Lemma 7.1. Let $k > 1/2$, $s > (N-1)/2$. Then for any $\varepsilon > 0$ and any $0 < \delta \leqq 1/2$ there exsist positive constants $R_0 = R_0(\varepsilon, \delta, s, k, N)$, $C = C(s, k, N)$ such that

$$(7\ 1)\quad | R^{N-1} \int_{S^{N-1}} F(R(\omega' - \omega))h(\omega')d\omega' - h(\omega) \int_{\Pi_\omega} F(\eta)d\eta |$$
$$\leqq \varepsilon \|h\|_{L^\infty(S^{N-1})} \|F\|_{k,s} + C \sup_{|\omega - \omega'| < \delta} |h(\omega') - h(\omega)| \|F\|_{k,s}$$

for for all $R \geqq R_0$, $h \in C(S^{N-1})$, $F \in H^k_s(\mathbf{R}^N)$ and $\omega \in S^{N-1}$

Proof of Lemma 5.11. Immediate from Lemma 7 1.∎

Lemma 7 1 will be proved after the series of lemmmas.

Lemma 7.2. Let $k > 1/2$, $s > (N-1)/2$. Then for any $0 < \delta < 1$ there exists a positive constant $C = C(\delta, s, k, N)$ such that

$$(7.2)\qquad R^{N-1} \int_{|\omega - \omega'| > \delta} |F(R(\omega' - \omega))| d\omega' \leqq C \cdot R^{-s + ((N-1)/2)} \|F\|_{k,s}$$

for all $R \geqq 1$, $F \in \mathcal{S}(\mathbf{R}^N)$ and $\omega \in S^{N-1}$

Proof. For each $\omega \in S^{N-1}$ there exists a $\chi_\omega \in C^\infty(\mathbf{R}^N)$ such that

$$0 \leq \chi_\omega \leq 1, \qquad \operatorname{supp} \chi_\omega \subset \{ \xi \in \mathbf{R}^N ;\ |\hat{\xi} - \omega| > \delta/2,\ |\xi| > 1/2 \},$$

$$\chi_\omega = 1 \ \text{ on } \{ \xi \in \mathbf{R}^N ;\ |\hat{\xi} - \omega| \geq \delta,\ |\xi| \geq 1 \},\ (\hat{\xi} := \xi/|\xi|),$$

$$\sup_{\omega \in S^{N-1},\, \xi \in \mathbf{R}^N} |D_\xi^\gamma \chi_\omega(\xi)| < \infty \quad \text{for each } \gamma$$

Let $dS_R(\xi)$ be the Lebesgue measure on $S_R := \{\xi \in \mathbf{R}^N;\ |\xi| = R\}$ Then

$$R^{N-1} \int_{|\omega - \omega'| > \delta} |F(R(\omega' - \omega))| d\omega' = \int_{|\hat{\xi} - \omega| > \delta} |F(\xi - R\omega)| dS_R(\xi)$$

$$\leq \| \chi_\omega(\xi) F(\xi - R\omega) \|_{L^1(S_R)}$$

(7.3)
$$\leq C_1 \cdot R^{-s + ((N-1)/2)} \| \langle \xi \rangle^s \chi_\omega(\xi) F(\xi - R\omega) \|_{L^2(S_R)}$$

$$\leq C_2 \cdot R^{-s + ((N-1)/2)} \| \langle \xi \rangle^s \chi_\omega(\xi) F(\xi - R\omega) \|_{H^k(\mathbf{R}^N)}$$

$$\leq C_2 \cdot R^{-s + ((N-1)/2)} \| \langle D_\xi \rangle^k \langle \xi \rangle^s \langle \xi - R\omega \rangle^{-s} \chi_\omega \langle D_\xi \rangle^{-k} \|_{B(L^2)} \times$$

$$\times \| \langle \xi - R\omega \rangle^s F(\xi - R\omega) \|_{H^k},$$

where in the second step we have used the Schwarz inequality, and in the last step but one we have used the fact that

(7 4)
$$\| f(\xi) \|_{L^2(S_R)} \leq C \| f \|_{H^k(\mathbf{R}^N)}$$

for all $R \geq 1$ and $f \in H^k(\mathbf{R}^N)$, where C is independent of R (cf. Proposition (2.1) of [G-M]) Note that

$$\| \langle \xi - R\omega \rangle^s F(\xi - R\omega) \|_{H^k} = \| \langle \xi \rangle^s F(\xi) \|_{H^k} \leq C \| F \|_{H_s^k}.$$

Since $\omega\ \xi \leq (1 - (\delta^2/8)) |\xi|$ for $\xi \in \operatorname{supp} \chi_\omega$, we have

$$|\xi - R\omega|^2 \geqq (1-(\delta^2/8))(|\xi|-R)^2 + (\delta^2/8)(|\xi|^2+R^2) \geqq (\delta^2/8)\langle\xi\rangle^2$$

for $\xi \in \mathrm{supp}\,\chi_*$, $\omega \in S^{N-1}$ and $R \geqq 1$, and so we obtain for any multi-index γ

(7.5) $$\sup_{R\geq 1,\ \omega\in S^{N-1},\ \xi\in R^N} |D_\xi^\gamma \langle\xi\rangle^s \langle\xi-R\omega\rangle^{-s}\chi_*(\xi)| < \infty$$

This means that $\|\langle D_\xi\rangle^k \langle\xi\rangle^s\langle\xi-R\omega\rangle^{-s}\chi_*\langle D_\xi\rangle^{-k}\|_{B(L^2(R^N))}$ is uniformly bounded for $R \geqq 1$ and $\omega \in S^{N-1}$ Thus we have obtained (7.2).∎

<u>Lemma 7.3</u>. Let $0 \leqq t \leqq 1$, $k-(1/2) > t$, $s-((N-1)/2) > t$. Then there exists a positive constant $C = C(t,s,k,N)$ such that

(7.6) $$\int_{|\eta|<\delta R} |F(\eta, \sqrt{R^2-\eta^2}-R) - F(\eta,0)|d\eta \leqq C\cdot\delta^t \|F\|_{k,s}$$

for all $R \geqq 1$, $0 < \delta \leqq 1/2$ and $F \in \mathcal{S}(R^N)$, where $\eta \in \mathbf{R}^{N-1}$

<u>Proof</u>. Using the Fourier transform, we have

$$F(\eta, \sqrt{R^2-\eta^2}-R) - F(\eta,0) = (2\pi)^{-N/2}\int_{-\infty}^{\infty}[e^{ix_N(\sqrt{R^2-\eta^2}-R)}-1]dx_N \times$$

(7 7) $$\times \int e^{ix'\cdot\eta}\hat{F}(x',x_N)dx' \qquad (x' \in R^{N-1})$$

Noting that $|e^{ir}-1| \leqq 2|r|^t$ for all $r \in \mathbf{R}$ and all $0 \leqq t \leqq 1$, and that $|x_N(\sqrt{R^2-\eta^2}-R)| = \eta^2(R+\sqrt{R^2-\eta^2})^{-1}|x_N| \leqq \delta|\eta||x_N|$ for $|\eta| \leqq \delta R$, we have

(7.8) $$|e^{ix_N(\sqrt{R^2-\eta^2}-R)}-1| \leqq 2\delta^t|\eta|^t|x_N|^t \quad \text{for } |\eta| \leqq \delta R.$$

Thus, by using the Schwarz inequality, we obtain

$$|F(\eta, \sqrt{R^2-\eta^2}-R) - F(\eta,0)|^2$$

(7.9) $$\leq C\cdot\delta^{2t}|\eta|^{2t}\int_{-\infty}^{\infty}\langle x_N\rangle^{2k}dx_N\left|\int e^{ix'\cdot\eta}\hat{F}(x',x_N)dx'\right|^2.$$

since $k-(1/2)>t$. (7.9) and the Schwarz inequality give

$$\int_{|\eta|<\delta R}|F(\eta,\sqrt{R^2-\eta^2}-R)-F(\eta,0)|d\eta$$

$$\leq C\cdot\left\{\int_{|\eta|<\delta R}\langle\eta\rangle^{2(s-t)}|F(\eta,\sqrt{R^2-\eta^2}-R)-F(\eta,0)|^2d\eta\right\}^{1/2}$$

(7 10) $$\leq C\cdot\delta^t\left\{\int_{-\infty}^{\infty}\langle x_N\rangle^{2k}dx_N\int\langle\eta\rangle^{2s}d\eta\left|\int e^{ix'\cdot\eta}\hat{F}(x',x_N)dx'\right|^2\right\}^{1/2}$$

$$\leq C\cdot\delta^t\left\{\int_{-\infty}^{\infty}\langle x_N\rangle^{2k}dx_N\int|\langle D_{x'}\rangle^s\hat{F}(x',x_N)|^2dx'\right\}^{1/2}$$

$$\leq C\cdot\delta^t\|F\|_{k,s},$$

where we have used the Parseval equality in $N-1$ variables in the third step.∎

<u>Lemma 7.4</u>. Let $k>1/2$, $s>(N-1)/2$. Then for any $\varepsilon>0$ there exists a positive constant $R_0=R_0(\varepsilon,s,k)\geq 1$ such that

(7.11) $$\left|R^{N-1}\int_{S^{N-1}}F(R(\omega'-\omega))d\omega'-\int_{\Pi_\omega}F(\eta)d\eta\right|\leq\varepsilon\|F\|_{k,s}$$

for all $R\geq R_0$, $F\in\mathcal{S}(R^N)$ and $\omega\in S^{N-1}$

<u>Proof</u>. Fix a sufficiently small positive constant δ, and let $\delta'=\delta\sqrt{1-(\delta^2/4)}$ We have

$$\int_{S^{N-1}}F(\xi-R\omega)\,dS_R(\xi)-\int_{\Pi_\omega}F(\eta)d\eta$$

$$=\int_{|\xi-\omega|>\delta}F(\xi-R\omega)\,dS_R(\xi)$$

(7 12)

$$+\left\{ \int_{|\xi-\omega|<\delta} F(\xi-R\omega)\, dS_R(\xi) - \int_{\substack{|\eta|<\delta' R\\ \eta\in\Pi_\omega}} F(\eta) \frac{1}{\sqrt{1-\frac{|\eta|^2}{|R|^2}}}\, d\eta \right\}$$

$$+ \int_{\substack{|\eta|<\delta' R\\ \eta\in\Pi_\omega}} F(\eta) \left[\frac{1}{\sqrt{1-\frac{|\eta|^2}{|R|^2}}} - 1\right] d\eta - \int_{\substack{|\eta|>\delta' R\\ \eta\in\Pi_\omega}} F(\eta)\, d\eta$$

$$= I_1 + I_2 + I_3 + J_4.$$

Applying Lemma 7.2 to I_1, we have

$$|I_1| \leq C_1 R^{-s+((N-1)/2)} \|F\|_{k,s}, \quad R \geqq 1, \tag{7.13}$$

where $C_1 = C_1(\delta, s, k, N) > 0$. Next we claim that

$$|I_2| \leq C_2 \delta'^{t} \|F\|_{k,s} \tag{7.14}$$

for any $t > 0$ satisfying $\mathrm{Min}\ \{ s-((N-1)/2),\ k-(1/2),\ 1\} > t$, where $C_2 = C_2(t, s, k, N) > 0$. Indeed, in the case $\omega = (0, \cdots, 0, 1)$, noting that

$$\int_{|\xi-\omega|<\delta} F(\xi-R\omega)\, dS_R(\xi) = \int_{|\eta|<\delta' R} F(\eta, \sqrt{R^2-\eta^2} - R) \frac{1}{\sqrt{1-\frac{|\eta|^2}{|R|^2}}}\, d\eta, \tag{7.15}$$

and that $\dfrac{1}{\sqrt{1-\frac{|\eta|^2}{|R|^2}}} \leqq 2$ for $|\eta| < \delta' R$ for sufficiently small δ,

and applying Lemma 7.3, we have (7.14). In other cases we have only to change the coordinates.

Noting that for $|\eta| < \delta' R$

$$\left| \frac{1}{\sqrt{1-\frac{|\eta|^2}{|R|^2}}} - 1 \right| \leq C \cdot \frac{|\eta|^2}{|R|^2} \leq C \delta'^2,$$

we have

$$|I_3| \leq C \cdot \delta'^2 \|\langle\eta\rangle^s F\|_{L^2(\Pi_\omega)} \leqq C_3 \delta'^2 \|F\|_{k,s}. \tag{7.16}$$

where $C_3=C_3(s,k,N)$. and we have used the trace theorem in the last step. I_4 is estimated as follows.

$$
(7\ 17)\qquad |I_4| \le \Big\{\int_{\substack{|\eta|>\delta' R\\ \eta\in\Pi_\omega}} \langle\eta\rangle^{-2s}d\eta\Big\}^{1/2}\Big\{\int_{\eta\in\Pi_\omega}\langle\eta\rangle^{2s}|F(\eta)|^2 d\eta\Big\}^{1/2}
$$

$$
\le C_4(\delta' R)^{-s+((N-1)/2)}\|F\|_{k,s},
$$

where $C_4=C_4(s,k,N)>0$. Thus we have the desired result by taking δ sufficiently small and then taking R sufficiently large.∎

Proof of Lemma 7.1. We may assume $F\in\mathcal{S}(\mathbf{R}^N)$ by the approximation, since $\mathcal{S}(\mathbf{R}^N)$ is dense in $H^k_s(\mathbf{R}^N)$ We have

$$
R^{N-1}\int_{S^{N-1}}F(R(\omega'-\omega))h(\omega')d\omega'
$$

$$
(7\ 18)\qquad \begin{aligned}
&= R^{N-1}\int_{S^{N-1}}F(R(\omega'-\omega))d\omega'\ h(\omega)\\
&+ R^{N-1}\int_{|\omega-\omega'|<\delta}F(R(\omega'-\omega))[h(\omega')-h(\omega)]d\omega'\\
&+ R^{N-1}\int_{|\omega-\omega'|>\delta}F(R(\omega'-\omega))[h(\omega')-h(\omega)]d\omega'\\
&= J_1+J_2+J_3.
\end{aligned}
$$

We first show that

$$
(7\ 19)\qquad R^{N-1}\int_{|\omega-\omega'|<\delta}|F(R(\omega'-\omega))|d\omega' \le C_1\|F\|_{k,s}.
$$

where $C_1=C_1(s,k,N)>0$. By the change of coordinates we may assume

$\omega=(0,\cdots,0,1)$ Then,

$$(7.20)\qquad R^{N-1}\int_{|\omega-\omega'|<\delta}|F(R(\omega'-\omega))|d\omega' \leqq C\int_{|\eta|<\delta' R}|F(\eta,\sqrt{R^2-\eta^2}-R)|d\eta,$$

where $\delta'=\delta\sqrt{1-(\delta^2/4)}$ (see (7.15)) By Lemma 7 3 we have

$$\left|\int_{|\eta|<\delta' R}|F(\eta,\sqrt{R^2-\eta^2}-R)|d\eta - \int_{|\eta|<\delta' R}|F(\eta,0)|d\eta\right|$$

$$(7.21)\qquad \leqq \int_{|\eta|<\delta' R}|F(\eta,\sqrt{R^2-\eta^2}-R) - F(\eta,0)|d\eta$$

$$\leqq C\ \|F\|_{k,s},$$

and by the trace theorem we have

$$(7.22)\qquad \int_{|\eta|<\delta' R}|F(\eta,0)|d\eta \leqq C\ \|F\|_{k,s}.$$

Thus, by (7.20)~(7.22), we obtain (7 19) (7 19) yields

$$(7.23)\qquad |J_2| \leqq C_1\cdot\sup_{|\omega-\omega'|<\delta}|h(\omega')-h(\omega)|\ \|F\|_{k,s}.$$

By Lemma 7.2 we have

$$(7.24)\qquad |J_3| \leqq C_2R^{-s+((N-1)/2)}\|h\|_{L^\infty}\|F\|_{k,s},$$

where $C_2=C_2(\delta,s,k,N)>0$.

By taking R sufficiently large in (7.24), the desired result follows from (7.23), (7.24) and Lemma 7 4.∎

Proof of Lemma 5.12. The proof is similar to that of Lemma 7 1, and so we give a sketch of the proof. We may assume $0<s<1$ Let $0<\delta\ll 1$ and let δ' be the same as in the proof of Lemma 7 4. We write

$$\int_{S^{N-1}} |F(\xi-R\omega)|^2 dS_R(\xi) - \int_{\Pi_\omega} |F(\eta)|^2 d\eta$$

(7.25)
$$\begin{aligned}
&= \int_{|\xi-\omega|>\delta} |F(\xi-R\omega)|^2 dS_R(\xi) \\
&\quad + \Big\{ \int_{|\xi-\omega|<\delta} |F(\xi-R\omega)|^2 dS_R(\xi) - \int_{\substack{|\eta|<\delta' R \\ \eta\in\Pi_\omega}} |F(\eta)|^2 \frac{1}{\sqrt{1-\frac{|\eta|^2}{|R|^2}}} d\eta \Big\} \\
&\quad + \int_{\substack{|\eta|<\delta' R \\ \eta\in\Pi_\omega}} |F(\eta)|^2 \Big[\frac{1}{\sqrt{1-\frac{|\eta|^2}{|R|^2}}} - 1 \Big] d\eta \\
&\quad - \int_{\substack{|\eta|>\delta' R \\ \eta\in\Pi_\omega}} |F(\eta)|^2 d\eta \\
&= I_1 + I_2 + I_3 - I_4.
\end{aligned}$$

In the same way as Lemma 7 2 we have

(7.26) $$|I_1| \leqq C_1\langle R\rangle^{-2s} \|\langle\xi\rangle^s \chi_\omega(\xi) F(\xi-R\omega)\|^2_{k,0} \leqq C_2 \langle R\rangle^{-2s} \|F\|^2_{k,s},$$

where $C_2=C_2(\delta,s,k,N)$ We next estimate I_2, and assume $\omega=(0,\cdots,0,1)$ Since $0<\delta<1$, we have by the trace theorem

$$\int_{|\eta|<\delta' R} |F(\eta,0)|^2 \frac{1}{\sqrt{1-\frac{|\eta|^2}{|R|^2}}} d\eta \leqq C_3 \|F\|^2_{k,\sigma}.$$

Furthermore, taking $t=s$ in (7 9), and integrating it w.r t. η, we get

$$\int_{|\eta|<\delta' R} |F(\eta, \sqrt{R^2-\eta^2}-R) - F(\eta,0)|^2 d\eta \leq C_4 \delta^{2s} \|F\|^2_{k,s}.$$

Therefore, by using the inequality

$$|\|f\|^2 - \|g\|^2| \leq \|f-g\|^2 + 2\|f-g\|\|g\|,$$

we obtain

$$(7.27) \qquad |I_2| \leq C_5 \delta^s \|F\|^2_{k,s}.$$

In a way similar to the proof of Lemma 7 4, I_3 and I_4 can be estimated as follows;

$$(7.28) \qquad |I_3| \leq C_6 \delta'^2 \|F\|^2_{k,s}, \qquad |I_4| \leq C_7 \langle \delta' R\rangle^{-2s} \|F\|^2_{k,s}.$$

Here note that $C_j = C_j(s,k,N)$ for $j=3,\cdots,7$ Thus we have

$$|I_1 + I_2 + I_3 - I_4| \leq C_8 \{C(\delta)\langle R\rangle^{-2s} + \delta^s + \langle \delta' R\rangle^{-2s}\} \|F\|^2_{k,s},$$

where $C_8 = C_8(s,k,N)$. which implies the desired result. ∎

§ 8 Proof of formula (2.10)

We shall prove here formula (2.10) of Theorem 2.3. The superscript * will be ommited in the proof. Let J be a compact interval of $(\lambda_{\beta\alpha}, \infty)\backslash\Lambda$ (see below (1 15) for $\lambda_{\beta\alpha} = \lambda^*_{\beta\alpha}$, $\Lambda = \Lambda^*$), and fix $f_\gamma \in L^2_s(R^N)$, $s > ½$, such that $E_{T_\gamma}(J) f_\gamma = f_\gamma$ for $\gamma = \alpha, \beta$, where $E_{T_\gamma}(\ )$ is the spectral measure of T_γ (see (1.9)) Considering in the momentum space, we can see that such f_γ's form a dense set in

$E_{T_\gamma}(J)L^2(R^N)$

We denote the resolvents of H, $h_c \otimes I + I \otimes (-(2n_c)^{-1}\Delta_{y_c})$, T_γ by $R(z)$, $R_\gamma(z)$ and $r_\gamma(z)$ $(D(\gamma)=c)$, respectively (see (1.7) for h_c) The following relation are obvious:

$$R_\gamma(z)P_\gamma = P_\gamma r_\gamma(z) \qquad \text{(see (1.9))} \tag{8.1}$$

Using the intertwining relation, we have

$$(S_{\beta\alpha}f_\alpha, f_\beta) = (W_{\alpha-}f_\alpha, W_{\beta+}f_\beta)$$

$$= \operatorname*{s-lim}_{t\to-\infty} (e^{itH}P_\alpha e^{-itT_\alpha}f_\alpha, W_{\beta+}f_\beta)$$

$$= \operatorname*{s-lim}_{t\to-\infty} (P_\alpha e^{-itT_\alpha}f_\alpha, W_{\beta+} e^{-itT_\beta}f_\beta) \tag{8.2}$$

$$= \lim_{\varepsilon\downarrow 0} 2\varepsilon \int_{-\infty}^{0} e^{2\varepsilon t}(P_\alpha e^{-itT_\alpha}f_\alpha, W_{\beta+} e^{-itT_\beta}f_\beta)dt.$$

Let $\chi(t)=1$ for $t<0$ and $=0$ for $t\geqq 0$. Then, for each $\varepsilon>0$, the inverse Fourier transforms of the vector-valued functions

$$P_\alpha\, e^{\varepsilon t-itT_\alpha}\chi(t)f_\alpha, \quad W_{\beta+}e^{\varepsilon t-itT_\beta}\chi(t)f_\beta$$

are $i(2\pi)^{-1/2}P_\alpha r_\alpha(\lambda-i\varepsilon)f_\alpha$, $i(2\pi)^{-1/2}W_{\beta+}r_\beta(\lambda-i\varepsilon)f_\beta$, respectively Therefore, applying the Parseval equality to the above integral in (8.2), we get

$$(S_{\beta\alpha}f_\alpha, f_\beta) = \lim_{\varepsilon\downarrow 0} \frac{\varepsilon}{\pi}\int_{-\infty}^{\infty}(P_\alpha r_\alpha(\lambda-i\varepsilon)f_\alpha, W_{\beta+}r_\beta(\lambda-i\varepsilon)f_\beta)d\lambda$$

$$= \lim_{\varepsilon\downarrow 0} \frac{\varepsilon}{\pi}\int_{J}(P_\alpha r_\alpha(\lambda-i\varepsilon)f_\alpha, W_{\beta+}r_\beta(\lambda-i\varepsilon)f_\beta)d\lambda. \tag{8.3}$$

where we have used $E_{T_\gamma}(J)f_\gamma = f_\gamma$ for $\gamma=\alpha,\ \beta$.

Set $u_\gamma = r_\gamma(\lambda - i\varepsilon)f_\gamma$ $(\gamma=\alpha,\ \beta)$ Similarly to the above, we obtain

$$(P_\alpha u_\alpha, W_{\beta+}u_\beta) = \lim_{\delta\downarrow 0}\frac{\delta}{\pi}\int_{-\infty}^{\infty}(R(\zeta+i\delta)P_\alpha u_\alpha, P_\beta r_\beta(\zeta+i\delta)u_\beta)d\zeta$$

$$(8.4)\qquad = \lim_{\delta\downarrow 0}\frac{\delta}{\pi}\int_J (R(\zeta+i\delta)P_\alpha u_\alpha, P_\beta r_\beta(\zeta+i\delta)u_\beta)d\zeta,$$

where we have used $\int_{-\infty}^{\infty}\|R(\zeta+i\delta)P_\alpha u_\alpha\|^2 d\zeta = \frac{\pi}{\delta}\|P_\alpha u_\alpha\|^2$, and $E_{T_\beta}(J)f_\beta = f_\beta$ in the last step. For $z\in \mathbf{C}\setminus\mathbf{R}$ we define $K(z) := -I_a + I_b R(z) I_a$ $(a=D(\alpha),\ b=D(\beta); I_a := V - V_{j3}$ for $a=\{i,(j,3)\})$ Then the following relation

$$R(\zeta+i\delta) = R_\alpha(\zeta+i\delta) + R_\beta(\zeta+i\delta)K(\zeta+i\delta)R_\alpha(\zeta+i\delta)$$

is obtained by using the resolvent equations:

$$R(z) = R_\alpha(z) - R(z)I_a R_\alpha(z), \qquad R(z) = R_\beta(z) - R_\beta(z)I_b R(z)$$

Substituting this in (8.4), we have

$$(P_\alpha u_\alpha, W_{\beta+}u_\beta)$$

$$(8.5)\quad = \lim_{\delta\downarrow 0}\frac{\delta}{\pi}\int_J (R_\alpha(\zeta+i\delta)P_\alpha u_\alpha, P_\beta r_\beta(\zeta+i\delta)u_\beta)d\zeta,$$

$$+\lim_{\delta\downarrow 0}\frac{\delta}{\pi}\int_J (K(\zeta+i\delta)P_\alpha r_\alpha(\zeta+i\delta)u_\alpha, P_\beta r_\beta(\zeta-i\delta)r_\beta(\zeta+i\delta)u_\beta)d\zeta,$$

where we have used (8.1) Reversing the argument used for showing (8.3), we see that the first term in the R.H.S. of (8.5) is equal to

$$\lim_{t\to+\infty} (P_\alpha e^{-itT_\alpha} u_\alpha, P_\beta e^{-itT_\beta} u_\beta) = (W_{\alpha+}u_\alpha, W_{\beta+}u_\beta)$$

$$= \delta_{\beta\alpha}(P_\alpha r_\alpha(\lambda-i\varepsilon)f_\alpha, P_\beta r_\beta(\lambda-i\varepsilon)f_\beta)$$

In the last step we have used the fact that Ran $W_{\alpha+} \perp$ Ran $W_{\beta+}$ (Ran = Range) for $\alpha \neq \beta$

We next consider the second term of (8.5) By the first resolvent equation we have

$$r_\alpha(\zeta+i\delta)u_\alpha = \frac{1}{(\zeta-\lambda)+i(\varepsilon+\delta)}\{r_\alpha(\zeta+i\delta) - r_\alpha(\lambda-i\varepsilon)\}f_\alpha,$$

$$r_\beta(\zeta+i\delta)r_\beta(\zeta-i\delta)u_\beta = (2i\delta)^{-1}\Big[\frac{1}{(\zeta-\lambda)+i(\varepsilon+\delta)}\{r_\beta(\zeta+i\delta) - r_\beta(\lambda-i\varepsilon)\}f_\beta$$

$$+ \frac{1}{(\zeta-\lambda)+i(\varepsilon-\delta)}\{r_\beta(\zeta-i\delta) - r_\beta(\lambda-i\varepsilon)\}f_\beta\Big]$$

Thus we obtain

$$(P_\alpha r_\alpha(\lambda-i\varepsilon)f_\alpha, W_{\beta+}r_\beta(\lambda-i\varepsilon)f_\beta)$$

$$= \delta_{\beta\alpha}(P_\alpha r_\alpha(\lambda-i\varepsilon)f_\alpha, P_\beta r_\beta(\lambda-i\varepsilon)f_\beta) +$$

$$+ \lim_{\delta\downarrow 0} \Big[\frac{i}{2\pi} \int_J \frac{1}{(\zeta-\lambda)^2+(\varepsilon+\delta)^2} \times$$

$$\times (K(\zeta+i\delta)P_-\{r_\alpha(\zeta+i\delta)-r_\alpha(\lambda-i\varepsilon)\}f_\alpha, P_\beta\{r_\beta(\zeta+i\delta)-r_\beta(\lambda-i\varepsilon)\}f_\beta)d\zeta$$

$$- \frac{i}{2\pi} \int_J \frac{1}{\{(\zeta-\lambda)+i(\varepsilon+\delta)\}\{(\zeta-\lambda)-i(\varepsilon-\delta)\}} \times$$

$$\times (K(\zeta+i\delta)P_-\{r_\alpha(\zeta+i\delta)-r_\alpha(\lambda-i\varepsilon)\}f_\alpha, P_\beta\{r_\beta(\zeta-i\delta)-r_\beta(\lambda-i\varepsilon)\}f_\beta)d\zeta\Big]$$

The norm limits $P_\beta{}^*K(\zeta+i0)P_\alpha:=\lim_{\delta\downarrow 0} P_\beta{}^*K(\zeta+i\delta)P_\alpha$, and

$r_\gamma(\zeta\pm i0):=\lim_{\delta\downarrow 0} r_\gamma(\zeta\pm i\delta)$ exist in $\mathbf{B}(L^2_{-s}(\mathbf{R}^N_{y_a}),L^2_s(\mathbf{R}^N_{y_b}))$,

$\mathbf{B}(L^2_s(\mathbf{R}^N_{y_c}),L^2_{-s}(\mathbf{R}^N_{y_c}))$, respectively, where $c=D(\gamma)$. and $s>\frac{1}{2}$

Indeed, the former follows from Lemmas 2.1, 2.2 and (V), the latter is well known (cf. [R-S]IV.XIII.8) Therefore we can write

$$(P_\alpha r_\alpha(\lambda-i\varepsilon)f_\alpha, W_{\beta+}r_\beta(\lambda-i\varepsilon)f_\beta)$$

$$=\delta_{\beta\alpha}(P_\alpha r_\alpha(\lambda-i\varepsilon)f_\alpha, P_\beta r_\beta(\lambda-i\varepsilon)f_\beta)+$$

$$+\frac{i}{2\pi}\int_J \frac{1}{(\zeta-\lambda)^2+\varepsilon^2}(h_+(\lambda,\zeta,\varepsilon)-h_-(\lambda,\zeta,\varepsilon))d\zeta,$$

$$h_\pm(\lambda,\zeta,\varepsilon):=$$

$$(K(\zeta+i0)P_\alpha\{r_\alpha(\zeta+i0)-r_\alpha(\lambda-i\varepsilon)\}f_\alpha, P_\beta\{r_\beta(\zeta\pm i0)-r_\beta(\lambda-i\varepsilon)\}f_\beta)$$

By substituting in (8.3) we have

$$(8.6) \qquad (S_{\beta\alpha}f_\alpha, f_\beta)=\delta_{\beta\alpha}(f_\alpha, f_\beta)+$$

$$+\lim_{\varepsilon\downarrow 0}\frac{i}{2\pi}\int_J d\lambda\int_J \frac{\varepsilon}{\pi\{(\zeta-\lambda)^2+\varepsilon^2\}}(h_+(\lambda,\zeta,\varepsilon)-h_-(\lambda,\zeta,\varepsilon))d\zeta$$

Since $h_\pm(\lambda,\zeta,\varepsilon)$ is continuous in $(\lambda,\zeta)\in J\times J$ for each $\varepsilon>0$ and

$$h_\pm(\lambda,\zeta):=\lim_{\varepsilon\downarrow 0} h_\pm(\lambda,\zeta,\varepsilon)$$

$$=(K(\zeta+i0)P_\alpha\{r_\alpha(\zeta+i0)-r_\alpha(\lambda-i0)\}f_\alpha, P_\beta\{r_\beta(\zeta\pm i0)-r_\beta(\lambda-i0)\}f_\beta)$$

uniformly for $(\lambda,\zeta)\in J\times J$, the limit in the R.H.S.of (8.6) converges to

$$\frac{i}{2\pi}\int_J (h_+(\lambda,\lambda)-h_-(\lambda,\lambda))d\lambda=\frac{i}{2\pi}\int_J h_+(\lambda,\lambda)d\lambda,$$

because of $h_-(\lambda,\lambda)=0$. Thus, by noting that

$$Z_\gamma^*(\lambda)Z_\gamma(\lambda)=\frac{1}{2\pi i}[r_\gamma(\lambda+i0)-r_\gamma(\lambda-i0)] \quad \text{(see (1.12))},$$

for $\gamma=\alpha,\ \beta$, we obtain

$$(S_{\beta\alpha}f_\alpha, f_\beta)=\delta_{\beta\alpha}(f_\alpha, f_\beta)+$$

$$+\ 2\pi i\int_J (K(\lambda+i0)P_\alpha Z_\alpha^*(\lambda)Z_\alpha(\lambda)f_\alpha, P_\beta Z_\beta^*(\lambda)Z_\beta(\lambda)f_\beta)d\lambda.$$

This implies (2.10). ∎

Department of Mathematics

Kyoto University

Acknowledgements

The author thanks Professor T. Ikebe and Professor A. Iwatsuka for reading the earlier version carefully and for helpful advices.

References

[A-B-G] W. O. Amrein, A. M. Berthier, V. Georgescu, On Mourre's approach to spectral theory, Helv Phys. Acta, 62 (1989), 1-20.

[Ag] S. Agmon, Lectures on exponential decay of solutions of second order elliptic equations, Princeton, Princeton University Press, 1982.

[A-J-S] W. O. Amrein, J. M. Jauch, K. B. Sinha, Scattering Theroy in Quantum Mechanics, Benjamin, Reading, 1977

[C-F-K-S] H. L. Cycon, R. G. Froese, W. Kirsch, B. Simon, Schrödinger operators, Springer, Berlin-Heidelberg-New York, 1987

[E] V. Enss, "Quantum scattering theory of two-and three-body systems with potentials of short and long range" in Schrödinger operators, ed. S.Graffi, Lecture Notes in Mathematics, Vol.1159, Springer, Berlin-Heidelberg-New York, 1985.

[F-H] R. Froese, I. Herbst, A new proof of the Mourre estimate, Duke Math. J., 49 (1982), 1075-1085.

[G] G. M. Graf, Phase space analysis of the charge transfer model, Helv. Phys. Acta, 63 (1990), 107-138.

[G-M] J. Ginibre, M. Moulin, Hilbert space approach to the quantum mechanical three-body problem, Ann. Inst. H. Poincaré, A 21 (1974), 97-145.

[Ha] G. A. Hagedorn, Asymptotic completeness for the impact parameter approximation, to three particle scattering, Ann. Inst. H. Poincaré, A 36 (1982), 19-40.

[Ho] J. S. Howland, Stationary scattering theory for time-dependent Hamiltonians, Math. Ann., 207 (1974), 315-335.

[J] A. Jensen, Propagation estimates for Schrödinger-type operators, Trans. AMS., 291 (1985), 129-144.

[J-M-P] A. Jensen, E. Mourre, P A. Perry, Multiple commutator estimates and resolvent smoothness in quantum scattering theory, Ann. Inst. H. Poincaré, A 41 (1984), 207-225.

[K] T. Kato, Perturbation Theory for Linear Operators, 2nd ed., Springer, Berlin-Heidelberg-New York, 1980.

[M] E. Mourre, Absence of singular continuous spectrum for certain self-adjoint operators, Comm. Math. Phys.,78 (1981), 391-408.

[P] P Perry, Propagation of states in dilation analytic potentials and asymptotic completeness, Comm. Math. Phys., 81 (1981),243-259.

[P-S-S] P. Perry, I. M. Sigal, B. Simon, Spectral analysis of N-body Schrödinger operators, Ann. Math., 114 (1981), 519-567

[R-S] M. Reed, B. Simon, Methods of Modern Mathematical Physics,I,II,III IV, Academic Press, 1972-1979.

[T] H. Tamura, Principle of limiting absorption for N-body Schrödinger operators, a remark on the commutator methods, Lett. Math. Phys., 17 (1989), 31-36.

[W] U. Wüller, Geometric methods in scattering theory of the charge transfer model, Duke Math. J., 62 (1991), 273-313.

[Y] K. Yajima, A multi-channel scattering theory for some time dependent Hamiltonians, charge transfer problem, Comm. Math. Phys., 75 (1980), 153-178.

[Yaf] D. R. Yafaev, The eikonal approximation and the asymptotics of the total scattering cross-section for the Schrödinger equation, Ann. Inst. H. Poincaré, A 44(1986), 397-425.