主論文

京都大学審査博士学位論文

Generalized Retarded Functions and Analytic Function in Momentum Space in Quantum Field Theory*

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The analytic n-point function in momentum space in quantum field theory is studied. Its different boundary values for real value of the argument are determined, and a necessary and sufficient condition for them to be obtainable from the Wightman functions is given. The conditions are relativistic covariance, support properties in coordinate space (retardedness), two-term identities for momentum below threshold (corresponding to spectrum conditions) and 4-term identities (Steinmann relations). The first three conditions are translatable into a statement about the domain of analyticity of the n-point function: it is analytic in a union of various extended tubes plus the points of contact of two neighboring tubes for real part of one momentum below threshold.

1. INTRODUCTION

The retarded functions (the vacuum expectation values of retarded products of field operators) in quantum field theory are, as is well known, boundary values of an analytic function in momentum space. In this paper, we will attempt a systematic investigation of this analytic function and its boundary values. Such an investigation has also been made independently by Ruelle,[1] Steinmann,[2] and Burgoyne.[3] The present work puts emphasis on the geometrical nature of the problem in contrast with the algebraic method of Steinmann and Burgoyne. The method of Ruelle has some common features with the present work but we believe that ours is more explicit and detailed.

First we consider the analytic function in the energy component only, and we easily obtain all its boundary values which include all the conventional retarded and advanced functions. These boundary values will be called generalized retarded functions (γ-function). Their number is 6, 32, 370, and 10932 for 3, 4, 5, and 6-fold in contrast with 6, 24, 120, 720, for the Wightman functions.

Using a generalization of the θ-function, we can express generalized retarded functions in terms of Wightman functions and the latter in terms of the former in a compact manner. Furthermore, we obtain necessary and sufficient conditions for generalized retarded functions to be obtainable from Wightman functions satisfying the usually considered conditions, namely (W1) relativistic covariance, (W2) local commutativity or anticommutativity, and (W3) certain mass spectrum conditions. The resulting conditions on the r -function are (R1) relativistic covariance, (R2) support properties in x-space (retardedness or advancedness), (R3) two-term identities in momentum space for momentum below threshold, (R4) 4-term identities. The 4-term identities have first been found by Steinmann[4] for the 4-point function.

The above mentioned analytic function can be extended to a covariant analytic function in all energy momentum components. The properties (R1), (R2), and (R3) are translatable into a statement about the domain of analyticity of this analytic function. Namely it is analytic in the union of various extended tubes plus points of contact of two neighbouring tubes for real parts of one momentum below threshold. We have not succeeded to translate (R4) into a statement about the domain of analyticity.

The time ordered function can also be expressed as a boundary value of the same analytic function. The boundary values must then be approached from a direction which depends on the value of the real part of the momenta.

All the results are valid for arbitrary types of fields, Bosons and Fermions.

In section 2 we collect our main results (Theorems 1 through 3), together with definitions of notations necessary for the statement of our results. In section 3, the properties of generalized θ-functions are studied and they are applied in sections 4 through 6 for the proof of our main results.

In section 7 we make a few remarks about the class of functions for which our results hold. If the behavior of Wightman functions for large energy momentum is not sufficiently good, we have been unable to obtain our full results. As for the behavior at large coordinate separation, the truncated Wightman functions are expected to tend to zero in contrast to the Wightman functions themselves. Hence the truncated functions are used extensively in this work and their properties are studied in Appendix B.

The spectrum condition assumed in the main text is the existence of a single lowest positive mass. The case of more general mass spectrum conditions is treated in Appendix A. We obtain two term identities for momentum below threshold and the corresponding analyticity. However, the sufficiency of this condition has not been fully established for a general mass spectrum condition .

In Appendix C, we collect definitions and known results concerning convex polyhedral cones which are extensively used in the main text.

2. NOTATIONS AND MAIN RESULTS

In this paper, we consider the quantum theory of several covariant fields $A_i(x)$ satisfying (1) the invariance under the inhomogeneous Lorentz group, (2) the local commutativity or anticommutativity and (3) spectrum conditions. As spectrum conditions, we assume (3a) the existence of the vacuum (the non-degenerate invariant state), (3b) the positiveness of energy, and (3c) the existence of a lowest positive mass m. In appendix A, we treat the case where (3c) is replaced by more complicated mass spectrum conditions.

The above conditions can be used in a most compact way[5] for the truncated vacuum expectation values as we will see in the following. The Wightman functions are denoted by

$$w_P(x) = \sigma_P(\Psi_0, A_{P(1)}(x_{P(1)})\cdots A_{P(n+1)}(x_{P(n+1)})\Psi_0) \tag{2.1}$$

where P denotes the permutation of $1\ldots(n+1)$, σ_P is the signature of the permutation of anticommuting fields[6] and

$$x = (x_1, \ldots, x_{n+1}) \tag{2.2}$$

Throughout this paper we shall take x_i as the argument of the field A_i.

The truncated Wightman functions are defined recursively by[7]

$$(\Psi_0, A_{i_1}(x_{i_1})\dots A_{i_m}(x_{i_m})\Psi_0) = (A_{i_1}(x_{i_1})\dots A_{i_m}(x_{i_m}))_T$$

$$+ \Sigma\,\sigma(A_{i_1}(x_{i_1})\dots)_T(A_{i_k}(x_{i_k})\dots)_T\dots \quad (2.3)$$

$$w_P^T(x) = \sigma_P(A_{P(1)}(x_{P(1)})\dots A_{P(n+1)}(x_{P(n+1)}))_T \quad (2.4)$$

where the summation extends over all grouping of points $x_1\dots x_m$, the A in each $(\ )_T$ of (2.3) are in the same order as on the left hand side, σ is the signature of the permutation of anticommuting fields which brings $A_{i_1}\dots A_{i_m}$ to the order of the A in that term and σ_P is as in (2.1). The purpose of this definition is to subtract from the Wightman functions in a symmetric manner the contributions from the vacuum intermediate states.

Because of the translational invariance of the theory, x can be taken modulo (1,..., 1) The 4n dimensional vector space formed by x modulo (1,..., 1) is denoted by X.

The Fourier transform of a Wightman function is denoted by

$$(2\pi)^4\,\delta\left(\sum_{i=1}^{n+1} q_i\right)\tilde{w}_P(q) = \int e^{i(q,x)}\,w_P(x)\,dx_1\dots dx_{n+1} \quad (2.5)$$

where

$$(q,x) = \sum_{i=1}^{n+1}(q_i,x_i) \quad (2.6)$$

and (q_i,x_i) is the conventional inner product in Minkowski space.[8] The 4n dimensional vector space formed by

$$q = (q_1, \ldots, q_{n+1}) \tag{2.7}$$

such that $\Sigma q_i = 0$ is denoted by Q. The $\tilde{w}_P(q)$ are functions of q in Q.

The $\tilde{w}_P^T(q)$ are defined in a similar manner, namely

$$\tilde{w}_P^T(q) = \int e^{i(q,x)} w_P^T(x)\, dx, \qquad q\epsilon Q \tag{2.8}$$

$$w_P^T(x) = (2\pi)^{-4n} \int e^{-i(q,x)} \tilde{w}_P^T(q)\, dq, x\epsilon X \tag{2.9}$$

where dx and dq are the volume elements of X and Q,

$$dx = dx_1 \ldots dx_n, \quad dq = \delta(\Sigma q_i)\, dq_i \ldots dq_{n+1} \tag{2.10}$$

In order to control the combinatorical difficulties for large n, it is essential to introduce a compact though somewhat involved notation. A set of integers is generally denoted by I, in particular the set $\{1, \ldots, n+1\}$ by $I(n+1)$ and

$$\{P(1), \ldots P(k)\} = I(P, k) \tag{2.11}$$

The set (of sets) $\{I(P,k);\ k = 1, \ldots n\}$ will be called $\mathcal{J}_P$. We define

$$q(I) = \sum_{\nu\epsilon I} q_\nu \tag{2.12}$$

Note that

$$q(I(n+1)) = 0, \quad q(I(n+1) - I) = -q(I) \tag{2.13}$$

$q(I)$, with $I\epsilon\mathcal{J}_P$ are the energy momentum vectors of intermediate states in the Wightman function w_P.

The peoperties of w_P^T which follow from the assumptions (1) - (3) on the theory are (see Appendix B)

(W1) The $w_P^T(x)$ are covariant functions of $x\epsilon X$.

(W2) If P' results from P by an interchange of the indices P(k) and P(k+1), and if $x_{P(k)}-x_{P(k+1)}$ is space-like, then $w_P^T(x) = w_{P'}^T(x)$.

(W3) $\tilde{w}_P^T(q) = 0$ unless $q(I)^2 \geqslant m$, $q^0(I) > 0$ for all $I\epsilon\mathcal{J}_P$.

We now turn to the main subject of the paper, the analytic function in momentum space. This function will be defined by Eq. (2.27) or in explicitly covariant form by (2.39). To show the equivalence of this definition with conventional usage, let us start from the customary definition of a retarded function for Bose fields:

$$r(x_1;x_2\ldots x_{n+1})=(-i)^n\Sigma\theta(x_1{}^0-x_{P(2)}{}^0)\ldots\theta(x_{P(n)}{}^0-x_{P(n+1)}{}^0)$$

$$(\Psi_0,\left[\ldots\left[\left[A_1(x_1),A_{P(2)}(x_{P(2)})\right],A_{P(3)}(x_{P(3)})\right]\ldots A_{P(n+1)}(x_{P(n+1)})\right]\Psi_0) \quad (2.14)$$

where the summation is over all permutations P of 2,...,n+1. Expanding the multiple commutators, this can be written as[9]

$$r(x_1;x_2\ldots x_{n+1})=\sum_{j}(-1)^{j-1}(-i)^n\sum_{P'(j)=1}\prod_{\nu=1}^{j-1}\theta(x^0_{P'(\nu+1)}-x^0_{P'(\nu)})$$

$$\prod_{\nu=j}^{n+1}\theta(x^0_{P'(\nu)}-x^0_{P'(\nu+1)})\;w_{P'}(x) \tag{2.15}$$

Because the time components appear explicitly in (2.15), we consider the n dimensional vector space T formed by the time component of $x\epsilon X$,

$$x^0=(x^0_1\ldots x_{n+1}0)\qquad \mathrm{mod}\ (1\ldots 1) \tag{2.16}$$

and the n dimensional vector space S formed by the energy component of $q\epsilon Q$. We use the following inner products,

$$q\cdot t=\sum_{i=1}^{n+1}q_i t_i,\qquad s\cdot x=\sum_{i=1}^{n+1}s_i x_i \tag{2.17}$$

$$s\cdot t=\sum_{i=1}^{n+1}s_i t_i \tag{2.18}$$

where $x\epsilon X$, $q\epsilon Q$, $t\,\epsilon T$, and $s\,\epsilon S$. The inner products in (2.17) are Minkowski vectors while the inner product in (2.18) is a number. The space Q is the dual of X relative to the inner product (2.6) and S is the dual of T relative to the inner product in (2.18). The complex vector spaces corresponding to X, Q, T, and S are denoted by Z, Z', U and V, respectively. (2.6), (2.17) and (2.18) are used also for these spaces.

If we define $t(I)$ by

$$t(I)_\nu = 1 \qquad \text{if } \nu \in I \tag{2.19}$$
$$\qquad = 0 \qquad \text{if } \nu \notin I$$

the $q(I)$ can be written as $q \cdot t(I)$. In a similar manner we define

$$s(ij)_\nu = \delta_{i\nu} - \delta_{j\nu} \tag{2.20}$$

which will be used to express $x_i - x_j$ as $s(ij) \cdot x$. Using the notation of (2.19), we can write the Fourier - Laplace transform of r as

$$\tilde{r}(v, \underset{\sim}{q}) = \sum_P \int dq^0 \tilde{w}_P(q)(2\pi)^{-n} \prod_{I \in \mathcal{J}(P)} \left[(v - q^0) \cdot t(I)\right]^{-1} \tag{2.21}$$

Here dq^0 is defined in an analogous way to dq in (2.10). The w_P in (2.15) and (2.21) can be replaced with the w_P^T as will be seen in Appendix B. Due to (W3), $\tilde{r}(v, \underset{\sim}{q})$ is analytic everywhere except at the cuts

$$\text{Im } v \cdot t(I) = 0, \quad \text{Re } v \cdot t(I) \geqslant \left(m^2 + (\underset{\sim}{q} \cdot t(I))^2\right)^{1/2} \tag{2.22}$$

If we fix the sign of every $\text{Im } v \cdot t(I)$, and let $\text{Im } v$ tends to zero, then $r(v, \underset{\sim}{q})$ approaches to one boundary value. Geometrically speaking, the family H_{n-1}^R of hyperplanes (in the space S of $\text{Im } v$) defined by

$$H_{n-1}^R = \left\{h(I) : I \subset I(n+1)\right\}, \quad h(I) = \left\{s ; s \cdot t(I) = 0\right\} \tag{2.23}$$

divide the entire space S into several convex polyhedral cones which we shall call C_i. If $\text{Im } v$ stays in the interior of one cone C_i, then the sign of Im $v \cdot t(I)$ stays constant, while if it moves from one cone to another the sign of some Im $v \cdot t(I)$ changes. Thus as Im v tends to zero from inside each cone C_i, $\tilde{r}(v,\underset{\sim}{q})$ approaches to one of its boundary values which we shall call $r_i(q)$. The $r_i(q)$ exhaust all boundary values of $\tilde{r}(v,\underset{\sim}{q})$. In particular we obtain the Fourier transform of the retarded function (2.15) as the boundary value corresponding to the cone Im $v \cdot t(I) \leqslant 0$ for $I = \{2\}, \{3\}, \ldots \{n+1\}$, i.e., for Im $v_i \leqslant 0$ for $i = 2, \ldots, n+1$.

We shall use the generalized θ-function:

$$\begin{aligned} \theta(t:c) &= 1 \quad \text{if } t \in C \\ &= 0 \quad \text{if } t \notin C \end{aligned} \tag{2.24}$$

If C is a pointed convex polyhedral cone[10] the Fourier-Laplace transform of θ,

$$\tilde{\theta}(v;C) = \int e^{iv \cdot t}\, \theta(t;C)\,dt \tag{2.25}$$

is a rational function of v. Its boundary value (considered as a distribution), as Im v tends to zero from within a cone C' of the space of Im v, is denoted by $\tilde{\theta}(s;C/C')$ and its inverse Fourier transform is denoted by $\theta(t;C/C')$. If C' is the positive polar[10] of C, then $\theta(t;C/C')$ is equal to $\theta(t;C)$. Similar definitions hold for $\theta(s;C)$, $\tilde{\theta}(u;C)$, $\tilde{\theta}(t;C/C')$ and $\theta(s;C/C')$. The properties of these functions will be studied in section 3.

As an example, let us consider the cones C_P in T defined by

$$C_P = \left\{x^0; x^0_{P(1)} \geqslant x^0_{P(2)} \geqslant \cdots \geqslant x^0_{P(n+1)}\right\} \tag{2.26}$$

Then (2.20) with w_P replaced by w_P^T can be written as

$$\tilde{r}(v, \underset{\sim}{q}) = \sum_P \int dq^0 \tilde{w}_P^T(q)\ \tilde{\theta}(v-q^0;\ C_P)(2\pi i)^{-n} \tag{2.27}$$

We remark that though the starting Eq. (2.14) referred to the Bose case, (2.27) is the appropriate definition of the retarded function for an arbitrary collection of local Bose and Fermi fields, i.e., theormes 1 and 2 below are true always.

Our first main Theorem lists the necessary and sufficient condition for the r_i to be obtainable from the w_P^T satisfying (W1) - (W3).

<u>Theorem 1.</u>[11] If $w_P^T(x)$ satisfies (W1) - (W3) then $r_i(x)$ defined by

$$r_i(x) = (-i)^n \sum_P \theta(x;\ C_P/C_i)\ w_P^T(x) \tag{2.28}$$

satisfies

(R1) $r_i(x)$ is a covariant function of $x \in X$.

(R2) $r_i(x) = 0$, if $x^0 \notin C_i^+$. (C^+ is the positive polar[10] of C.)

(R3) $\tilde{r}_i(q) = \tilde{r}_j(q)$, if dim $(C_i \cap C_j \cap h(I)) = n-1$[12] and $q(I)^2 < m^2$.

(R4) $r_{++}(x) - r_{+-}(x) - r_{-+}(x) + r_{--}(x) = 0$, if

$\dim(C_{++} \cap C_{+-} \cap C_{-+} \cap C_{--} \cap h(I) \cap h(I')) = n-2$[12]

$\sigma I \cap \sigma' I' \neq$ empty, $C_{\sigma\sigma'} \subset C(t(I))^{\sigma} \cap C(t(I'))^{\sigma'}$ $(\sigma, \sigma' = +$ or $-)$

Conversely if $r_i(x)$ satisfies (R1) - (R4), then $w_P^T(x)$ defined by

$$\widetilde{w}_P^T(q) = (i)^n \sum_i \theta(q^0; C_i/C_P)\widetilde{r}_i(q) \quad , \tag{2.29}$$

satisfies (W1) - (W3) and the original $r_i(x)$ is given by (2.28) in terms of this w_P^T.

Remarks: 1) Note that the conditions (R1), (R2), and (R3) in this theorem are almost dual to the conditions (W1), (W3), and (W2). In fact, (W2) can be rewritten in our notation as (W2') $w_P^T(x) = w_P^T(x)$ if $\dim(C_P \cap C_{P'} \cap h(ij)) = n - 1$[13] and if $(s(ij) \cdot x)^2 < 0$ where $h(ij)$ is the hyperplane orthogonal to $s(ij)$.

2) The support condition in x-space, (R2), expresses the retardedness in certain variables. Namely, if we denote the 1-facets[10] of C_i by $C(s_i^{\lambda})$, then (R2) is equivalent to

(R2') $r_i(x)=0$ unless $s_i^{\lambda} \cdot x \in \bar{V}_+$(the future light cone) for all λ.

Actually, r_i has in general more retardedness than (R2'), which, however, invariably contains alternative statements. This retardedness is, of course, implied by (R1) - (R4) but not immediately apparant.[14]

3) The condition (R4) has been first noted by Steinmann[4] for the four point function (n=3). The

intersection of two (n-1) planes[10] $h(I)$ and $h(I')$ $(I \neq I')$ is a (n-2)-plane.[10] This intersection is not contained in any other $h(I'')$ if and only if $\pm I$ and $\pm I'$ has non-empty intersection for any combination of the signs where we have denoted $I(n+1)-I$ by $-I$. If this is the case, the (n-2)-plane $h(I) \cap h(I')$ is divided into several polyhedral convex cones by $h(I'')(I'' \neq I, I')$ and corresponding to each of these cones, there are exactly 4 cones C_i which have that cone as a (n-2)-facet[10] and which are on different sides of (n-1)-planes $h(I)$ and $h(I')$. The condition (R4) gives a linear relation among the corresponding four r_i which are denoted by $r_{\sigma\sigma'}$ (σ, σ' = + or -).

Our second main task is to convert conditions (R1) - (R4) on r_i to a condition on the domain of analyticity of the analytic function in p-space. We have succeeded in this only for (R1) - (R3).

To state our result, we need further definitions. We define open convex cones V_i^Q in Q by

$$V_i^Q = \left\{q;\ q \cdot t(I) \in V_+,\ I \in \mathcal{I}_i\right\} \tag{2.30}$$

where V_+ is the interior of the future light cone and $\mathcal{I}_i$ is the set of $I \subset I(n+1)$ such that $C(t(I))$, $I \in \mathcal{I}_i$ constitute the 1-facets of C_i^+. (The $h(I)$, $I \in \mathcal{I}_i$ are boundary planes of C_i.) If C_i and C_j are neighbouring cones across the (n-1)-plane $h(I_0)$, (namely $\dim(C_i \cap C_j \cap h(I_0)) = n-1$), the interior of the set

$(\bar{V}_i^Q \cap \bar{V}_j^Q)$ is denoted by $S^Q(ij)$.

$$S^Q(ij) = \left\{ q;\ q \cdot t(I_0) = 0,\ q \cdot t(I) \in V_+ \text{ for } I \in \mathcal{I}_i \text{ or } \mathcal{I}_j \right.$$
$$\left. \text{and } I \neq I_0 \right\} \tag{2.31}$$

The tube $T(V_i^Q)$ is the subset of Z' defined by

$$T(V_i^Q) = \left\{ \zeta \in Z';\ \mathrm{Im}\ \zeta \in V_i^Q \right\} \tag{2.32}$$

The extended tube $T'(V_i^Q)$ is the union of images of $T(V_i^Q)$ under all complex proper Lorentz transformations. The corresponding definitions in X are

$$V_P^X = \left\{ x \in X;\ s(P(k), P(k+1)) \cdot x \in V_-,\ k=1, \ldots, n \right\} \tag{2.33}$$

$$S(P,k) = \left\{ x \in X;\ s(P(k), P(k+1)) \cdot x = 0, \right.$$
$$\left. s(P(m), P(m+1)) \cdot x \in V_- \text{ for } m \neq k \right\} \tag{2.34}$$

$$T(V_P^X) = \left\{ z \in Z;\ \mathrm{Im}\ z \in V_P^X \right\} \tag{2.35}$$

If the two cones C_P and $C_{P'}$ are neighbouring, namely if $P(i) = P'(i)$ for $i \neq k$, $k+1$ and $P(k) = P'(k+1)$, $P(k+1) = P'(k)$, then

$$S(P,k) = S(P',k) = \text{the interior of } \bar{V}_P^X \cap \bar{V}_{P'}^X. \tag{2.36}$$

We are now ready to state our second main Theorem.

<u>Theorem 2</u>. The $\tilde{r}_i(q)$ satisfying (R1) - (R3) are boundary values of one analytic function $\tilde{r}(\zeta)$ as ζ tends to q from inside the tube $T(V_i^Q)$. $\tilde{r}(\zeta)$ is analytic in the union of $T'(V_i^Q)$ for all possible i and in the sets

$$\Sigma(ij, m) = \left\{ \zeta \in Z';\ \mathrm{Im}\ \zeta \in S^Q(ij),\ (\mathrm{Re}\ \zeta \cdot t(I))^2 < m^2 \right\} \tag{2.37}$$

for all i, j, I such that C_i and C_j is neighbouring across $h(I)$. $\tilde{r}(\zeta)$ is analytic at a real point $\zeta = q$, if all $q(I)^2$ are smaller than m^2. Conversely if $r(\zeta)$ is analytic in the above region and has a certain boundedness property,[15] then its boundary values $\tilde{r}_i(q)$ satisfy (R2) and (R3).

This will be proved in section 6. For the sake of comparison we mention the corresponding Theorem for w_P^T.

Theorem 3. The $w_P^T(x)$ satisfying (W1) - (W3) are boundary values of one analytic function $w^T(z)$ as z tends to x from inside the tube $T(V_P^X)$. $w^T(z)$ is analytic in the union of $T'(V_P^X)$ for all possible P and in the sets

$$\Sigma(P,k) = \left\{ z \epsilon Z;\ \mathrm{Im}\ z \epsilon S(P,k),\ (\mathrm{Re}\ s(P(k),P(k+1)) \cdot x)^2 < 0 \right\} \tag{2.38}$$

$w^T(z)$ is analytic at a real point $z = x$ if all $s(ij) \cdot x$ are space-like. Conversely if $w^T(z)$ is analytic in the above region and satisfies a certain boundedness condition,[15] then its boundary values $w_P^T(x)$ satisfy (W2) and (W3).

Covariant formulas which express $\tilde{r}(\zeta)$ and $w(z)$ in terms of boundary values of the other are given by

$$\tilde{r}(\zeta) = (-i)^n \sum_{\alpha} \sum_{\nu} \int dx\ e^{i(\zeta, x)} \theta(x^0; C^X_{\alpha\nu} / \mathrm{Im}\,\zeta^0)\, \theta(x; \Delta^X_\alpha)\, w^T_{\alpha\nu}(x) \tag{2.39}$$

$$w^T(z) = (i)^n \sum_{\beta} \sum_{\nu} \int dq\ e^{-i(q,z)} \theta(q^0; C^Q_{\beta\nu} / \mathrm{Im}\ z^0)\, \theta(q; \Delta^Q_\beta(m))\, \tilde{r}_{\beta\nu}(q) \tag{2.40}$$

Here Δ_α^X and $\Delta_\beta^Q(m)$ designate various regions in X or Q where w(z) or $\tilde{r}(\zeta)$ have different number of boundary values. Namely we divide the space X into several Δ_α^X according to whether each $s(ij)\cdot x$ is space-like or time-like and the different regions are distinguished by subscript α. A similar definition holds for $\Delta_\beta^Q(m)$.

$$\Delta_\alpha^X = \left\{ x\epsilon X;\ \sigma_\alpha(ij)(s(ij)\cdot x)^2 > 0 \right\} \tag{2.41}$$

$$\Delta_\beta^Q(m) = \left\{ q\epsilon Q;\ \sigma_\beta(I)\ \left[(q\cdot t(I))^2 - m^2\right] > 0 \right\} \tag{2.42}$$

where σ_α and σ_β are + or −. For each region Δ_α^X vectors $(s(k\ell)\cdot x)$ with $\sigma_\alpha(k\ell) > 0$ can be either positive or negative time-like. To distinguish such possibilities we use the cones $C^X_{\alpha\nu}$ in T which are defined by

$$C^X_{\alpha\nu} = \left\{ t\epsilon T;\ (s(k\ell)\cdot t)\,\sigma_{\alpha\nu}(k\ell) > 0 \text{ for all } k,\ell \text{ such that } \sigma_\alpha(k\ell) > 0 \right\} \tag{2.43}$$

where as ν varies $\sigma_{\alpha\nu}(k\ell)$ exhaust all possibilities for consistent assignment of signs to $s(k\ell)\cdot t$. For example, if all $\sigma_\alpha(k\ell) > 0$, then $\{C_{\alpha\nu}\}$ coincides with $\{C_P\}$. In general, $C_{\alpha\nu}$ is a union of several C_P. C^Q_β are similarly defined and coincides with $\{C_i\}$ if $\sigma_\beta(I) > 0$ for all I. The summation over α in (2.39) extends over α such that the C_α are pointed. (In other words, if the $s(k\ell)$ for which $\sigma_\alpha(k\ell) > 0$ span S.) For each α, the summation over ν

extends over all possibilities. Similar prescription applies for the summations in (2.40). $\theta(x^0;C^x_{\alpha\nu}/\mathrm{Im}\,\zeta^0)$ is the $\theta(x^0;\ C^x_\alpha\ /C')$ where C' is determined by $\mathrm{Im}\,\zeta^0\epsilon C'$. It is invariant if $x\epsilon\,\Delta^X_\alpha$ and all $\mathrm{Im}\,\zeta\cdot t(I)$ are time- or light-like. $\theta(q^0;C^Q_{\beta\nu}/\mathrm{Im}\ z^0)$ is similarly defined.

w^T_α is the w^T_P with P such that $C_P\subset C^X_{\alpha\nu}$. Due to (W2), if x Δ^X_α, then the $w^T_P(x)$ are all equal for different P as long as C_P stays in one C^X_α. $r_{\beta\nu}$ is the r_i with i such that $C_i\subset C_{\beta\nu}$.

Finally we note that the vacuum expectation value of time ordered product, $\tau(x)$, and its Fourier transform $\tilde{\tau}(q)$ can be expressed as

$$\tau(x) = \lim_{z\to x,\ \mathrm{Im}\ z\epsilon V_T(x)} w(z) \tag{2.44}$$

$$(q) = \lim_{\zeta\to q\ \ \mathrm{Im}\,\zeta\,\epsilon\tilde{V}_T(q)} i^n\ r(\zeta) \tag{2.45}$$

where V_T and $\tilde{V}_T$ are defined by

$$V_T(x)=T(C_P),\ \text{if}\ x^0\epsilon C_P;\quad \tilde{V}_T(q)=T(C_i),\ \text{if}\ q^0\epsilon C_i \tag{2.46}$$

3. PROPERTIES OF GENERALIZED θ-FUNCTION

First let us consider the generalized θ-function defined by (2.24) for the special case of a simplex cone C.[10] Suppose 1-facets of C and C^+ are $t_1\ldots t_n$ and $s_1\ldots s_n$ where $s_i\cdot t_j=\delta_{ij}$. ($\det(t_i)\neq 0$.) Then we have

$$\theta(t;C) = \prod_{i=1}^{n} \theta(s_i \cdot t) \tag{3.1}$$

$$\tilde{\theta}(v;C) = i^n \left|\det(t_i)\right| \prod_{i=1}^{n} (v \cdot t_i)^{-1} \tag{3.2}$$

If we define associated simplex cones σC by

$$\sigma C = C(\sigma_1 t_1 \ldots \sigma_n t_n) \tag{3.3}$$

where the σ_i are ± 1, then, we have the formulas,

$$\tilde{\theta}(v;\sigma C) = \left(\prod_{i=1}^{n} \sigma_i\right)\tilde{\theta}(v;C) \tag{3.4}$$

$$\theta(t;C/\sigma C^+) = \left(\prod_{i=1}^{n} \sigma_i\right)\theta(t;\sigma C)^{16} \tag{3.5}$$

We note that the poles of $\tilde{\theta}(v;C)$ appears at $v \cdot t_i = 0$, $i = 1\ldots n$ and discontinuity of $\theta(t;C/\sigma C^+)$ appears at $s_i \cdot t=0$, $i=1\ldots n$.

We now turn to the case of general convex polyhedral cones C.

Lemma 1. Let C be a pointed polyhedral convex cone.[10] The integral in (2.25) defines an analytic function of v in the tube $T(C^+) = \{v; \mathrm{Im}\ v \in \text{interior of } C^+\}$ (which is non-empty). This analytic function is a rational function with simple poles at $v \cdot t=0$, for $t \in F_1(C)$ (the set of all 1-facets of C).

Lemma 2. (Addition theorem.) Let C and C_α be convex polyhedral cones such that C is the union of C_α and the C_α are mutually almost disjoint ($C=\bigcup_\alpha C_\alpha$, $\dim(C_\alpha \cap C_\beta) < n$ for $\alpha \neq \beta$). Then

$$\sum_\alpha \tilde\theta(v;C_\alpha)=\tilde\theta(v;C) \quad \text{if lin } C=0 \tag{3.6}$$

$$=0 \quad \text{if lin } C\neq 0 \quad \text{lin } C_\alpha=0 \tag{3.7}$$

$$\sum_\alpha \tilde\theta(u;C_\alpha^+)=\tilde\theta(u;C^+) \quad \text{if dim } C^+=n \tag{3.8}$$

$$=0 \quad \text{if dim } C_\alpha^+=n \text{ and dim } C^+\neq n \tag{3.9}$$

For the proof, we first note that if $v\epsilon T(C^+)$, then $\text{Im } v\cdot t>0$ for $t\epsilon C$ and as $t\to\infty$ within C, the integrand of (2.25) tends to zero exponentially. Hence it defines an analytic function of v. Next, we obviously have

$$\theta(t;C)=\sum_\alpha \theta(t;C_\alpha) \quad \text{almost everywhere} \tag{3.10}$$

Because $C_\alpha^+\supset C^+$ and C^+ is non-empty, the integral representation (2.25) can be applied to all $\tilde\theta(v;C_\alpha)$ and $\tilde\theta(v;C)$ if $v\epsilon T(C^+)$. Hence we get (3.6) from (3.10) as a relation between analytic functions. To prove that $\tilde\theta(v;C)$ is rational, we invoke the simplexial decomposition of C: $C=\bigcup_\alpha C_\alpha$. We already know that, for simplex cones C_α, $\tilde\theta(v;C_\alpha)$ is rational. Hence $\tilde\theta(v;C)$ is also rational by (3.6). Moreover, because $F_1(C_\alpha)\subset F_1(C)$ for <u>standard</u> simplexial decomposition and the latter is possible if lin C=0,[17] we see that the singularities of $\tilde\theta(v;C)$ occur only at $v\cdot t=0$, $t\epsilon F_1(C)$.

To prove (3.7), we first consider a special case where $C=\bigcup_\sigma C_\sigma$, $C_\sigma=C(\sigma_1 t_1\ldots\sigma_m t_m, t_{m+1}\ldots t_n)$, $\dim C_\sigma=n$, and $\sigma_i=\pm 1$. Since C_σ is simplex, we easily get (3.7) from (3.4). Using this result, we make generalizations in two steps.

First consider the case where $C = \bigcup_\sigma C_\sigma$, $C_\sigma = C(T_0 \cup T_\sigma)$, $\dim C(T_0) = n-m$, $\mathrm{lin}\, C(T_0) = 0$, $T_\sigma = \{\sigma_1 t_1 \ldots \sigma_m t_m\}$, $\sigma_i = \pm 1$ and $\dim C_\sigma = n$. We make a simplexial decomposition of $C(T_0)$ in $h(T_0)$: $C(T_0) = \bigcup_\beta C(T_\beta)$. Setting $C_{\beta\sigma} = C(T_\beta \cup T_\sigma)$ and $C_\beta = \bigcup_\sigma C_{\beta\sigma}$, and using (3.6) for $C_\sigma = \bigcup_\beta C_{\beta\sigma}$ and (3.7) for $C_\beta = \bigcup_\sigma C_{\beta\sigma}$, we have $\sum_\sigma \tilde{\theta}(v;C_\sigma) = \sum_\beta(\sum_\sigma \tilde{\theta}(v;C_{\beta\sigma})) = 0$. Finally for the most general case, let $C = \bigcup_\alpha C_\alpha$, $\mathrm{lin}\, C = m$, $L(C) = h(\Sigma)$, and $\Sigma = \{s_1 \ldots s_m\}$. Let $\Sigma_\sigma = \{\sigma_1 s_1 \ldots \sigma_m s_m\}$, $C_\sigma = C(\Sigma_\sigma)^+ \cap C$, and $C_{\alpha\sigma} = C_\alpha \cap C_\sigma$. Since $\mathrm{lin}\, C_\sigma = 0$ by construction, we have $\tilde{\theta}(v;C_\sigma) = \sum_\alpha \tilde{\theta}(v;C_{\alpha\sigma})$ due to (3.6). Since $\mathrm{lin}\, C_\alpha = 0$ by assumption, we have $\tilde{\theta}(v;C_\alpha) = \sum_\alpha \theta(v;C_{\alpha\sigma})$. By the (3.7) for the previously proved case, we have $\sum\tilde{\theta}(v;C_\sigma) = 0$. Combining these, we get the (3.7) for the most general case.

(3.8) and (3.9) can be proved at the same time. (If $\dim C \neq n$, $\tilde{\theta}(u;C) = 0$). First consider a special case where $C = C_1 \cup C_2$. $C_1 = C \cap C(-s)^+$, $C_2 = C \cap C(s)^+$, and s, $-s \notin C.^+$ The (n-1)-planes in $H_{n-1}(C^+)$[18] divides C_1^+ and C_2^+ into several convex cones. Let this decomposition be $C_1^+ = C^+ \cup (\bigcup_\alpha C_\alpha^+)$ and $C_2^+ = C^+ \cup (\bigcup_\beta C_\beta^+)$. Since C_1^+ and C_2^+ are pointed, we have from (3.6) $\tilde{\theta}(u;C_1^+) = \tilde{\theta}(u;C^+) + \sum\tilde{\theta}(u;C_\alpha^+)$ and $\tilde{\theta}(u;C_2{}^+) = \tilde{\theta}(u;C^+) + \sum\tilde{\theta}(u;C_\beta{}^+)$ Since $C_1{}^+$ $C_2{}^+$ is not pointed, we have from (3.7) $\tilde{\theta}(u;C^+) + \sum\tilde{\theta}(u;C_\alpha{}^+) + \sum\tilde{\theta}(u;C_\beta{}^+) = 0$. Hence we get (3.8) and (3.9), for this case. Next consider the case where C is cut into several C_α by a family of planes $h(s)^\perp$, $s \in S_0$.

By applying the previous result, every time one cuts C by a $h(s)^{\perp}$, one gets the (3.8) or (3.9) for $C=\bigcup C_\alpha$. Finally, consider the most general case $C=\bigcup C_\alpha$. The (n-1)-planes in $\bigcup_\alpha H_{n-1}(C_\alpha)$ cut C and C_α into several convex cones. Let this decomposition be $C_\alpha=\bigcup_i C_{\alpha i}$ and $C=\bigcup_{\alpha i} C_{\alpha i}$. Then by applying the previous result for C_α and C, we get (3.8) and (3.9). This completes the proof of Lemas 1 and 2.

Next we investigate the residue of $\tilde{\theta}$ at its pole. We define

$$R(v;C/t)=\lim_{v'\to v} v'\cdot t\,\tilde{\theta}(v';C) \qquad v\cdot t=0 \tag{3.11}$$

$$R(v;C/t_1\ldots t_m)=\lim_{v'\to v} v'\cdot t_m\, R(v';C/t_1\ldots t_{m-1}),$$

$$v\cdot t_i=0,\ i=1\ldots m \tag{3.12}$$

<u>Lemma 3</u>. $R(V;C/t_1)=i\epsilon(C;f_1)\tilde{\theta}_1(v;C_1{}')$ (3.13)

$$R(v;C/t_1\ldots t_n)=i^n \det(t_i) \prod_{m=1}^{n}\epsilon(C_{m-1};f_m) \tag{3.14}$$

where $C_m=C+h(t_1\ldots t_m)$, $C_0=C$, $f_m=C(t_m)+h(t_1\ldots t_{m-1})$,

$$\begin{aligned}\epsilon(C;f)&=1 \quad \text{if } f\epsilon F_m(C)^{17} \quad \text{for some } m\\ &=-1 \text{ if } -f\epsilon F_m(C) \quad \text{for some } m,\\ &=0 \text{ otherwise,}\end{aligned} \tag{3.15}$$

$\tilde{\theta}_1$ is the $\tilde{\theta}$ where the space T mod $h(t_1)$ is used instead of T and $h(t_1)^{\perp}$ instead of V. The volume element of T mod $h(t_1)$ in the definition of $\tilde{\theta}_1$ is so chosen that, if t_1, $t_2{}'\ldots t_n{}'$ span a parallelepiped of unit volume, $t_2{}'\ldots t_n{}'$ span the same in T mod $h(t_1)$.

To prove (3.13), we note that $R(v;C/t_1)$ is a rational function of v in $h(t_1)^{\perp}$ in V. We can calculate R by

$$-2\pi i\delta(v \cdot t_1)\ R(v;C/t_1)=\lim_{\epsilon\to 0}\left[\tilde{\theta}(v+i\epsilon;C)-\tilde{\theta}(v-i\epsilon;C)\right] \tag{3.16}$$

where $\mathrm{Im}\ v \cdot t_1=0$ and $\epsilon \cdot t_1>0$. From Lemma 1, we have

$$R(v;C/t_1)=0 \quad \text{unless } t_1 \text{ or } -t_1\epsilon F_1(C) \tag{3.17}$$

Suppose $C(\sigma t_1)\epsilon F_1(C)$ $(\sigma=\pm)$. Due to (3.17) and the addition theorem (3.6), we can adjoin to C or cut off from C any convex cones whose 1-facets do not contain $\pm t_1$ without changing $R(v;C/t_1)$. By this process, we can shift all (n-1)-facets of C not containing $\pm t_1$, to one facet f. Suppose $f\perp s$ and $s \cdot t_1>0$. Denoting $C_1=C+h(t_1)$, $C'=C_1\cap C(s)^+$, $C''=C_1\cap C(-s)^+$, we have

$$R(v;C/t_1)=R(v;C'/t_1)$$

On the other hand we know from (3.7) that $\tilde{\theta}(v-i\sigma\epsilon;C')$ $=-\tilde{\theta}(v-i\sigma\epsilon;C'')$. From these we obtain

$$-2\pi i\delta(v \cdot t_1)R(v;C/t_1)=\lim_{0}\sigma\left[\tilde{\theta}(v+i\sigma\epsilon;C')+\tilde{\theta}(v-i\sigma\epsilon;C'')\right] \tag{3.18}$$

Since R is rational function, we can easily find an open set O (relative to $h(t_1)^{\perp}$) in domain of analyticity of R and ϵ satisfying $\epsilon \cdot t_1>0$, such that $\sigma\epsilon+\mathrm{Im}\ v\epsilon C'$ and $-\sigma\epsilon+\mathrm{Im}\ v\epsilon C''^{+}$ when $v\epsilon O$. We can use the integral representation for both $\tilde{\theta}$ in (3.18) for such v and ϵ, and we obtain

$$-2\pi i\delta(v \cdot t_1)R(v;C/t_1) = \int e^{iv \cdot t}\,\theta(t;C_1)dt$$

Thus (3.13) is true for $v \in 0$. Since both sides of (3.13) are rational, it holds everywhere.

By repeated application of (3.13), we obtain

$$\left(\prod_{m=1}^{n} 2\pi\delta(s \cdot t_m)\right)R(v;C/t_1 \ldots t_n) = i^n \prod_{m=1}^{n} \epsilon(C_{m-1}/v_m) \int e^{is \cdot t} dt.$$

which implies (3.14). This completes the proof of Lemma 3.

We now discuss the boundary values of $\tilde{\theta}$.

Lemma 4. The boundary value of $\tilde{\theta}$

$$\tilde{\theta}(s;C/s') = \lim_{k \to +0} \tilde{\theta}(s+iks';C) \tag{3.19}$$

is the same for all s' in the interior of any one cone C' of $\Gamma(H_{n-1}(C^+))$.[19]

This is obvious if one recalls that $\tilde{\theta}(v;C)$ is rational and its poles appear only when Im v is on one of planes in $H_{n-1}(C^+) = H_1(C)^{\perp}$.

This justifies the notation $\tilde{\theta}(s;C/C')$ instead of $\tilde{\theta}(s;C/s')$ as long as C' is a cone of $\Gamma(H_{n-1}(C^+))$ or contained in such a cone.

Lemma 5. The Fourier transform of $\tilde{\theta}(s;C/C')$,

$$\theta(t;C/C') = \int e^{-is \cdot t}\,\theta(s;C/C')ds\,(2\pi)^{-n} \tag{3.20}$$

is a function taking integral values (almost everywhere) and with discontinuities only at planes belonging to $H_{n-1}(C)$. Furthermore

$$\theta(t;C/C')=0 \qquad \text{if } t\notin C'^{+} \tag{3.21}$$

$$\theta(t;C/s')=0 \qquad \text{if } t\in \text{interior of } C \text{ and } s'\notin C^{+} \tag{3.22}$$

To prove the first part of the Lemma, we note that this is true for simplex C (cf. (3.5)). For arbitrary C, we see by a simplexial decomposition $C=\Sigma C_\alpha$, that discontinuities occur on (n-1)-planes. Furthermore, if a (n-1)-plane $h\notin H_{n-1}(C)$, then by Lemma C2, we can make this decomposition in such a way that $h\notin H_{n-1}(C_\alpha)$ for any α. Hence discontinuities occur only on planes of $H_{n-1}(C)$.

To prove (3.21),[20] we note that if $t\notin C'^{+}$ then there is a $s_1\in C'$ such that $s_1\cdot t<0$. Using a basis $s_1\ldots s_n$ in S,

$$\theta(t;C/C')=\int e^{-i\Sigma\rho_j s_j\cdot t}\left|\det(s_j)\right|\widetilde{\theta}(\Sigma\rho_j s_j;C/C')\prod d\rho_j$$

Since $\widetilde{\theta}$ is analytic for $\mathrm{Im}\,\rho_1>0$ with fixed real ρ_j, $j\geqslant 2$, we have (3.21) by contour deformation in the ρ_1-integration.

To prove (3.22), we make a simplexial decomposition of C^+: $C^+=\bigcup C_\alpha{}^+$. Obviously $s'\notin C_\alpha{}^+$. Since $\widetilde{\theta}(v;C)=\Sigma\widetilde{\theta}(v;C_\alpha)$ due to (3.8), we have $\theta(t;C/s')=\Sigma\theta(t;C_\alpha/s')$. If s' happens to be on some plane of $H_{n-1}(C^+)$ there is always another s" near s' which is not on any plane of $H_{n-1}(C_\alpha{}^+)$ nor in C^+ and satisfies $\theta(t;C/s')=\theta(t;C/s'')$. ($s'\notin H_{n-1}(C^+)$.) For simplex C_α, we see from (3.5) that $\theta(t;C/s')=0$ if $t\in C\subset C_\alpha$ and $s''\notin C_\alpha{}^+$. Hence we have (3.22).

Finally we prove the following inversion formula,

<u>Lemma 6</u>. If dim C=n, lin C=0 and $H\supset H_{n-1}(C^+)$, then

$$\sum_{C'\in\Gamma(H)}\widetilde{\theta}(v;C')\theta(t;C/C')=\widetilde{\theta}(v;C^+) \tag{3.23}$$

To prove this, we first consider the case where C is simplex. Since $H \subset H_{n-1}(C^+)$, each $C' \in \Gamma(H)$ is contained in some $C_\sigma{}^+$. By Lemma 4, (3.6), (3.4) and (3.5), we obtain

$$\begin{aligned}\sum_{C'} \tilde{\theta}(v;C')\theta(t;C/C') &= \sum_{\sigma}\Big(\sum_{C' \subset C_\sigma{}^+} \tilde{\theta}(v;C')\theta(t;C/C_\sigma{}^+)\Big) \\ &= \sum_{\sigma} \tilde{\theta}(v;C_\sigma{}^+)\theta(t;C/C_\sigma{}^+) \\ &= \sum_{\sigma} \tilde{\theta}(v;C^+)\theta(t;C_\sigma) = \tilde{\theta}(v;C^+)\end{aligned}$$

For general C, we make a standard simplexial decomposition $C = \bigcup_\alpha C_\alpha$. Since $H_{n-1}(C_\alpha{}^+) \subset H_{n-1}(C^+)$, we can use (3.23) for every C_α. Hence by using (3.6) and (3.8), we obtain (3.23) for the general case.

4. THE NECESSITY PROOF OF THEOREM 1

To prove (R1), we rewrite definition (2.28) in a form similar to (2.39). Namely, using the notation (2.41) - (2.43), we see that $C^X_{\alpha\nu}$ is sum of several C_P. Moreover, due to (W2), if $x \in \Delta^X_\alpha$ then the $w_P(x)$ are equal for various P as far as C_P stays in one C^X_α. Hence using (3.6), we get

$$r_i(x) = (-i)^n \sum_\alpha \theta(x;\Delta^X_\alpha) \sum_\nu \theta(x^0;C^X_{\alpha\nu}/C_i) w^T_{\alpha\nu}(x). \qquad (4.1)$$

$\theta(x^0;C^X_{\alpha\nu}/C_i)$ is invariant as long as $x \in \Delta^X_\alpha$ because its discontinuity occurs only at $s(k\ell) \cdot x^0 = 0$ with k, ℓ such that $(s(k\ell) \cdot x)^2 > 0$ and otherwise it stays constant. Since $w^T_{\alpha\nu}(x)$ is covariant due to (W1) for w^T_P, we have (R1).

(R2) is an obvious consequence of (3.21).

To prove (R3), we note that the difference $\theta(s;C_P/C_i)-\theta(s;C_P/C_j)$ for neighbouring C_i and C_j is the boundary value of $R(v;C_P/t(I))$ multiplied by $\pm 2\pi i\delta(s\cdot t(I))$ (cf. (3.16)). Hence, due to (3.13), only terms with those P for which $\pm C(t(I))\epsilon F_1(C_P)$ survive and, due to the presence of the above δ-function and (W3), w_P^T vanishes if $\pm q\cdot t(I)$ is one of its intermediate momentum. (Note that $(q(I))^2 < m^2$.) Since $\pm C(t(I))\epsilon F_1(C_P)$ implies that $\pm q(I)$ is an intermediate momentum of w_P^T, we have (R3).

To prove (R4), we first note that, since $I(P,k)$, $k=1\ldots n$ is totally ordered by set inclusion, if $\sigma I \wedge \sigma' I' \neq$ empty then $\pm q(I)$ and $\pm q(I')$ can not be intermediate state for one w_P^T simultaneously. Thus by Lemma 4

$$\theta(x;C_P/C_{+\sigma'})=\theta(x;C_P/C_{-\sigma'}) \quad \text{if } \pm C(t(I))\notin F_1(C_P)$$

$$\theta(x;C_P/C_{\sigma+})=\theta(x;C_P/C_{\sigma-}) \quad \text{if } \pm C(t(I'))\notin F_1(C_P)$$

Since one of these equalities is true for each C_P, we have (R4).

5. THE SUFFICIENCY PROOF OF THEOREM 1

First let us show that if r_i is obtained from w_P^T as in (2.28), then we get the w_P^T by (2.29). Namely we define

$$\tilde{w}^T(q/t)=(i)^n\sum_i\theta(q^0;C_i/t)\tilde{r}_i(q) \tag{5.1}$$

Then by substituting (2.28) into (5.1) we have

$$\tilde{w}^T(q/t)=\sum_P\int e^{iq\,x}\,w_P^T(x)\,(\sum_i\theta(q^0;C_i/t)\theta(x^0;C_P/C_i)dx$$

By (3.23) the summation in the parenthesis is equal to $\theta(q^0;C_P^+/t)$. (Note that $\{C_i\}=\Gamma(H_{n-1}^R)$ and $H_{n-1}^R=\bigcup_P H_{n-1}(C_P^+)\supset H_{n-1}(C_P^+)$.) We now have

$$\widetilde{w}^T(q/t)=\sum_P \theta(q^0;C_P^+/t)\widetilde{w}_P^T(q)$$

By (W2) $\widetilde{w}_P^T(q)=0$ if q^0 is not in the interior of C_P^+. If q^0 belongs to the interior of C_P^+ and $t\notin C_P$, then by (3.22) $\theta(q^0;C_P^+/t)=0$. If q^0 is in the interior of C_P^+ and $t\in C_P$, then $\theta(q^0;C_P^+/t)=\theta(q^0;C_P^+)=1$. Thus we have

$$\widetilde{w}^T(q/t)=\widetilde{w}_P^T(q) \quad \text{if } t\in C_P \tag{5.2}$$

We now assume (R1) - (R4) for $r_i(x)$ and define

$$w^T(u;\underset{\sim}{x})=(-2\pi i)^{-n}\int dx^0\sum_i\widetilde{\theta}(u-x^0;C_i)r_i(x) \tag{5.3}$$

$$w^T(x/t)=\lim_{\lambda\to+0} w^T(x^0+i\lambda t;\underset{\sim}{x}) \tag{5.4}$$

We denote $\bigcup_i H_1(C_i)^{\perp}=\bigcup_i H_{n-1}(C_i^+)$ by H_{n-1}^{RW} and $\bigcup_P H_{n-1}(C_P)$ by H_{n-1}^W. We easily see that $H_{n-1}^W\subset H_{n-1}^{RW}$ and in fact H_{n-1}^{RW} is much larger set than H_{n-1}^W in general.

If we denote the cones in $\Gamma(H_{n-1}^{RW})$ by $C_{P\gamma}$ where $C_P=\bigcup_\gamma C_{P\gamma}$ and the $w^T(x/t)$ with t in the interior of $C_{P\gamma}$ by $w_{P\gamma}^T(x)$, then by Lemma 4, $W_{P\gamma}^T(x)$ is independent of the choice of t in $C_{P\gamma}$. However, it depends on γ in general.

By Lemma 1, $w^T(u,\underset{\sim}{x})$ has singularities for Im $u\in h\subset H_{n-1}^{RW}$ in general. Hence in order to be able to define w_P^T from $w^T(u,\underset{\sim}{x})$, we have to show that the jump across the cut on

Im $u\epsilon h$ for $w^T(u,\underset{\sim}{x})$ vanishes if $h\epsilon H^{RW}_{n-1}$ and $h\notin H^{W}_{n-1}$. (The $w^T(u,\underset{\sim}{x})$ constructed from r_i of the form (2.28) is regular there.) This follows from (R4) in the following way.

By (3.13), what we have to show is

$$\sum_i \int R(u-x^0;C_i/s)\,\delta(s\cdot(u-x^0))\,r_i(x)\,dx^0=0 \tag{5.5}$$

for Im $s\cdot u=0$, $h(s)^{\perp}\epsilon H^{RW}_{n-1}$ and $h(s)^{\perp}\notin H^{W}_{n-1}$. This is equivalent to

$$\sum_{i=1}^{n} \epsilon((C_i)_{m-1};f_m)\,r_i(x)=0 \tag{5.6}$$

for all $s_2 \ldots s_n$, where $(C_i)_m=C_i+h_m$, $f_m=C(s_m)+h_{m-1}$, and $h_m=h(s,s_2\ldots s_m)$. The necessity follows from (3.14). For the sufficiency proof, we expand the rational function $R(u;C_i/s)$ into partial fractions first with respect to u_1 (the first component of u). Each expansion coefficient is the residue of R at the pole of that partial fraction and is a rational function of u given by some $R(u;C/s,s_2)$. Repeating this process, we arrive at a formula of the type[21]

$$R(u;C_i/s)=\text{const.}\Sigma\Big(\prod_{m=1}^{n} \epsilon((C_i)_{m-1};f_m)\Big)R(u)$$

where f_m and $(C_i)_{m-1}$ are defined as in Lemma 3, the summation is over $s_2 \ldots s_n$ and $R(u)$ is a rational function of u depending on s, $s_2 \ldots s_n$. By substituting this into (5.5) and using (5.6), we see the sufficiency of (5.5).

Next we prove (5.6) from (R4). In (5.6), if $h_m \notin H_m(C_i)$, for all i and for one fixed m, then all ϵ vanishes and the equation is satisfied. Hence we now assume that $h_m \epsilon H_m(C_i)$ for some i, namely that h_m is a m-dimensional intersection of planes $h(I_1)\ldots h(I_{n-m})$.

We first show that there is one and only one C_i for a given σ_m, m=1...n such that

$$\epsilon((C_i)_{m-1}; f_m) = \sigma_m \tag{5.7}$$

where $\sigma_m = \pm 1$. If this is true then denoting the corresponding r_i by r_σ, we can rewrite (5.6) as

$$\sum_{\sigma_1 \ldots \sigma_n} \sigma_1 \ldots \sigma_n r_\sigma = 0 \tag{5.8}$$

To prove the above statement, we note that each h_m is divided by planes h(I) not containing h_m into several (closed) convex polyhedral cones, say $C_\alpha^{(m)}$. For each $C_\alpha^{(m)}$, there is at least one cone C_i for which $C_i \cap h_m = C_\alpha^{(m)}$. Furthermore, each h_m is divided by h_{m-1} into two sides: $h_m = h_m^+ \cup h_{m-1} \cup h_m^-$, where $\pm s_m \epsilon h_m^\pm$. For each $C_\alpha^{(m-1)}$ there are just two $C_\beta^{(m)}$ containing $C_\alpha^{(m-1)}$ (in its boundary), one on each side of h_{m-1}. Hence by induction we obtain the above statement. (Note that $C_\alpha^{(n)}$ coincides with C_i.) We also see that $C_\alpha^{(m)}$ can be characterized by the value σk, $k \leq m$. Hence we use the notation $C^{(m)}(\sigma_1 \ldots \sigma_m)$.

Next let us investigate h_m more closely. If I_a and I_b are proper non-empty subsets of I(n+1), and if

$\sigma_a I_a \neq \sigma_b I_b$ for $\sigma_a, \sigma_b = \pm$, then there are 5 mutually exclusive possibilities: (α1) $I_a \cap I_b$=empty, (α2) $I_a \subset I_b$ (α3) $I_a \supset I_b$, (α4) $I_a \cup I_b = I(n+1)$ or (β) $\sigma_a I_a \cap \sigma_b I_b$=non-empty for $\sigma_a, \sigma_b = \pm$ We now prove that there exists integers k and ℓ ($\ell < k < n$) and the set $\{I_\nu^{(m)}; \nu \leq m \leq k\}$ satisfying the conditions: (A1) $h_{n-m} = \bigcap_{\nu=1}^{m} h(I_\nu^{(m)})$, (A2) $I_\nu^{(m)}$ is a partial sum of $I_\mu^{(m')}$, $\mu = \nu \ldots m'$ where $m < m'$, (A3) $I_a = I_\mu^{(m)}$ and $I_b = I_\nu^{(m)}$ satisfy (α1) for $\mu, \nu < k$ and (α1) or (β) for $m = \nu = k$. In the latter case, (β) holds for $\mu = \ell$.

Suppose $I_\nu^{(m)}$ has been defined for $m < M$ satisfying (A1), (A2) and the condition (A3'): $I_\mu^{(m)}$ and $I_\nu^{(m)}$ fulfil (β). Then we will construct $I_\nu^{(M)}$ which satisfy (A1), (A2) and either (A3) or (A3'). If this can be done, then by induction there is some M=k for which (A3) is true for the first time or else we find mutually disjoint $I_\nu^{(n-1)}$ such that $h_1 = \bigcup_\nu h(I_\nu^{(n-1)})$. This latter possibility contradicts $h_1 \notin H_{n-1}^W$. To construct $I_\nu^{(M)}$, let $h_{n-M} = h_{n-M+1} \cup h(I)$. If $I \supset I_\mu^{(M-1)}$, we replace I by $I' = I - I_\mu^{(M-1)}$. After doing this replacement for each μ, I' and $I_\mu^{(M-1)}$ never satisfy (α3) nor (α4). (If M=2, (α4) may happen, but then we replace I by -I without harming other conditions.) Now if $I' \subset I_\mu^{(M-1)}$ (which happens only for one μ), we define $I_\nu^{(M)} = I_\nu^{(M-1)}$ for $\nu \neq \mu$, $I_\mu^{(M)} = I_\mu^{(M-1)} - I'$, and $I_M^{(M)} = I'$ and they will satisfy (A1), (A2) and (A3'). Otherwise we define $I_\mu^{(M)} = I_\mu^{(M-1)}$ and $I_M^{(M)} = I'$, and they will satisfy (A1), (A2) and (A3) or (A3').

We now claim that

$$\sum_{\sigma_{n-k+1},\,\sigma_{n-\ell+1}} \sigma_{n-k+1}\,\sigma_{n-\ell+1}\, r_\sigma = 0 \qquad (5.9)$$

To prove this, we consider an inner point P of $C^{(n-k)}_{\sigma_1\cdots\sigma_{n-k}}$ in h_{n-k}. In the neighbourhood of P, there are no planes h(I) except those containing h_{n-k}. We define the point

$$P(\epsilon_{n-k+1}\cdots\epsilon_m)=P(\epsilon_{n-k+1}\cdots\epsilon_{m-1})+\epsilon_m s'_m \qquad (5.10)$$

where $s'_m=s_m$ except $s'_{n-\ell+1}$ is chosen to satisfy $s'_{n-\ell+1}\cdot t(I_\mu^{(k)})=0$ for $\mu\neq\ell$ and $s'_{n-\ell+1}\in h^+_{n-\ell+1}$. Obviously $P(\epsilon_{n-k+1}\cdots\epsilon_m)\in h_m$. If we choose ϵ_m successively smaller enough, and if (sign ϵ_m) $=\sigma_m$, then $P(\ldots\epsilon_m)$ will be in the relative interior of $C_{\sigma_1\cdots\sigma_m}$. We now fix ϵ_m so that the point

$$P(\rho,\rho')=P(\rho\epsilon_{n-k+1}\cdots\rho'\epsilon_{n-\ell+1}\cdots\epsilon_n) \qquad (5.11)$$

is in the interior of $C_{\sigma_1\cdots\sigma_n}$ for $\rho=\sigma_{n-k+1}$, and $\rho'=\sigma_{n-\ell+1}$. We also define $C(\rho\rho')=C_i$, $r(\rho\rho')=r_i$ if $P(\rho\rho')$ is in the interior of C_i. We now prove that $r_{\rho+}-r_{\rho-}$ is constant in ρ. This will prove (5.9).

For this purpose, we consider the segment $L_+=\{P(\rho,+1);\,|\rho|\leq 1$ and $L_-=\{P(\rho,-1);\,|\rho|\leq 1\}$ and consider the question: where L_+ and L_- meet the boundaries of C_i? Since L_+ and L_- are parallel to $s_{n-k+1}\in h_{n-k+1}$, they will never meet planes containing h_{n-k+1}, namely planes h(I) where I is any partial sum of $I_\mu^{(k-1)}$. On the other

hand, if the ϵ are sufficiently small, $L_{\pm}$ are near P and will never meet with planes not containing h_{n-k}. Thus, the only planes $h(I)$ which $L_{\pm}$ meets are for $I=I_k^{(k)}+\sum I_\mu^{(k)}$ 22
where the summation is any partial sum of $I_\mu^{(k)}$ such that $I_k^{(k)}$ and $I_\mu^{(k)}$ have the property (α1). L_+ and L_- may meet more than one planes $h(I)$ at one time. In such a case we change the choice of ϵ slightly and then $I_\mu^{(k)}$ will meet only one plane at a time. Since $s_{n-\ell+1}\cdot t(I_\mu^{(k)})=0$ for $\mu\neq\ell$, and $\mu=\ell$ does not appear in the summation in the definition of I, L_+ and L_- meet $h(I)$ at the same time.

For each fixed I, we fix ρ^+ and ρ^- such that $P(\rho^\sigma,\sigma')$ is on the same side of $h(I)$ as $P(\sigma,\sigma')$ and sufficiently near to $h(I)$. We now prove

$$r(\rho^+,+1)-r(\rho^+,-1)=r(\rho^-,+1)-r(\rho^-,-1)$$

by proving that $r(\rho^+,\rho')-r(\rho^-,\rho')$ is constant in $\rho'(\,|\rho'|\leq 1)$.

Let the segment $\{P(\rho^\sigma,\rho');\ |\rho'|\leq 1\}$ be L'_σ. We investigate planes $h(I')$ which L'_σ meets. Since the L'_σ are near P, $h(I')$ should contain h_{n-k}. Since L'_σ are parallel to $s_{n-\ell+1}$ and since $s_{n-\ell+1}\cdot t(I_\mu^{(k)})=0$ for $\mu\neq\ell$, I cannot be a partial sum of $I_\mu^{(k)}$, $\mu\neq\ell$. Hence $I'=I_\ell^{(k)}+\sum I_\mu^{(k)}$ where summation is any partial sum of $I_\mu^{(k)}$ with $\mu\neq\ell,k$. Suppose $P(\rho^\sigma,\rho'^{\sigma'})$ is sufficiently near to $h(I')$ and on the same side of $h(I')$ as $P(\rho^\sigma,\sigma')$. Then what we would like to prove is

$$r(\rho^+, \rho'^+)-r(\rho^-, \rho'^+)=r(\rho^+, \rho'^-)-r(\rho^-, \rho'^-)$$

Because I and I' satisfies (β), this is nothing but (R4). Thus we have succeeded in proving that $w^T(u;\underset{\sim}{x})$ has no cut across the plane Im $s \cdot u=0$ unless $h(s)^{\perp} \in H^W_{n-1}$.

We now prove the properties (W1) - (W3) for w^T. First (W1) becomes obvious if we write $\tilde{w}^T_P(q)$ as

$$\tilde{w}^T_P(q)=i^n\sum_\beta\theta(q;\Delta^Q_\beta(m))(\sum_\nu\theta(q^0;C^Q_{\beta\nu}/t)\tilde{r}_{\beta\nu}(q)) \quad t\epsilon C_P \quad (5.12)$$

where notations are as in (2.40) and the proof is similar to that of (4.1).

To prove (W2) or equivalently (W2'), we calculate by Lemma 3 the jump of $w^T(u;\underset{\sim}{x})$ across the cut Im $s \cdot u=0$,

$$i^n\sum_i\int R(u-x^0;C_i/s(\mu\nu))\delta(s(\mu\nu)\cdot(u-x^0))r_i(x)dx^0, \quad \text{Im } s(\mu\nu)\cdot u=0 \quad (5.13)$$

If $\pm s(\mu\nu)$ is not a 1-facet of C_i, then R vanishes. If $\pm s(\mu\nu)$ is a 1-facet of C_i and if $(\text{Re } s(\mu\nu)\cdot x)^2 < 0$, $r_i(x)$ vanishes because of (R2'). Thus (5.13) vanishes if $(\text{Re } s(\mu\nu)\cdot u)^2 < 0$, which proves (W2).

To prove (W3), we first note that if $q^0 \notin C_P{}^+$, then $q^0 \notin C^+_{P\gamma}$ for at least one γ, and therefore $\tilde{w}^T_P(q)=\tilde{w}^T_{P\gamma}(q)$ vanishes because each $\theta(q^0;C_i/C_{P\gamma})$ vanishes due to (3.21). Suppose $q^0 \epsilon C^+_P$ and $(q\cdot t(I))^2 < m^2$ for at least one $I \epsilon \mathcal{J}_P$. We will prove $\tilde{w}^T_P(q)=0$ for this case by using the following Lemma,

Lemma 7. If $(q \cdot t(I))^2 < m^2$ for one $I \in \mathcal{J}_P$ and $\Delta_\beta^Q(m)$ contains q, then each cone $C_{\beta\nu}^Q$ contains points outside of C_P^+.

If this Lemma is true, then for any point $q^0 \epsilon C_P^+$ there is a point $q^{0\prime}$ outside the cone C_P^+ which can be connected with q^0 by a continuous line without crossing boundary planes of any $C_{\beta\nu}^Q$. For such a $q^{0\prime}$, $\theta(q^0; C_{\beta\nu}^Q/C_{P\gamma}) = \theta(q^{0\prime}; C_{\beta\nu}^Q/C_{P\gamma})$ by Lemma 4. Since $\theta(q^{0\prime}; C_{\beta\nu}^Q/C_{P\gamma})$ is a sum of $\theta(q^{0\prime}; C_i/C_{P\gamma})$ by (3.6) and the latter vanishes, we have $\tilde{w}_P^T(q)=0$.

To prove Lemma 7, it suffices to prove that if, $q(I)^2 < m^2$ for at least one $I \epsilon \mathcal{J}_P$ and a polyhedral convex cone $C = \bigcap_{I' \epsilon \mathcal{J}} C(t(I'))^+$ is contained in C_P^+, then there is at least one $I' \epsilon \mathcal{J}$ for which $q(I')^2 < m^2$. To prove this, we note that $C \subset C_P$ implies (Lemma C1) that

$$\lambda(I)t(I) = \sum_{I' \epsilon \mathcal{J}} \lambda(I,I')t(I') \quad \text{for } I \epsilon \mathcal{J}_P \tag{5.14}$$

where $\lambda(I)$ and $\lambda(I,I')$ are positive integers. By comparing any fixed component on both sides of (5.15), we easily see

$$\lambda(I) \leqslant \sum_{I'} \lambda(I,I') \tag{5.15}$$

If $q(I')^2 \geqslant m^2$ for all $I' \epsilon \mathcal{J}$, and if $q^0 \epsilon C$, then each $q(I')$ is positive time-like and we have[23]

$$(q(I)^2)^{1/2} \geqslant \sum_{I'} \lambda(I)^{-1} \lambda(I,I') \; (q(I')^2)^{1/2} \geqslant m^2$$

which contradicts with the assumption. This completes the proof of Lemma 7.

Finally we show that $w_P^T(x)$ satisfies (2.28). Since $w_P^T(x)=w_{P\gamma}^T(x)$ for any γ, we obtain, due to (3.6),

$$\sum_P \theta(x^0;C_P/C_i)w_P^T(x)=\sum_{P\gamma}\theta(x^0;C_{P\gamma}/C_i)w_{P\gamma}^T(x)$$

Substituting the definition of w_P^T (x) into this equation and using (3.23) we obtain

$$\sum_P \theta(x^0;C_P/C_i)w_P^T(x)=\sum_{i'}\theta(x^0;C_{i'}^+/C_i)r_{i'}(x)$$

By using (R2) and (3.22), the terms with $i'\neq i$ vanishes. By $\theta(x^0;C_i^+/C_i)=\theta(x^0;C_i{}^+)$, the remaining term is identical with $r_i(x)$.

6. PROOF OF THEOREM 2

The Fourier transform of $r_i(x)$,

$$\tilde{r}_i(\zeta)=\int e^{i(\zeta,x)}r_i(x)dx \tag{6.1}$$

is analytic for $\zeta\epsilon T(V_i^Q)$ due to (R2). Conversely, if $\tilde{r}_i(\zeta)$ is analytic in $T(V_i^Q)$ and satisfy certain boundedness condition, then its boundary value $\tilde{r}_i(q)$ has the property (R2). Since $\tilde{r}_i(\zeta)$ is covariant, due to (R1), it is analytic in the extended tube $T'(V_i^Q)$ by the theorem of Hall and Wightman.[24]

We will now prove from the property (R3), that

$$\lim_{\epsilon\to+0}\tilde{r}_i(\zeta+i\epsilon q)=\lim_{\epsilon\to+0}\tilde{r}_j(\zeta-i\epsilon q) \tag{6.2}$$

where C_i and C_j are neighbouring across the plane h(I),

$$\zeta\,\epsilon\Sigma(ij,m) \tag{6.3}$$

$q \in Q$, $(q \cdot t(I))^2 > 0$ and $q^0 \cdot t(I) > 0$. If this is proved, then by the edge of wedge theorem,[25] $\tilde{r}_i$ and $\tilde{r}_j$ are analytic at $\Sigma(ij,m)$[25] and identical with each other, and therefore the theorem 2 is proved.

To prove (6.2), we denote the boundary values in Eq. (6.2) by $\tilde{r}_i(\zeta)$ and $\tilde{r}_j(\zeta)$. By taking the Fourier transform of (3.16) and (3.13), we obtain

$$\theta(x^0;C/C_i)-\theta(x^0;C/C_j)=\sigma\epsilon(C;C(t(I)))\theta_1((x^0)_I;C_I/C_{ij}) \tag{6.4}$$

where $C_I = C+h(t(I))$ (as a set in T mod $h(t(I))$), $C_{ij}=C_i \cap C_j$ ($\epsilon h(I)=h(t(I))^{\perp}$), $(x^0)_I$ is x^0 taken mod $h(t(I))$, θ_1 is as described in Lemma 3, and σ is defined by $C(t(I))^{\sigma} \supset C_i$. Using the addition theorem (3.6) for the left hand side of (6.4) we easily see

$$\begin{aligned}&\epsilon(\cup C_p;C(t(I)))\theta_1((x^0)_I;(\cup C_p)_I/C_{ij})\\ &\qquad =\Sigma\epsilon(C_p;C(t(I)))\theta_1((x^0)_I;(C_p)_I/C_{ij})\end{aligned} \tag{6.5}$$

where $\cup C_p$ is any partial sum of C_p and is assumed to be a polyhedral convex cone.

Using the integral representation (6.1) for $\tilde{r}_i(\zeta)$ and $\tilde{r}_j(\zeta)$ with $\zeta \in \Sigma(ij,m)$, we obtain by (6.4)

$$\begin{aligned}\tilde{r}_i(\zeta)-\tilde{r}_j(\zeta)=\int e^{i(\zeta,x)} dx\Big(\sum_{\alpha\nu}\epsilon(C_{\alpha\nu};C(t(I)))\Big)\theta_1((x^0)_I;(C_{\alpha\nu})_I/C_{ij})\\ \theta(x;\Delta_\alpha^X)w_{\alpha\nu}^T(x)\end{aligned} \tag{6.6}$$

Since $C_{\alpha\nu}$ is a partial sum of C_P, we can rewrite (6.6) using (6.5) as

$$\tilde{r}_i(\zeta)-\tilde{r}_j(\zeta)=\int e^{i(\zeta,x)}dx \sum_P \theta_1((x^0)_I;(C_P)_I/C_{ij})\epsilon(C_P;C(t(I)))w_P^T(x) \tag{6.7}$$

We now introduce a basis $t(I)$, $t_2 \ldots t_n$ in T and make the transformation of variables $x \to y$, through $x=t(I)\otimes y_1+\sum_{i=2}^{n} t_i\otimes y_i$. ($y$ and y_i are Minkowski vectors.) Then θ_1 in (6.7) is independent of y_1 and if $t(I)\epsilon F_1(C_P)$, the Fourier transform of $w_P^T(x)$ in y_1 with fixed y_i, $i \geq 2$,

$$w_P^T(p;y_2\ldots y_n)=\int e^{i(p,y_1)}w_P^T(y_1;y_2\ldots y_n)dy_1$$

$$=(2\pi)^{n-1}\int \exp(\sum_{i=2}^{n}(q,y_i)t_i)\delta(p-q\cdot t(I))\tilde{w}_P^T(q)dq$$

vanishes for $p^2 < m^2$ due to (W3). On the other hand if $t(I) \notin F_1(C_P)$, then $\epsilon(C_P;C(t(I)))$ vanishes by definition (3.15) and hence we have $\tilde{r}_i(\zeta)=\tilde{r}_j(\zeta)$ for $\zeta\epsilon\Sigma(ij,m)$.

We note that (2.39) is obtained from (6.1) because if $\zeta\epsilon T(V_i^Q)$ then $\mathrm{Im}\,\zeta^0\epsilon C_i$. Unlike (6.1), (2.39) holds in all $T(V_i^Q)$.

Finally we add the proof of (2.45). By definition

$$\tau(x)=\sum_P \theta(x^0;C_P)w_P(x)$$

Using $\theta(x^0;C_P)=\theta(x^0;C_P/C_P{}^+)$, we obtain

$$\tilde{\tau}(q)=(2\pi)^{-n}\sum_P\int\tilde{\theta}(v-q^0;C_P/C_P{}^+)\tilde{w}_P(q)dq^0$$

We now assume that $s = \mathrm{Re}\, v \in C_i$. If $C_P^+ \supset C_i$, the replacement of C_P^+ by C_i can be done trivially. On the other hand if $C_P^+ \not\supset C_i$, then $s \cdot t(I) < 0$ for at least one $I \in \mathcal{J}_P$ and due to (W3) $\int \tilde{\theta}(v-q^0;C_P)\tilde{w}_P(q)\,dq^0$ will be analytic. Hence we can again replace C_P^+ by C_i. Thus we have the formula (2.45)

7. ADDITIONAL REMARKS

To make the Theorem 1 in section 2 precise, one has to state the class of distributions to which w_P^T and r_i belong.[16] We do not attempt to make a precise statement as to the class of distributions for which our proof holds, but we would like to make some remarks pertinent to this point.

The behavior for large value of space time coordinate can be estimated by physical arguments and it is expected that w_P^T decreases exponentially in space-like directions and according to a power law in time-like directions. This behavior will be inherited by r_i. Hence the assumption that the multiplication of $\tilde{r}_i(q)$ by $\theta(q^0;C_i/t)$ is well-defined is a reasonable one.

We have shown that w_P^T and w_P yield the same r_i. We have also shown that w_P^T can be obtained from r_i by an inversion formula. The reason why w_P cannot be obtained by the same inversion formula is the following. w_P will (in general) approach to non-zero values for large separation of its coordinates due to the vacuum intermediate state. Because of this, expressions like $\theta(q^0;c_i/t)\tilde{w}_P(q)$ have ambiguity and especially the formula (3.23) cannot be used when multiplied by $\tilde{w}(q)$.[16] Thus, if we substitute (2.28) with w_P^T replaced by w_P into (5.1), we cannot change the order of summation over P and multiplication by $\theta(q^0;c_i/t)$ and hence we do not get $w_P(x)$. On the other hand if w_P^T behaves as we conjectured, then we will get w_P^T by (2.29). This is one of the reasons for using w_P^T instead of w_P.

We do not know much about the behavior of $\tilde{w}_P(q)$ for large energy momentum. If $w_P(q)$ does not decrease for large q, we have to use the subtraction method. It seems to be a non-trivial problem to extend our results to this case.

APPENDIX A. CASE OF MORE COMPLEX SPECTRUM CONDITIONS

We define $m(P,k)$ by the lowest upper bound of m such that

$$(\Psi_0, A_{P(1)}(x_{P(1)})\dots A_{P(k)}(x_{P(k)})(P(m)-P_0)$$
$$A_{P(k+1)}(x_{P(k+1)})\dots A_{P(n+1)}(x_{P(n+1)})\Psi_0) \qquad (A.1)$$

vanishes identically where $P(m)$ is the projection into states with mass below m and P_0 is the projection into the vacuum Ψ_0. We first prove

$$m(p,k)=m(P',k) \quad \text{if} \quad I(P,k)=I(P',k) \qquad (A.2)$$

Suppose (A.1) vanishes identically for P and $m < m(P,k)$. Then (A.1) vanishes for P' and $m < m(P,k)$ if the points $x_{P(1)} \cdots x_{P(k)}$ and $x_{P(k+1)} \cdots x_{P(n+1)}$ are space-like to each other within each group. We now note that (A.1) for P' as a distribution in the difference variables $\xi_j = x_{P'(j)} - x_{P'(j+1)}$ is a boundary value of a function which is analytic for $\mathrm{Im}\, \xi_j \in V_-$. Hence[27] (A.1) for P' also vanishes identically for $m < m(P,k)$.

Because of (A.2) we can define

$$m(I) = m(P,k) \quad \text{if } I = I(P,k) \tag{A.3}$$

We now assume the following:

Assumption A. $m(I)$ with fixed I is the same for all n such that $w_P(x) \neq 0$. In addition

$$m(I(n+1)-I) = m(I) \tag{A.4}$$

$$\sum_j \lambda_j m(I_j) \geqslant m(I) \quad \text{if } t(I) = \sum_j \lambda_j t(I_j) \tag{A.5}$$

(A.4) is obviously true for Hermitian fields. The idea behind (A.5) is the following. The state $\Psi = \prod_j \prod_{\nu=1}^{\lambda_j} \prod_{i \in I_j} A_i(x_{j\nu i})^* \Psi_0$ will have the same quantum numbers (which is associated with fields, additive, and zero for vacuum state)[28] as the states $\Psi' = \prod_{i \in I} A_i(x_i)^* \Psi_0$ and $\Psi'' = \prod_{i \in -I} A_i(x_i) \Psi_0$. By definition of $m(I_j)$ there is a state Ψ_j with mass around $m(I_j)$ such that $(\Psi_j, \prod_{i \in I_j} A_i(x_i)^* \Psi_0) \neq 0$. Assuming assymptotic conditions, we write Ψ_j in the form

$\Psi_j = F_j(A^{in})\Psi_0$. Then the state $\Psi''' = \prod_j (F_j(A^{in}))^{\lambda_j}\Psi_0$ will have the same quantum number as Ψ and the mass around $\sum_j \lambda_j m(I_j)$. Then, assuming no accidental cancellation, (Ψ, Ψ') and (Ψ, Ψ'') will not vanish identically, and we see that (A.5) is a reasonable assumption.

We note that for n=2 (A.5) takes the form

$$m_i + m_j \geqslant m_k \geqslant |m_i - m_j| \tag{A.6}$$

where (ijk) is any permutation of 123).[29]

As will be proved in Appendix B, w_P^T will satisfy (under the assumption B)

(W3') $\tilde{w}_P^T(q)=0$ unless $q \cdot t(I) \in V_+$ and $(q \cdot t(I))^2 \geqslant m(I)^2$ for all $I \in \mathcal{J}_P$.

By the same proof as for (R3), we obtain

(R3') $\tilde{r}_i(q) = \tilde{r}_j(q)$ if C_i and C_j are neighbouring across $h(I)$ and if $(q \cdot t(I))^2 < m(I)^2$.

We also get the analyticity of $\tilde{r}(\zeta)$ at

$$\Sigma(ij;\{m(I)\}) = \{\zeta \in Z'; \ \mathrm{Im}\,\zeta \in S(ij), \ (\mathrm{Re}\ q \cdot t(I))^2 < m(I)^2 \tag{A.7}$$

The sufficiency of (R3') for (W3') will be established in the same way as in section 5 if the following is true (cf. Lemma 7),

(M1) If $(q \cdot t(I))^2 < m(I)^2$ for at least one $I \in \mathcal{J}_P$ and $\Delta_\beta^Q(\{m(I)\})$ contains q, then each cone $C_{\beta\nu}^Q$ contains points outside of C_P^+, where

$$\Delta_\beta^Q(\{m(I)\})=\{q\epsilon Q;\ \sigma_\beta(I)((q\cdot t(I))^2-m(I)^2)>0\} \quad (A.8)$$

This Lemma follows from (A.5) in the same way as the proof of Lemma 7, if the following statement is true,

(M2) The $\lambda(I)$ in (5.13) can be taken as 1. Namely if $C=\bigcap_{I'\epsilon\mathcal{J}} C(t(I'))^+$ and $C\subset C_P$, then

$$t(I)=\sum_{I'\epsilon\mathcal{J}}\lambda(I,I')t(I') \quad \text{for } I\epsilon\mathcal{J}_P \quad (A.9)$$

where $\lambda(I,I')$ is an integer.

We have been unable to prove this for general n, but for $n\leqslant 4$ (n=4 corresponds to the 5 point function) (M2) can be verified easily.

Summing up we have the following theorem.

<u>Theorem A</u>. If w_P^T satisfies (W1), (W2), and (W3'), then r_i satisfies (R1), (R2), (R3') and (R4). The converse is true if (M1) holds (which is the case for $n\leqslant 4$). $\tilde{r}(\zeta)$ is analytic in the union of extended tubes $T'(V_i^Q)$ and at the points of $\Sigma(ij;\{m(I)\})$.

APPENDIX B. TRUNCATED VACUUM EXPECTATION VALUES

First we prove a Lemma which will be used in later discussion. Let $B(x_1\ldots x_n)$ and $C(y_1\ldots y_m)$ be products of fields $B_i(x_i)$ and $C_i(y_i)$ respectively. If the theory satisfies (2) in section 2, $B(x_1\ldots x_n)$ and $C(y_1\ldots y_n)$ either commute or anticommute if all the x_i-y_j are space-like.

<u>Lemma B.</u> If $B(x_1\ldots x_n)$ and $C(y_1\ldots y_n)$ anticommute for space-like x_i-y_j, then the vacuum expectation value of

either $B(x_1 \dots x_n)$ or $C(y_1 \dots y_m)$ vanishes identically.[30]

For the proof, by the theorem 3 of our previous paper[31] which has been proved there under the assumption of (1), (3a) and (3b) (but not (2)) we have

$$\lim_{\lambda \to \infty} (\Psi_0, BU(\lambda a, 1)C\Psi_0) = (\Psi_0, B\Psi_0)(\Psi_0, C\Psi_0)$$

$$\lim_{\lambda \to \infty} (\Psi_0, CU(-\lambda a, 1)B\Psi_0) = (\Psi_0, B\Psi_0)(\Psi_0, C\Psi_0)$$

where $U(\lambda a, 1)$ is the unitary operator for the translation by λa, and a is any space-like vector. If B and C anti-commute for space-like $x_i - y_j$, then for sufficiently large λ

$$(\Psi_0, BU(\lambda a)C\Psi_0) = -(\Psi_0, CU(-\lambda a)B\Psi_0)$$

Hence we have

$$(\Psi_0, B\Psi_0) \cdot (\Psi_0, C\Psi_0) = 0 \tag{B.1}$$

We now consider the truncated vacuum expectation values defined recursively by (2.3). We note that, although the definition of sign σ of each term in (2.3) refers to the order of the factors in that term, σ is actually independent of their order or else that term vanishes identically due to the above Lemma.

We define

$$w(i_1 \dots i_k) = (\Psi_0, A_{i_1}(x_{i_1}) \dots A_{i_k}(x_{i_k})\Psi_0)\sigma(i_1 \dots i_k) \tag{B.2}$$

$$(i_1 \dots i_k)_T = (A_{i_1}(x_{i_1}) \dots A_{i_k}(x_{i_k}))_T \, \sigma(i_1 \dots i_k) \tag{B.3}$$

$\sigma(i_1 \dots i_k)$ is the sign which one obtains if one commutes fields from the natural order to the order $i_1, \dots i_k$

for totally space-like configuration of x_i. σ_P of (2.1) is $\sigma(P(1)\ldots P(n+1))$.

The definition (2.3) now becomes

$$w(i_1\ldots i_m)=(i_1\ldots i_m)_T+\sum_G \sigma_G(i_1\ldots)_T(i_k\ldots)_T\ldots \qquad (B.4)$$

where the order of the i in $(\quad)_T$ is as in w, and the summation is over all groupings G of $i_1\ldots i_m$. σ_G is

$$\sigma_G=\sigma\cdot\sigma(i_1\ldots i_m)\,\sigma(i_1\ldots)\,\sigma(i_k\ldots)\ldots$$

$$=\sigma(i_1\ldots,i_k\ldots,\ldots)\,\sigma(i_1\ldots)\,\sigma(i_k\ldots \qquad (B.5)$$

In this form we see that σ_G depends only on the grouping and not on the order of $i_1\ldots i_m$ in w. Note that, by Lemma B, $\sigma(i_1\ldots,i_k\ldots,\ldots)$ is independent of the order of the groups $(i_1\ldots),(i_k\ldots),\ldots$ unless that term vanishes identically.

The spectrum condition of Appendix A can be written as

(W3") $\tilde{w}(i_1\ldots i_m)=0$ unless $q(i_1\ldots i_k)\in V_+$, $q(i_1\ldots i_k)^2\geqslant m(\,i_1\ldots i_k\,)^2$

for all $k\leqslant m$ or $q(i_1\ldots i_k)=0$ for some $k\leqslant m$

$$\tilde{w}(i_1\ldots i_m)=\sigma(i_1\ldots i_m)\,\sigma(i_1\ldots i_k)\,\sigma(i_{k+1}\ldots i_m)\tilde{w}(i_1\ldots i_k)\tilde{w}(i_{k+1}\ldots i_m)$$

if $q(i_1\ldots i_k)=0$.

The notations are:

$$\tilde{w}(i_1\ldots i_m)=\int\exp\, i(\Sigma(q_i\;,x_i\;))w(i_1\ldots i_m)dx_{i_1}\ldots dx_{i_m} \qquad (B.6)$$

$$q(i_1\ldots i_m)=q_{i_1}+\ldots+q_{i_m} \qquad (B.7)$$

Note that $\tilde{w}$ contains δ-function, in contrast to our former definition of $\tilde{w}_P$.

We now strengthen the Assumption A a little.

<u>Assumption B</u>. If $t(I)=\sum_j \lambda_j t(I_j)$,

$$\begin{aligned} m(I) &\leqslant \sum_j \lambda_j m'(I_j) \quad \text{unless } m'(I_j)=0 \text{ for all } j, \\ &\leqslant \min_j m(I_j) \quad \text{if} \quad m'(I_j)=0 \text{ for all } j. \end{aligned} \tag{B.8}$$

where

$$\begin{aligned} m'(\{i_1 \ldots i_m\}) &= m(\{i_1 \ldots i_m\}) \quad \text{if } w(i_1 \ldots i_m)=0 \\ &= 0 \qquad\qquad\qquad \text{otherwise.} \end{aligned} \tag{B.9}$$

The idea behind this assumption is the same as for Assumption A.

We now prove the following theorem.

<u>Theorem B</u>. If $w(i_1 \ldots i_m)$ satisfies conditions (W1), (W2), and (W3"), then $(i_1 \ldots i_m)_T$ satisfies (W1), (W2), and (W3'). The converse is also true. (We make the Assumption B.)

For the proof, the equivalence of (W1) for $w(i_1 \ldots i_m)$ and $(i_1 \ldots i_m)_T$ is obvious, because the defining equation has a unique solution in both directions. In addition, because (W2) is the requirement of symmetry in i and j when $x_i - x_j$ is space-like, and because (B.4) is a completely symmetric definition, the equivalence of (W2) for $w(i_1 \ldots i_m)$ and $(i_1 \ldots i_m)_T$ is also obvious. (It is important here that σ_G is independent of the order of $i_1 \ldots i_m$.)

We now prove the equivalence of (W3") and (W3'). First suppose $q(i_1 \ldots i_\ell)^2 < m(i_1 \ldots i_\ell)^2$. Then by Assumpsion B, for any grouping of $i_1 \ldots i_\ell$, either there is a group for which $q(i_k \ldots)^2 < m'(i_k \ldots)^2$ or else $q(i_k \ldots)^2 < m(i_k \ldots)^2$ for all groups. From this we easily see that (W3') implies (W3"). To prove the converse, we define

$$(i_1 \ldots i_m)_0 = \sigma(i_1 \ldots i_m)(\Psi_0, A_{i_1}(x_{i_1})(1-P_0)\ldots(1-P_0)A_{i_m}(x_{i_m})\Psi_0) \tag{B.10}$$

In the same way as in our previous paper,[32] we can derive

$$(i_1 \ldots i_n)_T = (i_1 \ldots i_n)_0 - \sum_{\text{con}} \sigma_G(i_1 \ldots)_T \ldots \tag{B.11}$$

where the summation is over all connected groupings.[33] We can now apply the same argument as above to (B.11) and easily see that (W3") implies (W3').

Finally we prove that r_i defined from w_P^T and w_P are the same. We show that the term from the summation over G in (2.3) cancels out in (2.28). Consider one fixed grouping $(i_1 \ldots i_k), (j_1 \ldots j_\ell), \ldots$ We note that there are several w_P which contribute to the same term of the form $(x_{i_1} \ldots x_{i_k})_T (x_{j_1} \ldots x_{j_\ell})_T \ldots$ The union of the C_P for such P is the cone

$$C_G = \{ t \in T;\ t_{i_1} \geq \ldots \geq t_{i_k}, \quad t_{j_1} \geq \ldots \geq t_{j_\ell}, \ldots \}$$

This cone is obviously not pointed. Since σ_G is independent of P, we see from (3.7) that the contributions from various P cancels out.

APPENDIX C. CONVEX POLYHEDRAL CONES[34]

Consider a real n dimensional vector space T and its dual S. A k dimensional linear subspace is called k-plane. The linear subspace generated by a subset T_1 is denoted by $h(T_1)$. For example, $h(\{t_1,\ldots t_m\}) = \left\{\sum_{i=1}^{m} \rho_i t_i ; \rho_i \text{ real}\right\}$. The orthogonal compliment of h is denoted by $h^\perp$. (If $h \epsilon T$, then $h^\perp \epsilon S$. If H is a family of planes h, then $H^\perp$ means the family of planes $h^\perp$. The convex polyhedral cone generated by $t_1,\ldots t_m$ is denoted by

$$C(t_1,\ldots t_m) = \left\{\sum_{i=1}^{m} \lambda_i t_i ; \lambda_i \geq 0\right\} \tag{C.1}$$

The positive polar C^+ and the negative polar C^- of a convex cone C is defined by

$$C^+ = \{s \epsilon S ; s\cdot t \geq 0, \ t \epsilon C\}, \ C^- = \{s \epsilon S ; s\cdot t \leq 0, \ t \epsilon C\} \tag{C.2}$$

The polars of a polyhedral convex cone in T are again polyhedral convex cones in S. The positive polar of the positive polar is the original cone. Note that

$$C(t_1\ldots t_m)^+ = \{s \epsilon S ; s\cdot t_i \geq 0, \ i=1,\ldots,m\} \tag{C.3}$$

$$h(t_1\ldots t_m) = C(\pm t_1 \ldots \pm t_m), \ h^+ = h^- = h^\perp \tag{C.4}$$

We call $h(C)$ the dimensionality space of the cone C and its dimension the simension of the Cone C. A polyhedral convex cone C has non-empty interior if and only if $\dim C=n$. The maximum linear subspace contained in C is called the linearity space of C and its dimension is called the linearity of C. (Notation: $L(C)$ and $\operatorname{lin} C$.) If $\operatorname{lin} C=0$, C is called pointed. C is pointed if and only if there is a $(n-1)$-plane intersecting with the cone C only at the origin. We have the following relations,

$$h(C^+)=h(C^-)=L(C)^\perp, \quad L(C^+)=L(C^-)=h(C)^\perp \tag{C.5}$$

$$\dim C + \operatorname{lin} C^+=\dim C^+ + \operatorname{lin} C = n \tag{C.6}$$

By (C.6) C is pointed if and only if C^+ has non-empty interior.

An extremum subset X of C is the set such that t_1, $t_2 \in C$ and $\alpha t_1+\beta t_2 \in X$ for some positive α and β with $\alpha+\beta=1$ necessarily imply $t_1, t_2 \in X$. Any convex extremum subset of C is again a polyhedral convex cone and is called k-facet where k is its dimension. If $\dim C=n$, the $(n-1)$-facets of C form the boundary of C. If $\operatorname{lin} C=0$, the 1-facets of C generate C. If $k+1<\dim C$, a k-facet F is a k-facet of some $(k+1)$-facet G and the intersection of such G is F. If f^+ is a k-facet of C^+, f is called k-corner of C. 1-facet is sometimes called extreme half-line and 1-corner is sometimes called supporting half-space.

We denote the set of all k-facets of C by $F_k(C)$, the set of all h(f) with f in $F_k(C)$ by $H_k(C)$ and the set of all k-corners by $F_k^+(C)$.

The sum C+C' is the set of all sums t+t' for $t \in C$ and $t' \in C'$. It is again a polyhedral convex cone. Note that $C(T_1 \cup T_2) = C(T_1) + C(T_2)$ where T_i are subsets of T. The intersection $C \cap C'$ is also a polyhedral convex cone. The C's form a lattice with the operations + and $\cap$. C^+'s form its dual. Namely,

$$(C_1 \cap C_2) + C_3 = (C_1 + C_3) \cap (C_2 + C_3), \quad (C_1 + C_2) \cap C_3 = (C_1 \cap C_3) + (C_2 \cap C_3) \tag{C.7}$$

$$(C+C')^+ = C^+ \cap C'^+, \qquad (C \cap C')^+ = C^+ + C'^+ \tag{C.8}$$

(Note that C can be replaced by h because of (C.4)).
The set of $-t$ for all $t \in C$ is denoted by $-C$.

If every element s of a set Σ is expressible as a positive linear combination $s = \Sigma\lambda(\nu)s(\nu)$ $(\lambda(\nu) \geq 0)$ of elements $s(\nu)$ of a subset Σ', then Σ' is called a positive basis of Σ. If every s in Σ is expressible as $s = \pm\Sigma\lambda(\nu)s(\nu)$ $(\lambda(\nu) \geq 0)$, then Σ' is called a c-basis of Σ. A c-basis of Σ which does not contain any sub-c-basis is called c-minimal. If $C(\Sigma)$ for a finite set Σ is pointed, Σ has a unique c-minimal positive basis. If a finite set Σ is c-minimal, $C(\Sigma)$ is pointed and $F_1(C(\Sigma))$ consists of $C(s)$, $s \in \Sigma$.

We now state a Lemma which is equivalent to the statement $(C^+)^+=C$.

Lemma C1. If $s\cdot t_1\geqslant 0,\ldots s\cdot t_m\geqslant 0$ imply $s\cdot t\geqslant 0$, then $t=\sum_i \lambda_i t_i$ with some non-negative λ_i.

Given a family of (n-1)-planes $H=\left\{h(s)^\perp ;s\in\Sigma\right\}$. If $h(\Sigma)$ is the total space S, then the planes in H will divide the entire space T into several pointed polyhedral convex cones with non-empty interior. We denote the set of all these convex cones by $\Gamma(H)$. Let $\Sigma_0=\left\{\pm s;s\in\Sigma\right\}$ and Σ_α be distinct c-minimal c-basis of Σ_0. Then $\Gamma(H)=\left\{C(\Sigma_\alpha)^+\right\}$. If we denote the set of k-planes generated by a subset of Σ by $\Pi_k(\Sigma)$, then $H_k(C(\Sigma_\alpha))\subset\Pi_k(\Sigma)$ and $H_k(C)\subset\Pi_{n-k}(\Sigma)^\perp$ for any $C\in\Gamma(H)$.

A cone $C(t_1\ldots t_n)$ with dimension n and linearity 0 is called a simplex cone. Its polar is also a simplex cone. If $s_i\cdot t_j=\delta_{ij}$, then $C(t_1\ldots t_n)^+=C(s_1\ldots s_n)$. Any polyhedral convex cone with dimension n can be decomposed into a union of almost disjoint simplex cones C_α

$$C=\bigcup_\alpha C_\alpha,\quad C_\alpha\text{:simplex, }\dim C_\alpha\cap C_\beta< n\text{ for }\alpha\neq\beta \tag{C.9}$$

If $F_1(C_\alpha)\subset F_1(C)$ for all α, this decomposition is called a standard simplexial decomposition. We now prove the following Lemma.

Lemma C2. If dim C=n and lin C=0, C has a standard simplexial decomposition. Furthermore, for any given plane h not belonging to $H_{n-1}(C)$, there is a standard simplexial decomposition (C.9) for which $h \notin H_{n-1}(C_\alpha)$ for any α.

For the proof of the first half, take any 1-facet f_1 and consider all polyhedral convex cones C_α generated by f_1 and any (n-1)-facet f^α_{n-1} not containing f_1. We easily see that $C = \bigcup C_\alpha$, dim $(C_\alpha \cap C_\beta) < n$ for $\alpha \neq \beta$, and $F_1(C_\alpha) \subset F_1(C)$. Hence by induction on the number of 1-facets, we get the first half. Moreover we get the second half by always taking a 1-facet f_1 not containing the given plane h. Note that if $f_1 \not\subset h$ and if there is only one facet not containing f_1, then any standard simplexial decomposition after that stage will have the property that $h \notin H_{n-1}(C_\alpha)$. Note also that if there is only one (n-1)-facet not containing f_1 for every 1-facet f_1, then the cone is simplex.

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FOOTNOTES AND REFERENCES

1. D. Ruelle, Thesis (Brussels, 1959).
2. O. Steinmann, Wightmanfunktionen und Retardierte Kommutatoren II., preprint.
3. N. Burgoyne, private communication. Also see H. Araki and N. Burgoyne, to be published.
4. O. Steinmann, Uber den Zusammenhang zwischen den Wightmanfunktionen und den retardierten Kommutatoren, preprint.
5. The mass spectrum condition for Wightman function is stated in (W2") of Appendix B. It is more complicated because of the presence of the vacuum intermediate state. Also see discussion of section 7.
6. It is meant that σ_P is the sign change which one obtains if one changes the order of the fields from the natural order 1, 2,...n+1 to (the order) P(1), P(2),...P(n+1) for totally space-like configuration of x_i. See Appendix B.
7. R. Haag, Phys. Rev. 112, 669 (1958). See also Appendix B for more detail. (2.3) corresponds to Ursell's expansion in statistical mechanics. H. P. Ursell, Proc. Cambridge Phil. Soc. 23, 685 (1927).

8. The signature of the metric is $(1,-1,-1,-1)$.

9. To prove (2.15), we note that the vacuum expectation value of the multiple commutator for each P in (2.14) contains a fixed $w_{P'}(x)$ in (2.15) if and only if

$$P^{-1}P'(1) > P^{-1}P'(2) > \ldots > P^{-1}P'(j-1),$$
$$P^{-1}P'(j+1) < \ldots < P^{-1}P'(n+1).$$

Since $(j-1)$ A's always come to the left of $A_1(x_1)$, the $w_{P'}(x)$ in all these terms have a common sign $(-1)^{j-1}$. Summing up θ-functions over all P satisfying the above equation, we get (2.15).

10. For the definition, see Appendix C.

11. To be precise, we have to specify the class of distributions to which $w_P(x)$ and $r_i(x)$ belong. The point is that a product like $\theta(x;C_P/C_i)w_P(x)$ or $\theta(q;C_i/C_P)r_i(q)$ has to be well-defined and the integral over dq^0 or dx^0 has to be convergent. In this paper we do not attempt any thorough discussion of this point, though we shall make a few remarks in section 7. See also footnote (16).

12. This means C_i and C_j are neighbouring cones with their common $(n-1)$-facet lying on $h(I)$. The cones in (R4) will be explained below.

13. C_P and $C_{P'}$ are neighbouring cones with their common $(n-1)$-facet lying on $h(ij)$.

14. For example, take $r_{12}(x_1 \ldots x_4)$ for the 4-fold case. (See H. Araki and N. Burgoyne, loc. cit.) This vanishes unless x_1 is advanced over x_3 and x_4 and x_2 is advanced over <u>either</u> x_3 <u>or</u> x_4. (R2) says that it vanishes unless (x_1-x_3), (x_1-x_4), and $(x_1+x_2-x_3-x_4)\epsilon\bar{V}_+$. Of course the latter and (R4) imply the former.

15. cf. L. Schwartz, Transformation de Laplace des Distribution, Med. Lunds Mat. Sem., Suppl. (1952), p.196.

16. Note that $\theta(t;C/\delta C^+)$ is defined only almost everywhere. The equation (3.5) should be taken in this sense. The product like $\theta(t)w(t)$ is meaningful only when $w(t)$ belongs to a certain class of distribution. See L. Schwartz, Seminaire Schwartz-Levy, 1956-57, No. 3, Faculté des Sciences de Paris.

17. cf. Lemma C2 in Appendix C.

18. $F_m(C)$ is the set of all m-facets of C and $H_m(C)$ is the set of dimensionality spaces of all m-facets of C: $H_m(C) = \{h(f); f\epsilon F_m(C)\}$.

19. $\Gamma(H)$ is the set of all convex polyhedral cones obtained by division of the whole space by (n-1)-planes belonging to H. See Appendix C.

20. Another proof can be obtained by using a standard simplexial decomposition of C: $C=\bigcup C_\alpha$. Then

$$\theta(t;C/C')=\sum_\alpha \theta(t;C_\alpha/C')$$

Since $F_1(C_\alpha) \subset F_1(C)$ and $C' \in \Gamma(H_1(C)^{\perp})$, C' is contained in one of $\delta C_\alpha^{\ +}$ (defined by (3.3)). By (3.5), if $t \notin C'^{+} \supset \delta C_\alpha$, then $\theta(t;C_\alpha/C') = \theta(t;C_\alpha/\delta C_\alpha^{+}) = 0$.

21. We are only interested in the coefficients.

22. Since h(I) should contain h_{n-k}, q(I)=0 should be derived from $q(I_\mu^{(k)})=0$, $\mu=1\ldots k$. (cf. Lemma Cl) One can easily find that I should contain the whole or no part of $I_\mu^{(k)}$ for each $\mu \neq k$, and I cannot contain $I_k^{(k)}$ and $I_\mu^{(k)}$ at the same time if they fulfill (β). Furthermore, since $h(I) \not\supset h_{n-k+1}$ and since $I_\mu^{(k)} = I_\mu^{(k-1)}$ for this case, I should contain $I_k^{(k)}$. Thus we have this result.

23. If a_i is positive time-like $\left[(\Sigma a_i)^2\right]^{1/2} \geqslant \Sigma(a_i^{\ 2})^{1/2}$. This is easily seen in the rest system of Σa_i.

24. D. Hall and A. Wightman, Mat. Fys. Medd. Dan. Vid. Selsk. 31, No. 5 (1957).

25. H. Bremmermann, R. Oehme, and J. G. Taylor, Phys. Rev. 109, 2178 (1958). J. G. Taylor, Annals of Physics, 5, No. 4, 391 (1958). F. J. Dyson, Phys. Rev. 110, 579 (1958). L. Garding and A. Beurling, to appear.

26. cf. H. Epstein, A Generalization of the Edge of Wedge Theorem, preprint.

27. By the theorem of Hall and Wightman, the analytic function in question is analytic in a Jost point

(R.Jost, Helv. Phys. Acta 30, 409 (1957)), where we have proved that (A.1) vanishes. Hence it vanishes identically. We could use also edge of wedge theorem (instead of Jost points) taking 0 as the analytic function approaching to the same boundary value from the other side.

28. For multiplicative quantum numbers of the form $(-1)^n$, one can take n mod 2.

29. We thank Professor A. S. Wightman for an illuminating explanation of the relevance of (A.6) for the sufficiency of the condition of the type (R.3).

30. We assume (1), (2), (3a), (3b) and (3c) for the theory. However, we do not make assumptions about the connection between commutation relation among different fields and the type of fields.

cf. H.Araki, "On the Connection of Spin and Commutation Relations between Different Fields".

31. H. Araki, Annals of Physics, in press. Theorem 3 in that paper is expressed in terms of w^T. However, the properties used for w^T in the proof are the

covariance and the existence of lowest positive mass in that intermediate state where $U(\lambda a,1)$ is inserted. $(\Psi_0, BU(\lambda a,1)C\Psi_0) - (\Psi_0,B\Psi_0)(\Psi_0,C\Psi_0)$ clearly has these properties.

32. H. Araki, loc. cit. Eqs. (2.11) through (2.16).

33. If each group of a grouping G occupies consecutive positions in $(i_1 \ldots i_n)$, then G is called a division of $(i_1 \ldots i_n)$. If a grouping is a subgrouping of a proper division, then it is called a disconnected grouping. Otherwise, a grouping is called a connected grouping. Thus for a connected grouping, numbers in one group are interlocked in $(i_1 \ldots i_n)$, with those in another group.

34. cf. M. Gerstenhaber, Activity Analysis of Production and Allocation, (John Wiley and Sons, Inc. New York 1951) Chapter 18.