学位申請論文

井田大輔

FOUNDATION OF MANY—SHELL SYSTEM IN GENERAL RELATIVITY

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Contents

I. INTRODUCTION

The component of the universe is often assumed to be the dust, namely the ideal fluid with zero pressure, when we consider the structure formation such as that of galaxies, cluster of galaxies or large-scale structures. A difficulty of the dust fluid model is that the shell crossing singularities develop from rather generic initial boundary conditions. In other words, the neighboring fluid lines of dust will cross each other, where the energy density diverges. The strong gravitational field of shell crossing may trigger the subsequent structure formation, however, there is no systematic method to compute the evolution of space-time after such shell crossing singularities. Hence it might be important to understand the dynamics at the shell crossing and the subsequent time development of the system in cosmological or astrophysical problems.

The shell crossing leads to the infinite mass density, so that the treatment of the matter field as a dust fluid becomes inappropriate. In order to continue the solution, we have to determine the nature of the shell crossing by imposing some physical assumptions on the dust fluid. One natural assumption might be that the dust fluid consists of a cold baryonic gas or collisionless particles with zero velocity dispersion. Since the analytic treatment of such a system is quite difficult, the N-body simulation is often adopted to deal with the complicated dynamics of particles.

The subject here is the analytic treatment of the dust universe even after the shell crossing singularities. We consider the spherically symmetric case as a first step, since, unlike the vacuum case, there is sufficient physical degrees of freedom even in this case. The Tolman-Bondi space-time is an exact solution of the Einstein equation, which describes the dynamics of the dust fluid with spherical symmetry, however this also has the above-mentioned difficulty of the shell crossing singularity. Our main idea is to discritize the dust distribution into many spherically symmetric thin dust shells. We expect that this many-shell model well approximates the Tolman-Bondi model if the number of shells are taken sufficiently large.

The dynamics of each dust shell in isolation can be treated as a singular hypersurface in the Schwarzschild background, which was formulated by Israel [10] more than thirty years ago. Let us consider a time-like hypersurface at which the metric is continuous and the first derivative of the metric is not continuous. Since the metric is roughly regarded as the gravitational potential, its first derivative is just the gravitational force. Hence this situation can be interpretated as that some matter field is confined within the surface; in fact, the integrated version of the Einstein equation implies that the stress-energy tensor has δ-function-like singularity at the hypersurface. More precisely, the property of the matter confined within the hypersurface is completely determined by the first fundamental form (the induced metric) and the difference of the second fundamental form (or extrinsic curvature) of the hypersurface measured on both sides. The motion of a spherically symmetric thin shell is described in a simple way. In particular, the equation of motion of a thin dust shell in vacuum space-time can be solved analytically. This is an advantage of the spherical thin shell model.

Another advantage of the many-shell model is that it may possibly describe more generic physical situation than the Tolman-Bondi space-time, since each shell can have arbitrary energy and momentum, while those of the Tolman-Bondi space-time should be continuous. This means that the many-shell model may treat the shell crossing singularities. The elementary process of the shell crossing might be the collision of two thin shells. We therefore have to specify the motion of shells after the collision, which is equivalent to determining the interaction between particles constructing thin shells. One possibility is that two shells marges into a shell. However, what we consider here is the situation in which the shell is composed of collisionless particles. In this case, two shells will freely pass through each other; we shall call such shells *transparent shells*. Hence our first task is to realize such a situation. Then, the shell crossing may be described as successive collisions of thin shells.

The collision of massless shells in spherically symmetric space-time was studied by Dray and 't Hooft [8] and Redmount [20]. They constructed this model by cut and past of four distinct Schwarzschild space-times and derived the junction condition known as the Dray-'t

Hooft-Redmount (DTR) relation [8,20,19]. Since the world-surface of a massless shell is the null hypersurface, there is little freedom of the dynamics, while in the case of massive shells the problem is more complicated. The collision of massive shells have been considered by Núñez, de Oliveira and Salim [17], though this problem has not yet been solved.

Once the junction condition for colliding pair of shells is obtained, the time evolution of the many-shell system can be completely calculated in principle. This method enables to compute the nonlinear stage of the dust universe even after the shell crossing. Each shell however refers to a distinct coordinate system from others, which is not suitable for putting the initial condition and for interpretation of the result. Our next task therefore is to find the coordinate system in which we can easily see the correspondence between the many-shell system and the dust-filled universe.

In Sec. II, the treatment of a self-gravitating thin shell is reviewed. In Sec. III, the collision of two spherically symmetric shells is investigated, where the junction condition is obtained. In Sec. IV, the correspondence between the many-shell system and the Tolman-Bondi space-time is considered, and prescriptions for putting initial conditions are shown. Conclusions are given in Sec. V. Newton's constant of gravitation and the speed of light are taken to be unity in what follows.

II. DYNAMICS OF A THIN SHELL

We here briefly review Israel's formulation for singular hypersurfaces and rederive the equation of motion of a thin shell in the spherically symmetric space-time.

A. Self–Gravitating Thin Shell

Let a timelike hypersurface $\mathscr{S}$ divide a space-time $(\mathscr{M}, g_{ij})$ into two subsets, $\mathscr{V}_I$ and $\mathscr{V}_J$, and let $\mathscr{S}$ be the common boundary of $\mathscr{V}_I$ and $\mathscr{V}_J$. The metric g_{ij} is required to be continuous, but the components of its first derivative may possess a finite difference across $\mathscr{S}$ with respect to a suitable coordinate system, so the energy-momentum tensor will have a δ-function-like singularity there.

Let h_{ij} be the first fundamental form of the hypersurface $\mathscr{S}$, and let $\overset{I}{K}$ be the second fundamental form with respect to the imbedding of $\mathscr{S}$ into $\mathscr{V}_I$:

$$h_{ij} = g_{ij} - n_i n_j, \tag{1}$$

$$K_{Iij} = -\tfrac{1}{2}\mathscr{L}_n h_{ij}\Big|_I \tag{2}$$

Here n_i is the unit normal form to $\mathscr{S}$ directed from $\mathscr{V}_I$ to $\mathscr{V}_J$, $\mathscr{L}_n$ is the Lie derivative with respect to n^i The first fundamental form h_{ij} is the induced metric on $\mathscr{S}$, so this should be determined uniquely, while K_{Iij} and K_{Jij} may differ.

When the hypersurface $\mathscr{S}$ is singular in the above sense, $\mathscr{S}$ may represent a thin shell, i.e., there is additional matter on $\mathscr{S}$ Here we introduce the surface stress-energy tensor S_{ij} on $\mathscr{S}$ which represents the matter on $\mathscr{S}$:

$$S_{ij} = \frac{1}{8\pi}\left\{K_{ij} - K h_{ij}\right\}_I^J, \tag{3}$$

where $\{Q\}_I^J = Q_J - Q_I$ is defined for any tensor field $Q_{I(J)}$ on $\mathscr{V}_{I(J)}$.

The Gauss and the Codazzi-Mainardi equations lead to

$$\mathscr{R} + K_{Iij}K_I{}^{ij} - K_I{}^2 = -2G_{Iij}n^i n^j \tag{4}$$

and

$$\partial_i K_I - \mathscr{D}_j K_{Ii}{}^j = h_i{}^j G_{Ijk} n^k, \tag{5}$$

where $\mathscr{R}$ and $\mathscr{D}$ denote the scalar curvature and the Riemannian connection of $(\mathscr{S}, h_{ij})$, respectively, and $G_{Iij} (= R_{Iij} - 1/2\ R_I g_{ij})$ the Einstein tensor evaluated in $\mathscr{V}_I$. By substructing Eq. (4) (Eq. (5)) for $\mathscr{V}_I$ from that for $\mathscr{V}_J$, we find the relationships

$$(K_{Iij} + K_{Jij}) S^{ij} = -\frac{1}{4\pi} \{G_{ij}\}_I^J n^i n^j \tag{6}$$

and

$$\mathscr{D}_j S_i{}^j = -\frac{1}{8\pi} h_i{}^j \{G_{jk}\}_I^J n^k \tag{7}$$

The formulas (6) and (7) are used later to derive the law of conservation of the proper mass of the shell and the equation of motion of the shell.

As a background space-time, we consider the Reissner-Nordström–de Sitter (RNdS) space-time, which is the static solution of the Einstein-Maxwell equations possibly with a cosmological term. The metric is

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 H_{ij} dx^i dx^j, \tag{8}$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2 \tag{9}$$

and H_{ij} denotes the standard metric of the two-sphere:

$$H = d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \tag{10}$$

The vector potential of the Maxwell field is

$$A_i = -\frac{Q}{r} \delta_i^t. \tag{11}$$

The metric (8) describes the black hole with the mass M, the electric charge Q and the cosmological constant Λ.

The stress-energy tensor becomes

$$T^i_j = \frac{Q^2}{8\pi r^4}(-\delta^i_t\delta^t_j - \delta^i_r\delta^r_j + \delta^i_\vartheta\delta^\vartheta_j + \delta^i_\varphi\delta^\varphi_j) \tag{12}$$

There is a curvature singularity at $\{r = 0\}$ unless $M = Q = 0$, and a null hypersurface $\{r > 0; f(r) = 0\}$ is a Killing horizon.

B. Equation of Motion of a Thin Shell With Spherical Symmetry

The equation of motion of a spherically symmetric thin shell becomes extremely simple. Let $\mathscr{V}_{I(J)}$ be an open set of $(M_{I(J)}, Q_{I(J)}, \Lambda)$–RNdS space-time. The shell is constructed by gluing $\mathscr{V}_I$ and $\mathscr{V}_J$ at the symmetric timelike hypersurface $\mathscr{S}$ Here we introduce the coordinate system $\{t_{I(J)}, r, \vartheta, \varphi\}$ for $\mathscr{V}_{I(J)}$, which is the static coordinate system of the RNdS space-time, where r, ϑ and φ each assumes the same value on both sides of $\mathscr{S}$, so omission of the subscripts I, J does not lead to confusion.

Due to the symmetry of the $\{\vartheta, \varphi\}$–plane, the surface stress-energy tensor takes the form of the ideal fluid:

$$S_{ij} = \sigma u_i u_j + P H_{ij}, \tag{13}$$

where u_i is the future-directed unit vector tangent to the fluid lines, namely the four-velocity. In what follows, the energy condition

$$\sigma > 0 \tag{14}$$

is imposed. In each coordinate system, the four-velocity is expressed as

$$u^i = \left(\frac{d}{d\tau}\right)^i = \left(\frac{dt_{I(J)}}{d\tau}\right)\delta^i_{t_{I(J)}} + \left(\frac{dr}{d\tau}\right)\delta^i_r \tag{15}$$

where τ is the proper time along the fluid line. Let n_I be the unit spacelike one-form normal to $\mathscr{S}$, which is directed from $\mathscr{V}_I$ to $\mathscr{V}_J$:

$$n_i = -\left(\frac{dr}{d\tau}\right)\delta^{t_I}_i + \left(\frac{dt_I}{d\tau}\right)\delta^r_i \tag{16}$$

All the non-vanishing components of K_I are

$$K_{Iij}u^i u^j = \left[f_I(r)\frac{dt_I}{d\tau}\right]^{-1}\left[\frac{d^2r}{d\tau^2}+\frac{f_I'(r)}{2}\right], \tag{17}$$

and

$$K_{I\vartheta}^{\vartheta} = K_{I\varphi}^{\varphi} = -\frac{f_I(r)}{r}\frac{dt_I}{d\tau}, \tag{18}$$

where $f_I = 1-2M_I/r+Q_I^2/r^2-\Lambda r^2/3$. By evaluating the components of Eq. (3), we obtain

$$\left(f_J\frac{dt_J}{d\tau}\right)^{-1}\left(\frac{d^2r}{d\tau^2}+\frac{f_J'}{2}\right)-\left(f_I\frac{dt_I}{d\tau}\right)^{-1}\left(\frac{d^2r}{d\tau^2}+\frac{f_I'}{2}\right)=4\pi(\sigma+2P), \tag{19}$$

and

$$f_I\frac{dt_I}{d\tau}-f_J\frac{dt_J}{d\tau}=4\pi\sigma r. \tag{20}$$

For RNdS metric, The r.h.s. of Eq. (7) vanishes, since $G_{Iij}n^j = -(Q_I^2/r^4+\Lambda)n_i$, which implies $h_i^{\ j}G_{Ij}^{\ \ k}n_k = 0$. Then Eq. (7) leads to

$$\frac{dm}{d\tau} = -8\pi Pr\frac{dr}{d\tau}, \tag{21}$$

where $m = 4\pi\sigma r^2$ is the proper mass of the shell. Given the equation of state of the shell $P = P(\sigma)$, the proper mass m becomes a function of r In particular, in the case of dust ($P = 0$), Eq. (21) leads to the law of conservation of the proper mass of the shell: $m =$ const.

By substituting Eqs.(17), (18) and the relation $G_{Iij}n^in^j = -(Q_I^2/r^4+\Lambda)$ into Eq. (6), we obtain

$$\left(f_J\frac{dt_J}{d\tau}\right)^{-1}\left(\frac{d^2r}{d\tau^2}+\frac{f_J'}{2}\right)+\left(f_I\frac{dt_I}{d\tau}\right)^{-1}\left(\frac{d^2r}{d\tau^2}+\frac{f_I'}{2}\right) = \frac{Q_J^2-Q_I^2}{mr^2}+\frac{2P}{\sigma r}\left(f_I\frac{dt_I}{d\tau}+f_J\frac{dt_J}{d\tau}\right) \tag{22}$$

From Eqs.(19), (20) and (22), we obtain

$$f_I\frac{dt_I}{d\tau} = \frac{m_-}{m}+\frac{m}{2r}-\frac{q_+q_-}{mr}, \tag{23}$$

$$f_J\frac{dt_J}{d\tau} = \frac{m_-}{m}-\frac{m}{2r}-\frac{q_+q_-}{mr}, \tag{24}$$

and the equation of motion of a shell

$$\frac{d^2r}{d\tau^2} = \frac{m_- q_+ q_- / m^2 - m_+}{2r^2} - \frac{(m^2 - q_+^2)(m^2 - q_-^2)}{4m^2r^3} + \frac{\Lambda}{3}r - \frac{2\pi P}{r}\left[m - \frac{(q_+q_- - 2m_-r)^2}{m^3}\right], \tag{25}$$

where $m_\pm := m_2 \pm m_1$ and $q_\pm := q_2 \pm q_1$. Furthermore, by substituting Eq. (23) or (24) into the normalization condition: $u_i u^i = -1$, we obtain the first integral of Eq. (25), or the law of conservation of energy:

$$\left(\frac{dr}{d\tau}\right)^2 = -1 + \mathscr{E}^2 + \frac{m_+ - \mathscr{E} q_+ q_- / m}{r} + \frac{(m^2 - q_+^2)(m^2 - q_-^2)}{4m^2r^2} + \frac{\Lambda}{3}r^2, \tag{26}$$

where $\mathscr{E} = m_-/m$ is the specific energy of the shell.

Equation (26) corresponds to the energy equation for the shell. In order to close the system of differential equations, we need one more equation, namely the equation of state which gives the relation between σ and P One special interest is in the sphericall massive shell composed of collisionless particle. In this case, σ and P can be given in the form

$$\sigma = \frac{m_0}{4\pi r^2}\sqrt{1 + \ell^2/r^2}, \tag{27}$$

$$P = \frac{\ell^2}{2(r^2 + \ell^2)}\sigma, \tag{28}$$

where m_0 and ℓ are constant. The constant m_0 corresponds to the conserved mass of shell, while ℓ is the specific angular momentum of a particle on the shell. The detailed derivation of the above expressions for σ and P is given in Appendix A. It is worthwhile to note that when ℓ vanishes, P also vanishes. This means that the dust–shell is composed of non-rotating collisionless particles. Note that Eqs. (27) and (28) satisfy the conservation law (21).

C. Motion of a Dust-Shell

In this section, we consider the motion of a dust–shell in Schwarzschild background. The equation of motion (25) can be solved analytically when the shell is composed of dust.

In general, the shell $\overset{I}{\mathscr{S}}$ is assumed to divide the regions $\mathscr{V}_I$ and $\mathscr{V}_{I+1}$. The label I in the expression $\overset{I}{Q}$ means that it is associated with the I^{th} shell, and the label I in Q_I means that it is defined in the region $\mathscr{V}_I$. The equation (26) for a dust-shell is

$$\frac{d\overset{I}{r}}{d\overset{I}{\tau}} = \overset{I}{\epsilon}\overset{I}{V}{}^{1/2}(\overset{I}{r}). \tag{29}$$

$$\overset{I}{V}(\overset{I}{r}) = -1 + \overset{I}{\mathscr{E}}{}^2 + \frac{\overset{I}{m}_+}{\overset{I}{r}} + \frac{\overset{I}{m}{}^2}{4\overset{I}{r}{}^2}, \tag{30}$$

where $\overset{I}{\epsilon} = \pm 1$. In the dust-shell case, $\overset{I}{m}$ and hence $\overset{I}{\mathscr{E}}$, are constants. The solution becomes

$$\overset{I}{\epsilon}(\overset{I}{\tau} - \overset{I}{\tau}_0) = \frac{\overset{I}{r}(2\overset{I}{m}_+\overset{I}{r} - \overset{I}{m}{}^2)}{3\overset{I}{m}_+{}^2}\overset{I}{V}{}^{1/2}(\overset{I}{r}), \ (\overset{I}{\mathscr{E}}{}^2 = 1) \tag{31}$$

$$\overset{I}{\epsilon}(\overset{I}{\tau} - \overset{I}{\tau}_0)$$
$$= \frac{\overset{I}{r}\overset{I}{V}{}^{1/2}(\overset{I}{r})}{\overset{I}{\mathscr{E}}{}^2 - 1} - \frac{\overset{I}{m}_+}{2(\overset{I}{\mathscr{E}}{}^2 - 1)^{3/2}}\ln\left|\frac{\overset{I}{m}_+}{2(\overset{I}{\mathscr{E}}{}^2 - 1)} + \overset{I}{r} + \overset{I}{r}\left(\frac{\overset{I}{V}(\overset{I}{r})}{\overset{I}{\mathscr{E}}{}^2 - 1}\right)^{1/2}\right|, \ (\overset{I}{\mathscr{E}}{}^2 > 1) \tag{32}$$

$$\overset{I}{\epsilon}(\overset{I}{\tau} - \overset{I}{\tau}_0)$$
$$= \frac{\overset{I}{r}\overset{I}{V}{}^{1/2}(\overset{I}{r})}{1 - \overset{I}{\mathscr{E}}{}^2} - \frac{\overset{I}{m}_+}{2(1 - \overset{I}{\mathscr{E}}{}^2)^{3/2}}\arcsin\frac{2(1 - \overset{I}{\mathscr{E}}{}^2)\overset{I}{r} + \overset{I}{m}_+}{[\overset{I}{m}_+{}^2 - 2(1 - \overset{I}{\mathscr{E}}{}^2)\overset{I}{m}{}^2]^{1/2}}, \ (\overset{I}{\mathscr{E}}{}^2 < 1) \tag{33}$$

according to the value of $\overset{I}{\mathscr{E}}{}^2$, where $\overset{I}{\tau}_0$ is the integration constant. The specific energy $\overset{I}{\mathscr{E}}$ determines the motion of the dust–shell: $\overset{I}{\mathscr{E}}{}^2 < 1$ for bound motion, $\overset{I}{\mathscr{E}}{}^2 = 1$ for marginally bound motion, and $\overset{I}{\mathscr{E}}{}^2 > 1$ for unbound motion. For $\overset{I}{\mathscr{E}}{}^2 \leq 1$, the allowed region for the motion of the dust–shell is not restricted. On the other hand, when $\overset{I}{\mathscr{E}}{}^2$ is smaller than unity, the areal radius of the dust shell assumes the maximal value given by

$$\overset{I}{r}_{max} = \frac{1}{1 - \overset{I}{\mathscr{E}}{}^2}\left\{\tfrac{1}{2}\overset{I}{m}_+ + \tfrac{1}{2}[\overset{I}{m}_+{}^2 + \overset{I}{m}{}^2(1 - \overset{I}{\mathscr{E}}{}^2)]^{1/2}\right\}, \ (\overset{I}{\mathscr{E}}{}^2 < 1). \tag{34}$$

The motion of a I^{th} dust–shell is expressed in terms of two static coordinate systems $\{\overset{I}{t}_-, \overset{I}{r}\}$ and $\{\overset{I}{t}_+, \overset{I}{r}\}$, where $\overset{I}{t}_- := t_I$ and $\overset{I}{t}_+ := t_{I+1}$. From Eqs. (23), (24) and (26), we can obtain the motion of I^{th} dust–shell in these coordinate systems. It is governed by the following differential equations

$$\frac{d\overset{I}{t}_\pm}{d\overset{I}{r}} = \overset{I}{\epsilon}\frac{\overset{I}{r}\overset{I}{\mathscr{E}} \mp \overset{I}{m}/2}{(\overset{I}{r} - \overset{I}{m}_+ \mp \overset{I}{m}\overset{I}{\mathscr{E}})\overset{I}{V}{}^{1/2}(\overset{I}{r})}. \tag{35}$$

These can be integrated according to the value of $\overset{I}{\mathscr{E}}^2$ as follows:

$$\overset{I}{\epsilon}(\overset{I}{t}_\pm - \overset{I}{t}_{0\pm}) = \int^{\overset{I}{r}} dr \left[\left(\overset{I}{\mathscr{E}}(r+\overset{I}{m}_+) \pm \overset{I}{m}(\overset{I}{\mathscr{E}}^2 - 1/2)\right) + (\overset{I}{m}_+ \pm \overset{I}{m}\overset{I}{\mathscr{E}})\left(\overset{I}{\mathscr{E}}\overset{I}{m}_+ \pm \overset{I}{m}(\overset{I}{\mathscr{E}}^2-1/2)\right)\frac{1}{r-\overset{I}{m}_+ \mp \overset{I}{m}\overset{I}{\mathscr{E}}}\right] \times \frac{1}{[(1-\overset{I}{\mathscr{E}}^2)r^2 + \overset{I}{m}_+ r + \overset{I}{m}^2/4]^{1/2}} = \overset{I}{\mathscr{F}}_\pm(\overset{I}{r}) + \overset{I}{\mathscr{G}}_\pm(\overset{I}{r}), \tag{36}$$

where $\overset{I}{t}_{0\pm}$ is the integration constant. The functions $\overset{I}{\mathscr{F}}_\pm(\overset{I}{r})$ and $\overset{I}{\mathscr{G}}_\pm(\overset{I}{r})$ are defined by

$$\overset{I}{\mathscr{F}}_\pm(\overset{I}{r}) = \frac{\overset{I}{r}[\overset{I}{\mathscr{E}}(2\overset{I}{m}_+\overset{I}{r} + 6\overset{I}{m}_+^2 - \overset{I}{m}^2) \pm 3\overset{I}{m}\overset{I}{m}_+]V^{1/2}(\overset{I}{r})}{3\overset{I}{m}_+^2}, \quad (\overset{I}{\mathscr{E}}^2 = 1) \tag{37}$$

$$\overset{I}{\mathscr{F}}_\pm(\overset{I}{r}) = \frac{\overset{I}{\mathscr{E}}\overset{I}{r}V^{1/2}(\overset{I}{r})}{1-\overset{I}{\mathscr{E}}^2} + \frac{(1/2-\overset{I}{\mathscr{E}}^2)[\overset{I}{m}_+\overset{I}{\mathscr{E}} \mp (1-\overset{I}{\mathscr{E}}^2)\overset{I}{m}]}{(1-\overset{I}{\mathscr{E}}^2)^{3/2}} \ln\left|\frac{(1-\overset{I}{\mathscr{E}}^2)\overset{I}{r} + \overset{I}{m}_+/2}{(1-\overset{I}{\mathscr{E}}^2)^{1/2}} + \overset{I}{r}V^{1/2}(\overset{I}{r})\right|, \quad (\overset{I}{\mathscr{E}}^2 < 1) \tag{38}$$

$$\overset{I}{\mathscr{F}}_\pm(\overset{I}{r}) = \frac{\overset{I}{\mathscr{E}}\overset{I}{r}V^{1/2}(\overset{I}{r})}{1-\overset{I}{\mathscr{E}}^2} - \frac{(\overset{I}{\mathscr{E}}^2-1/2)[\overset{I}{m}_+\overset{I}{\mathscr{E}} \pm (\overset{I}{\mathscr{E}}^2-1)\overset{I}{m}]}{(\overset{I}{\mathscr{E}}^2-1)^{3/2}} \arcsin\left(\frac{\overset{I}{m}_+ - 2(\overset{I}{\mathscr{E}}^2-1)\overset{I}{r}}{[\overset{I}{m}_+^2 + \overset{I}{m}^2(\overset{I}{\mathscr{E}}^2-1)]^{1/2}}\right), \quad (\overset{I}{\mathscr{E}}^2 > 1) \tag{39}$$

and

$$\overset{I}{\mathscr{G}}_\pm(\overset{I}{r}) = \frac{(\overset{I}{m}_+ \pm \overset{I}{m}\overset{I}{\mathscr{E}})[\overset{I}{m}_+\overset{I}{\mathscr{E}} \pm \overset{I}{m}(\overset{I}{\mathscr{E}}^2-1/2)]}{\overset{I}{\mathscr{P}}_\pm^{1/2}} \times \ln\left|(2\overset{I}{\mathscr{E}}^2-1)\overset{I}{m}_+ \pm 2\overset{I}{m}\overset{I}{\mathscr{E}}(\overset{I}{\mathscr{E}}^2-1) + \frac{2\overset{I}{\mathscr{P}}_\pm - 2\overset{I}{\mathscr{P}}_\pm^{1/2}\overset{I}{r}V^{1/2}(\overset{I}{r})}{\overset{I}{r} - \overset{I}{m}_+ \mp \overset{I}{m}\overset{I}{\mathscr{E}}}\right|, \tag{40}$$

where

$$\overset{I}{\mathscr{P}}_\pm = (\overset{I}{\mathscr{E}}^2-1)(\overset{I}{m}_+ \pm \overset{I}{m}\overset{I}{\mathscr{E}})^2 + \overset{I}{m}_+(\overset{I}{m}_+ \pm \overset{I}{m}\overset{I}{\mathscr{E}}) + \overset{I}{m}^2/4 = (\overset{I}{m}_+ \pm \overset{I}{m}\overset{I}{\mathscr{E}})^2 V(\overset{I}{m}_+ \pm \overset{I}{m}\overset{I}{\mathscr{E}}) > 0. \tag{41}$$

III. COLLISION OF TWO SHELLS

A. Transparent Shells

We consider here the situation in which two shells $\overset{I}{\mathscr{S}}$ and $\overset{I+1}{\mathscr{S}}$ with respective proper masses $\overset{I}{m}$ and $\overset{I+1}{m}$, collide with each other at the two-sphere p with radius r_0. Then, the world-surfaces of two shells will divide space-time into four regions, $\mathscr{V}_a$ ($a = I, I+1, I+2, (I+1)'$). Suppose that $\overset{I}{\mathscr{S}}$ divides $\mathscr{V}_I$ and $\mathscr{V}_{I+1}$, $\overset{I+1}{\mathscr{S}}$ divides $\mathscr{V}_{I+1}$ and $\mathscr{V}_{I+2}$, $\overset{I}{\mathscr{S}}{}'$ divides $\mathscr{V}_I$ and $\mathscr{V}_{(I+1)'}$, and $\overset{I+1}{\mathscr{S}}{}'$ divides $\mathscr{V}_{(I+1)'}$ and $\mathscr{V}_{I+2}$.

FIGURES

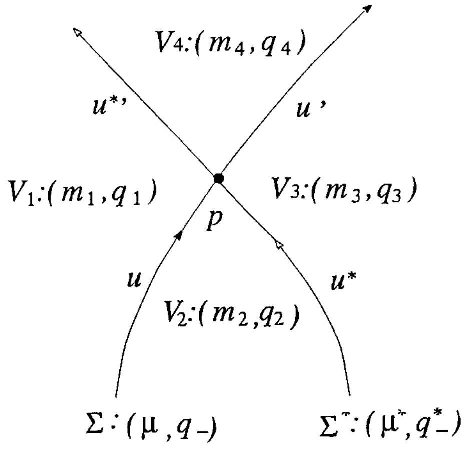

FIG. 1. The configuration of $\overset{I}{\mathscr{S}}$, $\overset{I+1}{\mathscr{S}}$ and $\mathscr{V}_a$ $(a = I-1, I, I+1, I')$.

We require that the charge of each shell does not change and that each proper mass is continuous throughout the collision. Each region $\mathscr{V}_a$ has the static coordinate system $\{t_a, r, \theta, \phi\}$ of the RNdS space-time with the parameters M_a, Q_a and Λ.

We now consider the problem of determining $M_{I'}$, $Q_{I'}$ and the velocity of each shell, when M_a, Q_a $(a = I, I+1, I+2)$, Λ, $\overset{I}{m}$ and $\overset{I+1}{m}$ are given. The charge of each shell is obtained by simply substracting Q_a of inside region from Q_a of the outside region, according to Gauss's law. Let $\mathscr{V}_I$ be the inside region and $\mathscr{V}_{I+2}$ the outside region without loss of generality. Then $\overset{I}{\mathscr{S}}$ and $\overset{I+1}{\mathscr{S}}$ have the charges $\overset{I}{q}_- = Q_{I+1} - Q_I$ and $\overset{I+1}{q}_- = Q_{I+2} - Q_{I+1}$, respectively. Since they must be conserved throughout the crossing, $Q_{(I+1)'} = Q_I + \overset{I+1}{q}_- = Q_I - Q_{I+1} + Q_{I+2}$. The mass parameter $M_{(I+1)'}$ of $\mathscr{V}_{(I+1)'}$ is not known until the velocity of a shell after the collision is determined.

Let $\overset{I}{\mathscr{S}}$ ($\overset{I+1}{\mathscr{S}}$) have velocity $\overset{I}{u} = d/d\overset{I}{\tau}$ ($\overset{I+1}{u} = d/d\overset{I+1}{\tau}$) and the outward normal $\overset{I}{n}$ ($\overset{I+1}{n}$), and let $\overset{I}{\mathscr{S}}{}'$ ($\overset{I+1}{\mathscr{S}}{}'$) have velocity $\overset{I}{u}{}' = d/d\overset{I}{\tau}{}'$ ($\overset{I+1}{u}{}' = d/d\overset{I+1}{\tau}{}'$), and the outward normal $\overset{I}{n}{}'$ ($\overset{I+1}{n}{}'$),where both $\overset{I}{u}$ and $\overset{I+1}{u}$ are future pointing. Applying Eqs. (23), (24) and (26), we can determine the velocity and normal of each shell before the collision up to two sign ambiguities. When $\overset{I}{u}, \overset{I}{n}$, $\overset{I+1}{u}$ and $\overset{I+1}{n}$ are written in the forms

$$\overset{I}{u} = (dt_I/d\overset{I}{\tau})\partial_{t_I} + (dr/d\overset{I}{\tau})\partial_r, \tag{42}$$

$$= (dt_{I+1}/d\overset{I}{\tau})\partial_{t_{I+1}} + (dr/d\overset{I}{\tau})\partial_r, \tag{43}$$

$$\overset{I}{n} = -(dr/d\overset{I}{\tau})dt_I + (dt_I/d\overset{I}{\tau})dr \tag{44}$$

$$= -(dr/d\overset{I}{\tau})dt_{I+1} + (dt_{I+1}/d\overset{I}{\tau})dr, \tag{45}$$

$$\overset{I+1}{u} = (dt_{I+1}/d\overset{I+1}{\tau})\partial_{t_{I+1}} + (dr/d\overset{I+1}{\tau})\partial_r, \tag{46}$$

$$= (dt_{I+2}/d\overset{I+1}{\tau})\partial_{t_{I+2}} + (dr/d\overset{I+1}{\tau})\partial_r, \tag{47}$$

$$\overset{I+1}{n} = -(dr/d\overset{I+1}{\tau})dt_{I+1} + (dt_{I+1}/d\overset{I+1}{\tau})dr \tag{48}$$

$$= -(dr/d\overset{I+1}{\tau})dt_{I+2} + (dt_{I+2}/d\overset{I+1}{\tau})dr, \tag{49}$$

the respective components are expressed as

$$dt_I/d\overset{I}{\tau} = (f_I - f_{I+1} + \overset{I}{\nu}{}^2)/2\overset{I}{\nu} f_I, \tag{50}$$

$$dt_{I+1}/d\overset{I}{\tau} = (f_I - f_{I+1} - \overset{I}{\nu}{}^2)/2\overset{I}{\nu} f_{I+1}, \tag{51}$$

$$dr/d\overset{I}{\tau} = \overset{I}{\epsilon}[\overset{I}{\nu}{}^4 - 2(f_I + f_{I+1})\overset{I}{\nu}{}^2 + (f_I - f_{I+1})^2]^{1/2}/2\overset{I}{\nu}, \tag{52}$$

$$dt_{I+1}/d\overset{I+1}{\tau} = (f_{I+1} - f_{I+2} + \overset{I+1}{\nu}{}^2)/2\overset{I+1}{\nu} f_{I+1}, \tag{53}$$

$$dt_{I+2}/d\overset{I+1}{\tau} = (f_{I+1} - f_{I+2} - \overset{I+1}{\nu}{}^2)/2\overset{I+1}{\nu} f_{I+2}, \tag{54}$$

$$dr/d\overset{I+1}{\tau} = \overset{I+1}{\epsilon}[\overset{I+1}{\nu}{}^4 - 2(f_{I+2} + f_{I+1})\overset{I+1}{\nu}{}^2 + (f_{I+2} - f_{I+1})^2]^{1/2}/2\overset{I+1}{\nu}, \tag{55}$$

where $\overset{I}{\nu} := \overset{I}{m}/r$, $\overset{I+1}{\nu} := \overset{I+1}{m}/r$ and $f_a := 1 - 2M_a/r + Q_a^2/r^2 - \Lambda r^2/3$, $(a = I, I+1, I+2)$, and $\overset{I}{\epsilon}$ ($\overset{I+1}{\epsilon}$) is the sign factor with respect to the velocity of the shell $\overset{I}{\mathscr{S}}$ ($\overset{I+1}{\mathscr{S}}$); namely, if $\epsilon = 1$ then $\mathscr{S}$ is expanding, while otherwise it is contracting.

We now need a boundary condition to determine $M_{(I+1)'}$ and the velocity of each shell after the collision. We impose here the *condition of transparent shells*, by which we mean that the velocities of $\overset{I}{\mathscr{S}}$ and $\overset{I+1}{\mathscr{S}}$ are conserved throughout the collision; namely, $\overset{I}{u}$ and $\overset{I+1}{u}$ continuously join $\overset{I+1}{u}{}'$ and $\overset{I}{u}{}'$ at p, respectively. (There is no notion indicating the continuity of vector fields across a singular hypersurface a priori. However, when a continuous metric is given as in the present situation, we may naturally introduce this notion by identifying the normal vectors on both sides of the singular hypersurface.) This condition is equivalent to that of Núñez et al [17]. Other conditions are, of course, possible. However this one is geometrically invariant and has a simple physical meaning, and thus it would be among the most natural assumptions. Furthermore, this uniquely determines $M_{(I+1)'}$, as we show in the following. To realize this condition, we evaluate u at p with respect to the coordinate system $\{t_{I+2}, r\}$ (coordinates θ and ϕ are ignored). We decompose u into the directions $\overset{I+1}{u}$ and $\overset{I+1}{n}$ Since the metric g on $\overset{I+1}{\mathscr{S}}$ has the form $g_{ij} = -\overset{I+1}{u}_i \overset{I+1}{u}_j + \overset{I+1}{n}_i \overset{I+1}{n}_j + r^2 H_{ij}$, $\overset{I}{u}$ can be expressed in the form $\overset{I}{u}{}^i = -\overset{I}{u}_j \overset{I+1}{u}{}^j \overset{I+1}{u}{}^i + \overset{I}{u}_j \overset{I+1}{n}{}^j \overset{I+1}{n}{}^i$ on $\overset{I+1}{\mathscr{S}}$ Since both $\overset{I+1}{u}$ and $\overset{I+1}{n}$ can be expressed with respect to the coordinate system $\{t_{I+2}, r\}$, we obtain the desired expression for u at p. As we have required, $\overset{I+1}{u}{}'$ continuously joins $\overset{I}{u}$ at p. Thus $\overset{I+1}{u}{}' = (dt_{I+2}/d\overset{I+1}{\tau}{}')\partial_{t_{I+2}} + (dr/d\overset{I+1}{\tau}{}')\partial_r$ becomes

$$\overset{I+1}{u}{}' = \left\{ f_{I+1} \frac{dt_{I+1}}{d\overset{I}{\tau}} \frac{dt_{I+1}}{d\overset{I+1}{\tau}} \frac{dt_{I+2}}{d\overset{I+1}{\tau}} - f_{I+1}^{-1} \frac{dr}{d\overset{I}{\tau}} \frac{dr}{d\overset{I+1}{\tau}} \frac{dt_{I+2}}{d\overset{I+1}{\tau}} - f_{I+2}^{-1} \left[\frac{dt_{I+1}}{d\overset{I}{\tau}} \left(\frac{dr}{d\overset{I+1}{\tau}} \right)^2 - \frac{dr}{d\overset{I}{\tau}} \frac{dt_{I+1}}{d\overset{I+1}{\tau}} \frac{dr}{d\overset{I+1}{\tau}} \right] \right\} \partial_{t_{I+2}}$$
$$+ \left\{ f_{I+1} \frac{dt_{I+1}}{d\overset{I}{\tau}} \frac{dt_{I+1}}{d\overset{I+1}{\tau}} \frac{dr}{d\overset{I+1}{\tau}} - f_{I+2} \left[\frac{dt_{I+1}}{d\overset{I}{\tau}} \frac{dt_{I+2}}{d\overset{I+1}{\tau}} \frac{dr}{d\overset{I+1}{\tau}} - \frac{dr}{d\overset{I}{\tau}} \frac{dt_{I+1}}{d\overset{I+1}{\tau}} \frac{dt_{I+2}}{d\overset{I+1}{\tau}} \right] - f_{I+1}^{-1} \frac{dr}{d\overset{I}{\tau}} \left(\frac{dr}{d\overset{I+1}{\tau}} \right)^2 \right\} \partial_r. \quad (56)$$

Therefore, using Eqs. (50)–(55), we obtain the relationships

$$\begin{aligned} dt_{I+2}/d\overset{I+1}{\tau}{}' = {} & (4\overset{I}{\nu} f_{I+1} f_{I+2})^{-1} \{ (\overset{I}{\nu}{}^2 - f_I + f_{I+1})(\overset{I+1}{\nu}{}^2 - f_{I+1} - f_{I+2}) \\ & - \overset{I}{\epsilon}\overset{I+1}{\epsilon} [\overset{I}{\nu}{}^4 - 2(f_I + f_{I+1})\overset{I}{\nu}{}^2 + (f_I - f_{I+1})^2]^{1/2} \\ & \times [\overset{I+1}{\nu}{}^4 - 2(f_{I+2} + f_{I+1})\overset{I+1}{\nu}{}^2 + (f_{I+2} - f_{I+1})^2]^{1/2} \}, \end{aligned} \quad (57)$$

and

$$\begin{aligned} dr/d\overset{I+1}{\tau}{}' = {} & (\overset{I+1}{\epsilon}/4\overset{I}{\nu} f_{I+1}) \\ & \times [(f_I - f_{I+1} - \overset{I}{\nu}{}^2)\{\overset{I+1}{\nu}{}^4 - 2(f_{I+2} + f_{I+1})\overset{I+1}{\nu}{}^2 + (f_{I+2} - f_{I+1})^2\}^{1/2} \\ & + (f_{I+2} + f_{I+1} - \overset{I+1}{\nu}{}^2)\{\overset{I}{\nu}{}^4 - 2(f_I + f_{I+1})\overset{I}{\nu}{}^2 + (f_I - f_{I+1})^2\}^{1/2}]. \end{aligned} \quad (58)$$

Finally, applying Eqs. (24) and (57), we obtain an expression for $M_{(I+1)'}$ after some manupulations:

$$
\begin{aligned}
M_{(I+1)'} = &-(\overset{I}{\nu}^2 + \overset{I+1}{\nu}^2) r_0/4 + (f_{I+1} - f_I - f_{I+2}) r_0/4 \\
&- (\overset{I}{\nu}^2 - f_I)(\overset{I+1}{\nu}^2 - f_{I+2}) r_0/4 f_{I+1} + r/2 + Q^2_{(I+1)'}/2r_0 - \Lambda r_0^3/6 \\
&-(\overset{I}{\epsilon}\overset{I+1}{\epsilon} r_0/4 f_{I+1})[\overset{I}{\nu}^4 - 2(f_I + f_{I+1})\overset{I}{\nu}^2 + (f_I - f_{I+1})^2]^{1/2} \\
&\times[\overset{I+1}{\nu}^4 - 2(f_{I+2} + f_{I+1})\overset{I+1}{\nu}^2 + (f_{I+2} - f_{I+1})^2]^{1/2},
\end{aligned}
\tag{59}
$$

or equivalently,

$$
\begin{aligned}
f_{(I+1)'} :&= 1 - 2M_{(I+1)'}/r_0 + Q^2_{(I+1)'}/r_0^2 - \Lambda r_0^2/3 \\
&= (\overset{I}{\nu}^2 + \overset{I+1}{\nu}^2)/2 + (f_I - f_{I+1} + f_{I+2})/2 + (\overset{I}{\nu}^2 - f_I)(\overset{I+1}{\nu}^2 - f_{I+2})/2f_{I+1} \\
&\quad + (\overset{I}{\epsilon}\overset{I+1}{\epsilon}/2f_{I+1})[\overset{I}{\nu}^4 - 2(f_I + f_{I+1})\overset{I}{\nu}^2 + (f_I - f_{I+1})^2]^{1/2} \\
&\quad \times [\overset{I+1}{\nu}^4 - 2(f_{I+2} + f_{I+1})\overset{I+1}{\nu}^2 + (f_{I+2} - f_{I+1})^2]^{1/2}.
\end{aligned}
\tag{60}
$$

We can obtain $\overset{I+1}{u}'$, or equivalently $dt_I/d\overset{I+1}{\tau}'$ and $dr/d\overset{I+1}{\tau}'$, in a similar manner. However, we have only to substitute as $t_I \leftrightarrow t_{I+2}$, $f_I \leftrightarrow f_{I+2}$, $\overset{I}{\tau} \leftrightarrow \overset{I+1}{\tau}$, $\overset{I}{\nu} \leftrightarrow \overset{I+1}{\nu}$ in Eqs. (57) and (58). Both $M_{(I+1)'}$ and $f_{(I+1)'}$ are of course invariant under such substitution. Equation (60) is a generalization of the DTR relation to the case of timelike shells, which is uniquely determined under the condition that two shells are transparent.

B. Neutral Null Shell Limit

We consider here the neutral null shell limit and rederive the original DTR relation. It has been pointed out by Núñez et al. [17] that it is possible for $f_I + f_{I+2} = f_{I+1} + f_{(I+1)'}$ or $f_{(I+1)'} = f_{I+1}$ to hold instead of the DTR relation $f_I f_{I+2} = f_{I+1} f_{(I+1)'}$ in this limit. We show in the following that the former can occur only when one of the shells violates the energy condition and that the latter is irrelevant.

The neutral null shell limit of, say $\mathscr{S}$, is formally achieved by setting $P = 0$ and taking $m, q \to 0$. Let λ be the affine parameter of $\mathscr{S}$, which can be constructed by the formal substitution $\lambda = \tau/m$. The tangent null vector $d/d\lambda$ is future pointing only when the limit $m \to 0$ is taken with keeping $m > 0$.

By taking the limits of Eqs. (23), (24) and (26), we obtain the components of each velocity of the two neutral null shells with respect to the coordinate system $\{t_I, r\}$:

$$\overset{I}{\mathscr{S}} \ : \ dt_{I+1}/d\overset{I}{\lambda} = \ (f_I - f_{I+1})r/2f_{I+1}, \tag{61}$$

$$dr/d\overset{I}{\lambda} = \ \overset{I}{\epsilon}|f_I - f_{I+1}|r/2, \tag{62}$$

$$\overset{I+1}{\mathscr{S}} : dt_{I+1}/d\overset{I+1}{\lambda} = (f_{I+1} - f_{I+2})r/2f_{I+1}, \tag{63}$$

$$dr/d\overset{I+1}{\lambda} = \ \overset{I+1}{\epsilon}|f_{I+1} - f_{I+2}|r/2, \tag{64}$$

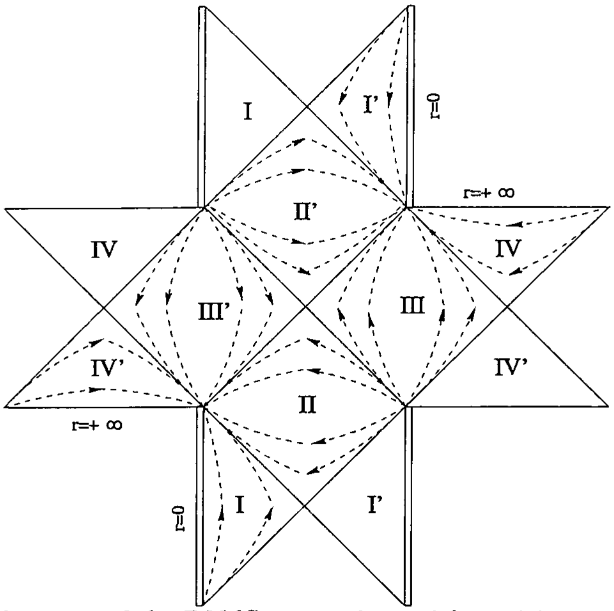

FIG. 2. The Penrose diagram of the RNdS space-time with positive cosmological constant when three kinds of Killing horizon exist. Double lines are the curvature singularities, dashed lines the hypersurfaces: $r =$ const and the directions in which t increses are represented by the arrows on the dashed lines.

Let the point of collision be located in block I or III in Fig. 2 with respect to the region $\mathscr{V}_{I+1}$, which is characterized by properties that (P1)t_{I+1} strictly increases in the future null directions, (P2)r strictly increases (decreases) in the out-going (in-going) null directions, and (P3)f_{I+1} is positive. Then, the inequalities $dt_{I+1}/d\overset{I}{\lambda} > 0$ and $dt_{I+1}/d\overset{I+1}{\lambda} > 0$ hold by the property (P1), and thus $f_I > f_{I+1} > f_{I+2}$ is obtained from Eqs. (61) and (63) with the property (P3). Furthermore, the property (P2) leads to $\overset{I}{\epsilon} = 1$ and $\overset{I+1}{\epsilon} = -1$. Similary, we can show that the relations $f_I < f_{I+1}$, $f_{I+1} > f_{I+2}$ and $\overset{I}{\epsilon} = \overset{I+1}{\epsilon} = 1$ hold both in II and

IV, the relations $f_I < f_{I+1} < f_{I+2}$, $\overset{I}{\epsilon} = -1$ and $\overset{I+1}{\epsilon} = 1$ hold both in I′ and III′, and the relations $f_I > f_{I+1}$, $f_{I+1} < f_{I+2}$ and $\overset{I}{\epsilon} = \overset{I+1}{\epsilon} = -1$ hold both in II′ and IV′ The four cases listed above are exhaustive, even if $\mathscr{V}_{I+1}$ has degenerate horizons, negative or vanishing mass and/or Λ. In all cases, the following relationship holds:

$$\overset{I}{\epsilon}\overset{I+1}{\epsilon}|f_I - f_{I+1}||f_{I+2} - f_{I+1}| = (f_I - f_{I+1})(f_{I+2} - f_{I+1}). \tag{65}$$

Then, Eq. (60) reduces to the well-known DTR relation, [8,20]

$$f_{I+1} f_{(I+1)'} = f_I f_{I+2}. \tag{66}$$

Equation (65) is violated only if one of $d/d\overset{I}{\lambda}$ and $d/d\overset{I+1}{\lambda}$ is past pointing; i.e., one of shells does not satisfy the energy condition. Then the following relationships [instead of Eqs. (65) and (66)] hold:

$$\overset{I}{\epsilon}\overset{I+1}{\epsilon}|f_I - f_{I+1}||f_{I+2} - f_{I+1}| = -(f_I - f_{I+1})(f_{I+2} - f_{I+1}), \tag{67}$$

and

$$f_{I+1} + f_{(I+1)'} = f_I + f_{I+2}. \tag{68}$$

C. Collisions Near Horizons

When two neutral null shells collide near the Cauchy horizon in the Reissner-Nordström (RN) space-time, it is seen from the DTR relation that the resulting mass, $M_{(I+1)'}$, is very large. This is a simple model of the mass inflation. [19]

We treat here the collision near the horizon of two timelike shells. For example, consider a collision slightly outside the inner black hole horizon of $\mathscr{V}_{I+1}$, $r = r_- := M_{I+1} - (M_{I+1}^2 - Q_{I+1}^2)^{1/2}$ in the RN space-time with $M_{I+1} > Q_{I+1}$ (see Fig. 3). This restriction is not essential and the following discussion may be applied to any horizon in general.

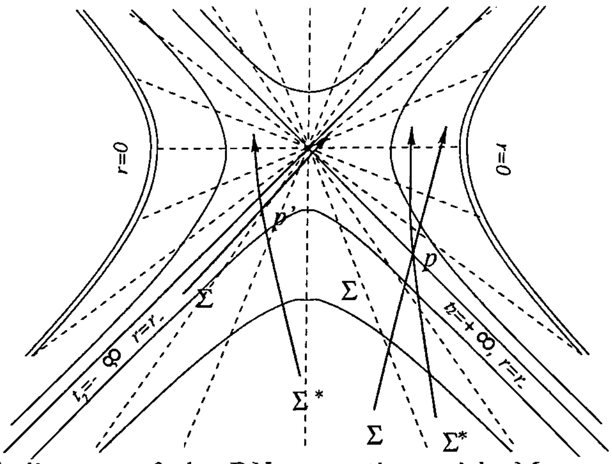

FIG. 3. The Kruskal diagram of the RN space-time with $M_{I+1} > Q_{I+1}$. The solid lines denote hypersurfaces ($r = \text{const}$), the double lines the singularities ($r = 0$), and the dashed lines hypersurfaces ($t_{I+1} = \text{const}$). Mass inflation does not occur at p, while it may occur at p', where a falling shell and another escaping one collide.

Let the radius of the collision be $r_0 = r_- + M_{I+1}\delta$, where δ is a small positive parameter. The quantity $f_{I+1} = O(\delta)$ is negative, since the collision is assumed to occur outside the inner horizon. When δ is sufficiently small, both $dr/d\overset{I}{\tau}$ and $dr/d\overset{I+1}{\tau}$ are positive, so $\overset{I}{\epsilon} = \overset{I+1}{\epsilon} = -1$. Assume that both shells fall into the same horizon, i.e. both $dt_{I+1}/d\overset{I}{\tau}$ and $dt_{I+1}/d\overset{I+1}{\tau}$ are possitive (possibly both negative). Then, from Eqs. (51) and (53), we obtain the following inequalities:

$$f_I - \overset{I}{\nu}^2 < f_{I+1}, \tag{69}$$

$$f_{I+2} - \overset{I+1}{\nu}^2 > f_{I+1}. \tag{70}$$

The condition $f_{I+1} < 0$ leads to the result that the l.h.s. of Eq. (69) is negative:

$$f_I - \overset{I}{\nu}^2 < 0. \tag{71}$$

While the l.h.s. of Eq. (70) may be negative, however, in the limit $\delta \to +0$, this is not possible. To show this, note that $dt_{I+1}/d\overset{I+1}{\tau}$ must diverge in this limit. If $f_{I+2} - \overset{I+1}{\nu}^2 \leq -f_{I+1}$ is kept, then $dt_{I+1}/d\overset{I+1}{\tau} \leq 1/\overset{I+1}{\nu}$ holds by Eq. (53). This leads to a contradiction, so we conclude that $f_{I+2} - \overset{I+1}{\nu}^2 > -f_{I+1}$ holds in the limit $\delta \to +0$. In particular,

$$f_{I+2} - \overset{I+1}{\nu}^2 > 0, \tag{72}$$

is obtained.

On the other hand, Eq. (60) becomes

$$f_{(I+1)'} = [(\overset{I}{\nu}^2 - f_I)(\overset{I+1}{\nu}^2 - f_{I+2}) + \overset{I}{\epsilon}\overset{I+1}{\epsilon}|\overset{I}{\nu}^2 - f_I||\overset{I+1}{\nu}^2 - f_{I+2}|]/2f_{I+1} + O(1), \tag{73}$$

where we have singled out the only possible singular terms. However, these singular terms cancel when $\overset{I}{\epsilon} = \overset{I+1}{\epsilon} = -1$ under the conditions (71) and (72). Thus, $f_{(I+1)'}$ remains finite in the limit $\delta \to +0$. We obtain a similar result in the case $\delta < 0$. Moreover, $f_{(I+1)'}$ converges to the same value in both limits:

$$\lim_{\delta\to 0} f_{(I+1)'} = \frac{(f_I f_{I+2} - \overset{I}{\nu}^2\overset{I+1}{\nu}^2)(f_I + f_{I+2} - \overset{I}{\nu}^2 - \overset{I+1}{\nu}^2)}{(f_I - \overset{I}{\nu}^2)(f_{I+2} - \overset{I+1}{\nu}^2)}. \tag{74}$$

This expression would be valid even if the horizon were an outer black hole or cosmological horizon, or a degenerate or non-degenerate horizon. The quantity $M_{(I+1)'}$ also converges to a finite value, which suggests that phenomena like mass inflation do not occur just in the presence of a massive flow falling into the horizon. From a physical viewpoint, the result is natural, since timelike flows of matter themselves do not suffer the infinite blue-shift, unlike null flows or gravitational waves.

Secondly, we assume that one of the shells does not fall into the inner black hole but instead tries to escape from it (either $dt/d\overset{I}{\tau}$ or $dt/d\overset{I+1}{\tau}$ is negative). This implies that this shell would fall into the other horizon in the limit $\delta \to 0$. Then a similar argument leads to

$$f_{(I+1)'} = (f_I - \overset{I}{\nu}^2)(f_{I+2} - \overset{I+1}{\nu}^2)/f_{I+1} + O(1) \to -\infty. \ (f_{I+1} \to 0), \tag{75}$$

This means that $M_{(I+1)'}$ can in general take arbitrarily large positive values in this case. The collision of neutral null shells rather corresponds to this case; it is not sufficient to take the limit $\overset{I}{\nu}, \overset{I+1}{\nu}, \overset{I}{q}, \overset{I+1}{q}_- \to 0$ in Eq. (74) to obtain the neutral null shell limit, since the conditions $dt/d\overset{I}{\tau} > 0$ and $dt/d\overset{I+1}{\tau} > 0$ are no longer satisfied in the case of the collision of null shells.

Thus, we obtain another conclusion that mass inflation occurs when a falling shell and another escaping one collide near the horizon.

Here we examine the relative velocity of colliding shells. Define the Lorentz factor between $\overset{I}{\mathscr{S}}$ and $\overset{I+1}{\mathscr{S}}$ by $\gamma := -\overset{I}{u}_i \overset{I+1}{u}^i$, which may be related to the relative velocity v by

$\gamma = (1 - v^2)^{-1/2}$ When two falling shells collide near the horizon, we obtain

$$\gamma = \overset{I}{\nu}(f_{I+2} - \overset{I+1}{\nu}{}^2)/2\overset{I+1}{\nu}(\overset{I}{\nu}{}^2 - f_I) + \overset{I+1}{\nu}(\overset{I}{\nu}{}^2 - f_I)/2\overset{I}{\nu}(f_{I+2} - \overset{I+1}{\nu}{}^2) + O(|\delta|), \tag{76}$$

while when a falling shell and another escaping shell collide, we have

$$\gamma = (\overset{I}{\nu}{}^2 - f_I)(f_{I+2} - \overset{I+1}{\nu}{}^2)/2\overset{I}{\nu}\overset{I+1}{\nu} f_{I+1} + O(1). \tag{77}$$

The former remains finite, while the latter in general diverges with $\delta \to 0$. This is another feature distinguishing two types of the collision near the horizon.

D. Collision of Two Dust–Shells

So far, the treatment of colliding shells are disscussed in general cases. We shall however investigate a simpler case of colliding dust–shells to make physical meaning clear.

It is convenient to introduce the following quantities defined at the moment of the collision:

$$\overset{I}{p} := \overset{I}{m} \frac{dr}{d\overset{I}{\tau}}\bigg|_{r=r_0}, \tag{78}$$

$$\overset{I}{w} := \frac{\overset{I}{m}{}^2}{2r_0}, \tag{79}$$

$$\overset{I}{E} := M_{I+1} - M_I, \tag{80}$$

and

$$\overset{I}{e}_\pm := \overset{I}{m}\overset{I}{\mathscr{E}} \mp \overset{I}{w}. \tag{81}$$

The quantity $\overset{I}{p}$ corresponds to the three-momentum, $\overset{I}{w}$ the self-gravity, and $\overset{I}{E}$ the energy of the I^{th} dust–shell. The equation (26) can be expressed in the form

$$\overset{I}{m}{}^2 \left(1 - \frac{2M_{I+1}}{r_0}\right) = \overset{I}{e}_+{}^2 - \overset{I}{p}{}^2, \tag{82}$$

or

$$\overset{I}{m}{}^2 \left(1 - \frac{2M_I}{r_0}\right) = \overset{I}{e}_-{}^2 - \overset{I}{p}{}^2, \tag{83}$$

and the Eqs. (23),(24) become

$$\left(1-\frac{2M_I}{r_0}\right)\frac{dt_I}{d\overset{I}{\tau}} = \overset{I}{m}\overset{I}{e}_-, \tag{84}$$

$$\left(1-\frac{2M_{I+1}}{r_0}\right)\frac{dt_{I+1}}{d\overset{I}{\tau}} = \overset{I}{m}\overset{I}{e}_+. \tag{85}$$

The velocities of the dust–shells $\overset{I}{\mathscr{S}}{}'$ and $\overset{I+1}{\mathscr{S}}{}'$ can be written as

$$\overset{I}{u}{}'^i = -\overset{I}{u}_j\overset{I+1}{u}{}^j\overset{I+1}{u}{}^i + \overset{I}{u}_j\overset{I+1}{n}{}^j\overset{I+1}{n}{}^i, \tag{86}$$

$$\overset{I+1}{u}{}'^i = -\overset{I}{u}_j\overset{I+1}{u}{}^j\overset{I}{u}{}^i + \overset{I+1}{u}_j\overset{I}{n}{}^j\overset{I}{n}{}^i. \tag{87}$$

Using the coordinate system of $\mathscr{V}_{I+1}$, we obtain

$$\overset{I}{u}_j\overset{I+1}{u}{}^j = \frac{\overset{I}{p}\overset{I+1}{p} - \overset{I}{e}_+\overset{I+1}{e}_-}{\overset{I}{m}\overset{I+1}{m}(1-2M_{I+1}/r_0)}, \tag{88}$$

$$\overset{I}{u}_j\overset{I+1}{n}{}^j = -\overset{I+1}{u}_j\overset{I}{n}{}^j = \frac{\overset{I+1}{e}_-\overset{I}{p} - \overset{I}{e}_+\overset{I+1}{p}}{\overset{I}{m}\overset{I+1}{m}(1-2M_{I+1}/r_0)} \tag{89}$$

The components of the velocities of the dust–shells $\overset{I}{\mathscr{S}}{}'$ and $\overset{I+1}{\mathscr{S}}{}'$ become

$$\left(1-\frac{2M_{I+2}}{r_0}\right)\frac{dt_{I+2}}{d\overset{I+1}{\tau}{}'} = \frac{1}{\overset{I}{m}}\left(\overset{I}{E} - \overset{I}{w} + 2(\overset{I}{w}\overset{I+1}{w})^{1/2}\overset{I}{u}_i\overset{I+1}{u}{}^i\right), \tag{90}$$

$$\frac{dr}{d\overset{I+1}{\tau}{}'} = \frac{1}{\overset{I}{m}}\left(\overset{I}{p} - 2(\overset{I}{w}\overset{I+1}{w})^{1/2}\overset{I}{u}_i\overset{I+1}{n}{}^i\right), \tag{91}$$

and

$$\left(1-\frac{2M_I}{r_0}\right)\frac{dt_I}{d\overset{I}{\tau}{}'} = \frac{1}{\overset{I+1}{m}}\left(\overset{I+1}{E} + \overset{I+1}{w} - 2(\overset{I}{w}\overset{I+1}{w})^{1/2}\overset{I}{u}_i\overset{I+1}{u}{}^i\right), \tag{92}$$

$$\frac{dr}{d\overset{I}{\tau}{}'} = \frac{1}{\overset{I+1}{m}}\left(\overset{I+1}{p} - 2(\overset{I}{w}\overset{I+1}{w})^{1/2}\overset{I}{u}_i\overset{I+1}{n}{}^i\right) \tag{93}$$

From the above expressions, we find that the energy transfer is expressed as

$$\Delta\overset{I,I+1}{E} = -2(\overset{I}{w}\overset{I+1}{w})^{1/2}\overset{I}{u}_i\overset{I+1}{u}{}^i, \tag{94}$$

and that the three momenta after the collision is obtained from the formula

$$\overset{I}{p}{}' = \overset{I+1}{p} + \Delta\overset{I,I+1}{p}, \tag{95}$$

$$\overset{I+1}{p}{}' = \overset{I}{p} + \Delta\overset{I,I+1}{p}, \tag{96}$$

where

$$\Delta \overset{I,I+1}{p} = -2(\overset{I}{w}\overset{I+1}{w})^{1/2}\overset{I}{u}_i \overset{I+1}{n}{}^i, \tag{97}$$

Note that $\overset{I}{u}_i \overset{I+1}{u}{}^i$ should be negative. Hence the energy transfer $\Delta \overset{I,I+1}{E}$ is always positive, which means that the $\overset{I}{\mathscr{S}}$ always releases energy, while $\overset{I+1}{\mathscr{S}}$ gains energy. In addition, $\overset{I}{u}_i \overset{I+1}{u}{}^i$ should be positive at p to ensure that the two dust–shells collide. Thus, the three-momenta of both shells always decrease: $\Delta \overset{I,I+1}{p} < 0$.

The decrement of three-momenta due to the collision may be recognized as to be a kind of Ricci focusing effet. Let us consider an irrotational congruence of timelike geodesics. The increasing rate of volume of a spacelike section of this congruence with respect to proper time of a timelike geodesic in it is called expansion of this congruence. As is well known, when this congruence goes through a region filled by matter satisfying the strong energy condition, its expansion necessarily decreases. This is called Ricci focusing effect.

A dust shell is regarded as a spacelike section of a congruence of timelike geodesics. A collision between two dust–shells means that a congruence corresponding to the trajectory of one shell goes through the other shell satisfying the stron energy condition. Hence we expect that the congruence corresponding to one shell suffers a kind of Ricci focusing effect in the collision. For the congruence corresponding to the other shell, the same effect is also expected.

However, we should note that volume of a spacelike section corresponding to a shell vanishes identically, since we have assumed that the shell is infinietely thin. Hence, strictly speaking, there is no Ricci focusing effect in ordinary sense. However, since the area of the section does not vanish, we may consider the focusing effect with respect to the increasing rate of this area. The area of the spacelike section is given by $4\pi r^2$. The increasing rate θ of this area is given by

$$\theta = \frac{2}{r}\frac{dr}{d\tau} \tag{98}$$

From the above equation, we can see that the three-momentum $p = mdr/d\tau$ of the shell is essentially the same as the increasing rate θ of the area, where m is the proper mass of the shell. Hence noting that the areal radius of a shell is unchanged through the collision, the decrement of the three-momentum due to the collision may be regarded as a kind of Ricci focusing effect.

Here it is worthwhile to note that there is a relationship

$$\Delta \overset{I,I+1}{E}{}^2 - \Delta \overset{I,I+1}{p}{}^2 = 4\overset{I}{w}\overset{I+1}{w} \tag{99}$$

This shows that $\Delta \overset{I,I+1}{E}$ has a lower bound.

IV. MANY DUST–SHELL SYSTEM

The many dust–shell system can be regarded as a discretized version of the Tolman–Bondi model, which describes spherically symmetric dust universes. We shall show that the initial condition of the many dust–shell system is fixed in terms of the Toman–Bondi solution. We describe a manner to set up the initial condition for the Toman–Bondi solution. Then, we give a prescription to identigy the initial condition for the Toman solution with that for the many–shell system.

A. Initial condition for Tolman–Bondi space-time

The metric of Tolman-Bondi solution is written in the form

$$ds^2_{TB} = -dT^2 + \frac{(\partial r/\partial R)^2}{1-\mathscr{K}(R)}dR^2 + r^2(T,R)H_{ij}dx^i dx^j, \tag{100}$$

where $\mathscr{K}(R) < 1$ is an arbitrary function. The Einstein equation gives

$$\left(\frac{\partial r}{\partial T}\right)^2 = -\mathscr{K}(R) + \frac{2M(R)}{r}, \tag{101}$$

and

$$\rho(T,R) = \frac{M'(R)}{4\pi R^2(\partial r/\partial R)}, \tag{102}$$

where $M(R)$ is the Misner–Sharp mass function and $\rho(T,R)$ is the energy density of the dust. The differential equation (101) can be integrated. The solution is

$$r = \left(\frac{9M(R)}{2}\right)^{1/3}[T - T_B(R)]^{2/3},\ (\mathscr{K}(R) = 0) \tag{103}$$

$$r = \frac{M(R)}{-\mathscr{K}(R)}(\cosh\eta - 1),\ T - T_B(R) = \frac{M(R)}{[-\mathscr{K}(R)]^{3/2}}(\sinh\eta - \eta),\ (\mathscr{K}(R) < 0) \tag{104}$$

$$r = \frac{M(R)}{\mathscr{K}(R)}(1-\cos\eta),\ T - T_B(R) = \frac{M(R)}{[\mathscr{K}(R)]^{3/2}}(\eta - \sin\eta),\ (0 < \mathscr{K}(R) < 1) \tag{105}$$

where $T_B(R)$ is an arbitrary function which corresponds to the moment of the big bang singularity. The Tolman–Bondi solution is characterized by two arbitrary functions, namely the Misner–Sharp mass function $M(R)$ and the specific energy

$$\mathscr{E}(R) = (1 - \mathscr{K}(R))^{1/2} \tag{106}$$

There is a scaling freedom of the comoving radial coordinate R. Assuming that the areal radius r is a monotonic function of R at the initial time $T = T^*$, we fix this freedom by imposing $R = r$ at $T = T^*$. Then, the mass function can be written in terms of the initial energy density $\rho^*(R)$ as

$$M(R) = \int_0^R 4\pi R^2 \rho^*(R) dR. \tag{107}$$

In other words, the mass function can be fixed by the initial condition. The remaining degree of freedom is encoded in the function $\mathscr{K}(R)$ or $T_B(R)$. These two functions depend on each other through the relationship for $\mathscr{K}(R) = 0$

$$T_B(R) = T^* - \left(\frac{2R^3}{9M(R)}\right)^{1/2}, \ (\mathscr{K}(R) = 0) \tag{108}$$

for $\mathscr{K}(R) < 0$

$$T_B(R) = T^* - \frac{M(R)}{(-\mathscr{K}(R))^{3/2}}(\sinh\eta - \eta), \ (\mathscr{K}(R) < 0) \tag{109}$$

where

$$\eta = \operatorname{arccosh}\left(1 - \frac{\mathscr{K}(R)}{M(R)}R\right), \ (0 < \mathscr{K}(R) < 1) \tag{110}$$

and for $0 < \mathscr{K}(R) < 1$

$$T_B(R) = T^* - \frac{M(R)}{\mathscr{K}(R)^{3/2}}(\eta - \sin\eta), \tag{111}$$

where

$$\eta = \arccos\left(1 - \frac{\mathscr{K}(R)}{M(R)}R\right) \tag{112}$$

We give two methods to fix these arbitrary functions.

In the first method, we consider the expanding universe (unperturbed Hubble flow) as a background. At the initial time $T = T^*$, the fluid lines of dust is assumed to conincide with that of the unperturbed Hubble flow:

$$\frac{\partial r}{\partial T} = \left(-\mathscr{K}(R) + \frac{2M(R)}{r}\right) = H^* r, \ (T = T^*) \tag{113}$$

where $H^* = \text{const.}$ is the Hubble parameter of the background universe. Since r is equal to R initially, it turns out that the function $\mathscr{K}(R)$ has the form

$$\mathscr{K}(R) = \frac{2M(R)}{R} - H^{*2}R^2 \tag{114}$$

The second method is just to set $T_B(R) = 0$, which means that the big bang is simultaneous.

B. Motion of Dust–Shells in Synchronous Comoving Coordinate System

It is convenient to work in the synchromous comoving coordinate system to see the correspondance between the dust–shell universe and the Tolman-Bondi solution.

Let us consider the coordinate transformation from the static coordinate system $\{t_I, r\}$ into the synchronous comoving coordinate system $\{T_I, R_I\}$ of the Schwarzschild metric defined by

$$dt_I = \frac{[1-\mathscr{K}_I(R_I)]^{1/2}}{1-2M_I/r} dT_I + \frac{1}{(1-2M_I/r)[1-\mathscr{K}_I(R_I)]^{1/2}} \left(\frac{\partial r}{\partial T_I}\right)\left(\frac{\partial r}{\partial R_I}\right) dR_I, \tag{115}$$

$$dr = \left(\frac{\partial r}{\partial T_I}\right) dT_I + \left(\frac{\partial r}{\partial R_I}\right) dR_I, \tag{116}$$

with the condition

$$\frac{\partial r}{\partial T_I} = \epsilon_I \left(\frac{2M_I}{r} - \mathscr{K}_I(R_I)\right)^{1/2} \tag{117}$$

where $\mathscr{K}_I(R)$ is an arbitrary function and $\epsilon_I = \pm 1$.

The metric in the synchronous comoving coordinate system becomes

$$ds_I^2 = -dT_I^2 + \frac{(\partial r/\partial R_I)^2}{1-\mathscr{K}_I} dR_I^2 + r^2(T_I, R_I) H_{ij} dx^i dx^j \tag{118}$$

The inverse transformation of Eqs. (115), (116) is

$$dT_I = (1 - \mathscr{K}_I(R_I))^{1/2} dt_I - \epsilon_I \frac{(2M_I/r - \mathscr{K}_I(R_I))^{1/2}}{1 - 2M_I/r} dr, \tag{119}$$

$$dR_I = \left(\frac{\partial r}{\partial R_I}\right)^{-1} \left[-\epsilon_I (1 - \mathscr{K}_I(R_I))^{1/2} (2M_I/r - \mathscr{K}_I(R_I))^{1/2} dt_I + \frac{1 - \mathscr{K}_I(R_I)}{1 - 2M_I/r} dr\right] \tag{120}$$

The equation (119) can be easily integrated when $\mathscr{K}_I(R_I) = \mathscr{K}_I = \text{const.}$:

$$T_I - T_{0I} = (1 - \mathscr{K}_I)^{1/2} t_I - \epsilon_I \left[\mathscr{H}_I(r) + \mathscr{I}_I(r)\right], \tag{121}$$

where T_{0I} is an integration constant and the functions $\mathscr{H}_I(r)$ and $\mathscr{I}_I(r)$ are defined by

$$\mathscr{H}_I(r) = 2(2M_I r)^{1/2}, \ (\mathscr{K}_I = 0) \tag{122}$$

$$\begin{aligned} \mathscr{H}_I(r) &= (-\mathscr{K}_I r^2 + 2M_I r)^{1/2} \\ &+ \frac{M_I(1 - 2\mathscr{K}_I)}{(-\mathscr{K}_I)^{1/2}} \ln\left|M_I - \mathscr{K}_I r + \left[\mathscr{K}_I(\mathscr{K}_I r^2 - 2M_I r)\right]^{1/2}\right|, \ (\mathscr{K}_I < 0) \end{aligned} \tag{123}$$

$$\mathscr{H}_I(r) = (-\mathscr{K}_I r^2 + 2M_I r)^{1/2} - \frac{M_I(1 - 2\mathscr{K}_I)}{\mathscr{K}_I^{1/2}} \arcsin\left(1 - \frac{\mathscr{K}_I r}{M_I}\right), \ (\mathscr{K}_I > 0) \tag{124}$$

and

$$\begin{aligned} \mathscr{I}_I(r) &= 2M_I(1 - \mathscr{K}_I)^{1/2} \\ &\times \ln\left|\tfrac{1}{2} - \mathscr{K}_I + \frac{2M_I(1 - \mathscr{K}_I) - [(1 - \mathscr{K}_I)(-\mathscr{K}_I r^2 + 2M_I r)]^{1/2}}{r - 2M_I}\right|. \end{aligned} \tag{125}$$

C. Matching Condition of Two Synchronous Comoving Coordinate Systems

Next, let us consider the trajectries of $(I-1)^{\text{st}}$ and I^{th} dust–shells in terms of the synchronous comoving coordinate system of I^{th} region $\mathscr{V}_I$. Substituting Eq. (36) into Eq. (121) we obtain

$$\overset{I-1}{T}_I(\overset{I-1}{r}) = T_{0I} + (1 - \mathscr{K}_I)^{1/2} \left[\overset{I-1}{t}_{0+} + \overset{I-1}{\epsilon} \left(\overset{I-1}{\mathscr{F}}_+(\overset{I-1}{r}) + \overset{I-1}{\mathscr{G}}_+(\overset{I-1}{r})\right)\right] - \epsilon_I \left[\mathscr{H}_I(\overset{I-1}{r}) + \mathscr{I}_I(\overset{I-1}{r})\right] \tag{126}$$

$$\overset{I}{T}_I(\overset{I}{r}) = T_{0I} + (1 - \mathscr{K}_I)^{1/2} \left[\overset{I}{t}_{0-} + \overset{I}{\epsilon} \left(\overset{I}{\mathscr{F}}_-(\overset{I}{r}) + \overset{I}{\mathscr{G}}_-(\overset{I}{r})\right)\right] - \epsilon_I \left[\mathscr{H}_I(\overset{I}{r}) + \mathscr{I}_I(\overset{I}{r})\right] \tag{127}$$

One does not have to know the comoving radial coordinate R_I of the dust–shells to decide whether $(I-1)^{\rm st}$ and $I^{\rm th}$ shells collide; the collision between these shells occurs when the solution of the following equation exists:

$$\overset{I-1}{T}_I(r) = \overset{I}{T}_I(r). \tag{128}$$

To match the two synchronous comoving coordinate times refered by $\overset{I-1}{\mathscr{S}}$ and $\overset{I}{\mathscr{S}}$, we have to give the relationship between $\overset{I-1}{t}_{0+}$ and $\overset{I}{t}_{0-}$, which is obtained by the equation

$$\overset{I-1}{T}_I(\overset{I-1}{r}{}^*) = \overset{I}{T}_I(\overset{I}{r}{}^*), \tag{129}$$

where $\overset{I}{r}{}^*$ denotes the areal radius of $\overset{I}{\mathscr{S}}$ at $T_I = T^*$ Then from Eqs. (126) and (127), w obtain

$$\begin{aligned} \overset{I-1}{t}_{0+} &= -\overset{I-1}{\epsilon}\left(\overset{I-1}{\mathscr{F}}_+(\overset{I-1}{r}) + \overset{I-1}{\mathscr{G}}_+(\overset{I-1}{r})\right) \\ &\quad +(1-\mathscr{K}_I)^{-1/2}\left[T^* - T_{0I} + \epsilon_I\left(\mathscr{H}_I(\overset{I-1}{r}) + \mathscr{I}_I(\overset{I-1}{r})\right)\right], && (130) \\ \overset{I}{t}_{0-} &= -\overset{I}{\epsilon}\left(\overset{I}{\mathscr{F}}_-(\overset{I}{r}) + \overset{I}{\mathscr{G}}_-(\overset{I}{r})\right) \\ &\quad +(1-\mathscr{K}_I)^{-1/2}\left[T^* - T_{0I} + \epsilon_I\left(\mathscr{H}_I(\overset{I}{r}) + \mathscr{I}_I(\overset{I}{r})\right)\right] && (131) \\ & && (132) \end{aligned}$$

These gives the relationship

$$\begin{aligned} \overset{I-1}{t}_{0+} - \overset{I}{t}_{0-} &= -\overset{I-1}{\epsilon}\left(\overset{I-1}{\mathscr{F}}_+(\overset{I-1}{r}) + \overset{I-1}{\mathscr{G}}_+(\overset{I-1}{r})\right) + \overset{I}{\epsilon}\left(\overset{I}{\mathscr{F}}_-(\overset{I}{r}) + \overset{I}{\mathscr{G}}_-(\overset{I}{r})\right) \\ &\quad +\epsilon_I(1-\mathscr{K}_I)^{-1/2}\left(\mathscr{H}_I(\overset{I-1}{r}) + \mathscr{I}_I(\overset{I-1}{r}) - \mathscr{H}_I(\overset{I}{r}) - \mathscr{I}_I(\overset{I}{r})\right) \end{aligned} \tag{133}$$

D. Initial Condition for Dust–Shell Universe

The initial data for the many dust–shell system is set up by using the initial data of the Tolman-Bondi solution. In terms of $\mathscr{M}(R)$ and $\mathscr{K}(R)$ of the Tolman-Bondi solution, the initial data for the dust–shell system is given in the manner

$$M_1 = 0,\ M_{I+1} = 2M(\overset{I}{r}{}^*) - M_I,\ (I = 1, 2, 3, \cdots) \tag{134}$$

$$\overset{I}{\mathscr{E}} = (1 - \mathscr{K}(\overset{I}{r}{}^*))^{1/2} \tag{135}$$

Since $\overset{I}{\mathscr{E}} = (M_{I+1} - M_I)/\overset{I}{m}$, we find

$$M(\overset{I}{r}{}^*) - M_I = \tfrac{1}{2}\overset{I}{m}(1 - \mathscr{K}(\overset{I}{r}{}^*))^{1/2} \tag{136}$$

We consider the following parameter α

$$\alpha = \frac{2M_2}{\overset{1}{r}{}^*} = \frac{2\overset{1}{m}_+}{\overset{1}{r}{}^*} = \frac{4M(\overset{1}{r}{}^*)}{\overset{1}{r}{}^*}. \tag{137}$$

A smaller α guarantees higher accuracy of the N-body approximation. For given α, we obtain $\overset{1}{r}{}^*$ by definition. Then we obtain the proper mass of the first shell

$$\overset{1}{m} = \frac{2M(\overset{1}{r}{}^*)}{\sqrt{1 - \mathscr{K}(\overset{1}{r}{}^*)}}. \tag{138}$$

We assume that $\overset{I}{m} = \overset{1}{m}$ for all shells.

Next let us consider the synchronous comoving corrdinate system associated with this initial data. This is give by

$$\mathscr{K}_{I+1} = (1 - \overset{I}{\mathscr{E}}{}^2)^{1/2} \tag{139}$$

and

$$\epsilon_{I+1} = \overset{I}{\epsilon}. \tag{140}$$

We may set $T_{0I} = 0$ for all I. The integration constants associated with the motion of the dust-shells $\overset{I}{t}_\pm$ are fixed by the conditions

$$\overset{I-1}{T}_I(\overset{I-1}{r}{}^*) = \overset{I}{T}_I(\overset{I}{r}{}^*) = T^* \tag{141}$$

for all I. Then we obtain

$$\begin{aligned} \overset{I-1}{t}_{0+} = & -\overset{I-1}{\epsilon}\left(\overset{I-1}{\mathscr{F}}_+(\overset{I-1}{r}) + \overset{I-1}{\mathscr{G}}_+(\overset{I-1}{r})\right) \\ & +(1 - \mathscr{K}_I)^{-1/2}\left[T^* + \epsilon_I\left(\mathscr{H}_I(\overset{I-1}{r}{}^*) + \mathscr{I}_I(\overset{I-1}{r}{}^*)\right)\right], \end{aligned} \tag{142}$$

$$\begin{aligned} \overset{I}{t}_{0-} = & -\overset{I}{\epsilon}\left(\overset{I}{\mathscr{F}}_-(\overset{I}{r}{}^*) + \overset{I}{\mathscr{G}}_-(\overset{I}{r}{}^*)\right) \\ & +(1 - \mathscr{K}_I)^{-1/2}\left[T^* + \epsilon_I\left(\mathscr{H}_I(\overset{I}{r}{}^*) + \mathscr{I}_I(\overset{I}{r}{}^*)\right)\right] \end{aligned} \tag{143}$$

V. CONCLUSIONS

We have considered a model of the inhomogeneous universe composed of many gravitating thin shells of dust. This model can treat the time evolution of the dust universe even after the shell crossing singularities arise. The problem of the shell crossing has been resolved here by introducing the transparent shells, which will represent the shells of collisionless particles.

The junction condition for colliding transparent shells has a simple form [Eq. (60)], and it reduces to the DTR relation in the limit of the massless shells.

We have investigated the relativistic effect of the shell-collision. The mass parameter of the region after the shell-collision depend on the relative velocity of colliding shells. This implies that mass inflation phenomena may not occur even when two massive shells collide near the event horizon, which shows good contranst to the case of the collision of massless shells.

The collision of two dust shells can be characterized by the energy and momentum transfer between shells, which are determined by the generalized DTR relation. Whenever two shells collide, the momentum of each shell necessarily decrieses. This can be regarded as a kind of the Ricci focusing effect.

The many dust–shell system has been described in terms of the synchronous comoving coordinate system. This method shows clear correspondence between the shell system and the analytic solution (Tolman–Bondi metric) of the Einstein equation, so that suitable for the set up of the initial data and for the interpretation of results.

This note gives the analytic approach of the many–shell system, and we can now be able to perform the N–shell simulation, which is currently under investigation.

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APPENDIX A: EQUATION OF STATE FOR A SPHERICAL SHELL OF COLLISIONLESS PARTICLES

We shall derive the equation of state for a spherical shell composed of collisionless particles. The stress-energy tensor for collisionless particles each of which has the rest mass μ is given by [24]

$$T^{ij} = \sum_{n=1}^{N} \int (-g)^{-1/2} \mu U^i U^j \delta^4(\mathrm{x} - \mathrm{x}_n(\lambda)) d\lambda, \tag{A1}$$

where U^i should be a tangent vector of a timelike geodesic. We implicitly assume the limit of $N \to \infty$ fixing $m_0 := N\mu$, which corresponds to the collisionless-particle system. We consider a spherical shell composed of such collisionless particles. The line element of the spherically symmetric space-time is written in the form

$$ds^2 = -A^2(T,R)dT^2 + B^2(T,R)dR^2 + r^2(T,R)(d\vartheta^2 + \sin^2\vartheta d\varphi^2). \tag{A2}$$

The shell is assumed to respect the spherical symmetry. The coordinate system is chosen such that $U^R = 0$. It is convenient to introduce the following tetrad basis:

$$E_i^{(0)} = -A\delta_i^0 \tag{A3}$$

$$E_i^{(1)} = B\delta_i^1 \tag{A4}$$

$$E_i^{(2)} = r\delta_i^2 \tag{A5}$$

$$E_i^{(3)} = r\sin\vartheta\delta_i^3 \tag{A6}$$

Then the angular components of the tangent vector $U^{(A)} := E_i^{(A)} U^i$ of the timelike geodesic are given by

$$U^{(2)} = \frac{\ell}{r}\cos\psi, \tag{A7}$$

$$U^{(3)} = \frac{\ell}{r}\sin\psi, \tag{A8}$$

where ℓ and ψ are constant. The other components are given by

$$U^{(0)} = -\sqrt{1+\left(U^{(2)}\right)^2+\left(U^{(3)}\right)^2} = -\sqrt{1+\ell^2/r^2}, \tag{A9}$$

$$U^{(1)} = 0. \tag{A10}$$

The constant ℓ corresponds to the absolute value of the specific angular momentum of the particle. The particles composing a spherical shell may have different values of ψ but ℓ should be identical for all particles; if the value of ℓ for each particle differes, these particles will not remain on a same shell.

The tetrad components of the stress-energy tensor is written in the form

$$T^{(A)(B)} = \sum_{i=1}^{N}\int \frac{\mu U^{(A)}U^{(B)}}{r^2 AB\sin\vartheta}\delta(T-T_s(\lambda))\delta(R-R_s)\delta(\vartheta-\vartheta_i(\lambda))\delta(\varphi-\varphi_i(\lambda))d\lambda \tag{A11}$$

where R_s is constant corresponding to the location of the shell and T_s is the time coordinate on the shell. First we perform the summation for particles at the same point $\{\vartheta,\varphi\} = \{\vartheta_{\mathscr{A}},\varphi_{\mathscr{A}}\}$. The number of particles at $\{\vartheta,\varphi\} = \{\vartheta_{\mathscr{A}},\varphi_{\mathscr{A}}\}$ is denoted by $N_{\mathscr{A}}$. Then we ontain

$$T^{(A)(B)} = \sum_{\mathscr{A}=1}^{N/N_{\mathscr{A}}}\int \frac{N_{\mathscr{A}}\mu\langle U^{(A)}U^{(B)}\rangle}{r^2 AB\sin\vartheta}\delta(T-T_s(\lambda))\delta(R-R_s)\delta(\vartheta-\vartheta_i(\lambda))\delta(\varphi-\varphi_i(\lambda))d\lambda, \tag{A12}$$

where from the assumption of the spherical symmetry

$$\langle U^{(A)}U^{(B)}\rangle = \frac{1}{N_{\mathscr{A}}}\sum_{n=1}^{N_{\mathscr{A}}} U_n^{(A)}U_n^{(B)} = \frac{1}{2\pi}\int_0^{2\pi} U^{(A)}U^{(B)}d\psi \tag{A13}$$

and $U_n^{(A)} := E_i^{(A)}dx_n^i/d\lambda$. Averaging over the sphere, the non-vanishing components of the stress-energy tensor are

$$\langle T^{(0)(0)}\rangle_\Omega = \frac{m_0}{4\pi}\int \frac{1+\ell^2/r^2}{r^2 AB}\delta(T-T_s(\lambda))\delta(R-R_s)d\lambda, \tag{A14}$$

$$\langle T^{(2)(2)}\rangle_\Omega = \langle T^{(3)(3)}\rangle_\Omega = \frac{m_0}{4\pi}\int \frac{\ell^2}{2r^4 AB}\delta(T-T_s(\lambda))\delta(R-R_s)d\lambda, \tag{A15}$$

where

$$\langle Q\rangle_\Omega = \frac{1}{4\pi}\int\int Q\sin\vartheta d\vartheta d\varphi \tag{A16}$$

is defined.

The unit tangent vector of the shell and its unit normal is given by

$$u^i = \frac{U^0}{\sqrt{1+\ell^2/r^2}}\delta^i_0, \tag{A17}$$

$$n_i = \frac{ABU^0}{\sqrt{1+\ell^2/r^2}}\delta^1_i. \tag{A18}$$

The surface stress-energy tensor of the shell is given by

$$S^{(A)(B)} = \int_{R_s-0}^{R_s+0} \langle T^{(A)(B)} \rangle_\Omega n_i dx^i \tag{A19}$$

Hence we find

$$S^{(0)(0)} = \frac{m_0}{4\pi r^2}\sqrt{1+\ell^2/r^2}, \tag{A20}$$

$$S^{(2)(2)} = S^{(3)(3)} = \frac{m_0}{4\pi r^2}\frac{\ell^2/r^2}{2\sqrt{1+\ell^2/r^2}}. \tag{A21}$$

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