学位申請論文

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主論文 1

ENTROPY OF SUBSHIFTS AND THE MACAEV NORM

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ABSTRACT. We obtain the exact value of Voiculescu's invariant $k_{\infty}^{-}(\tau)$, which is an obstruction of the existence of quasicentral approximate units relative to the Macaev ideal in perturbation theory, for a tuple τ of operators in the following two classes: (1) creation operators associated with a subshift, which are used to define Matsumoto algebras, (2) unitaries in the left regular representation of a finitely generated group.

1. INTRODUCTION

In the remarkable serial works [Voi1], [Voi2], [Voi3] and [DV] on perturbation of Hilbert space operators, Voiculescu investigated a numerical invariant $k_{\Phi}(\tau)$ for a family τ of bounded linear operators on a separable Hilbert space, where $k_{\Phi}(\tau)$ is the obstruction of the existence of quasicentral approximate units relative to the normed ideal $\mathfrak{S}_{\Phi}^{(0)}$ corresponding to a symmetric norming function Φ, (see definitions in Section 2). The invariant $k_{\Phi}(\tau)$ is considered to be a kind of dimension of τ with respect to the normed ideal $\mathfrak{S}_{\Phi}^{(0)}$ (see [Voi1] and [DV]).

In the present paper, we study the invariant $k_{\Phi}(\tau)$ for the Macaev ideal, which is denoted by $k_{\infty}^{-}(\tau)$. It is known that $k_{\infty}^{-}(\tau)$ possesses several remarkable properties: for instance, $k_{\infty}^{-}(\tau)$ is always finite and $k_{\Phi}(\tau) = 0$ if $\mathfrak{S}_{\Phi}^{(0)}$ is strictly larger than the Macaev ideal. In [Voi3], Voiculescu investigated the invariant $k_{\infty}^{-}(\tau)$ for several examples. He proved that $k_{\infty}^{-}(\tau) = \log N$ for an N-tuple τ of isometries in extensions of the Cuntz algebra $\mathcal{O}_N$. Here, $\log N$ can be interpreted as the value of the topological entropy of the N-full shift. Inspired by this result, we show that $k_{\infty}^{-}(\tau) = h_{\text{top}}(X)$ for a general subshift X with a certain condition, where $h_{\text{top}}(X)$ is the topological entropy of X and τ is the family of creation operators on the Fock space associated with the subshift X, which is used to define the Matsumoto algebra associated with X (e.g. see [Mat]). In particular, we show that $k_{\infty}^{-}(\tau) = h_{\text{top}}(X)$ holds for every almost sofic shift X (cf. [Pet]).

Let Γ be a countable finitely generated group and S its generating set. We also study $k_{\infty}^{-}((\lambda_a)_{a\in S})$, where λ is the left regular representation of Γ. For the related topic, see [Voi5], in which a relation between $k_{\infty}^{-}((\lambda_a)_{a\in S})$ and the entropy of random walks on groups is discussed. By using a method introduced in [Oka], we can compute the exact value of $k_{\infty}^{-}((\lambda_a)_{a\in S})$ for certain amalgamated free product groups. Voiculescu proved that $\log N \leq k_{\infty}^{-}((\lambda_a)_{a\in S}) \leq \log(2N-1)$ holds for the free group $\mathbb{F}_N$ with the canonical generating set S ([Voi3, Proposition 3.7. (a)]). As a particular case of our results, we show that $k_{\infty}^{-}((\lambda_a)_{a\in S}) = \log(2N-1)$ actually holds.

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2. Preliminary

Let H be a separable infinite dimensional Hilbert space. By $\mathbb{B}(H)$, $\mathbb{K}(H)$, $\mathbb{F}(H)$ and $\mathbb{F}(H)_1^+$, we denote the bounded linear operators, the compact operators, the finite rank operators and the finite rank positive contractions on H, respectively.

We begin by recalling some facts concerning normed ideal in [GK]. Let c_0 be the set of real valued sequences $\xi = (\xi_j)_{j\in\mathbb{N}}$ with $\lim_{j\to\infty}\xi_j = 0$, and $c_{0,0}$ the subspace of c_0 consisting of the sequences with finite support. A function Φ on $c_{0,0}$ is said to be a *symmetric norming function* if Φ satisfies:

(1) Φ is a norm on $c_{0,0}$;
(2) $\Phi((1,0,0,\dots)) = 1$;
(3) $\Phi((\xi_j)_{j\in\mathbb{N}}) = \Phi((|\xi_{\pi(j)}|)_{j\in\mathbb{N}})$ for any bijection $\pi : \mathbb{N} \to \mathbb{N}$.

For $\xi = (\xi_j)_{j\in\mathbb{N}} \in c_0$, we define

$$\Phi(\xi) = \lim_{n\to\infty} \Phi(\xi^*(n)) \in [0,\infty],$$

where $\xi^*(n) = (\xi_1^*,\dots,\xi_n^*,0,0,\dots) \in c_{0,0}$ and $\xi_1^* \geq \xi_2^* \geq \cdots$ is the decreasing rearrangement of the absolute value $(|\xi_j|)_{j\in\mathbb{N}}$. If $T \in \mathbb{K}(H)$ and Φ is a symmetric norming function, then let us denote

$$||T||_\Phi = \Phi((s_j(T))_{j\in\mathbb{N}}),$$

where $(s_j(T))_{j\in\mathbb{N}}$ is the singular numbers of T. We define two symmetrically normed ideals

$$\mathfrak{S}_\Phi = \{T \in \mathbb{K}(H) \mid ||T||_\Phi < \infty\},$$

and $\mathfrak{S}_\Phi^{(0)}$ by the closure of $\mathbb{F}(H)$ with respect to the norm $||\cdot||_\Phi$. Note that $\mathfrak{S}_\Phi^{(0)}$ does not coincide with $\mathfrak{S}_\Phi$ in general. If $\mathfrak{S}$ is a *symmetrically normed ideal*, i.e. $\mathfrak{S}$ is a ideal of $\mathbb{B}(H)$ and a Banach space with respect to the norm $||\ ||_\mathfrak{S}$ satisfying:

(1) $||XTY||_\mathfrak{S} \leq ||X||\ \ ||T||_\mathfrak{S}\ \ ||Y||$ for $T \in \mathfrak{S}$ and $X, Y \in \mathbb{B}(H)$,
(2) $||T||_\mathfrak{S} = ||T||$ if T is of rank one,

where $||\ \ ||$ is the operator norm in $\mathbb{B}(H)$, then there exists a unique symmetric norming function Φ such that $||T||_\mathfrak{S} = ||T||_\Phi$ for $T \in \mathbb{F}(H)$ and $\mathfrak{S}_\Phi^{(0)} \subseteq \mathfrak{S} \subseteq \mathfrak{S}_\Phi$.

We introduce some symmetrically normed ideals. For $1 < p \leq \infty$, the symmetrically normed ideal $\mathcal{C}_p^-(H)$ is given by the symmetric norming function

$$\Phi_p^-(\xi) = \sum_{j=1}^{\infty} \frac{\xi_j^*}{j^{1-1/p}}.$$

We define $\mathcal{C}_p^-(H) = \mathfrak{S}_{\Phi_p^-}^{(0)}$. We remark that it coincides with $\mathfrak{S}_{\Phi_p^-}$ For $1 \leq p < \infty$, the symmetrically normed ideal $\mathcal{C}_p^+(H)$ is given by the symmetric norming function

$$\Phi_p^+(\xi) = \sup_{n\in\mathbb{N}} \frac{\sum_{j=1}^n \xi_j^*}{\sum_{j=1}^n j^{1/p}}.$$

We define $\mathcal{C}_p^+(H) = \mathfrak{S}_{\Phi_p^+}$. However $\mathfrak{S}_{\Phi_p^+}^{(0)}$ is strictly smaller than $\mathcal{C}_p^+(H)$. For $1 \leq p < q < r \leq \infty$, we have

$$\mathcal{C}_p(H) \subsetneqq \mathcal{C}_q^-(H) \subsetneqq \mathcal{C}_q(H) \subsetneqq \mathcal{C}_q^+(H) \subsetneqq \mathcal{C}_r(H),$$

where $\mathcal{C}_p(H)$ is the Schatten p class.

For a given symmetric norming function Φ, which is not equivalent to the l^1-norm, there is a symmetric norming function Φ^* such that $\mathfrak{S}_{\Phi^*}$ is the dual of $\mathfrak{S}_\Phi^{(0)}$, where the dual pairing is given by the bilinear form $(T,S) \mapsto \mathrm{Tr}(TS)$. If $1/p + 1/q = 1$, then $\mathcal{C}_p(H)^* \simeq \mathcal{C}_q(H)$ and $\mathcal{C}_p^-(H)^* \simeq \mathcal{C}_q^+(H)$. In particular, $\mathcal{C}_\infty^-(H)$ and $\mathcal{C}_1^+(H)$ are called the *Macaev ideal* and the *dual Macaev ideal*, respectively.

Let $\mathfrak{S}_\Phi^{(0)}$ be a symmetrically normed ideal with a symmetric norming function Φ. If $\tau = (T_1, \ldots, T_N)$ is an N-tuple of bounded linear operators, then the number $k_\Phi(\tau)$ is defined by

$$k_\Phi(\tau) = \liminf_{u \in \mathbb{F}(H)_1^+} \max_{1 \leq a \leq N} ||[u, T_a]||_\Phi,$$

where the inferior limit is taken with respect to the natural order on $\mathbb{F}(H)_1^+$ and $[A,B] = AB - BA$. Throughout this paper, we denote $|| \ ||_{\Phi_p^-}$ by $||\cdot||_p^-$ and $k_{\Phi_p^-}$ by k_p^- A relation between the invariant k_Φ and the existence of quasicentral approximate units relative to the symmetrically normed ideal $\mathfrak{S}_\Phi^{(0)}$ is discussed in [Voi1]. A quasicentral approximate unit for $\tau = (T_1, \ldots, T_N)$ relative to $\mathfrak{S}_\Phi^{(0)}$ is a sequence $\{u_n\}_{n=1}^\infty \subseteq \mathbb{F}(H)_1^+$ such that $u_n \nearrow I$ and $\lim_{n\to\infty} ||[u_n, T_a]||_\Phi = 0$ for $1 \leq a \leq N$. Note that for an N-tuple $\tau = (T_1, \ldots, T_N)$, there exists a quasicentral approximate unit for τ relative to $\mathfrak{S}_\Phi^{(0)}$ if and only if $k_\Phi(\tau) = 0$ (e.g. see [Voi2, Lemma 1.1]).

We use the following propositions to prove our theorem.

Proposition 2.1 ([Voi1, Proposition 1.1]). *Let $\tau = (T_1, \ldots, T_N) \in \mathbb{B}(H)^N$ and $\mathfrak{S}_\Phi^{(0)}$ be a symmetrically normed ideal with a symmetric norming function Φ. If we take a sequence $\{u_n\}_{n=1}^\infty \subseteq \mathbb{F}(H)_1^+$ with* w-$\lim_{n\to\infty} u_n = I$, *then*

$$k_\Phi(\tau) \leq \liminf_{n\to\infty} \max_{1 \leq a \leq N} ||[u_n, T_a]||_\Phi.$$

Proposition 2.2 ([Voi3, Proposition 2.1]). *Let $\tau = (T_1, \ldots, T_N) \in \mathbb{B}(H)^N$ and $X_a \in \mathcal{C}_1^+(H)$ for $a = 1, \ldots, N$. If*

$$\sum_{a=1}^N [X_a, T_a] \in \mathcal{C}_1(H) + \mathbb{B}(H)_+,$$

then we have

$$\left| \mathrm{Tr}\left(\sum_{a=1}^N [X_a, T_a] \right) \right| \leq k_\infty^-(\tau) \sum_{a=1}^N ||X_a||_1^{\tilde{+}},$$

where $||X_a||_1^{\tilde{+}} = \inf_{Y \in \mathbb{F}(H)} ||X_a - Y||_{\Phi_1^+}$.

The following proposition was shown in the proof of [GK, Theorem 14.1].

Proposition 2.3. *For $T \in \mathcal{C}_1^+(H)$, we have*

$$||T||_1^{\tilde{+}} = \limsup_{n\to\infty} \frac{\sum_{j=1}^n s_j(T)}{\sum_{j=1}^n 1/j}$$

3. Subshifts and Macaev Norm

Let $\mathcal{A}$ be a finite set with the discrete topology, which we call the *alphabet*, and $\mathcal{A}^{\mathbb{Z}}$ the two-sided infinite product space $\prod_{i=-\infty}^{\infty} \mathcal{A}$ endowed with the product topology. The *shift map* σ on $\mathcal{A}^{\mathbb{Z}}$ is given by $(\sigma(x))_i = x_{i+1}$ for $i \in \mathbb{Z}$. The pair $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ is called the *full shift*. In particular, if the cardinality of the alphabet $\mathcal{A}$ is N, then we call it the *N-full shift*.

Let X be a shift invariant closed subset of $\mathcal{A}^{\mathbb{Z}}$. The topological dynamical system (X, σ_X) is called a *subshift* of $\mathcal{A}^{\mathbb{Z}}$, where σ_X is the restriction of the shift map σ. We sometimes denote the subshift (X, σ_X) by X for short. A *word* over $\mathcal{A}$ is a finite sequence $w = (a_1, \dots, a_n)$ with $a_i \in \mathcal{A}$. For $x \in \mathcal{A}^{\mathbb{Z}}$ and a word $w = (a_1, \dots, a_n)$, we say that w *occurs in* x if there is an index i such that $x_i = a_1, \dots, x_{i+n-1} = a_n$. The empty word occurs in every $x \in \mathcal{A}^{\mathbb{Z}}$ by convention. Let $\mathcal{F}$ be a collection of words over $\mathcal{A}^{\mathbb{Z}}$ We define the subshift $X_{\mathcal{F}}$ to be the subset of sequences in $\mathcal{A}^{\mathbb{Z}}$ in which *no* word in $\mathcal{F}$ occurs. It is well-known that any subshift X of $\mathcal{A}^{\mathbb{Z}}$ is given by $X_{\mathcal{F}}$ for some collection $\mathcal{F}$ of forbidden words over $\mathcal{A}^{\mathbb{Z}}$. Note that for $\mathcal{F} = \emptyset$, the subshift $X_{\mathcal{F}}$ is the full shift $\mathcal{A}^{\mathbb{Z}}$

Let X be a subshift of $\mathcal{A}^{\mathbb{Z}}$ We denote by $\mathcal{W}_n(X)$ the set of all words with length n that occur in X and we set

$$\mathcal{W}(X) = \bigcup_{n=0}^{\infty} \mathcal{W}_n(X).$$

Let $\varphi : \mathcal{W}_{m+n+1}(X) \to \mathcal{A}$ be a map, which we call a *block map*. The extension of φ from X to $\mathcal{A}^{\mathbb{Z}}$ is defined by $(x_i)_{i\in\mathbb{Z}} \mapsto (y_i)_{i\in\mathbb{Z}}$, where

$$y_i = \varphi((x_{i-m}, x_{i-m+1}, \dots, x_{i+n})).$$

We also denote this extension by φ and call it a *sliding block code*. Let X, Y be two subshifts and $\varphi : X \to Y$ a sliding block code. If φ is one-to-one, then φ is called an *embedding* of X into Y and we denote $X \subseteq Y$ If φ has an *inverse*, i.e. a sliding block code $\psi : Y \to X$ such that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$, then two subshifts X and Y are *topologically conjugate*.

The *topological entropy* of a subshift X is defined by

$$h_{\mathrm{top}}(X) = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{W}_n(X)|,$$

where $|\mathcal{W}_n(X)|$ is the cardinality of $\mathcal{W}_n(X)$. The reader is referred to [LM] for an introduction to symbolic dynamics.

For a given subshift X, we next construct the creation operators on the Fock space associated with X (cf. [Mat]). Let $\{\xi_a\}_{a\in\mathcal{A}}$ be an orthonormal basis of N-dimensional Hilbert space $\mathbb{C}^N$, where N is the cardinality of $\mathcal{A}$. For $w = (a_1, \dots, a_n) \in \mathcal{W}_n(X)$, we denote $\xi_w = \xi_{a_1} \otimes \cdots \otimes \xi_{a_n}$. We define the Fock space $\mathcal{F}_X$ for a subshift X by

$$\mathcal{F}_X = \mathbb{C}\xi_0 \oplus \bigoplus_{n\in\mathbb{N}} \mathrm{span}\{\xi_w \mid w \in \mathcal{W}_n(X)\},$$

where ξ_0 is the vacuum vector. The creation operator T_a on $\mathcal{F}_X$ for $a \in \mathcal{A}$ is given by

$$\begin{aligned} T_a\xi_0 &= \xi_a, \\ T_a\xi_w &= \begin{cases} \xi_a \otimes \xi_w & \text{if } aw \in \mathcal{W}(X), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Note that T_a is a partial isometry such that

$$P_0 + \sum_{a \in \mathcal{A}} T_a T_a^* = 1,$$

where P_0 is the rank one projection onto $\mathbb{C}\xi_0$. We denote by P_n the projection onto the subspace spanned by ξ_w for all $w \in \mathcal{W}_n(X)$. For $w = (a_1, \ldots, a_n) \in \mathcal{W}_n(X)$, we set $T_w = T_{a_1} \cdots T_{a_n}$ The following proposition is essentially proved in [Voi3].

Proposition 3.1. *If $\tau = (T_a)_{a \in \mathcal{A}}$, then we have*

$$k_\infty^-(\tau) \le h_{\mathrm{top}}(X).$$

Proof. We first assume that the topological entropy of X is non-zero. Let us denote $h = h_{\mathrm{top}}(X)$. By definition, for a given $\epsilon > 1$, there exists $K \in \mathbb{N}$ such that for any $n \ge K$, we have

$$\frac{1}{n} \log |\mathcal{W}_n(X)| < \varepsilon h.$$

Thus

$$|\mathcal{W}_n(X)| < e^{n\epsilon h},$$

for all $n \ge K$. We set

$$X_n = \sum_{j=0}^{n-1} \left(1 - \frac{j}{n}\right) P_j.$$

One can show that

$$\|[X_n, T_a]\| \le \frac{1}{n}.$$

Since

$$r_n = \mathrm{rank}([X_n, T_a]) \le \sum_{j=1}^{n} |\mathcal{W}_j(X)| \le \sum_{j=1}^{K-1} |\mathcal{W}_j(X)| + \sum_{j=K}^{n} e^{j\epsilon h}$$

for $n \ge K$, we obtain

$$k_\infty^-(\tau) \le \limsup_{n \to \infty} \max_{a \in \mathcal{A}} \|[X_n, T_a]\|_\infty^- \le \limsup_{n \to \infty} \frac{\sum_{j=1}^{r_n} 1/j}{n} \le \varepsilon h.$$

In the case of $h = 0$, for any $\varepsilon > 0$, we have

$$|\mathcal{W}_n(X)| < e^{n\varepsilon}$$

for sufficiently large n. By the same argument, we can get

$$k_\infty^-(\tau) \le \limsup_{n \to \infty} \max_{a \in \mathcal{A}} \|[X_n, T_a]\|_\infty^- \le \varepsilon,$$

for arbitrary $\epsilon > 0$. □

Next we obtain the lower bound of $k_\infty^-(\tau)$ by using Proposition 2.2. Before it, we prepare some notations. For any $m \in \mathbb{Z}$ and $w = (a_1, \ldots, a_n) \in \mathcal{W}_n(X)$, let us denote

$${}_m[w] = \{(x_i)_{i \in \mathbb{Z}} \in X \mid x_m = a_1, \ldots, x_{m+n-1} = a_n\}.$$

We sometimes denote the cylinder set ${}_0[w]$ by $[w]$ for short. Let μ be a shift invariant probability measure on X. The following holds:

(1) $\sum_{a \in \mathcal{A}} \mu([a]) = 1$;
(2) $\mu([a_1, \ldots, a_n]) = \sum_{a_0 \in \mathcal{A}} \mu([a_0, a_1, \ldots, a_n])$;
(3) $\mu([a_1, \ldots, a_n]) = \sum_{a_{n+1} \in \mathcal{A}} \mu([a_1, \ldots, a_n, a_{n+1}])$.

For any partition $\beta = (B_1, \dots, B_n)$ of X, we define a function on X by

$$I_\mu(\beta) = -\sum_{B\in\beta} \log \mu(B)\chi_B,$$

where χ_B is the characteristic function of B. Let $\beta_1, \dots, \beta_k$ be partitions of X. The partition $\bigvee_{i=1}^k \beta_i$ is defined by

$$\left\{ \bigcap_{i=1}^k B_i \mid B_i \in \beta_i, 1 \le i \le k \right\}.$$

The value

$$H_\mu(\beta) = -\sum_{B\in\beta} \mu(B)\log\mu(B)$$

is called the *entropy of the partition* β. We define

$$h_\mu(\beta, \sigma_X) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\bigvee_{i=0}^{n-1} \sigma_X^{-i}(\beta)).$$

The *entropy* of (X, σ_X, μ) is defined by

$$h_\mu(\sigma_X) = \sup\{h_\mu(\beta, \sigma_X) \mid H_\mu(\beta) < \infty\}.$$

Note that $h_\mu(\sigma_X) \le h_{\mathrm{top}}(X)$ in general. A shift invariant probability measure μ is said to be a *maximal measure* if $h_{\mathrm{top}}(X) = h_\mu(\sigma_X)$. The reader is referred to [DGS] for details.

Theorem 3.2. *Let $\tau = (T_a)_{a\in\mathcal{A}}$ be the creation operators for a subshift X. If there exists a shift invariant probability measure μ on X such that for any $\varepsilon > 0$ we have*

$$\sum_{n=0}^\infty \mu\left(\left\{x \in X : \left|\frac{1}{n+1} I_\mu\left(\bigvee_{i=0}^n \sigma_X^{-i}\beta\right)(x) - h_\mu(\sigma_X)\right| > \varepsilon\right\}\right) < \infty,$$

where β is the generating partition $\{[a]\}_{a\in\mathcal{A}}$ of X, then

$$h_\mu(\sigma_X) \le k_\infty^-(\tau).$$

In particular, if we can take a maximal measure μ with the above condition, then we have

$$k_\infty^-(\tau) = h_{\mathrm{top}}(X).$$

Proof. Let μ be a shift invariant probability measure on X. For $a \in \mathcal{A}$, we set

$$X_a = \sum_{n\ge0} \sum_{w\in\mathcal{W}_n(X)} \mu([aw]) T_w P_0 T_{aw}^*.$$

Then

$$\begin{aligned}\sum_{a\in\mathcal{A}} T_a X_a &= \sum_{n\ge0}\sum_{a\in\mathcal{A}}\sum_{w\in\mathcal{W}_n(X)} \mu([aw]) T_{aw} P_0 T_{aw}^* \\ &= \sum_{n\ge1}\sum_{w\in\mathcal{W}_n(X)} \mu([w]) T_w P_0 T_w^*,\end{aligned}$$

and

$$\begin{aligned}\sum_{a\in\mathcal{A}} X_a T_a &= \sum_{n\geq 0}\sum_{w\in\mathcal{W}_n(X)}\left(\sum_{a\in\mathcal{A}}\mu([aw])\right)T_w P_0 T_w^* \\ &= \sum_{n\geq 0}\sum_{w\in\mathcal{W}_n(X)}\mu([w])T_w P_0 T_w^*.\end{aligned}$$

Hence we have

$$\sum_{a\in\mathcal{A}}[X_a, T_a] = P_0.$$

We assume that $h_\mu(\sigma_X)\neq 0$ and denote it by h for short. To apply Proposition 2.2, we need an estimate of $||X_a||_1^+$ Fix $\varepsilon > 0$ and $a\in\mathcal{A}$. We set

$$D_n = \left\{w\in\mathcal{W}_n(X) \mid e^{-(n+1)(h+\varepsilon)} \leq \mu([aw]) \leq e^{-(n+1)(h-\varepsilon)}\right\},$$

and

$$\varepsilon_n = \sum_{w\in\mathcal{W}_n(X)\setminus D_n}\mu([aw]).$$

If μ satisfies the assumption, then we have

$$\sum_{n\geq 0}\varepsilon_n < \infty. \tag{$\star$}$$

Note that $s_j(X_a) = s_j(X_aT_a)$ for all $j\in\mathbb{N}$. Thus we have $||X_a||_1^+ = ||X_aT_a||_1^+$ We put

$$\widetilde{X_a} = \sum_{n\geq 0}\sum_{w\in D_n}\mu([aw])T_w P_0 T_w^*.$$

We remark that for each $j\in\mathbb{N}$, there are $n\in\mathbb{N}$, $w\in\mathcal{W}_n(X)$ such that $s_j(X_aT_a) = \mu([aw])$. By ($\star$), we obtain

$$\begin{aligned}||X_a||_1^+ = ||X_aT_a||_1^+ &= \limsup_{n\to\infty}\frac{\sum_{j=1}^n s_j(X_aT_a)}{\sum_{j=1}^n 1/j} \\ &\leq ||\widetilde{X_a}||_1^+ + \limsup_{n\to\infty}\frac{\sum_{k=0}^\infty \varepsilon_k}{\sum_{j=1}^n 1/j} = ||\widetilde{X_a}||_1^+\end{aligned}$$

Hence it suffices to give an estimate of $||\widetilde{X_a}||_1^+$ Let $d_n = \sum_{j=0}^n |D_j|$, where $|D_j|$ is the cardinality of D_j. One can easily check that

$$||\widetilde{X_a}||_1^+ \leq \limsup_{n\to\infty}\frac{\sum_{j=1}^{d_n} s_j(\widetilde{X_a})}{\sum_{j=1}^{d_n} 1/j}.$$

Note that if $s_j(\widetilde{X_a}) = \mu([aw])$ for some $w\in D_n$, then we have

$$e^{-(n+1)(h+\varepsilon)} \leq s_j(\widetilde{X_a}) = \mu([aw]) \leq e^{-(n+1)(h-\varepsilon)}$$

Assume that there are $m > n$ such that $s_j(\widetilde{X_a}) = \mu([aw])$ for some $w\in D_m$ and $j\leq d_n$. Then it holds that

$$e^{-(m+1)(h-\varepsilon)} \geq e^{-(n+1)(h+\varepsilon)} \tag{$\star\star$}$$

Indeed, if $e^{-(m+1)(h-\varepsilon)} < e^{-(n+1)(h+\varepsilon)}$, then

$$s_j(\widetilde{X_a}) = \mu([aw]) \leq e^{-(m+1)(h-\varepsilon)} < e^{-(n+1)(h+\varepsilon)} \leq \mu([au]),$$

for all $u \in D_k$ $(1 \le k \le n)$. However, by our assumption, we have $\mu([av]) \le s_j(\widetilde{X_a}) = \mu([aw])$ for some $v \in D_l$ and $1 \le l \le n$. This is a contradiction.

Hence, by $(\star\star)$, we have

$$m+1 \le (n+1)\frac{h+\varepsilon}{h-\varepsilon}$$

Let $k \in \mathbb{N}$ with

$$(n+1)\frac{h+\varepsilon}{h-\varepsilon} - 1 < k+1 \le (n+1)\frac{h+\varepsilon}{h-\varepsilon}.$$

Since

$$\begin{aligned}
\frac{\sum_{j=1}^{d_n} s_j(\widetilde{X_a})}{\sum_{j=1}^{d_n} 1/j} &\le \frac{\sum_{i=0}^{k}\sum_{w\in D_i}\mu([aw])}{\log d_n} \\
&\le \frac{\sum_{i=0}^{k}\mu([a])}{\log d_n} \\
&\le \frac{n+1}{\log d_n}\,\frac{h+\varepsilon}{h-\varepsilon}\mu([a]),
\end{aligned}$$

we obtain

$$||\widetilde{X_a}||_1^{\widetilde{+}} \le \limsup_{n\to\infty}\frac{n+1}{\log d_n}\cdot\frac{h+\varepsilon}{h-\varepsilon}\mu([a]).$$

Moreover, because

$$\mu([a]) = \sum_{w\in D_n}\mu([aw]) + \sum_{w\in \mathcal{W}_n(X)\setminus D_n}\mu([aw]) \le |D_n|e^{-(n+1)(h-\varepsilon)} + \varepsilon_n,$$

we have

$$(\mu([a]) - \varepsilon_n)\, e^{(n+1)(h-\varepsilon)} \le |D_n|.$$

Note that $\varepsilon_n \to 0$ $(n\to\infty)$ by $(\star)$. Therefore

$$\begin{aligned}
||\widetilde{X_a}||_1^{\widetilde{+}} &\le \limsup_{n\to\infty}\frac{n+1}{\log|D_n|}\,\frac{h+\varepsilon}{h-\varepsilon}\mu([a]) \\
&\le \limsup_{n\to\infty}\frac{n+1}{\log(\mu([a])-\varepsilon_n)+(n+1)(h-\varepsilon)}\cdot\frac{h+\varepsilon}{h-\varepsilon}\mu([a]) \\
&= \frac{h+\varepsilon}{(h-\varepsilon)^2}\mu([a]).
\end{aligned}$$

Since ε is arbitrary, we have

$$||X_a||_1^{\widetilde{+}} \le \frac{1}{h}\mu([a]).$$

By Proposition 2.2, the proof is complete. □

We now give some examples of subshifts with a maximal measure satisfying the condition in Theorem 3.2.

Corollary 3.3. *Let A be a 0-1 $N\times N$ matrix. We denote by Σ_A the Markov shift associated with A, i.e.*

$$\Sigma_A = \{(a_i)_{i\in\mathbb{Z}} \in S^{\mathbb{Z}} \mid A(a_i, a_{i+1}) = 1\},$$

where $S = \{1,\dots,N\}$ is an alphabet. If $\tau = (T_a)_{a\in S}$ is the creation operators for the Markov shift Σ_A, then we have

$$k_\infty^-(\tau) = h_{\mathrm{top}}(\Sigma_A).$$

Proof. It suffices to show that the unique maximal measure of Σ_A satisfies the condition in Theorem 3.2. For simplicity, we may assume that A is irreducible with the Perron value α. Note that the topological entropy $h_{top}(\Sigma_A)$ is equal to $\log\alpha$. If l and r are the left and right Perron vectors with $\sum_{a=1}^{N} l_a r_a = 1$, then the unique maximal measure μ is given by

$$\mu([a_0, a_1, \dots, a_n]) = \frac{l_{a_0} r_{a_n}}{\alpha^n},$$

where $(a_0, a_1, \dots, a_n) \in \mathcal{W}_{n+1}(\Sigma_A)$ (e.g. see [Kit]). For any $\varepsilon > 0$, there exists $K \in \mathbf{N}$ such that for any $n \geq K$, we have

$$\left|\frac{\log l_a r_b \alpha}{n+1}\right| < \varepsilon,$$

for all $1 \leq a, b \leq N$. Therefore for any $w \in \mathcal{W}_{n+1}(\Sigma_A)$, we have

$$\left|-\frac{1}{n+1}\log\mu([w]) - \log\alpha\right| < \varepsilon,$$

for all $n \geq K$, i.e. the maximal measure μ satisfies the condition in Theorem 3.2. □

More generally, there is a class of subshifts, which is called almost sofic (see [Pet]). A subshift X is said to be *almost sofic* if for any $\varepsilon > 0$, there is an SFT $\Sigma \subseteq X$ such that $h_{top}(X) - \varepsilon < h_{top}(\Sigma)$, where a *shift of finite type* or SFT is a subshift that can be described by a finite set of forbidden words, i.e. a subshift having the form $X_{\mathcal{F}}$ for some finite set $\mathcal{F}$ of words.

Corollary 3.4. *If $\tau = (T_a)_{a\in\mathcal{A}}$ is the creation operators for an SFT Σ, then we have*

$$k_\infty^-(\tau) = h_{top}(\Sigma).$$

Proof. We recall that every SFT Σ is topologically conjugate to a Markov shift Σ_A associated with a 0-1 matrix A. Now we give a short proof of this result. Let Σ be an SFT that can be described by a finite set $\mathcal{F}$ of forbidden words. We may assume that all words in $\mathcal{F}$ have length $N+1$. We set $\mathcal{A}_\Sigma^{[N]} = \mathcal{W}_N(\Sigma)$ and the block map $\varphi : \mathcal{W}_N(\Sigma) \to \mathcal{A}_\Sigma^{[N]}$, $w \mapsto w$. We define the *N-th higher block code* $\beta_N : \Sigma \to (\mathcal{A}_\Sigma^{[N]})^{\mathbf{Z}}$ by

$$(\beta_N(x))_i = (x_i, \dots, x_{i+N-1}) \in \mathcal{A}_\Sigma^{[N]},$$

for $x = (x_i)_{i\in\mathbf{N}} \in \Sigma$. Note that β_N is the sliding block code with respect to φ. The subshift $\beta_N(\Sigma)$ is given by a Markov shift, i.e. there is a 0-1 matrix A with $\beta_N(\Sigma) = \Sigma_A$.

Let μ be the maximal measure of Σ_A. The maximal measure of Σ is given by $\nu = \mu\circ\beta_N$. We recall that μ is the Markov measure given by the left and right eigenvectors l, r and the eigenvalue α. For $w \in \mathcal{W}_n(\Sigma)$ with $n \geq N$, we have

$$\begin{aligned}\nu([w]) &= \mu([\varphi(w_{[1,N]}), \dots, \varphi(w_{[n-N+1,n]})]) \\ &= \frac{l_a r_b}{\alpha^{n-N}},\end{aligned}$$

where $a = \varphi(w_{[1,N]})$, $b = \varphi(w_{[n-N+1,n]})$ and $w_{[k,l]} = (w_k, \dots, w_l)$ for $k \leq l$. Hence one can show that the maximal measure ν of Σ satisfies the condition in Theorem 3.2 by the same argument as in the proof of Corollary 3.3. □

Corollary 3.5. *Let X be an almost sofic shift. If $\tau = (T_a)_{a\in\mathcal{A}}$ is the creation operators for X, then we have*

$$k_\infty^-(\tau) = h_{\rm top}(X).$$

Proof. Let $\varepsilon > 0$. Since X is almost sofic, there is an SFT $\Sigma \subsetneq X$ such that $h_{\rm top}(X) - \varepsilon < h_{\rm top}(\Sigma)$. Let $\varphi : \Sigma \to X$ be an embedding. Note that the subshift $\varphi(\Sigma)$ is also an SFT. Thus we may identify $\varphi(\Sigma)$ with Σ. Let μ be the unique maximal measure of Σ. For $a \in \mathcal{A}$, we set

$$X_a = \sum_{n\geq 0}\sum_{w} \mu([aw])T_w P_0 T_{aw}^*,$$

where w runs over all elements in $\mathcal{W}_n(\Sigma)$ with $aw \in \mathcal{W}(\Sigma)$. We have shown that the maximal measure μ of Σ satisfies the condition of Theorem 3.2 in the proof of Corollary 3.4. Hence by the same argument as in the proof of Theorem 3.2, we have

$$h_{\rm top}(\Sigma) \leq k_\infty^-(\tau).$$

Thus for arbitrary $\varepsilon > 0$, the following holds:

$$h_{\rm top}(X) - \varepsilon < h_{\rm top}(\Sigma) \leq k_\infty^-(\tau).$$

It therefore follows from Proposition 3.1 that $h_{\rm top}(X) = k_\infty^-(\tau)$ if X is an almost sofic shift. □

For $\beta > 1$, the β-transformation T_β on the interval $[0,1]$ is defined by the multiplication with β (mod 1), i.e. $T_\beta(x) = \beta x - [\beta x]$, where $[t]$ is the integer part of t. Let $N \in \mathbf{N}$ with $N - 1 < \beta \leq N$ and $\mathcal{A} = \{0, 1, \dots, N-1\}$. The *$\beta$-expansion* of $x \in [0,1]$ is a sequence $d(x,\beta) = \{d_i(x,\beta)\}_{i\in\mathbf{N}}$ of $\mathcal{A}$ determined by

$$d_i(x,\beta) = [\beta T_\beta^{i-1}(x)].$$

We set

$$\zeta_\beta = \sup_{x\in[0,1)} (d_i(x,\beta))_{i\in\mathbf{N}},$$

where the above supremum is taken in the lexicographical order, and we define the shift invariant closed subset Σ_β^+ of the full one-sided shift $\mathcal{A}^{\mathbf{N}}$ by

$$\Sigma_\beta^+ = \left\{x \in \mathcal{A}^{\mathbf{N}} \mid \sigma^i(x) \leq \zeta_\beta,\ i = 0, 1, \dots\right\},$$

where $\leq$ is the lexicographical order on $\mathcal{A}^{\mathbf{N}} = \{0, 1, \dots, N-1\}^{\mathbf{N}}$ The *β-shift* Σ_β is the natural extension given by

$$\Sigma_\beta = \left\{(x_i)_{i\in\mathbf{Z}} \in \mathcal{A}^{\mathbf{Z}} \mid (x_i)_{i\geq k} \in \Sigma_\beta^+,\ k \in \mathbf{Z}\right\}.$$

It is known that $h_{\rm top}(\Sigma_\beta) = \log\beta$, (see [Hof]).

The following result might be known among specialists. However, we give a proof here as we can not find it in the literature.

Proposition 3.6. *For $\beta > 1$, the β-shift Σ_β is an almost sofic shift.*

Proof. In [Par], it is shown that Σ_β is an SFT if and only if $d(1,\beta)$ is finite, i.e. there is $K \in \mathbf{N}$ such that $d_k(1,\beta) = 0$ for all $k \geq K$. Thus we may assume that $d(1,\beta)$ is not finite. Let $\zeta_\beta = (\xi_i)_{i\in\mathbf{N}}$. For $n \in \mathbf{N}$, there is $\beta(n) < \beta$ such that

$$1 = \frac{\xi_1}{\beta(n)} + \frac{\xi_2}{\beta(n)^2} + \cdots + \frac{\xi_n}{\beta(n)^n}.$$

In [Par, Theorem 5], it is proved that

$$\lim_{n\to\infty}\beta(n)=\beta.$$

Hence we may assume that $N-1\leq\beta(n)<\beta$ for sufficiently large n. Since the maximal element $\zeta_{\beta(n)}$ has the form

$$(\xi_1,\xi_2,\ldots,(\xi_n-1),\xi_1,\xi_2,\ldots,(\xi_n-1),\xi_1,\ldots),$$

we have $\zeta_{\beta(n)}<\zeta$, where $<$ is the lexicographical order. Therefore we obtain

$$\Sigma^+_{\beta(n)}\subseteq\Sigma^+_\beta\subseteq\{0,1,\ldots,N-1\}^{\mathbb{N}}$$

It follows that $\Sigma_{\beta(n)}$ is the shift invariant closed subset of Σ_β with topological entropy $\log\beta(n)$. Since $d(1,\beta(n))$ is finite, the subshift $\Sigma_{\beta(n)}$ is an SFT. It therefore follows form [Par, Theorem 5] that Σ_β is an almost sofic. $\square$

Hence it holds that $k^-_\infty(\tau)=h_{\rm top}(\Sigma_\beta)$ for every β-shift by Corollary 3.5.

Corollary 3.7. *Let Σ_β be the β-shift for $\beta>1$. If $\tau=(T_a)_{a\in A}$ is the creation operators for Σ_β, then we have*

$$k^-_\infty(\tau)=h_{\rm top}(\Sigma_\beta)=\log\beta.$$

4. Groups and Macaev Norm

We discuss a relation between groups and the Macaev norm. Let Γ be a countable finitely generated group, S a symmetric set of generators of Γ We denote by $|\ \ |_S$ the word length and by $\mathcal{W}_n(\Gamma,S)$ the set of elements in Γ with length n, with respect to the system of generators S. The *logarithmic volume* of a group Γ in a given system of generators S is the number

$$v_S=\lim_{n\to\infty}\frac{\log|\mathcal{W}_n(\Gamma,S)|}{n},$$

(cf. [Ver]). The following proposition can be proved in the same way as in the free group case [Voi3, Proposition 3.7. (a)].

Proposition 4.1. *Let Γ be a finitely generated group with a finite generating set S and λ the left regular representation of Γ If we set $\lambda_S=(\lambda_a)_{a\in S}$, then*

$$k^-_\infty(\lambda_S)\leq v_S.$$

Proof. Let us denote by P_n the projection onto the subspace $\overline{\rm span}\{\delta_g\in l^2(\Gamma)\mid |g|_S=n\}$. If we set

$$X_n=\sum_{j=0}^{n-1}\left(1-\frac{j}{n}\right)P_j,$$

then we have

$$||X_n\lambda_a-\lambda_aX_n||=||\lambda_a^*X_n\lambda_a-X_n||\leq\frac{1}{n}$$

for $a\in S$. Hence

$$k^-_\infty(\lambda_S)\leq\limsup_{n\to\infty}\max_{a\in S}|||[X_n,\lambda_a]|||^-_\infty\leq\lim_{n\to\infty}\frac{\log\sum_{j=0}^n|\mathcal{W}_n(\Gamma,S)|}{n}=v_S.$$

$\square$

Now we compute the exact value of $k_\infty^-(\lambda_S)$ for certain amalgamated free product groups.

Proposition 4.2. *Let A be a finite group, $G_1, \cdots, G_M$ nontrivial finite groups containing A as a subgroup and $H_1, \ldots, H_N$ the product group of the infinite cyclic group $\mathbb{Z}$ and the finite group A, $(N + M > 1)$. Let Γ be the amalgamated free product group of $G_1, \cdots, G_M, H_1, \cdots, H_N$ with amalgamation over A. Set $S = G_1 \cup \cdots \cup G_M \cup (S_1 \times A) \cup \cdots \cup (S_N \times A) \setminus \{e\}$, where S_j is the canonical generating set $\{x_j, x_j^{-1}\}$ of the infinite cyclic group $\mathbb{Z}$ and e is the group unit. Let λ be the left regular representation of Γ and $\lambda_S = (\lambda_a)_{a\in S}$. Then we have*

$$k_\infty^-(\lambda_S) = v_S.$$

In particular, for the free group $\mathbb{F}_N$ $(N \geq 2)$, we have

$$k_\infty^-(\lambda_S) = \log(2N-1).$$

Proof. By Proposition 4.1, it suffices to show that $v_S \leq k_\infty^-(\lambda_S)$. Let Ω_i be the set of the representatives of G_i/A with $e \in \Omega_i$ for $i = 1, \ldots, M$. We identify x_j with $(x_j, e) \in H_j$ for $j = 1, \ldots, N$, and set $\Omega_{M+j} = \{x_j, x_j^{-1}, e\}$. Let

$$\widetilde{S} = \bigcup_{i=1}^{M+N} \Omega_i \setminus \{e\}.$$

We define the 0-1 matrix A with index $\widetilde{S}$ by

$$A(a, b) = \begin{cases} 1 & \text{if } |ab|_S = 2; \\ 0 & \text{otherwise.} \end{cases}$$

One can easily check that the above matrix A is irreducible and the topological entropy $h_{\mathrm{top}}(\Sigma_A)$ of the Markov shift Σ_A coincides with the logarithmic volume v_S of Γ with respect to the generating set S.

We denote by Γ_0 the subset of Γ consisting of the group unit e and elements $a_1 \cdots a_n \in \Gamma$, $(n \in \mathbb{N})$ of the form

$$\begin{cases} a_k \in \Omega_{i_k} \setminus \{e\} & \text{for } k = 1, \ldots, n, \\ i_k \neq i_{k+1} & \text{if } 1 \leq i_k \leq M, \\ a_k = a_{k+1} & \text{if } M+1 \leq i_k \leq M+N, i_k = i_{k+1}. \end{cases}$$

Note that the subspace $l^2(\Gamma_0)$ can be identified with the Fock space $\mathcal{F}_A$ of the Markov shift Σ_A by the following correspondence:

$$\begin{aligned} \delta_e &\longleftrightarrow \xi_0, \\ \delta_{a_1\cdots a_n} &\longleftrightarrow \xi_{a_1} \otimes \cdots \otimes \xi_{a_n}. \end{aligned}$$

Let us denote by P_n the projection onto the subspace

$$\overline{\mathrm{span}}\{\delta_g \in l^2(\Gamma) \mid |g|_S = n\}.$$

For $a \in S$, we define the partial isometry $T_a \in \mathbb{B}(l^2(\Gamma))$ by

$$T_a = \sum_{n\geq 0} P_{n+1} \lambda_a P_n.$$

Under the identification with $\mathcal{F}_A$, the partial isometry $T_a|_{l^2(\Gamma_0)}$ for $a \in \widetilde{S}$ is the creation operator on $\mathcal{F}_A$, (cf. [Oka]). We also identify Γ_0 and $\mathcal{W}(\Sigma_A)$. For $w = a_1 \cdots a_n \in \Gamma_0$, we set $T_w = T_{a_1} \cdots T_{a_n}$. Let μ be the maximal measure of Σ_A. For $a \in \widetilde{S}$, we put

$$X_a = \sum_{n\geq 0} \sum_{w} \mu([aw]) T_w P_0 T_{aw}^*,$$

where w runs over all $w \in \Gamma_0$ with $|w|_S = n$ and $|aw|_S = |w|_S + 1$. For $a \in S \setminus \widetilde{S}$, we set $X_a = 0$. It can be easily checked that $[\lambda_a, X_a] = [T_a, X_a]$ for $a \in \widetilde{S}$. Therefore by the same proof as in the sufshift case, we obtain

$$v_S = h_{\mathrm{top}}(\Sigma_A) = k_\infty^-(\lambda_S).$$

$\square$

Remark 4.3. Let Γ be a finitely generated group with a finite generating set S. In [Voi5], Voiculescu proved that if the entropy $h(\Gamma, \mu)$ of a random walk μ on Γ with support S is non-zero, then $k_\infty^-((\lambda_a)_{a\in S})$ is non-zero. However the above proposition suggests that the volume v_S of Γ is more related to the invariant $k_\infty^-((\lambda_a)_{a\in S})$ rather than the entropy $h(\Gamma, \mu)$. It is an interesting problem to ask whether v_S being non-zero implies $k_\infty^-((\lambda_a)_{a\in S})$ being non-zero. We also remark here that there is a relation between v_S and $h(\Gamma, \mu)$: If $h(\Gamma, \mu) \neq 0$, then $v_S \neq 0$, (see [Ver, Theorem 1]). If the above mentioned problem was solved affirmatively, then it would follow from Proposition 4.1 that $k_\infty^-((\lambda_a)_{a\in S}) \neq 0$ if and only if $v_S \neq 0$, i.e. Γ has exponential growth.

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Cuntz-Krieger-Pimsner Algebras Associated with Amalgamated Free Product Groups

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Abstract

We give a construction of a nuclear C^*-algebra associated with an amalgamated free product of groups, generalizing Spielberg's construction of a certain Cuntz-Krieger algebra associated with a finitely generated free product of cyclic groups. Our nuclear C^*-algebras can be identified with certain Cuntz-Krieger-Pimsner algebras. We will also show that our algebras can be obtained by the crossed product construction of the canonical actions on the hyperbolic boundaries, which proves a special case of Adams' result about amenability of the boundary action for hyperbolic groups. We will also give an explicit formula of the K-groups of our algebras. Finally we will investigate the relationship between the KMS states of the generalized gauge actions on our C^* algebras and random walks on the groups.

1 Introduction

In [Cho], Choi proved that the reduced group C^*-algebra $C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ of the free product of cyclic groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ is embedded in $\mathcal{O}_2$. Consequently, this shows that $C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ is a non-nuclear exact C^*-algebra, (see S. Wassermann [Was] for a good introduction to exact C^*-algebras). Spielberg generalized it to finitely generated free products of cyclic groups in [Spi]. Namely, he constructed a certain action on a compact space and proved that some Cuntz-Krieger algebras (see [CK]) can be obtained by the crossed product construction for the action. For a related topic, see W. Szymański and S. Zhang's work [SZ].

More generally, the above mentioned compact space coincides with Gromov's notion of the boundaries of hyperbolic groups (e.g. see [GH]). In [Ada], Adams proved that the action of any discrete hyperbolic group Γ on the hyperbolic boundary $\partial\Gamma$ is amenable in the sense of Anantharaman-Delaroche [Ana]. It follows from [Ana] that the corresponding crossed product $C(\partial\Gamma) \rtimes_r \Gamma$ is nuclear, and this implies that $C_r^*(\Gamma)$ is an exact C^*-algebra.

Although we know that $C(\partial\Gamma) \rtimes_r \Gamma$ is nuclear for a general discrete hyperbolic group Γ as mentioned above, there are only few things known about this C^*-algebra. So one of our purposes is to generalize Spielberg's construction to some finitely generated amalgamated free product Γ and to give detailed description of the algebra $C(\partial\Gamma) \rtimes_r \Gamma$. More precisely, let I be a finite index set and G_i be a group containing a copy of a finite group H as a subgroup for $i \in I$. We always assume that each G_i is either a finite group or $\mathbb{Z} \times H$. Let $\Gamma = *_H G_i$ be the amalgamated free product group. We will construct a nuclear C^*-algebra $\mathcal{O}_\Gamma$ associated with Γ by mimicking the construction for Cuntz-Krieger algebras with respect to the full Fock space in M. Enomoto, M. Fujii and Y. Watatani [EFW1] and D. E. Evans [Eva]. This generalizes Spielberg's construction.

First we show that $\mathcal{O}_\Gamma$ has a certain universal property as in the case of the Cuntz-Krieger algebras, which allows several descriptions of $\mathcal{O}_\Gamma$. For example, it turns out that $\mathcal{O}_\Gamma$ is a Cuntz-Krieger-Pimsner algebra, introduced by Pimsner in [Pim2] and studied by several authors, e.g. T. Kajiwara, C. Pinzari and Y. Watatani [KPW]. We will also show that $\mathcal{O}_\Gamma$ can be obtained by the crossed product construction. Namely, we will introduce a boundary space Ω with a natural Γ-action, which coincides with the boundary of the associated tree (see [Ser], [W1]). Then we will prove that $C(\Omega) \rtimes_r \Gamma$ is isomorphic to $\mathcal{O}_\Gamma$. Since the hyperbolic boundary $\partial\Gamma$ coincides with Ω and the two actions of Γ on $\partial\Gamma$ and Ω are conjugate, $\mathcal{O}_\Gamma$ is also isomorphic to $C(\partial\Gamma) \rtimes_r \Gamma$, and depends only on the group structure of Γ. As a consequence, we give a proof to Adams' theorem in this special case.

Next, we will consider the K-groups of $\mathcal{O}_\Gamma$. In [Pim1], Pimsner gave a certain exact sequence of KK-groups of the crossed product by groups acting on trees. However, it is not a trivial task to apply Pimsner's exact sequence to $C(\partial\Gamma) \rtimes_r \Gamma$ and obtain its K-groups. We will give explicit formulae of the K-groups of $\mathcal{O}_\Gamma$ following the method used for the Cuntz-Krieger algebras instead of using $C(\partial\Gamma) \rtimes_r \Gamma$ We can compute the K-groups of $C(\partial\Gamma) \rtimes_r \Gamma$ for concrete examples. They are completely determined by the representation theory of H and the actions of H on G_i/H (the space of right cosets) by left multiplication.

Finally we will prove that KMS states on $\mathcal{O}_\Gamma$ for generalized gauge actions arise from harmonic measures on the Poisson boundary with respect to random walks on the discrete group Γ. Consequently, for special cases, we can determine easily the type of factor $\mathcal{O}_\Gamma''$ for the corresponding unique KMS state of the gauge action by essentially the same arguments in M. Enomoto, M. Fujii and Y. Watatani [EFW2], which generalized J. Ramagge and G. Robertson's result [RR].

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2 Preliminaries

In this section, we collect basic facts used in the present article. We begin by reviewing the Cuntz-Krieger-Pimsner algebras in [Pim2]. Let A be a C^*-algebra and X be a Hilbert bimodule over A, which means that X is a right Hilbert A-module with an injective $*$-homomorphism of A to $\mathcal{L}(X)$, where $\mathcal{L}(X)$ is the C^*-algebra of all adjointable A-linear operators on X. We assume that X is full, that is, $\{\langle x,y\rangle_A \mid x,y \in X\}$ generates A as a C^*-algebra, where $\langle\cdot,\cdot\rangle_A$ is the A-valued inner product on X. We further assume that X has a finite basis $\{u_1,\dots,u_n\}$, which means that $x = \sum_{i=1}^n u_i\langle u_i,x\rangle_A$ for any $x \in X$. We fix a basis $\{u_1,\dots,u_n\}$ of X. Let $\mathcal{F}(X) = A \oplus \bigoplus_{n\geq 1} X^{(n)}$ be the full Fock space over X, where $X^{(n)}$ is the n-fold tensor product $X \otimes_A X \otimes_A \cdots \otimes_A X$. Note that $\mathcal{F}(X)$ is naturally equipped with Hilbert A-bimodule structure. For each $x \in X$, the operator $T_x : \mathcal{F}(X) \to \mathcal{F}(X)$ is defined by

$$\begin{aligned} T_x(x_1\otimes\cdots\otimes x_n) &= x\otimes x_1\otimes\cdots\otimes x_n, \\ T_x(a) &= xa, \end{aligned}$$

for $x, x_1,\dots,x_n \in X$ and $a \in A$. Note that $T_x \in \mathcal{L}(\mathcal{F}(X))$ satisfies the following relations

$$\begin{aligned} T_x^*T_y &= \langle x,y\rangle_A, \qquad x,y\in X, \\ aT_xb &= T_{axb}, \qquad x\in X, a,b\in A. \end{aligned}$$

Let π be the quotient map of $\mathcal{L}(\mathcal{F}(X))$ onto $\mathcal{L}(\mathcal{F}(X))/\mathcal{K}(\mathcal{F}(X))$ where $\mathcal{K}(\mathcal{F}(X))$ is the C^*-algebra of all compact operators of $\mathcal{L}(\mathcal{F}(X))$. We denote $S_x = \pi(T_x)$ for $x \in X$. Then we define the Cuntz-Krieger-Pimsner algebra $\mathcal{O}_X$ to be

$$\mathcal{O}_X = C^*(S_x \mid x \in X).$$

Since X is full, a copy of A acting by left multiplication on $\mathcal{F}(X)$ is contained in $\mathcal{O}_X$. Furthermore we have the relation

$$\sum_{i=1}^n S_{u_i}S_{u_i}^* = 1. \tag{†}$$

On the other hand, $\mathcal{O}_X$ is characterized as the universal C^*-algebra generated by A and S_x, satisfying the above relations [Pim2, Theorem 3.12]. More precisely, we have

Theorem 2.1 ([Pim2, Theorem 3.12]) *Let X be a full Hilbert A-bimodule and $\mathcal{O}_X$ be the corresponding Cuntz-Krieger-Pimsner algebra. Suppose that $\{u_1,\cdots,u_n\}$ is a finite* *basis for X. If B is a C^*-algebra generated by $\{s_x\}_{x\in X}$ satisfying*

$$\begin{aligned} s_x + s_y &= s_{x+y}, && x \in X, \\ a s_x b &= s_{axb}, && x \in X, a, b \in A, \\ s_x^* s_y &= \langle x, y\rangle_A, && x, y \in X, \\ \sum_{i=1}^{n} s_{u_i} s_{u_i}^* &= 1. \end{aligned}$$

Then there exists a unique surjective $*$-homomorphism from O_X onto $C^*(s_x)$ that maps S_x to s_x.

Next we recall the notion of amenability for discrete C^*-dynamical systems introduced by C. Anantharaman-Delaroche in [Ana]. Let (A, G, α) be a C^*-dynamical system, where A is a C^*-algebra, G is a group and α is an action of G on A. An A-valued function h on G is said to be of *positive type* if the matrix $[\alpha_{s_i}(h(s_i^{-1}s_j))] \in M_n(A)$ is positive for any $s_1, \ldots, s_n \in G$. We assume that G is discrete. Then α is said to be *amenable* if there exists a net $(h_i)_{i\in I} \subset C_c(G, Z(A''))$ of functions of positive type such that

$$\begin{cases} h_i(e) \leq 1 & \text{for } i \in I, \\ \lim_i h_i(s) = 1 & \text{for } s \in G, \end{cases}$$

where the limit is taken in the σ-weak topology in the enveloping von Neumann algebra A'' of A. We remark that this is one of several equivalent conditions given in [Ana, Théorème 3.3]. We will use the following theorems without a proof.

Theorem 2.2 **([Ana, Théorème 4.5])** *Let (A, G, α) be a C^*-dynamical system such that A is nuclear and G is discrete. Then the following are equivalent:*
1) The full C^*-crossed product $A \rtimes_\alpha G$ is nuclear;
2) The reduced C^*-crossed product $A \rtimes_{\alpha r} G$ is nuclear;
3) The W^*-crossed product $A'' \rtimes_{\alpha w} G$ is injective;
4) The action α of G on A is amenable.

Theorem 2.3 **([Ana, Théorème 4.8])** *Let (A, G, α) be an amenable C^*-dynamical system such that G is discrete. Then the natural quotient map from $A \rtimes_\alpha G$ onto $A \rtimes_{\alpha r} G$ is an isomorphism.*

Finally, we review the notion of the strong boundary actions in [LS]. Let Γ be a discrete group acting by homeomorphisms on a compact Hausdorff space Ω. Suppose that Ω has at least three points. The action of Γ on Ω is said to be a *strong boundary action* if for every pair U, V of non-empty open subsets of Ω there exists $\gamma \in \Gamma$ such that $\gamma U^c \subset V$ The action of Γ on Ω is said to be *topologically free* in the sense of [AS] if the fixed point set of each non-trivial element of Γ has empty interior.

Theorem 2.4 ([LS, **Theorem** 5]) *Let (Ω, Γ) be a strong boundary action where Ω is compact. We further assume that the action is topologically free. Then $C(\Omega) \rtimes_r \Gamma$ is purely infinite and simple.*

3 A motivating example

Before introducing our algebras, we present a simple case of Spielberg's construction for $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$ with generators a and b as a motivating example. See also [RS]. The Cayley graph of $\mathbb{F}_2$ is a homogeneous tree of degree 4. The boundary Ω of the tree in the sense of [Fre] (see also [Fur]) can be thought of as the set of all infinite reduced words $\omega = x_1x_2x_3\cdots$, where $x_i \in S = \{a, b, a^{-1}, b^{-1}\}$. Note that Ω is compact in the relative topology of the product topology of $\prod_{\mathbb{N}} S$. In an appendix, several facts about trees are collected for the convenience of the reader, (see also [FN]). Left multiplication of $\mathbb{F}_2$ on Ω induces an action of $\mathbb{F}_2$ on $C(\Omega)$. For $x \in \mathbb{F}_2$, let $\Omega(x)$ be the set of infinite words beginning with x. We identify the implementing unitaries in the full crossed product $C(\Omega) \rtimes \mathbb{F}_2$ with elements of $\mathbb{F}_2$. Let p_x denote the projection defined by the characteristic function $\chi_{\Omega(x)} \in C(\Omega)$. Note that for each $x \in S$,

$$p_x + xp_{x^{-1}}x^{-1} = 1,$$

$$p_a + p_{a^{-1}} + p_b + p_{b^{-1}} = 1,$$

hold. For $x \in S$, let $S_x \in C(\Omega) \rtimes \mathbb{F}_2$ be a partial isometry

$$S_x = x(1 - p_{x^{-1}}).$$

Then we have

$$\begin{aligned} S_x^* S_y = x^{-1}p_x p_y y = \delta_{x,y} S_x^* S_x = \delta_{x,y}(1 - p_{x^{-1}}), \\ S_x S_x^* = x(1 - p_{x^{-1}})x^{-1} = p_x, \\ S_x^* S_x = 1 - p_{x^{-1}} = \sum_{y \neq x^{-1}} S_y S_y^*. \end{aligned}$$

These relations show that the partial isometries S_x generate the Cuntz-Krieger algebra $\mathcal{O}_A$ [CK], where

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

On the other hand, we can recover the generators of $C(\Omega) \rtimes \mathbb{F}_2$ by setting

$$x = S_x + S_{x^{-1}}^* \quad \text{and} \quad p_x = S_x S_x^*.$$

Hence we have $C(\Omega) \rtimes \mathbb{F}_2 \simeq \mathcal{O}_A$.

Next we recall the Fock space realization of the Cuntz-Krieger algebras, (e.g. see [Eva], [EFW1]). Let $\{e_a, e_b, e_{a^{-1}}, e_{b^{-1}}\}$ be a basis of $\mathbb{C}^4$ We define the Fock space associated with the matrix A by

$$\mathcal{F}_A = \mathbb{C}e_0 \oplus \bigoplus_{n\geq 1} (\overline{\mathrm{span}}\{e_{x_1} \otimes \cdots \otimes e_{x_n} \mid A(x_i, x_{i+1}) = 1\}),$$

where e_0 is the vacuum vector. For any $x \in S$, let T_x be the creation operator on $\mathcal{F}$, given by

$$\begin{aligned} T_x e_0 &= e_x, \\ T_x(e_{x_1} \otimes \cdots \otimes e_{x_n}) &= \begin{cases} e_x \otimes e_{x_1} \otimes \cdots \otimes e_{x_n} & \text{if } A(x, x_1) = 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Let p_0 be the rank one projection on the vacuum vector e_0. Note that we have

$$T_a T_a^* + T_b T_b^* + T_{a^{-1}} T_{a^{-1}}^* + T_{b^{-1}} T_{b^{-1}}^* + p_0 = 1.$$

If π is the quotient map of $\mathcal{B}(\mathcal{F})$ onto the Calkin algebra $\mathcal{Q}(\mathcal{F})$, then the C^*-algebra generated by the partial isometries $\{\pi(T_a), \pi(T_b), \pi(T_{a^{-1}}), \pi(T_{b^{-1}})\}$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_A$.

Now we look at this construction from another point of view. We can perform the following natural identification:

$$\mathcal{F} \ni \begin{array}{ccc} e_0 & \longleftrightarrow & \delta_e \\ e_{x_1} \otimes \cdots \otimes e_{x_n} & \longleftrightarrow & \delta_{x_1 \cdots x_n} \end{array} \in l^2(\mathbb{F}_2).$$

Under this identification, the creation operator T_x on $l^2(\mathbb{F}_2)$ can be expressed as

$$\begin{aligned} T_x \delta_e &= \lambda_x \delta_e, \\ T_x \delta_{x_1 \cdots x_n} &= \begin{cases} \lambda_x \delta_{x_1 \cdots x_n} & \text{if } x \neq x_1^{-1}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

where λ is the left regular representation of $\mathbb{F}_2$.

For a reduced word $x_1 \cdots x_n \in \mathbb{F}_2$, we define the length function $|\ |$ on $\mathbb{F}_2$ by $|x_1 \cdots x_n| = n$. Let p_n be the projection onto the closed linear span of $\{\delta_\gamma \in l^2(\mathbb{F}_2) \mid |\gamma| = n\}$. Then we can express T_x for $x \in S$ by

$$T_x = \sum_{n\geq 0} p_{n+1} \lambda_x p_n.$$

Note that this expression makes sense for every finitely generated group. In the next section, we generalize this construction to amalgamated free product groups.

4 Construction of a nuclear C^*-algebra $\mathcal{O}_\Gamma$

In what follows, we always assume that I is a finite index set and G_i is a group containing a copy of a finite group H as a subgroup for $i \in I$. Moreover, we assume that each G_i is either a finite group or $\mathbb{Z} \times H$. We set $I_0 = \{i \in I \mid |G_i| < \infty\}$. Let $\Gamma = *_H G_i$ be the amalgamated free product.

First we introduce a "length function" $|\ \ |$ on each G_i. If $i \in I_0$, we set $|g| = 1$ for any $g \in G_i \setminus H$ and $|h| = 0$ for any $h \in H$. If $i \in I \setminus I_0$ we set $|(a_i^n, h)| = |n|$ for any $(a_i^n, h) \in G_i = \mathbb{Z} \times H$ where a_i is a generator of $\mathbb{Z}$. Now we extend the length function to Γ Let Ω_i be a set of left representatives of G_i/H with $e \in \Omega_i$. If $\gamma \in \Gamma$ is written uniquely as $g_1 \cdots g_n h$, where $g_1 \in \Omega_{i_1}, \dots, g_n \in \Omega_{i_n}$ with $i_1 \neq i_2, \dots, i_{n-1} \neq i_n$(we write simply $i_1 \neq \cdots \neq i_n$), then we define

$$|\gamma| = \sum_{k=1}^{n} |g_k|.$$

Let p_n be the projection of $l^2(\Gamma)$ onto $l^2(\Gamma_n)$ for each n, where $\Gamma_n = \{\gamma \in \Gamma \mid |\gamma| = n\}$. We define partial isometries and unitary operators on $l^2(\Gamma)$ by

$$\begin{cases} T_g = \sum_{n \geq 0} p_{n+1} \lambda_g p_n & \text{if } g \in \bigcup_{i \in I} G_i \setminus H, \\ V_h = \lambda_h & \text{if } h \in H, \end{cases}$$

where λ is the left regular representation of Γ. Let π be the quotient map of $\mathcal{B}(l^2(\Gamma))$ onto $\mathcal{B}(l^2(\Gamma))/\mathcal{K}(l^2(\Gamma))$, where $\mathcal{B}(l^2(\Gamma))$ is the C^*-algebra of all bounded linear operators on $l^2(\Gamma)$ and $\mathcal{K}(l^2(\Gamma))$ is the C^*-subalgebra of all compact operators of $\mathcal{B}(l^2(\Gamma))$. We set $\pi(T_g) = S_g$ and $\pi(V_h) = U_h$. For $\gamma \in \Gamma$, we define S_γ by

$$S_\gamma = S_{g_1} \cdots S_{g_n},$$

where $\gamma = g_1 \cdots g_n$ for some $g_1 \in G_{i_1} \setminus H, \dots, g_n \in G_{i_n} \setminus H$ with $i_1 \neq \cdots \neq i_n$. Note that S_γ does not depend on the expression $\gamma = g_1 \cdots g_n$. We denote the initial projections of S_γ by $Q_\gamma = S_\gamma^* S_\gamma$ and the range projections by $P_\gamma = S_\gamma S_\gamma^*$ for $\gamma \in \Gamma$.

We collect several relations, which the family $\{S_g, U_h \mid g \in \bigcup_{i \in I} G_i \setminus H, h \in H\}$ satisfies.

For $g, g' \in \bigcup_i G_i \setminus H$ with $|g| = |g'| = 1$ and $h \in H$,

$$S_{gh} = S_g U_h, \qquad S_{hg} = U_h S_g, \tag{1}$$

$$P_g P_{g'} = \begin{cases} P_g = P_{g'} & \text{if } gH = g'H, \\ 0 & \text{if } gH \neq g'H. \end{cases} \tag{2}$$

Moreover, if $g \in G_i \setminus H$ and $i \in I_0$, then

$$Q_g = \sum_{\substack{j \in I_0 \\ j \neq i}} \sum_{g' \in \Omega_j \setminus \{e\}} P_{g'} + \sum_{j \in I \setminus I_0} P_{a_j} + P_{a_j^{-1}}, \tag{3}$$

and if $g = a_i^{\pm 1}$ and $i \in I \setminus I_0$, then

$$Q_{a_i^{\pm 1}} = \sum_{j \in I_0} \sum_{g' \in \Omega_j \setminus \{e\}} P_{g'} + \sum_{\substack{j \in I \setminus I_0 \\ j \neq i}} \left(P_{a_j} + P_{a_j^{-1}} \right) + P_{a_i^{\pm 1}}. \tag{3$'$}$$

Finally,

$$1 = \sum_{i \in I_0} \sum_{g \in \Omega_i \setminus \{e\}} P_g + \sum_{i \in I \setminus I_0} \left(P_{a_i} + P_{a_i^{-1}} \right) \tag{4}$$

Indeed, (1) follows from the relations $T_{gh} = T_g V_h$ and $T_{hg} = V_h T_g$. From the definition, we have $T_{g'}^* T_g = \sum_{n \geq 0} p_n \lambda_{g'}^* p_{n+1} \lambda_g p_n$. This can be non-zero if and only if $|g'^{-1} g| = 0$, i.e. $g'^{-1} g \in H$. We have (2) immediately. The relation

$$1 = \sum_{i \in I_0} \sum_{g \in \Omega_i} T_g T_g^* + \sum_{i \in I \setminus I_0} \left(T_{a_i} T_{a_i}^* + T_{a_i^{-1}} T_{a_i^{-1}}^* \right) + p_0,$$

implies (4). By multiplying S_g^* on the left and S_g on the right of equation (4) respectively, we obtain (3).

Moreover, the following condition holds: Let $P_i = \sum_{g \in \Omega_i} P_g$ for $i \in I_0$, and $P_i = P_{a_i} + P_{a_i^{-1}}$ for $i \in I \setminus I_0$. For every $i \in I$, we have

$$C^*(H) \simeq C^* \left(P_i U_h P_i \mid h \in H \right). \tag{5}$$

Indeed, since the unitary representation $P_i' V_h P_i'$ contains the left regular representation of H with infinite multiplicity, where P_i' is some projection with $\pi(P_i') = P_i$. we have relation (5).

Now we consider the universal C^*-algebra generated by the family $\{S_g, U_h \mid g \in \bigcup_{i \in I} G_i \setminus H, h \in H\}$ satisfying (1), (2), (3) and (4). We denote it by $\mathcal{O}_\Gamma$. Here, the universality means that if another family $\{s_g, u_h\}$ satisfies (1), (2), (3) and (4), then there exists a surjective $*$-homomorphism ϕ of $\mathcal{O}_\Gamma$ onto $C^*(s_g, u_h)$ such that $\phi(S_g) = s_g$ and $\phi(U_h) = u_h$. Summing up the above, we employ the following definitions and notation:

Definition 4.1 *Let I be a finite index set and G_i be a group containing a copy of a finite group H as a subgroup for $i \in I$. Suppose that each G_i is either a finite group or $\mathbb{Z} \times H$. Let I_0 be the subset of I such that G_i is finite for all $i \in I_0$. We denote the amalgamated free product $*_H G_i$ by Γ*

We fix a set Ω_i of left representatives of G_i/H with $e \in \Omega_i$ and a set X_i of representatives of $H \backslash G_i / H$ which is contained in Ω_i. Let (a_i, e) be a generator of G_i for $i \in I \setminus I_0$. We write a_i, for short. Here we choose $\Omega_i = X_i = \{a_i^n \mid n \in \mathbb{N}\}$. We exclude the case where $\bigcup_i \Omega_i \setminus \{e\}$ has only one or two points.

We define the corresponding universal C^*-algebra $\mathcal{O}_\Gamma$ generated by partial isometries S_g for $g \in \bigcup_{i \in I} G_i \setminus H$ and unitaries U_h for $h \in H$ satisfying (1), (2), (3) and (4).

We set for $\gamma \in \Gamma$,

$$Q_\gamma = S_\gamma^* \ S_\gamma, \ \ P_\gamma = S_\gamma \ S_\gamma^*,$$
$$P_i = \textstyle\sum_{g\in\Omega_i} P_g \qquad \text{if } i \in I_0,$$
$$P_i = P_{a_i} + P_{a_i^{-1}} \quad \text{if } i \in I \setminus I_0.$$

For convenience, we set for any integer n,

$$\Gamma_n = \{\gamma \in \Gamma \mid |\gamma| = n\},$$

$$\Delta_n = \{\gamma \in \Gamma_n \mid \gamma = \gamma_1 \cdots \gamma_n, \gamma_k \in \Omega_{i_k}, i_1 \neq \cdots \neq i_n\}.$$

We also set $\Delta = \bigcup_{n\geq 1} \Delta_n$.

Lemma 4.2 *For* $i \in I$ *and* $h \in H$,

$$U_h P_i = P_i U_h.$$

Proof. Use the above relations (2). □

Lemma 4.3 *Let* $\gamma_1, \gamma_2 \in \Gamma$. *Suppose that* $S_{\gamma_1}^* S_{\gamma_2} \neq 0$.
If $|\gamma_1| = |\gamma_2|$, *then* $S_{\gamma_1}^* S_{\gamma_2} = Q_g U_h$ *for some* $g \in \bigcup_{i\in I} G_i, h \in H$.
If $|\gamma_1| > |\gamma_2|$, *then* $S_{\gamma_1}^* S_{\gamma_2} = S_\gamma^*$ *for some* $\gamma \in \Gamma$ *with* $|\gamma| = |\gamma_1| - |\gamma_2|$.
If $|\gamma_1| < |\gamma_2|$, *then* $S_{\gamma_1}^* S_{\gamma_2} = S_\gamma$ *for some* $\gamma \in \Gamma$ *with* $|\gamma| = |\gamma_2| - |\gamma_1|$.

Proof. By (2), we obtain the lemma. □

Corollary 4.4

$$\mathcal{O}_\Gamma = \overline{\text{span}}\{\, S_\mu P_i S_\nu^* \mid \mu, \nu \in \Gamma, i \in I\,\}.$$

Proof. This follows from the previous lemma. □

Next we consider the gauge action of $\mathcal{O}_\Gamma$. Namely, if $z \in \mathbb{T}$ then the family $\{\, zS_g, U_h \,\}$ also satisfies (1), (2), (3), (4) and generates $\mathcal{O}_\Gamma$. The universality gives an automorphism α_z on $\mathcal{O}_\Gamma$ such that $\alpha_z(S_g) = zS_g$ and $\alpha_z(U_h) = U_h$. In fact, α is a continuous action of $\mathbb{T}$ on $\mathcal{O}_\Gamma$, which is called *the gauge action.* Let dz be the normalized Haar measure on $\mathbb{T}$ and we define a conditional expectation Φ of $\mathcal{O}_\Gamma$ onto the fixed-point algebra $\mathcal{O}_\Gamma^{\mathbb{T}} = \{\, a \in \mathcal{O}_\Gamma \mid \alpha_z(a) = a, \text{for } z \in \mathbf{T}\,\}$ by

$$\Phi(a) = \int_{\mathbb{T}} \alpha_z(a)\, dz, \qquad \text{for } a \in \mathcal{O}_\Gamma.$$

Lemma 4.5 *The fixed-point algebra* $\mathcal{O}_\Gamma^{\mathbb{T}}$ *is an AF-algebra.*

Proof. For each $i \in I$, set

$$\mathcal{F}_n^i = \overline{\operatorname{span}}\{ S_\mu P_i S_\nu^* \mid \mu, \nu \in \Gamma_n \}.$$

We can find systems of matrix units in $\mathcal{F}_n^i$, parameterized by $\mu, \nu \in \Delta_n$, as follows:

$$e^i_{\mu,\nu} = S_\mu P_i S_\nu^*.$$

Indeed, using the previous lemma, we compute

$$e^i_{\mu_1,\nu_1} e^i_{\mu_2,\nu_2} = \delta_{\nu_1,\mu_2} S_{\mu_1} P_i Q_{\nu_1} P_i S_{\nu_2}^* = \delta_{\nu_1,\mu_2} e^i_{\mu_1,\nu_2}.$$

Thus we obtain the identifications

$$\mathcal{F}_n^i \simeq M_{N(n,i)}(\mathbf{C}) \otimes e^i_{\mu,\mu} \mathcal{F}_n^i e^i_{\mu,\mu},$$

for some integer $N(n,i)$ and some $\mu \in \Delta_n$. Moreover, for ξ, η,

$$e^i_{\mu,\mu} \left(S_\xi P_i S_\eta^* \right) e^i_{\mu,\mu} = \begin{cases} S_\mu P_i U_h P_i S_\mu^* & \text{if } \quad \xi, \eta \in \mu H, \\ 0 & \text{otherwise.} \end{cases}$$

for some $h \in H$. Note that $C^*(S_\mu P_i U_h P_i S_\mu^* \mid h \in H)$ is isomorphic to $C^*(P_i U_h P_i \mid h \in H)$ via the map $x \mapsto S_\mu^* x S_\mu$. Therefore the relation (5) gives

$$\mathcal{F}_n^i \simeq M_k(\mathbf{C}) \otimes \overline{\operatorname{span}}\{ S_\mu P_i U_h P_i S_\mu^* \mid h \in H \} \simeq M_k(\mathbf{C}) \otimes C^*(H).$$

Note that $\{\mathcal{F}_n^i \mid i \in I\}$ are mutually orthogonal and

$$\mathcal{F}_n = \oplus_{i \in I} \mathcal{F}_n^i$$

is a finite-dimensional C^*-algebra.

The relation (2) gives $\mathcal{F}_n \hookrightarrow \mathcal{F}_{n+1}$. Hence,

$$\mathcal{F} = \overline{\bigcup_{n \geq 0} \mathcal{F}_n}$$

is an AF-algebra. Therefore it suffices to show that $\mathcal{F} = \mathcal{O}_\Gamma^{\mathbf{T}}$. It is trivial that $\mathcal{F} \subseteq \mathcal{O}_\Gamma^{\mathbf{T}}$. On the other hand, we can approximate any $a \in \mathcal{O}_\Gamma^{\mathbf{T}}$ by a linear combination of elements of the form $S_\mu P_i S_\nu^*$. Since $\Phi(a) = a$, a can be approximated by a linear combination of elements of the form $S_\mu P_i S_\nu^*$ with $|\mu| = |\nu|$. Thus $a \in \mathcal{F}$ □

We need another lemma to prove the uniqueness of $\mathcal{O}_\Gamma$.

Lemma 4.6 *Suppose that $i_0 \in I$ and W consists of finitely many elements $(\mu, h) \in \Delta \times H$ such that the last word of μ is not contained in Ω_{i_0} and $W \cap H = \emptyset$. Then there exists $\gamma = g_0 \cdots g_n$ with $g_k \in \Omega_{i_k}$ and $i_0 \neq \cdots \neq i_n \neq i_0$ such that for any $(\mu, h) \in W$, $\mu h \gamma$ never have the form $\gamma\gamma'$ for some $\gamma' \in \Gamma$.*

Proof. Let $i_0 \in I$ and W be a finite subset of $\Delta \times H$ as above. We first assume that $|I| \geq 3$. Then we can choose $x \in \Omega_{i_0}, y \in \Omega_j$ and $z \in \Omega_{j'}$ such that $j \neq i_0 \neq j'$ and $j \neq j'$ For sufficiently long word

$$\gamma = (xy)(xz)(xyxy)(xzxz)(xyxyxy)(xzxzxz)\cdots(\cdots z),$$

we are done. We next assume that $|I| = 2$. Since we exclude the case where $\Omega_1 \cup \Omega_2 \setminus \{e\}$ has only one or two elements, we can choose at least three distinct points $x \in \Omega_{i_0}, y \in \Omega_j$ and $z \in \Omega_{j'}$. If $i_0 \neq j = j'$ we set

$$\gamma = (xy)(xz)(xyxy)(xzxz)(xyxyxy)(xzxzxz)\cdots(\cdots z),$$

as well. If $i_0 = j \neq j'$ we set

$$\gamma = (xz)(yz)(xzxz)(yzyz)(xzxzxz)(yzyzyz)\cdots(\cdots z).$$

Then if γ has the desired properties, we are done. Now assume that there exist some $(\mu, h) \in W$ such that $\mu h \gamma = \gamma\gamma'$ for some γ' Fix such an element $(\mu, h) \in W$ By hypothesis, we can choose $\delta \in \Delta$ with $|\gamma'| \leq |\delta|$ such that the last word of δ does not belong to Ω_{i_0} and δ does not have the form $\gamma'\delta'$ for some δ' Set $\tilde{\gamma} = \gamma\delta$. Then $\mu h\tilde{\gamma}$ does not have the form $\gamma\gamma''$ for any γ'' Indeed,

$$\mu h\tilde{\gamma} = \mu h\gamma\delta = \gamma\gamma'\delta \neq \tilde{\gamma}\gamma'',$$

for some γ'' Since W is finite, we can obtain a desired element γ by replacing $\tilde{\gamma}$, inductively. □

We now obtain the uniqueness theorem for $\mathcal{O}_\Gamma$.

Theorem 4.7 *Let $\{ s_g, u_h \}$ be another family of partial isometries and unitaries satisfying (1), (2), (3) and (4). Assume that*

$$C^*(H) \simeq C^*(p_i u_h p_i \mid h \in H),$$

where $p_i = \sum_{g\in\Omega_i\setminus\{e\}} s_g s_g^*$ for $i \in I_0$ and $p_i = s_{a_i}s_{a_i}^* + s_{a_i^{-1}}s_{a_i^{-1}}^*$ for $i \in I \setminus I_0$. Then the canonical surjective $*$-homomorphism π of $\mathcal{O}_\Gamma$ onto $C^*(s_g, u_h)$ is faithful.

Proof. To prove the theorem, it is enough to show that (a) π is faithful on the fixed-point algebra $\mathcal{O}_\Gamma^{\mathbb{T}}$, and (b) $\|\pi(\Phi(a))\| \leq \|\pi(a)\|$ for all $a \in \mathcal{O}_\Gamma$ thanks to [BKR, Lemma 2.2].

To establish (a), it suffices to show that π is faithful on $\mathcal{F}_n$ for all $n \geq 0$. By the proof of Lemma 4.5, we have

$$\mathcal{F}_n^i = M_{N(n,i)}(\mathbb{C}) \otimes C^*(H),$$

for some integer $N(n,i)$. Note that $s_g s_g^*$ is non-zero. Hence π is injective on $M_{N(n,i)}(\mathbb{C})$. By the other hypothesis, π is injective on $C^*(H)$.

Next we will show (b). It is enough to check (b) for

$$a = \sum_{\mu,\nu\in F} \sum_{j\in J} C^j_{\mu,\nu} S_\mu P_j S_\nu^*,$$

where F is a finite subset of Γ and J is a subset of I. For $n = \max\{|\mu| \mid \mu \in F\}$, we have

$$\Phi(a) = \sum_{\{\mu,\nu\in F \mid |\mu|=|\nu|\}} \sum_{j\in J} C^j_{\mu,\nu} S_\mu P_j S_\nu^* \in \mathcal{F}_n.$$

Now by changing F if necessary, we may assume that $\min\{|\mu|,|\nu|\} = n$ for every pair $\mu,\nu \in F$ with $C^j_{\mu,\nu} \neq 0$. Since $\mathcal{F}_n = \oplus_i \mathcal{F}_n^i$, there exists some $i_0 \in J$ such that

$$\|\pi(\Phi(a))\| = \|\sum_{|\mu|=|\nu|} C^{i_0}_{\mu,\nu} s_\mu p_{i_0} s_\nu^*\|.$$

By changing F such that $F \subset \Delta$ again, we may further assume that

$$\|\pi(\Phi(a))\| = \|\sum_{\substack{\mu,\nu\in F\\ |\mu|=|\nu|}} \sum_{h\in F'} C^{i_0}_{\mu,\nu,h} s_\mu p_{i_0} u_h p_{i_0} s_\nu^*\|$$

where F' consists of elements of H, (perhaps with multiplicity). By applying the preceding lemma to

$$W = \{(\mu',h) \in \Delta \times H \mid \mu' \text{ is subword of } \mu \in F, h^{-1} \in F'\},$$

we have $\gamma \in \Delta$ satisfying the property in the previous lemma. Then we define a projection

$$Q = \sum_{\tau\in\Delta_n} s_\tau s_\gamma p_{i_0} s_\gamma^* s_\tau^*.$$

By hypothesis, Q is non-zero.

If $\mu,\nu \in \Delta_n$ then

$$Q\left(s_\mu p_{i_0} s_\nu^*\right) Q = s_\mu s_\gamma p_{i_0} s_\gamma^* p_{i_0} s_\gamma p_{i_0} s_\gamma^* s_\nu^* = s_\mu s_\gamma p_{i_0} s_\gamma^* s_\nu^*$$

is non-zero. Therefore $s_\mu(s_\gamma p_{i_0} s_\gamma^*) s_\nu^*$ is also a family of matrix units parameterized by $\mu,\nu \in \Delta_n$. Hence the same arguments as in the proof of Lemma 4.5 give

$$\pi(\mathcal{F}_n^{i_0}) \simeq M_{N(n,i_0)}(\mathbb{C}) \otimes C^*\left(s_\mu s_\gamma p_{i_0} u_h p_{i_0} s_\gamma^* s_\mu^* \mid h \in H\right)$$

By hypothesis, we deduce that $b \mapsto Q\pi(b)Q$ is faithful on $\mathcal{F}_n^{i_0}$. In particular, we conclude that $\|\pi(\Phi(a))\| = \|Q\pi(\Phi(a))Q\|$.

We next claim that $Q\pi(\Phi(a))Q = Q\pi(a)Q$. We fix $\mu, \nu \in F$ If $|\mu| \neq |\nu|$ then one of μ, ν has length n and the other is longer; say $|\mu| = n$ and $|\nu| > n$. Then

$$Q\left(s_\mu p_{i_0} u_h p_{i_0} s_\nu^*\right) Q = s_\mu s_\gamma p_{i_0} s_\gamma^* p_{i_0} u_h p_{i_0} s_\nu^* \left(\sum_{\tau \in \Delta_n} s_\tau s_\gamma p_{i_0} s_\gamma^* s_\tau^*\right)$$

Since $|\nu| > |\tau|$, this can have a non-zero summand only if $\nu = \tau\nu'$ for some ν' However $s_\gamma^* u_h s_\nu^* s_\tau s_\gamma = s_\gamma^* u_h s_{\nu'}^* s_\gamma$, and $s_{\nu' h^{-1}\gamma}^* s_\gamma$ is non-zero only if $\nu' h^{-1}\gamma$ has the form $\gamma\gamma'$ This is impossible by the choice of γ. Therefore we have $Q\left(s_\mu p_{i_0} s_\nu\right) Q = 0$ if $|\mu| \neq |\nu|$, namely $Q\pi(\Phi(a))Q = Q\pi(a)Q$. Hence we can finish proving (b):

$$\|\pi(\Phi(a))\| = \|Q\pi(\Phi(a))Q\| = \|Q\pi(a)Q\| \leq \|\pi(a)\|.$$

Therefore [BKR, Lemma 2.2] gives the theorem. □

By essentially the same arguments, we can prove the following.

Corollary 4.8 *Let $\{t_g, v_h\}$ and $\{s_g, u_h\}$ be two families of partial isometries and unitaries satisfying (1), (2), (3) and (4). Suppose that the map $p_i v_h p_i \mapsto q_i u_h q_i$ gives an isomorphism:*

$$C^*(p_i v_h p_i \mid h \in H) \simeq C^*(q_i v_h q_i \mid h \in H),$$

where $p_i = \sum_{g \in \Omega_i \setminus \{e\}} t_g t_g^*$, $q_i = \sum_{g \in \Omega_i \setminus \{e\}} s_g s_g^*$ and so on. Then the canonical map gives the isomorphism between $C^*(t_g, v_h)$ and $C^*(s_g, u_h)$.

Before closing this section, we will show that our algebra $\mathcal{O}_\Gamma$ is isomorphic to a certain Cuntz-Krieger-Pimsner algebra. Let $A = C^*\left(P_i U_h P_i \mid h \in H, i \in I\right) \simeq \bigoplus_{i \in I} C_r^*(H)$. We define a Hilbert A-bimodule X as follows:

$$X = \overline{\operatorname{span}}\{\, S_g P_i \mid g \in \bigcup_{j \neq i} G_j,\ |g| = 1,\ i \in I\,\}$$

with respect to the inner product $\langle S_g P_i, S_{g'} P_j \rangle = P_i S_g^* S_{g'} P_j \in A$. In terms of the groups, the A-A bimodule structure can be described as follows: we set

$$A = \bigoplus_{i \in I} A_i = \bigoplus_{i \in I} \mathbb{C}[H],$$

and define an A-bimodule $\mathcal{H}_i$ by

$$\mathcal{H}_i = \mathbb{C}[\{g \in \bigcup_{j \neq i} G_j \mid |g| = 1\}]$$

with left and right A-multiplications such that for $a = (h_i)_{i \in I} \in A$ and $g \in G_j \setminus H \subset \mathcal{H}_i$,

$$a \cdot g = h_j g \quad \text{and} \quad g \cdot a = g h_i,$$

and with respect to the inner product

$$\langle g, g' \rangle_{\mathcal{H}_i} = \begin{cases} g^{-1}g' \in A_i & \text{if } g^{-1}g' \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Then we define the A-bimodule X by

$$X = \bigoplus_{i \in I} \mathcal{H}_i,$$

and we obtain the CKP-algebra $\mathcal{O}_X$.

Proposition 4.9 *Assume that A and X are as above. Then*

$$\mathcal{O}_\Gamma \simeq \mathcal{O}_X.$$

Proof. We fix a finite basis $u(g,i) = g \in \mathcal{H}_i$ for $g \in \Omega_j, i \in I$ with $j \neq i, |g| = 1$. Then we have $\mathcal{O}_X = C^*(S_{u(g,i)})$. Let $s_{u(g,i)} = S_g P_i$ in $\mathcal{O}_\Gamma$. Note that we have $\mathcal{O}_\Gamma = C^*(s_{u(g,i)})$. The relation (4) corresponds to the relations (†) of the CKP-algebras. The family $\{s_{u(g,i)}\}$ therefore satisfies the relations of the CKP-algebras. Since the CKP-algebra has universal properties, there exists a canonical surjective $*$-homomorphism of $\mathcal{O}_X$ onto $\mathcal{O}_\Gamma$. Conversely, let $s_g = \sum_{i\in I} S_{u(g,i)}$ and $u_h = \oplus_{i \in I} h$ for $h \in H$ in $\mathcal{O}_X$, and then we have $\mathcal{O}_X = C^*(s_g, u_h)$. By the universality of $\mathcal{O}_\Gamma$, we can also obtain a canonical surjective $*$-homomorphism of $\mathcal{O}_\Gamma$ onto $\mathcal{O}_X$. These maps are mutual inverses. Indeed,

$$\begin{array}{lllll} S_g & \mapsto & \sum_{i\in I} S_{u(g,i)} & \mapsto & \sum_{i \in I} S_g P_i = S_g, \\ U_h & \mapsto & \bigoplus_{i \in I} h & \mapsto & \sum_{i\in I} P_i U_h P_i = U_h. \end{array}$$

□

5 Crossed product algebras associated with $\mathcal{O}_\Gamma$

In this section, we will show that $\mathcal{O}_\Gamma$ is isomorphic to a crossed product algebra. We first define a "boundary space" We set

$$\tilde{\Lambda} = \{ (\gamma_n)_{n \geq 0} \mid \gamma_n \in \Gamma, |\gamma_n| + 1 = |\gamma_{n+1}|, |\gamma_n^{-1}\gamma_{n+1}| = 1 \text{ for a sufficiently large } n \geq 0 \}.$$

We introduce the following equivalence relation $\sim$; $(\gamma_n)_{n\geq 0}, (\gamma'_n)_{n \geq 0} \in \tilde{\Lambda}$ are equivalent if there exists some $k \in \mathbb{Z}$ such that $\gamma_n H = \gamma'_{n+k} H$ for a sufficiently large n. Then we define $\Lambda = \tilde{\Lambda}/\sim$ We denote the equivalent class of $(\gamma_n)_{n \geq 0}$ by $[\gamma_n]_{n \geq 0}$.

Before we define an action of Γ on Λ, we construct another space Ω to introduce a compact space structure, on which Γ acts continuously. Let Ω denote the set of sequences $x : \mathbb{N} \to \Gamma$ such that

$$\begin{cases} x(n) \in \Omega_{i_n} \setminus \{e\} & \text{for } n \geq 1, \\ x(n) \in \{a_{i_n}^{\pm 1}\} & \text{if } i_n \in I \setminus I_0, \\ i_n \neq i_{n+1} & \text{if } i_n \in I_0, \\ x(n) = x(n+1) & \text{if } i_n \in I \setminus I_0, i_n = i_{n+1}. \end{cases}$$

Note that Ω is a compact Hausdorff subspace of $\prod_{\mathbf{N}}(\bigcup_i \Omega_i \setminus \{e\})$. We introduce a map ϕ between Λ and Ω; for $x = (x(n))_{n\geq 1} \in \Omega$, we define a map $\phi(x) = [\gamma_n] \in \Lambda$ by

$$\begin{aligned} \gamma_0 &= e \qquad \text{if } n = 0, \\ \gamma_n &= x(1)\cdots x(n), \qquad \text{if } n \geq 1. \end{aligned}$$

Lemma 5.1 *The above map ϕ is a bijection from Λ onto Ω and hence Λ inherits a compact space structure via ϕ.*

Proof. For $x = (x(n)) \neq x' = (x'(n))$, there exists an integer k such that $x(k) \neq x'(k)$. If $\phi(x) = [\gamma_n]$ and $\phi(x') = [\gamma_n']$, then $\gamma_k H \neq \gamma_k' H$. Hence we have injectivity of ϕ. Next we will show surjectivity. Let $[\gamma_n] \in \Sigma$. We may take a representative (γ_n) satisfying $|\gamma_n| = n$. Now we assume that γ_n is uniquely expressed as $\gamma_n = g_1\cdots g_n h, \gamma_{n+1} = g_1'\cdots g_{n+1}'h'$ for $g_k \in \Omega_{i_k}, g_k' \in \Omega_{j_k}, h, h' \in H$. Since $|\gamma_n^{-1}\gamma_{n+1}| = 1$, we have

$$h^{-1}g_n^{-1}\cdots g_1^{-1}g_1'\cdots g_{n+1}'h' = g,$$

for some $g \notin H$ with $|g| = 1$. Inductively, we have $g_1 = g_1', \ldots, g_n = g_n'$. Hence we can assume that $\gamma_n = g_1\cdots g_n$. We set $x(n) = g_n$ and get $\phi((x(n))) = [\gamma_n]$. □

Next we define an action of Γ on Λ. Let $[\gamma_n]_{n\geq 0} \in \Lambda$. For $\gamma \in \Gamma$, define

$$\gamma \cdot [\gamma_n]_{n\geq 0} = [\gamma\gamma_n]_{n\geq 0}.$$

We will show that this is a continuous action of Γ on Λ. Let $[\gamma_n],[\gamma_n'] \in \Lambda$ such that $(\gamma_n) \sim (\gamma_n')$ and $\gamma \in \Gamma$. Since there exists some integer k such that $\gamma_n H = \gamma_{n+k}' H$ for sufficiently large integers n, we have $\gamma\gamma_n H = \gamma\gamma_{n+k}' H$. Hence this is well-defined. To show that γ is continuous, we consider how γ acts on Ω via the map ϕ. For $g \in \Omega_i$ with $|g| = 1$ and $x = (x(n))_{n\geq 1} \in \Omega$,

$$(g\cdot x)(1) = \begin{cases} g & \text{if } i \neq i_1, \\ g_1 & \text{if } i = i_1,\ gx(1) \notin H,\ i \in I_0, \\ & \text{and } gx(1) = g_1h_1\ (g_1 \in \Omega_{i_1}, h_1 \in H), \\ g & \text{if } i = i_1,\ gx(1) \notin H,\ i \in I \setminus I_0, \\ g_2 & \text{if } i = i_1,\ gx(1) \in H,\ i \in I_0, \\ & \text{and } gx(1) = h_1,\ h_1x(2) = g_2h_2(g_2 \in \Omega_{i_2}, h_1, h_2 \in H), \\ x(2) & \text{if } i = i_1,\ gx(1) \in H,\ i \in I \setminus I_0, \end{cases}$$

and for $n > 1$,

$$(g\ \ x)(n) = \begin{cases} x(n-1) & \text{if } i \neq i_1, \\ g_n & \text{if } i = i_1,\ gx(1) \notin H, \\ & \text{and } h_{n-1}x(n) = g_nh_n\ (g_n \in \Omega_{i_n}, h_n \in H), \\ x(n-1) & \text{if } i = i_1,\ gx(1) \notin H,\ i \in I \setminus I_0, \\ g_{n+1} & \text{if } i = i_1,\ gx(1) \in H, \\ & \text{and } h_nx(n+1) = g_{n+1}h_{n+1}, (g_{n+1} \in \Omega_{i_{n+1}}, h_{n+1} \in H), \\ x(n+1) & \text{if } i = i_1,\ gx(1) \in H,\ i \in I \setminus I_0. \end{cases}$$

For $h \in H$,

$$(h\ \ x)(n) = \begin{cases} g_1 & \text{if } n = 1, \\ & \text{and } hx(1) = g_1 h_1,\ (g_1 \in \Omega_{i_1}, h_n \in H), \\ g_n & \text{if } n > 1, \\ & \text{and } h_{n-1}x(n) = g_n h_n,\ (g_n \in \Omega_{i_n}, h_n \in H). \end{cases}$$

Then one can check easily that the pull-back of any open set of Ω by γ is also an open set of Ω. Thus we have proved that γ is a homeomorphism on Λ. The equations

$$(\gamma\gamma')[\gamma_n] = [\gamma\gamma'\gamma_n] = \gamma([\gamma'\gamma_n]) = \gamma \circ \gamma'[\gamma_n],$$

imply associativity.

Therefore we have obtained the following:

Lemma 5.2 *The above space Ω is a compact Hausdorff space and Γ acts on Ω continuously.*

The following result is the main theorem of this section.

Theorem 5.3 *Assume that Ω and the action of Γ on Ω are as above. Then we have the identifications*

$$\mathcal{O}_\Gamma \simeq C(\Omega) \rtimes \Gamma \simeq C(\Omega) \rtimes_r \Gamma.$$

Proof. We first consider the full crossed product $C(\Omega) \rtimes \Gamma$ Let $Y_i = \{ (x(n)) \mid x(1) \in \Omega_i \} \subset \Omega$ be clopen sets for $i \in I$. Note that if $i \in I_0$, then Y_i is the disjoint union of the clopen sets $\{ g(\Omega \setminus Y_i) \mid g \in \Omega_i \setminus \{e\} \}$, and if $i \in I \setminus I_0$, then $Y_i = Y_i^+ \cup Y_i^-$ where $Y_i^\pm = \{ (x(n)) \mid x(1) = a_i^\pm \}$. Let $p_i = \chi_{\Omega \setminus Y_i}$ and $p_i^\pm = \chi_{Y_i^\pm}$. We define $T_g = gp_i$ for $g \in G_i \setminus H$ and $i \in I_0$ and $T_{a_i^{\pm 1}} = a_i^{\pm 1}(p_i + p_i^\pm)$ for $i \in I \setminus I_0$. Let $V_h = h$ for $h \in H$. Then the family $\{T_g, V_h\}$ satisfies the relations (1), (2), (3) and (4). Indeed, we can first check that $h \in H$ commutes with p_i and $p_i^{\pm 1}$. So the relation (1) holds. Let $g \in G_i \setminus H$ and $g' \in G_j \setminus H$ with $i, j \in I_0$. Then

$$T_g^* T_{g'} = p_i g^{-1} g' p_j = g^{-1} \chi_{g(\Omega \setminus Y_i)} \chi_{g'(\Omega \setminus Y_j)} g' = \delta_{i,j} \delta_{gH, g'H} p_i g^{-1} g'$$

Moreover it follows from $\Omega \setminus Y_i = \bigcup_{j \neq i} Y_j$ that

$$\begin{aligned} T_g^* T_g = \chi_{\Omega \setminus Y_i} &= \sum_{j \neq i} \chi_{Y_j} \\ &= \sum_{j \in I_0, j \neq i} \sum_{g \in \Omega_j \setminus \{e\}} \chi_{g(\Omega \setminus Y_j)} + \sum_{j \in I \setminus I_0} \chi_{a_j(\Omega \setminus Y_j)} + \chi_{a_j^{-1}(\Omega \setminus Y_j)} \\ &= \sum_{j \in I_0, j \neq i} \sum_{g \in \Omega_j \setminus \{e\}} g p_j g^{-1} + \sum_{j \in I \setminus I_0} p_j^+ + p_j^- \\ &= \sum_{j \in I_0, j \neq i} \sum_{g \in \Omega_j \setminus \{e\}} T_g T_g^* + \sum_{j \in I \setminus I_0} T_{a_j} T_{a_j}^* + T_{a_j^{-1}} T_{a_j^{-1}}^*. \end{aligned}$$

For all other cases, we can also check the relations (2) and (3) by similar calculations. Since Ω is the disjoint union of Y_i, we have (4). Note that $g, p_i, p_i^{\pm} \in C^*(T_g, V_h)$. Moreover, since the family $\{\gamma(\Omega \backslash Y_i) \mid \gamma \in \Gamma, i \in I\} \cup \{\gamma Y_i^{\pm} \mid \gamma \in \Gamma, i \in I \backslash I_0\}$ generates the topology of Ω, we have $C(\Omega) \rtimes \Gamma = C^*(T_g, V_h)$. By the universality of $\mathcal{O}_\Gamma$, there exists a canonical surjective $*$-homomorphism of $\mathcal{O}_\Gamma$ onto $C(\Omega) \rtimes \Gamma$, sending S_g to T_g and U_h to V_h.

Conversely, let $q_i = \sum_{j \neq i} P_j$ and $q_i^{\pm} = S_{a_i^{\pm 1}} S^*_{a_i^{\pm 1}}$. Let

$$\begin{cases} w_g = S_g + \sum_{g' \in \Omega_i \backslash H \cup g^{-1}H} S_{gg'} S^*_{g'} + S^*_g & \text{for } g \in G_i \setminus H, i \in I_0, \\ w_{a_i} = S_{a_i} + S^*_{a_i^{-1}} & \text{for } i \in I \setminus I_0, \\ w_h = U_h & \text{for } h \in H. \end{cases}$$

We will check that w_g are unitaries for $g \in G_i \setminus H$ with $i \in I_0$. If $g' \in \Omega_i \setminus H \cup g^{-1}H$, then $gg'H = \gamma H$ for some $\gamma \in \Omega_i \setminus \{e, g\}$. Hence

$$\begin{aligned} & w_g w_g^* \\ = & \left(S_g + \sum_{g' \in \Omega_i \backslash H \cup g^{-1}H} S_{gg'} S^*_{g'} + S^*_{g^{-1}}\right) \left(S_g + \sum_{g' \in \Omega_i \backslash H \cup g^{-1}H} S_{gg'} S^*_{g'} + S^*_{g^{-1}}\right)^* \\ = & S_g S_g^* + \sum_{g' \in \Omega_i \backslash H \cup g^{-1}H} S_{gg'} S^*_{g'} S_{g'} S^*_{gg'} + S^*_{g^{-1}} S_{g^{-1}} \\ = & P_g + \sum_{g' \in \Omega_i \backslash \{e, g\}} P_{g'} + Q_g = 1. \end{aligned}$$

Similarly, we have $w_g^* w_g = 1$. For the other case, we can check in the same way.

If $i \in I_0, \tau \in \Omega_i \setminus \{e\}$ then

$$\begin{aligned} & \sum_{g \in \Omega_i} w_g q_i w_g^* \\ = & \sum_{g \in \Omega_i} \left(S_g + \sum_{g' \in \Omega_i \backslash H \cup g^{-1}H} S_{gg'} S^*_{g'} + S^*_{g^{-1}}\right) S^*_\tau S_\tau w_g^* \\ = & \sum_{g \in \Omega_i} S_g S^*_\tau S_\tau \left(S^*_g + \sum_{g' \in \Omega_i \backslash H \cup g^{-1}H} S_g S^*_{gg'} + S_{g^{-1}}\right) \\ = & \sum_{g \in \Omega_i} S_g S^*_\tau S_\tau S^*_g = 1. \end{aligned}$$

For $i \in I \setminus I_0$, we have $q_i^+ + w_{a_i} q_i^- w^*_{a_i} = 1$ and $q_i^+ + q_i^- + q_i = 1$ as well. Therefore the conjugates of the family $\{q_i, q_i^{\pm}\}$ by the elements of Γ generate a commutative C^*-algebra. This is the image of a representation of $C(\Omega)$. Therefore (q_i, w) gives a covariant representation of the C^*-dynamical system $(C(\Omega), \Gamma)$. Note that (q_i, w_g) generates $\mathcal{O}_\Gamma$. Hence by the universality of the full crossed product $C(\Omega) \rtimes \Gamma$, there exists a canonical surjective $*$-homomorphism of $C(\Omega) \rtimes \Gamma$ onto $\mathcal{O}_\Gamma$. It is easy to show that the above two $*$-homomorphisms are the inverses of each other.

$$\begin{array}{lclcl} S_g & \mapsto & gp_i & \mapsto & w_g Q_g = S_g, \\ S_{a_i^{\pm 1}} & \mapsto & a_i^{\pm 1}(p_i + p_i^{\pm}) & \mapsto & w_{a_i^{\pm 1}}(Q_{a_i^{\pm 1}} + P_{a_i^{\pm 1}}) = S_{a_i^{\pm 1}}, \\ U_h & \mapsto & h & \mapsto & U_h. \end{array}$$

We have shown the identification $\mathcal{O}_\Gamma \simeq C(\Omega) \rtimes \Gamma$. Since there exists a canonical surjective map of $C(\Omega) \rtimes \Gamma$ onto $C(\Omega) \rtimes_r \Gamma$, we have a surjective $*$-homomorphism of $\mathcal{O}_\Gamma$ onto $C(\Omega) \rtimes_r \Gamma$. Let $C(\Omega) \rtimes_r \Gamma = C^*(\tilde{\pi}(p_i), \lambda)$ where $\tilde{\pi}$ is the induced representation on the Hilbert space $l^2(\Gamma, \mathcal{H})$ by the universal representation π of $C(\Omega)$ on a Hilbert space $\mathcal{H}$ and λ is the unitary representation of Γ on $l^2(\Gamma, \mathcal{H})$ such that $(\lambda_s x)(t) = x(s^{-1}t)$ for $x \in l^2(\Gamma, \mathcal{H})$. By the uniqueness theorem for $\mathcal{O}_\Gamma$, it suffices to check

$$C^*\left(\tilde{\pi}(\chi_{Y_i})\lambda_h \tilde{\pi}(\chi_{Y_i})\right) \simeq C^*(H).$$

But the unitary representation $\tilde{\pi}(\chi_{Y_i})\lambda_h\tilde{\pi}(\chi_{Y_i})$ is quasi-equivalent to the left regular representation of H. This completes the proof of the theorem. □

In [Ser], Serre defined the tree G_T, on which Γ acts. In an appendix, we will give the definition of the tree $G_T = (V, E)$ where V is the set of vertices and E is the set of edges. We denote the corresponding natural boundary by ∂G_T. We also show how to construct boundaries of trees in the appendix. (See Furstenberg [Fur] and Freudenthal [Fre] for details.)

Proposition 5.4 *The space ∂G_T is homeomorphic to Ω and the above two actions of Γ on ∂G_T and Ω are conjugate.*

Proof. We define a map ψ from ∂G_T to Ω. First we assume that $I = \{1, 2\}$. The corresponding tree G_T consists of the vertex set $V = \Gamma/G_1 \coprod \Gamma/G_2$ and the edge set $E = \Gamma/H$. For $\omega \in \partial G_T$, we can identify ω with an infinite chain $\{G_{i_1}, g_1 G_{i_2}, g_1 g_2 G_{i_3}, \dots\}$ with $g_k \in \Omega_{i_k} \setminus \{e\}$ and $i_1 \neq i_2 \neq \cdots$ Then we define $\psi(\omega) = [x(n) = g_{i_n}]$. We will recall the definition of the corresponding tree G_T, in general, on the appendix, (see [Ser]). Similarly, we can identify $\omega \in \partial G_T$ with an infinite chain $\{G_0, G_{i_1}, g_1 G_0, g_1 G_{i_2}, g_1 g_2 G_0, \dots\}$. Moreover we may ignore vertices γG_0 for an infinite chain ω,

$$\{G_0, G_{i_1}, (g_1 G_0 \to \text{ignoring}), g_1 G_{i_2}, (g_1 g_2 G_0 \to \text{ignoring}), g_1 g_2 G_{i_3}, \dots\}.$$

Therefore, we define a map ψ of ∂G_T to Ω by

$$\psi(\omega) = [x(n) = g_n].$$

The pull-back by ψ of any open set of ∂G_T is an open set on Ω. It follows that ψ is a homeomorphism. The two actions on ∂G_T and Ω are defined by left multiplication. So it immediately follows that these actions are conjugate. $\square$

It is known that Γ is a hyperbolic group (see a proof in the appendix, where we recall the notion of hyperbolicity for finitely generated groups as introduced by Gromov e.g. see [GH]). Let $S = \{\bigcup_{i\in I} G_i\}$ and $G(\Gamma, S)$ be the Cayley graph of Γ with the word metric d. Let $\partial\Gamma$ be the hyperbolic boundary.

Proposition 5.5 *The hyperbolic boundary $\partial\Gamma$ is homeomorphic to Ω and the actions of Γ are conjugate.*

Proof. We can define a map ψ from Ω to $\partial\Gamma$ by $(x(n)) \mapsto [x_n = x(1)\cdots x(n)]$. Indeed, since $\langle x_n \,|\, x_m\rangle = \min\{n, m\} \to \infty\ (n, m \to \infty)$, it is well-defined. For $x \neq y$ in Ω, there exists k such that $x(k) \neq y(k)$. Then $\langle\psi(x)|\psi(y)\rangle \leq k+1$, which shows injectivity. Let $(x_n) \in \partial\Gamma$. Suppose that $x_n = g_{n(1)}\cdots g_{n(k_n)}h_n$ for some $g_l \in \bigcup_i \Omega_i \setminus \{e\}$ with $n(1) \neq \cdots \neq n(k_n)$. If $g_{n(1)} = g_{m(1)}, \cdots, g_{n(l)} = g_{m(l)}$ and $g_{n(l+1)} \neq g_{m(l+1)}$, then we set $a_{n,m} = g_{n(1)}\cdots g_{n(l)} = g_{m(1)}\cdots g_{m(l)}$. So we have

$$\langle x_n \,|\, x_m\rangle \leq d(e, a_{n,m}) + 1 \to \infty\ (n, m \to \infty).$$

Therefore we can choose sequences $n_1 < n_2 < \cdots$, and $m_1 < m_2 < \cdots$, such that a_{n_k,m_k} is a sub-word of $a_{n_{k+1},m_{k+1}}$. Then a sequence $\{g_{n_k(1)}, \cdots, g_{n_k(l)}, g_{n_{k+1}(l+1)}, \cdots\}$ is mapped to (x_n) by ψ. We have proved that ψ is surjective. The pull-back of any open set in $\partial\Gamma$ is an open set in Ω. So ψ is continuous. Since $\Omega, \partial\Gamma$ are compact Hausdorff spaces, ψ is a homeomorphism. Again, the two actions on Ω and $\partial\Gamma$ are defined by left multiplication and hence are conjugate. $\square$

Remark Since the action of Γ on $\partial\Gamma$ depends only on the group structure of Γ in [GH], the above proposition shows that $\mathcal{O}_\Gamma$ is, up to isomorophism, independent of the choice of generators of Γ.

6 Nuclearity, simplicity and pure infiniteness of $\mathcal{O}_\Gamma$

We first begin by reviewing the crossed product $B \rtimes \mathbb{N}$ of a C^*-algebra B by a $*$-endomorphism; this construction was first introduced by Cuntz [C1] to describe the Cuntz algebra $\mathcal{O}_n$ as the crossed product of UHF algebras by $*$-endomorphisms. See Stacey's paper [Sta] for a more detailed discussion. Suppose that ρ is an injective $*$-endomorphism on a unital C^*-algebra B. Let $\overline{B}$ be the inductive limit $\varinjlim(B \xrightarrow{\rho} B)$ with the corresponding injective homomorphisms $\sigma_n : B \to \overline{B}$ $(n \in \mathbb{N})$. Let p be the projection $\sigma_0(1)$. There exists an automorphism $\bar{\rho}$ given by $\bar{\rho} \circ \sigma_n = \sigma_n \circ \rho$ with inverse $\sigma_n(b) \mapsto \sigma_{n+1}(b)$. Then the crossed product $B \rtimes_\rho \mathbb{N}$ is defined to be the hereditary C^*-algebra $p(\overline{B} \rtimes_{\bar{\rho}} \mathbb{Z})p$.

The map σ_0 induces an embedding of B into $\overline{B}$. Therefore the canonical embedding of $\overline{B}$ into $\overline{B} \rtimes_{\beta} \mathbf{Z}$ gives an embedding $\pi : B \to B \rtimes_{\rho} \mathbf{N}$. Moreover the compression by p of the implementing unitary is an isometry V belonging to $B \rtimes_{\rho} \mathbf{N}$ satisfying

$$V\pi(b)V^* = \pi(\rho(b)).$$

In fact, $B \rtimes_{\rho} \mathbf{N}$ is also the universal C^*-algebra generated by a copy $\pi(B)$ of B and an isometry V satisfying the above relation. If B is nuclear, then so is $B \rtimes_{\rho} \mathbf{N}$.

Proposition 6.1

$$\mathcal{O}_{\Gamma} \simeq \mathcal{O}_{\Gamma}^{\mathbf{T}} \rtimes_{\rho} \mathbf{N}$$

In particular, $\mathcal{O}_{\Gamma}$ is nuclear.

Proof. We fix $g_i \in G_i \setminus H$ for all $i \in I$. We can choose projections e_i which are sums of projections P_g such that $e_i \leq Q_{g_i}$ and $\sum_{i \in I} e_i = 1$. Then $V = \sum_{i \in I} S_{g_i} e_i$ is an isometry in $\mathcal{O}_{\Gamma}$.

We claim that $V\mathcal{O}_{\Gamma}^{\mathbf{T}}V^* \subseteq \mathcal{O}_{\Gamma}^{\mathbf{T}}$ and $\mathcal{O}_{\Gamma} = C^*\left(\mathcal{O}_{\Gamma}^{\mathbf{T}}, V\right)$. Let $a \in \mathcal{O}_{\Gamma}^{\mathbf{T}}$. It is obvious that $VaV^* \in \mathcal{O}_{\Gamma}^{\mathbf{T}}$ and $C^*\left(\mathcal{O}_{\Gamma}^{\mathbf{T}}, V\right) \subseteq \mathcal{O}_{\Gamma}$. To show the second claim, it suffices to check that $S_{\mu}P_iS_{\nu}^* \in \mathcal{O}_{\Gamma}$ for all μ, ν and i. If $|\mu| = |\nu|$, we have $S_{\mu}P_iS_{\nu}^* \in \mathcal{O}_{\Gamma}^{\mathbf{T}}$. If $|\mu| \neq |\nu|$, then we may assume $|\mu| < |\nu|$. Let $|\nu| - |\mu| = k$. Thus $S_{\mu}P_iS_{\nu}^* = (V^*)^k V^k S_{\mu}P_iS_{\nu}^*$ and $V^k S_{\mu}P_iS_{\nu}^* \in \mathcal{O}_{\Gamma}^{\mathbf{T}}$. This proves our claim.

We define a $*$-endomorphism ρ of $\mathcal{O}_{\Gamma}^{\mathbf{T}}$ by $\rho(a) = VaV^*$ for $a \in \mathcal{O}_{\Gamma}^{\mathbf{T}}$. Thanks to the universality of the crossed product $\mathcal{O}_{\Gamma}^{\mathbf{T}} \rtimes_{\rho} \mathbf{N}$, we obtain a canonical surjective $*$-homomorphism σ of $\mathcal{O}_{\Gamma}^{\mathbf{T}} \rtimes_{\rho} \mathbf{N}$ onto $C^*(\mathcal{O}_{\Gamma}^{\mathbf{T}}, V)$. Since $\mathcal{O}_{\Gamma}^{\mathbf{T}} \rtimes_{\rho} \mathbf{N}$ has the universal property, there also exists a gauge action β on $\mathcal{O}_{\Gamma}^{\mathbf{T}} \rtimes_{\rho} \mathbf{N}$. Let Ψ be the corresponding canonical conditional expectation of $\mathcal{O}_{\Gamma}^{\mathbf{T}} \rtimes_{\rho} \mathbf{N}$ onto $\mathcal{O}_{\Gamma}^{\mathbf{T}}$. Suppose that $a \in \ker\sigma$. Then $\sigma(a^*a) = 0$. Since $\alpha \circ \sigma = \sigma \circ \beta$, we have $\sigma \circ \Psi(a^*a) = 0$. The injectivity of σ on $\mathcal{O}_{\Gamma}^{\mathbf{T}}$ implies $\Psi(a^*a) = 0$ and hence $a^*a = 0$ and $a = 0$. It follows that $\mathcal{O}_{\Gamma} \simeq \mathcal{O}_{\Gamma}^{\mathbf{T}} \rtimes_{\rho} \mathbf{N}$. □

In section 2, we reviewed the notion of amenability for discrete group actions. The following is a special case of [Ada].

Corollary 6.2 *The action of Γ on $\partial\Gamma$ is amenable.*

Proof. This follows from Theorem 2.2 and the above proposition. □

We also have a partial result of [Kir], [D1], [D2] and [DS].

Corollary 6.3 *The reduced group C^*-algebra $C_r^*(\Gamma)$ is exact.*

Proof. It is well-known that every C^*-subalgebra of an exact C^*-algebra is exact; see Wassermann's monograph [Was]. Therefore the inclusion $C_r^*(\Gamma) \subset \mathcal{O}_{\Gamma}$ implies exactness. □

Finally we give a sufficient condition for the simplicity and pure infiniteness of $\mathcal{O}_{\Gamma}$.

Corollary 6.4 *Suppose that $\Gamma = *_H G_i$ satisfies the following condition:*
There exists at least one element $j \in I$ such that

$$\bigcap_{i \neq j} N_i = \{e\},$$

where $N_i = \bigcap_{g \in G_i} gHg^{-1}$
Then $\mathcal{O}_\Gamma$ is simple and purely infinite.

Proof. We first claim that for any $\mu \in \Delta$ and $|g| = 1$ with $|\mu g| = |\mu| + 1$,

$$\mu H \mu^{-1} \cap H \supseteq \mu g H g^{-1} \mu^{-1} \cap H.$$

Suppose that $\mu = \mu_1 \cdots \mu_n$ such that $\mu_k \in \Omega_{i_k}$ with $\mu_1 \neq \cdots \neq \mu_n$ and $g \in G_i$ with $i \neq i_n$. We first assume that $\mu = \mu_1$. If $\mu g h g^{-1} \mu^{-1} \in \mu g H g^{-1} \mu^{-1} \cap H$, then $ghg^{-1} \in \mu^{-1} H \mu \subseteq G_{i_1}$. Thus $ghg^{-1} \in G_i \cap G_{i_1}$ implies $ghg^{-1} \in H$. Next we assume that $|\mu| > 1$. If $\mu g h g^{-1} \mu^{-1} \in \mu g H g^{-1} \mu^{-1} \cap H$, then

$$\mu_2 \cdots \mu_n g h g^{-1} \mu_k^{-1} \cdots \mu_2^{-1} \in \mu_1^{-1} H \mu_1 \subseteq G_{i_1}.$$

Thus $|\mu_2 \cdots \mu_n g h g^{-1} \mu_k^{-1} \cdots \mu_2^{-1}| \leq 1$ implies $ghg^{-1} \in H$. This proves the claim.

Let $\{S_g, U_h\}$ be any family satisfying the relations (1), (2), (3) and (4). By the uniqueness theorem, it is enough to show that $C^*(P_i U_h P_i \mid h \in H) \simeq C^*(H)$ for any $i \in I$. We next claim that there exists $\nu \in \Gamma$ such that the initial letter of ν belongs to Ω_i and $\{U_h S_\nu\}_{h \in H}$ have mutually orthogonal ranges.

Let $g \in \Omega_i$. If $gHg^{-1} \cap H = \{e\}$, then it is enough to set $\nu = g$. Now suppose that there exists some $h \in gHg^{-1} \cap H$ with $h \neq e$. We first assume that $i = j$. By the hypothesis, there exists some $i_1 \in I$ such that $g^{-1} h g \notin N_{i_1}$ and $i \neq i_1$. Hence there exists $g_1 \in \Omega_{i_1}$ such that $g^{-1} h g \notin g_1 H g_1^{-1}$ and so $h \notin g g_1 H g_1^{-1} g^{-1}$ If $g g_1 H g_1^{-1} g^{-1} \cap H = \{e\}$, then it is enough to put $\nu = g g_1$. If not, we set $\gamma_1 = g_1 g_1'$ for some $g_1' \in \Omega_j$. By the first part of the proof, we have

$$gHg^{-1} \cap H \supsetneqq \mu \gamma_1 H \gamma_1^{-1} \mu^{-1} \cap H.$$

Since H is finite, we can inductively obtain $\gamma_1, \gamma_2, \dots \gamma_n$ satisfying

$$gHg^{-1} \cap H \supsetneqq g\gamma_1 H \gamma_1^{-1} g^{-1} \cap H \supsetneqq \cdots \supsetneqq g\gamma_1 \cdots \gamma_n H \gamma_n^{-1} \cdots \gamma_1^{-1} g^{-1} \cap H = \{e\}.$$

Then we set $\nu = g\gamma_1 \cdots \gamma_n$. If $i \neq j$, we can carry out the same arguments by replacing g by $\gamma = g g_j$ for some $g_j \in \Omega_j$. Hence from the identification $U_h S_\nu \leftrightarrow \delta_h \in l^2(H)$, it follows that the unitary representation $P_i U_h P_i$ is quasi-equivalent to the left regular representation of H. Thus $\mathcal{O}_\Gamma$ is simple.

In Section 5, we have proved that $\mathcal{O}_\Gamma \simeq C(\Omega) \rtimes_r \Gamma$ We show that the action of Γ on Ω is the strong boundary action (see Preliminaries). Let U, V be any non-empty open sets in Ω. There exists some open set $O = \{(x(n)) \in \Omega \mid x(1) = g_1, \cdots, x(k) = g_k\}$ which is contained in V We may also assume that U^c is an open of the form $\{(x(n)) \in \Omega \mid x(1) = \gamma_1, \cdots, x(m) = \gamma_m\}$. Let $\gamma = g_1 \cdots g_k \gamma_m^{-1} \cdots \gamma_1^{-1}$ Then we have $\gamma U^c \subset O \subset V$ Since $C(\Omega) \rtimes_r \Gamma$ is simple, it follows from [AS] that the action of Γ is topological free. Therefore it follows from Theorem 2.4 that $C(\Omega) \rtimes_r \Gamma$, namely $\mathcal{O}_\Gamma$, is purely infinite. □

Remark We gave a sufficient condition for $\mathcal{O}_\Gamma$ to be simple. However, we can completely determine the ideal structure of $\mathcal{O}_\Gamma$ with further effort. Indeed, we will obtain a matrix A_Γ to compute K-groups of $\mathcal{O}_\Gamma$ in the next section. The same argument as in [C2] also works for the ideal structure of $\mathcal{O}_\Gamma$. For Cuntz-Krieger algebras, we need to assume that corresponding matrices have the condition (II) of [C2] to apply the uniqueness theorem. Since we have another uniqueness theorem for our algebras, we can always apply the ideal structure theorem.

Let $\Sigma = I \times \{1, \ldots, r\}$ be a finite set, where r is the number of all irreducible unitary representations of H. For $x, y \in \Sigma$, we define $x \geq y$ if there exists a sequence $x_1, \ldots, x_m$ of elements in Σ such that $x_1 = x, x_m = y$ and $A_\Gamma(x_a, x_{a+1}) \neq 0 (a = 1, \ldots, m-1)$. We call x and y equivalent if $x \geq y \geq x$ and write Γ_{A_Γ} for the partially ordered set of equivalence classes of elements x in Σ for which $x \geq x$. A subset K of Γ_{A_Γ} is called hereditary if $\gamma_1 \geq \gamma_2$ and $\gamma_1 \in K$ implies $\gamma_2 \in K$. Let

$$\Sigma(K) = \{x \in \Sigma \mid x_1 \geq x \geq x_2 \quad \text{for some} \quad x_1, x_2 \in \bigcup_{\gamma \in K} \gamma\}.$$

We denote by I_K the closed ideal of $\mathcal{O}_\Gamma$ generated by projections $P(i, k)$, which is defined in the next section, for all $(i, k) \in \Sigma(K)$.

Theorem 6.5 (**[C2, Theorem 2.5.]**) *The map $K \mapsto I_K$ is an inclusion preserving bijection of the set of hereditary subsets of Γ_{A_Γ} onto the set of closed ideals of $\mathcal{O}_\Gamma$.*

7 K-theory for $\mathcal{O}_\Gamma$

In this section we give explicit formulae of the K-groups of $\mathcal{O}_\Gamma$. We have described $\mathcal{O}_\Gamma$ as the crossed product $\mathcal{O}_\Gamma^{\mathbb{T}} \rtimes \mathbb{N}$ in Section 6. So to apply the Pimsner-Voiculescu exact sequence [PV], we need to compute the K-groups of the AF-algebra $\mathcal{O}_\Gamma^{\mathbb{T}}$. We assume that each G_i is finite for simplicity throughout this section. We can also compute the K-groups for general cases by essentially the same arguments. Recall that the fixed-point algebra is described as follows:

$$\mathcal{O}_\Gamma^{\mathbb{T}} = \overline{\bigcup_{n \geq 0} \mathcal{F}_n},$$

$$\mathcal{F}_n = \oplus_{i\in I}\mathcal{F}_n^i.$$

For each n, we consider a direct summand of $\mathcal{F}_n$, which is

$$\mathcal{F}_n^i = C^*(\, S_\mu P_i U_h P_i S_\nu^* \mid h \in H, |\mu| = |\nu| = n\,),$$

and the embedding $\mathcal{F}_n^i \hookrightarrow \mathcal{F}_{n+1}$ is given by

$$\begin{aligned} & S_\mu P_i U_h P_i S_\nu^* \\ &= \sum_{g\in\Omega_i\setminus\{e\}} S_\mu U_h (S_g Q_g S_g^*) S_\nu^* \\ &= \sum_g \sum_{i'\neq i} S_\mu S_{hg} P_{i'} S_{\nu g}^*. \end{aligned}$$

Let $\{\chi_1, \dots, \chi_r\}$ be the set of characters corresponding with all irreducible unitary representations of the finite group H with degrees $n_1, \dots, n_r$. Then we have the identification $C^*(H) \simeq M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$. We can write a unit p_k of the k-th component $M_{n_k}(\mathbb{C})$ of $C^*(H)$ as follows:

$$p_k = \frac{n_k}{|H|} \sum_{h\in H} \overline{\chi_k(h)} U_h.$$

Suppose that for $i \neq j$,

$$\mathcal{F}_n^i \simeq M_{N(n,i)}(\mathbb{C}) \otimes C^*(H),$$
$$\mathcal{F}_{n+1}^j \simeq M_{N(n+1,j)}(\mathbb{C}) \otimes C^*(H).$$

Now we compute each embedding of $\mathcal{F}_n^i \hookrightarrow \mathcal{F}_{n+1}^j$,

$$M_{N(n,i)}(\mathbb{C}) \otimes M_{n_i}(\mathbb{C}) \hookrightarrow M_{N(n+1,j)}(\mathbb{C}) \otimes M_{n_j}(\mathbb{C})$$

at the K-theory level. $P(i,k)$ denotes $P_i p_k P_i$. Let P be the projection $e\otimes 1$ in $M_{N(n,i)}(\mathbb{C}) \otimes M_{n_k}(\mathbb{C})$ given by

$$P = S_\mu P(i,k) S_\mu^* \qquad \text{for some } \mu \in \Delta_n,$$

where e is a minimal projection in the matrix algebras, and Q be the unit of $M_{N(n+1,j)}(\mathbb{C}) \otimes M_{n_l}(\mathbb{C})$ given by

$$Q = \sum_{\nu\in\Delta_{n+1}} S_\nu P(j,l) S_\nu^*.$$

At the K-theory level, we have $[P] = n_k[e]$. Hence it suffices to compute $\mathrm{tr}(PQ)/n_k$, where tr is the canonical trace in the matrix algebras.

$$
\begin{aligned}
\frac{\mathrm{tr}(PQ)}{n_k} &= \mathrm{tr}\left(\frac{1}{n_k}(S_\mu P(i,k)S_\mu^*)(\sum_{\nu\in\Delta_{n+1}} S_\nu P(j,l)S_\nu^*)\right) \\
&= \mathrm{tr}\left(\frac{1}{|H|}(\sum_{h\in H}\overline{\chi_k(h)}S_\mu U_h P_i S_\mu^*)(\sum_{\nu\in\Delta_{n+1}} S_\nu P(j,l)S_\nu^*)\right) \\
&= \frac{1}{|H|}\mathrm{tr}\left(\sum_{h\in H}\overline{\chi_k(h)}(\sum_{g\in\Omega_i\setminus\{e\}}\sum_{i'\neq i} S_\mu S_{hg}P_{i'}S_{\mu g}^*)(\sum_{\nu\in\Delta_{n+1}} S_\nu P(j,l)S_\nu^*)\right) \\
&= \frac{1}{|H|}\mathrm{tr}\left(\sum_{h\in H}\overline{\chi_k(h)}(\sum_{g\in\Omega_i\setminus\{e\}} S_\mu S_{hg}P(j,l)S_{\mu g}^*)\right) \\
&= \frac{1}{|H|}\sum_{g\in\Omega_i\setminus\{e\}}\sum_{h\in H(g)}\overline{\chi_k(h)}\mathrm{tr}\left(S_{\mu g}U_{g^{-1}hg}P(j,l)S_{\mu g}^*\right) \\
&= \frac{1}{|H|}\sum_{g\in\Omega_i\setminus\{e\}}\sum_{h\in H(g)}\overline{\chi_k(h)}\chi_l(g^{-1}hg),
\end{aligned}
$$

where $H(g)$ is the stabilizer of gH by the left multiplication of H.

Now fix $x \in X_i \setminus \{e\}$. Let $\{g \in \Omega_i \mid HgH = HxH\} = \{g_0 = x, g_1, \dots, g_{m-1}\}$. Then there exists $h_1, h'_1, \dots, h_{m-1}, h'_{m-1} \in H$ such that $h_1 x = g_1 h'_1, \dots, h_{m-1}x = g_{m-1}h'_{m-1}$. Note that $h_s H(x) h_s^{-1} = H(g_s)$ for $s = 1, \dots, m-1$. Since χ_k, χ_l are class functions, we have

$$
\begin{aligned}
\frac{\operatorname{tr}(PQ)}{n_k} &= \frac{1}{|H|}\sum_{x\in X_i}\left(\sum_{s=1}^{m-1}\sum_{h\in H(x)}\overline{\chi_k(h_s h h_s^{-1})}\chi_l(h_s' x^{-1}h_s^{-1}\quad h_s h h_s^{-1}\quad h_s x h_s'^{-1})\right)\\
&= \frac{1}{|H|}\sum_{x\in X_i}\left(\sum_{s=1}^{m-1}\sum_{h\in H(x)}\overline{\chi_k(h_s h h_s^{-1})}\chi_l(h_s' x^{-1}hxh_s'^{-1})\right)\\
&= \frac{1}{|H|}\sum_{x\in X_i}\left(\sum_{s=1}^{m-1}\sum_{h\in H(x)}\overline{\chi_k(h)}\chi_l(x^{-1}hx)\right)\\
&= \frac{1}{|H|}\sum_{x\in X_i}\left(\sum_{s=1}^{m-1}\sum_{h\in H(x)}\overline{\chi_k(h)}\chi_l^x(h)\right)\\
&= \sum_{x\in X_i}\left(\frac{|H(x)|}{|H|}\sum_{s=1}^{m-1}\langle\chi_k,\chi_l^x\rangle_{H(x)}\right)\\
&= \sum_{x\in X_i}\langle\chi_k,\chi_l^x\rangle_{H(x)},
\end{aligned}
$$

where

$$
\begin{aligned}
\chi_l^x(h) &= \chi_l\left(x^{-1}hx\right)\\
\langle\chi_k,\chi_l^x\rangle_{H(x)} &= \frac{1}{|H(x)|}\sum_{h\in H(x)}\overline{\chi_k(h)}\chi_l^x(h).
\end{aligned}
$$

Let $A_\Gamma((j,l),(i,k)) = \sum_{x\in X_i\setminus\{e\}}\langle\chi_k,\chi_l^x\rangle_{H(x)}$ for $i\neq j$ and $A_\Gamma((i,k),(i,l)) = 0$ for $1\leq k,l\leq r$ Then we describe the embedding $\mathcal{F}_n^i\hookrightarrow\mathcal{F}_{n+1}^j$ at the K-theory level by the matrix $[A_\Gamma((i,k),(j,l))]_{1\leq k,l\leq r}$. Let $A_\Gamma = [A_\Gamma((i,k),(j,l))]$. We have the following lemma.

Lemma 7.1

$$
K_0\left(\mathcal{O}_\Gamma^{\mathbb{T}}\right) = \varinjlim\left(\mathbb{Z}^N\xrightarrow{A_\Gamma}\mathbb{Z}^N\right)
$$

$$
K_1\left(\mathcal{O}_\Gamma^{\mathbb{T}}\right) = 0
$$

where $N = |I|r$

We can compute the K-groups of $\mathcal{O}_\Gamma$ by using the Pimsner-Voiculescu sequence with essentially the same argument as in the Cuntz-Krieger algebra case (see [C2]).

Theorem 7.2

$$
\begin{aligned}
K_0(\mathcal{O}_\Gamma) &= \mathbb{Z}^N/(1-A_\Gamma)\mathbb{Z}^N\\
K_1(\mathcal{O}_\Gamma) &= \operatorname{Ker}\{1-A_\Gamma:\mathbb{Z}^N\to\mathbb{Z}^N\}\quad \textit{on } \mathbb{Z}^N
\end{aligned}
$$

Proof. It suffices to compute the K-groups of $\overline{\mathcal{O}}_\gamma = \overline{\mathcal{O}}_\Gamma^{\mathbf{T}} \rtimes_\beta \mathbf{Z}$. We represent the inductive limit

$$\varinjlim \left(\mathbf{Z}^N \xrightarrow{A_\Gamma} \mathbf{Z}^N \right)$$

as the set of equivalence classes of $x = (x_1, x_2, \cdots)$ such that $x_k \in \mathbf{Z}^N$ with $x_{k+1} = A(x_k)$. If S is a partial isometry in $\mathcal{O}_\Gamma$ such that $\alpha_z(S) = zS$ and P is a projection in $\mathcal{O}_\Gamma^{\mathbf{T}}$ with $P \le S^*S$, then $[\rho(P)] = [VPV^*] = [(VS^*S)P(VS^*S)^*] = [SPS^*]$ in $K_0(\mathcal{O}_\Gamma^{\mathbf{T}})$. Recall that

$$p_k = \frac{n_k}{|H|} \sum_{h \in H} \overline{\chi_k(h)} U_h.$$

Let $P = S_\mu P(i,k) S_\mu^*$ for some $\mu \in \Delta_n$. If $\mu = \mu_1 \cdots \mu_n$, then

$$\begin{aligned}
&[\rho^{-1}(P)] \\
&= [S_{\mu_1}^* P S_{\mu_1}] \\
&= [\frac{n_k}{|H|} \sum_{h \in H} \overline{\chi_k(h)} \left(S_{\mu_2} \cdots S_{\mu_n} P_i U_h P_i S_{\mu_n} \cdots S_{\mu_2}^* \right)] \\
&= \cdots \\
&= \sum_{j \ne i} \sum_{l=1}^{r} n_i \left(\sum_{x \in X_i \setminus \{e\}} \langle \chi_k, \chi_l^x \rangle [e_l] \right),
\end{aligned}$$

where the e_l are non-zero minimal projections for $1 \le l \le r$ Thus it follows that ρ_*^{-1} is the shift on $K_0(\overline{\mathcal{O}}_\Gamma^{\mathbf{T}})$. We denote the shift by σ. If $x = (x_1, x_2, x_3, \cdots) \in K_0(\overline{\mathcal{O}}_\Gamma^{\mathbf{T}})$, then $\sigma(x) = (x_2, x_3, \cdots)$. By the Pimsner-Voiculescu exact sequence, there exists an exact sequence

$$0 \to K_1(\overline{\mathcal{O}}_\Gamma) \to K_0(\overline{\mathcal{O}}_\Gamma^{\mathbf{T}}) \to K_0(\overline{\mathcal{O}}_\Gamma^{\mathbf{T}}) \to K_0(\overline{\mathcal{O}}_\Gamma) \to 0.$$

It therefore follows that $K_0(\overline{\mathcal{O}}_\Gamma) = K_0(\overline{\mathcal{O}}_\Gamma^{\mathbf{T}})/(1-\sigma)K_0(\overline{\mathcal{O}}_\Gamma^{\mathbf{T}})$ and $K_1(\overline{\mathcal{O}}_\Gamma) = \ker(1-\sigma)$ on $K_0(\overline{\mathcal{O}}_\Gamma^{\mathbf{T}})$. □

Finally we consider some simple examples. First let $\Gamma = SL(2,\mathbf{Z}) = \mathbf{Z}_4 *_{\mathbf{Z}_2} \mathbf{Z}_6$. Let χ_1 be the unit character of $\mathbf{Z}_2$ and let χ_2 be the character such that $\chi_2(a) = -1$ where a is a generator of $\mathbf{Z}_2$. These are one-dimensional and exhaust all the irreducible characters. Then we have the corresponding matrix

$$A_\Gamma = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

Hence the corresponding K-groups are $K_0(\mathcal{O}_\Gamma) = 0$ and $K_1(\mathcal{O}_\Gamma) = 0$. In fact, $\mathcal{O}_{\mathbf{Z}_4 *_{\mathbf{Z}_2} \mathbf{Z}_6} \simeq \mathcal{O}_{\mathbf{Z}_2 * \mathbf{Z}_3} \oplus \mathcal{O}_{\mathbf{Z}_2 * \mathbf{Z}_3} \simeq \mathcal{O}_2 \oplus \mathcal{O}_2$.

Next let $\Gamma = \mathfrak{S}_4 *_{\mathfrak{S}_3} \mathfrak{S}_4$, $\tau = (1\,2)$ and $\sigma = (1\,2\,3)$. Note that $\mathfrak{S}_3 = \langle 1, \tau, \sigma \rangle$. $\mathfrak{S}_3$ has three irreducible characters:

	1	τ	σ
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Moreover, $\mathfrak{S}_3 \backslash \mathfrak{S}_4 / \mathfrak{S}_3$ has only two points; say $\mathfrak{S}_3$ and $\mathfrak{S}_3 x \mathfrak{S}_3$ with $x = (1\,2)(3\,4)$. Then we obtain the corresponding matrix

$$A_\Gamma = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \end{pmatrix}$$

Hence this gives $K_0(\mathcal{O}_\Gamma) = \mathbb{Z} \oplus \mathbb{Z}_4$ and $K_1(\mathcal{O}_\Gamma) = \mathbb{Z}$. In this case, Γ satisfies the condition of Theorem 6.3. So $\mathcal{O}_\Gamma$ is a simple, nuclear, purely infinite C^*-algebra.

8 KMS states on $\mathcal{O}_\Gamma$

In this section, we investigate the relationship between KMS states on $\mathcal{O}_\Gamma$ for generalized gauge actions and random walks on Γ. Throughout this section, we assume that all groups G_i are finite though we can carry out the same arguments if $G_i = \mathbb{Z} \times H$ for some $i \in I$. Let $\omega = (\omega_i)_{i \in I} \in \mathbb{R}_+^{|I|}$ By the universality of $\mathcal{O}_\Gamma$, we can define an automorphism α_t^ω for any $t \in \mathbb{R}$ on $\mathcal{O}_\Gamma$ by $\alpha_t^\omega(S_g) = e^{\sqrt{-1}\omega_i t} S_g$ for $g \in G_i \setminus H$ and $\alpha_t^\omega(U_h) = U_h$ for $h \in H$. Hence we obtain the $\mathbb{R}$-action α^ω on $\mathcal{O}_\Gamma$. We call it *the generalized gauge action* with respect to ω. We will only consider actions of these types and determine KMS states on $\mathcal{O}_\Gamma$ for these actions.

In [W1], Woess showed that our boundary Ω can be identified with the Poisson boundary of random walks satisfying certain conditions. The reader is referred to [W2] for a good survey of random walks.

Let μ be a probability measure on Γ and consider a random walk governed by μ, i.e. the transition probability from x to y given by

$$p(x, y) = \mu(x^{-1} y).$$

A random walk is said to be *irreducible* if for any $x, y \in \Gamma$, $p^{(n)}(x, y) \neq 0$ for some integer n, where

$$p^{(n)}(x, y) = \sum_{x_1, x_2, \ldots, x_{n-1} \in \Gamma} p(x, x_1) p(x_1, x_2) \cdots p(x_{n-1}, y).$$

A probability measure ν on Ω is said to be *stationary* with respect to μ if $\nu = \mu * \nu$, where $\mu * \nu$ is defined by

$$\int_{\Omega} f(\omega) d\mu * \nu(\omega) = \int_{\Omega} \int_{\mathrm{supp}\mu} f(g\omega) d\mu(g) d\nu(\omega), \quad \text{for} \quad f \in C(\Omega, \nu).$$

By [W1, Theorem 9.1], if a random walk governed by a probability measure μ on Γ is irreducible, then there exists a unique stationary probability measure ν on Ω with respect to μ. Moreover if μ has finite support, then the Poisson boundary coincides with (Ω, ν).

If ν is a probability measure on the compact space Ω, then we can define a state ϕ_ν by

$$\phi_\nu(X) = \textstyle\int_{\Omega} E(X) d\nu \quad \text{for} \quad X \in \mathcal{O}_\Gamma,$$

where E is the canonical conditional expectation of $C(\Omega) \rtimes_r \Gamma$ onto $C(\Omega)$.

One of our purposes in this section is to prove that there exists a random walk governed by a probability measure μ that induces the stationary measure ν on Ω such that the corresponding state ϕ_ν is the unique KMS state for α^ω. Namely,

Theorem 8.1 *Assume that the matrix A_Γ obtained in the preceding section is irreducible. For any $\omega = (\omega_i)_{i \in I} \in \mathbb{R}_+^{|I|}$, there exists a unique probability measure μ with the following properties:*

(i) *$supp(\mu) = \bigcup_{i \in I} G_i \setminus H$.*

(ii) *$\mu(gh) = \mu(g)$ for any $g \in \bigcup_{i \in I} G_i \setminus H$ and $h \in H$.*

(iii) *The corresponding unique stationary measure ν on Ω induces the unique KMS state ϕ_ν for α^ω and the corresponding inverse temperature β is also unique.*

We need the hypothesis of the irreducibility of the matrix A_Γ for the uniqueness of the KMS state. Though it is, in general, difficult to check the irreducibility of A_Γ, by Theorem 6.5, the condition of simplicity of $\mathcal{O}_\Gamma$ in Corollary 6.4 is also a sufficient condition for irreducibility of A_Γ. To obtain the theorem, we first present two lemmas.

Lemma 8.2 *Assume that ν is a probability measure on Ω. Then the corresponding state ϕ_ν is the KMS state for α^ω if and only if ν satisfies the following conditions:*

$$\nu(\Omega(x_1 \cdots x_m)) = \frac{e^{-\beta\omega_{i_1}} \cdots e^{-\beta\omega_{i_{m-1}}}}{[G_{i_m} : H] - 1 + e^{\beta\omega_{i_m}}},$$

for $x_k \in \Omega_{i_k}$ with $i_1 \neq \cdots \neq i_m$, where $\Omega(x_1 \cdots x_m)$ is the cylinder subset of Ω defined by

$$\Omega(x_1 \cdots x_m) = \{(x(n))_{n \geq 1} \in \Omega \mid x(1) = x_1, \ldots, x(m) = x_m\}.$$

Proof ϕ_ν is the KMS state for α^ω if and only if

$$\phi_\nu(S_\xi P_i U_h S_\eta^* \cdot S_\sigma P_j U_k S_\tau^*) = \phi(S_\sigma P_j U_k S_\tau^* \cdot \alpha^\omega_{\sqrt{-1}\beta}(S_\xi P_i U_h S_\eta^*)),$$

for any $\xi, \eta, \sigma, \tau \in \Delta, h, k \in H$ and $i, j \in I$.

We may assume that $|\xi| + |\sigma| = |\eta| + |\tau|$ and $|\eta| \geq |\sigma|$. Set $|\xi| = p, |\eta| = q, |\sigma| = s, |\tau| = t$ and let $\xi = \xi_1 \cdots \xi_p$, $\eta = \eta_1 \cdots \eta_q$ with $\xi_k \in \Omega_{i_k} \setminus \{e\}, \eta_l \in \Omega_{j_l} \setminus \{e\}$ and $i_1 \neq \cdots \neq i_p, j_1 \neq \cdots \neq j_q$. Then

$$\begin{aligned}
&\phi_\nu(S_\xi P_i U_h S_\eta^* \cdot S_\sigma P_j U_k S_\tau^*) = \delta_{\eta_1 \cdots \eta_s, \sigma} \delta_{\eta_{s+1}, j} \phi_\nu(S_\xi P_i U_h S_{\eta_{s+1} \cdots \eta_q}^* U_k S_\tau^*) \\
&= \delta_{\eta_1 \cdots \eta_s, \sigma} \delta_{\eta_{s+1}, j} \phi_\nu(S_{\xi h} P_i S_{\tau k^{-1} \eta_{s+1} \cdots \eta_q}) \\
&= \delta_{\eta_1 \cdots \eta_s, \sigma} \delta_{\eta_{s+1}, j} \delta_{\xi h, \tau k^{-1} \eta_{s+1} \cdots \eta_q} \sum_{x \in \Omega_i \setminus \{e\}} \nu(\Omega(\xi x)),
\end{aligned}$$

and

$$\begin{aligned}
&\phi_\nu(S_\sigma P_j U_k S_\tau^* \cdot \alpha^\omega_{\sqrt{-1}\beta}(S_\xi P_i U_h S_\eta^*)) \\
&= e^{-\beta\omega_{i_1}} \cdots e^{-\beta\omega_{i_p}} e^{\beta\omega_{j_1}} \cdots e^{\beta\omega_{j_q}} \phi_\nu(S_\sigma P_j U_k S_\tau^* \cdot S_\xi P_i U_h S_\eta^*) \\
&= e^{-\beta\omega_{i_1}} \cdots e^{-\beta\omega_{i_p}} e^{\beta\omega_{j_1}} \cdots e^{\beta\omega_{j_q}} \delta_{\tau, \xi_1 \cdots \xi_t} \delta_{\xi_{t+1}, j} \phi_\nu(S_{\sigma k \xi_{t+1} \cdots \xi_p h} P_i S_\eta^*) \\
&= e^{-\beta\omega_{i_1}} \cdots e^{-\beta\omega_{i_p}} e^{\beta\omega_{j_1}} \cdots e^{\beta\omega_{j_q}} \delta_{\tau, \xi_1 \cdots \xi_t} \delta_{\xi_{t+1}, j} \delta_{\sigma k \xi_{t+1} \cdots \xi_p h, \eta} \sum_{x \in \Omega_i \setminus \{e\}} \nu(\Omega(\eta x)),
\end{aligned}$$

where $\delta_{g,i} = 1$ only if $g \in G_i \setminus H$. Therefore the corresponding state ϕ_ν is the KMS state for α^ω if and only if ν satisfies the following conditions:

$$\nu(\Omega(\xi_1 \ldots \xi_p x)) = e^{-\beta\omega_{i_1}} \cdots e^{-\beta\omega_{i_p}} \nu(\Omega(x)),$$

for $x \in \Omega_i \setminus \{e\}$ with $i \neq i_p$.

Now we assume that ϕ_ν is the KMS state for α^ω. Then for $i \in I$,

$$\begin{aligned}
\nu(Y_i) = \phi_\nu(P_i) &= \sum_{g \in \Omega_i \setminus \{e\}} \phi_\nu(S_g S_g^*) \\
&= \sum_{g \in \Omega_i \setminus \{e\}} \phi_\nu(S_g^* \alpha^\omega_{\sqrt{-1}\beta}(S_g)) \\
&= e^{-\beta\omega_i} \sum_{g \in \Omega_i \setminus \{e\}} \phi_\nu(Q_g) \\
&= e^{-\beta\omega_i} \sum_{g \in \Omega_i \setminus \{e\}} \phi_\nu(1 - P_i) \\
&= e^{-\beta\omega_i}([G_i : H] - 1)(1 - \nu(Y_i)).
\end{aligned}$$

Hence,

$$\nu(Y_i) = \frac{[G_i : H] - 1}{[G_i : H] - 1 + e^{\beta\omega_i}}.$$

Moreover,

$$\begin{aligned}
&\nu(\Omega(x_1 \ldots x_m)) = \phi_\nu(S_{x_1} \cdots S_{x_m} S^*_{x_m} \cdots S^*_{x_1}) \\
&= \phi_\nu(S^*_{x_m} \cdots S^*_{x_1} \alpha^\omega_{\sqrt{-1}\beta}(S_{x_1} \cdots S_{x_m})) \\
&= e^{-\beta\omega_{i_1}} \cdots e^{-\beta\omega_{i_m}} \phi_\nu(Q_{x_m}) \\
&= e^{-\beta\omega_{i_1}} \cdots e^{-\beta\omega_{i_m}} (1 - \nu(\Omega(Y_{i_m}))) \\
&= \frac{e^{-\beta\omega_{i_1}} \cdots e^{-\beta\omega_{i_{m-1}}}}{[G_{i_m} : H] - 1 + e^{\beta\omega_{i_m}}}.
\end{aligned}$$

Conversely, suppose that a probability measure ν satisfies the condition of this lemma. By the first part of this proof, ϕ_ν is the KMS state for α^ω. □

Lemma 8.3 *Assume that ν is the unique stationary measure on Ω with respect to a random walk on Γ, governed by a probability measure μ with the conditions* (i), (ii) *in Theorem 8.1. Then ϕ_ν is a β-KMS state for α^ω if and only if μ satisfies the following conditions:*

$$\mu(g) = \frac{\prod_{j \neq i} C_j}{\sum_{k \in I}(g_k \prod_{l \neq k} C_l)} \qquad \textit{for} \quad g \in G_i \setminus H \quad \textit{and} \quad i \in I,$$

where $g_i = |G_i \setminus H|$ and $C_i = (1 - e^{-\beta\omega_i})g_i - (1 - e^{\beta\omega_i})|H|$ for $i \in I$.

Proof Assume that ϕ_ν is a β-KMS state for α^ω For any $f \in C(\Omega)$,

$$\begin{aligned}
&\iint f(\omega) d\nu(\omega) = \iint f(\omega) d\mu * \nu(\omega) \\
&= \iint f(g\omega) d\nu(\omega) d\mu(g) \\
&= \iint (\lambda^*_g f \lambda_g)(\omega) d\nu(\omega) d\mu(g) \\
&= \sum_{g \in \operatorname{supp}(\mu)} \mu(g) \phi_\nu(\lambda^*_g f \lambda_g) \\
&= \sum_{g \in \operatorname{supp}(\mu)} \mu(g) \phi_\nu(f \lambda_g \alpha^\omega_{\sqrt{-1}\beta}(\lambda^*_g)),
\end{aligned}$$

where $\mathcal{O}_\Gamma \simeq C(\Omega) \rtimes_r \Gamma = C^*(f, \lambda_\gamma \mid f \in C(\Omega), \gamma \in \Gamma)$.

Put $f = \chi_{\Omega(x)} = P_x$ for $i \in I$ and $x \in \Omega_i \setminus \{e\}$. Since $\lambda_g = S_g + \sum_{g' \in \Omega_{i'} \setminus H \cup g^{-1}H} S_{gg'} S_{g'}^* + S_{g^{-1}}^*$ for $g \in G_{i'} \setminus H$ and $i' \in I$, we have

$$1 = \sum_{gH = xH} \mu(g) e^{\beta\omega_i} + \sum_{g \in G_i \setminus H, gH \neq xH} \mu(g) + \sum_{g \in G_j \setminus H, j \neq i} \mu(g) e^{-\beta\omega_j}$$

for any $i \in I$ and $x \in \Omega_i \setminus \{e\}$. Let $x, y \in \Omega_i \setminus \{e\}$ with $xH \neq yH$. Then

$$1 = \sum_{gH = xH} \mu(g) e^{\beta\omega_i} + \sum_{gH \neq xH} \mu(g) + \sum_{g \in G_j \setminus H, j \neq i} \mu(g) e^{-\beta\omega_j},$$

$$1 = \sum_{gH = yH} \mu(g) e^{\beta\omega_i} + \sum_{gH \neq yH} \mu(g) + \sum_{g \in G_j \setminus H, j \neq i} \mu(g) e^{-\beta\omega_j}$$

By the above equations, we have $\mu(x) = \mu(y)$, and then it follows from hypothesis (ii) in Theorem 8.1 that $\mu(g) = \mu_i$ for any $g \in G_i \setminus H$. Therefore we have

$$1 = |H| e^{\beta\omega_i} \mu_i + (g_i - |H|)\mu_i + \sum_{j \neq i} g_j e^{-\beta\omega_j} \mu_j,$$

for any $i \in I$, where $g_i = |G_i \setminus H|$. Thus by considering the above equations for i and $j \in I$,

$$|H| e^{\beta\omega_i} \mu_i - |H| e^{\beta\omega_j} \mu_j + (g_i - |H|)\mu_i - (g_j - |H|)\mu_j + g_j e^{-\beta\omega_j} \mu_j - g_i e^{-\beta\omega_i} \mu_i = 0.$$

Hence we obtain the equation,

$$(|H| e^{\beta\omega_i} + g_i - |H| - g_i e^{-\beta\omega_i})\mu_i = (|H| e^{\beta\omega_j} + g_j - |H| - g_j e^{-\beta\omega_j})\mu_j.$$

Since $\mu(\bigcup_{i \in I} G_i \setminus H) = 1$, we have

$$g_i \mu_i + \sum_{j \neq i} g_j \frac{(1 - e^{-\beta\omega_i}) g_i - (1 - e^{-\beta\omega_i})|H|}{(1 - e^{-\beta\omega_j}) g_j - (1 - e^{-\beta\omega_j})|H|} \mu_i = 1.$$

We put $C_i = (1 - e^{-\beta\omega_i}) g_i - (1 - e^{-\beta\omega_i})|H|$ and then

$$(g_i + C_i \sum_{j \neq i} \frac{g_j}{C_j}) \mu_i = 1.$$

Therefore

$$\begin{aligned}
\mu_i &= \frac{1}{g_i + C_i \sum_{j \neq i} g_j / C_j} \\
&= \frac{\prod_{j \neq i} C_j}{g_i \prod_{j \neq i} C_j + \sum_{j \neq i} (g_j C_i \prod_{k \neq i,j} C_k)} \\
&= \frac{\prod_{j \neq i} C_j}{\sum_{k \in I} g_k \prod_{l \neq k} C_l}.
\end{aligned}$$

On the other hand, let ν be the probability measure on Ω satisfying the condition in Lemma 8.2. Then the corresponding state ϕ_ν is the KMS state. It is enough to check that $\mu * \nu = \nu$ by [W1]. Since

$$\nu(\Omega(x_1 \cdots x_n)) = e^{-\beta\omega_{i_1}} \cdots e^{-\beta\omega_{i_{n-1}}} \nu(\Omega(x_n)),$$

for $x_k \in \Omega_{i_k} \setminus \{e\}$ with $i_1 \neq \cdots \neq i_n$, we have

$$\begin{aligned}
\mu * \nu(\Omega(x_1 \cdots x_n)) &= \iint \chi_{\Omega(x_1 \cdots x_n)}(\omega) d\mu * \nu(\omega) \\
&= \sum_{g \in \mathrm{supp}\mu} \mu(g) \int (\lambda_g^* \chi_{\Omega(x_1 \cdots x_n)} \lambda_g)(\omega) d\nu(\omega) \\
&= \sum_{g \in G_{i_1} \setminus H, x_1 H = gH} \mu_{i_1} \phi_\nu(S_{x_2} \cdots S_{x_n} S_{x_n}^* \cdots S_{x_2}^*) \\
&\quad + \sum_{g \in G_{i_1} \setminus H, x_1 H \neq gH} \mu_{i_1} \phi_\nu(S_{g^{-1}x_1} S_{x_2} \cdots S_{x_n} S_{x_n}^* \cdots S_{x_2}^* S_{g^{-1}x_1}^*) \\
&\quad + \sum_{g \in G_i \setminus H, i \neq i_1} \mu_i \phi_\nu(S_{g^{-1}} S_{x_1} S_{x_2} \cdots S_{x_n} S_{x_n}^* \cdots S_{x_2}^* S_{x_1}^* S_{g^{-1}}^*) \\
&= \left(|H| e^{\beta\omega_{i_1}} \mu_{i_1} + (g_{i_1} - |H|)\mu_{i_1} + \sum_{i \neq i_1} g_i e^{-\beta\omega_i} \mu_i \right) \nu(\Omega(x_1 \cdots x_n)) \\
&= \nu(\Omega(x_1 \ldots x_n)).
\end{aligned}$$

□

To prove the uniqueness of KMS states of $\mathcal{O}_\Gamma$, we need the irreducibility of the matrix A_Γ (See [EFW2] for KMS states on Cuntz-Krieger algebras). Set an irreducible matrix $B = [B((i,k),(j,l))] = [e^{-\beta\omega_i} A_\Gamma^t((i,k),(j,l))]$. Let K_β be the set of all β-KMS states for the action α^ω We put

$$L_\beta = \{ y = [y(i,k)] \in \mathbb{R}^N \mid By = y, \quad y(i,k) \geq 0, \quad \sum_{i \in I} \sum_{k=1}^{r} n_k y(i,k) = 1 \}.$$

We now have the necessary ingredients for the proof of Theorem 8.1.

Proof of Theorem 8.1 We first prove the uniqueness of the corresponding inverse temperature. Let ϕ be a β-KMS state for α^ω. For $i \in I$,

$$\begin{aligned}
\phi(P_i) = \sum_{g \in \Omega_i \setminus \{e\}} \phi(S_g S_g^*) &= \sum_{g \in \Omega_i \setminus \{e\}} \phi(S_g^* \alpha^\omega_{\sqrt{-1}\beta}(S_g)) \\
&= e^{-\beta\omega_i} \sum_{g \in \Omega_i \setminus \{e\}} \phi(Q_g) \\
&= e^{-\beta\omega_i}([G_i : H] - 1)(1 - \phi(P_i)).
\end{aligned}$$

Thus $\phi(P_i) = \lambda_i(\beta)/(1+\lambda_i(\beta))$, where $\lambda_i(\beta) = e^{-\beta\omega_i}([G_i : H]-1)$. Since $\sum_{i\in I} P_i = 1$,

$$|I| - 1 = \sum_{i\in I} \frac{1}{1+\lambda_i(\beta)}.$$

The function $\sum_{i\in I} 1/(1+\lambda_i(\beta))$ is a monotone increasing continuous function such that

$$\sum_{i\in I} \frac{1}{1+\lambda_i(\beta)} = \begin{cases} \sum_{i\in I} 1/[G_i : H] & \text{if } \beta = 0, \\ |I| & \text{if } \beta \to \infty. \end{cases}$$

Since $\sum_{i\in I} 1/[G_i : H] \le |I|/2 \le |I| - 1$, there exists a unique β satisfying

$$|I| - 1 = \sum_{i\in I} \frac{1}{([G_i : H]-1)e^{-\beta\omega_i} + 1}$$

Therefore we obtain the uniqueness of the inverse temperature β.

We will next show the uniqueness of the KMS state ϕ_ν. We claim that K_β is in one-to-one correspondence with L_β. In fact, we define a map f from K_β to L_β by

$$f(\phi) = [\phi(P(i,k))/n_k].$$

Indeed,

$$\begin{aligned}
e^{\beta\omega_i}\phi(P(i,k)) &= \sum_{g\in\Omega_i\setminus\{e\}} \phi(p_k S_g \alpha^{\omega}_{\sqrt{-1}\beta}(S_g^*)) \\
&= \sum_{g\in\Omega_i\setminus\{e\}} \phi(S_g^* p_k S_g) \\
&= \frac{n_k}{|H|} \sum_{g\in\Omega_i\setminus\{e\}} \sum_{h\in H} \overline{\chi_k(h)}\phi(S_g^* U_h S_g) \\
&= \frac{n_k}{|H|} \sum_{g\in\Omega_i\setminus\{e\}} \sum_{h\in H(g)} \overline{\chi_k(h)}\phi(Q_g U_{g^{-1}hg}) \\
&= \frac{n_k}{|H|} \sum_{g\in\Omega_i\setminus\{e\}} \sum_{h\in H(g)} \overline{\chi_k(h)} \sum_{j\ne i} \phi(P_j U_{g^{-1}hg} P_j) \\
&= \frac{n_k}{|H|} \sum_{g\in\Omega_i\setminus\{e\}} \sum_{h\in H(g)} \overline{\chi_k(h)} \sum_{j\ne i} \sum_{l=1}^{r} \phi(P(j,l) U_{g^{-1}hg} P(j,l)).
\end{aligned}$$

Since ϕ is a trace on $C^*(P(j,l)U_h P(j,l) \mid h \in H) \simeq M_{n_l}(\mathbb{C})$ and $M_{n_l}(\mathbb{C})$ has a unique tracial state, we have

$$\phi(P(j,l)U_{g^{-1}hg}P(j,l)) = \chi_l(g^{-1}hg)\frac{\phi(P(j,l))}{n_l}$$

Therefore, by the same arguments as in the previous section, we obtain

$$
\begin{aligned}
& e^{\beta\omega_i}\phi(P(i,k)) \\
&= \frac{n_k}{|H|}\sum_{g\in\Omega_i\setminus\{e\}}\sum_{h\in H(g)}\overline{\chi_k(h)}\sum_{j\neq i}\sum_{l=1}^{r}\phi(P(j,l)U_{g^{-1}hg}P(j,l)) \\
&= n_k\sum_{x\in X_i\setminus\{e\}}\sum_{j\neq i}\sum_{l=1}^{r}(\chi_k,\chi_l^x)_{H(x)}\phi(P(j,l))/n_l \\
&= n_k\sum_{(j,l)}A_\Gamma((j,l),(i,k))\phi(P(j,l))/n_l.
\end{aligned}
$$

Hence this is well-defined.

Suppose that ν is the probability measure in Lemma 8.2 and ϕ_ν is the induced β-KMS state for α^ω. Set a vector $y=[y(i,k)=\phi_\nu(P(i,k))/n_k]$. Since y is strictly positive and B is irreducible, 1 is the eigenvalue which dominates the absolute value of all eigenvalue of B by the Perron-Frobenius theorem. It also follows from the Perron-Frobenius theorem that L_β has only one element. Hence f is surjective.

Let $\phi\in K_\beta$. For $\xi=\xi_{i_1}\cdots\xi_{i_n},\eta=\eta_{j_1}\cdots\eta_{j_n}$ with $i_1\neq\cdots\neq i_n, j_1\neq\cdots\neq j_n$, $h\in H$ and $i\in I$,

$$
\begin{aligned}
& e^{\beta\omega_{j_1}}\cdots e^{\beta\omega_{j_n}}\phi(S_\xi U_h P_i S_\eta^*) = \phi(S_\xi U_h P_i \alpha^\omega_{\sqrt{-1}\beta}(S_\eta^*)) \\
&= \phi(S_\eta^* S_\xi U_h P_i) = \delta_{\xi,\eta}\phi(U_h P_i) \\
&= \delta_{\xi,\eta}\sum_{k=1}^{r}\phi(U_h P(i,k)) = \delta_{\xi,\eta}\sum_{k=1}^{r}\chi_k(h)\phi(P(i,k))/n_k,
\end{aligned}
$$

because ϕ is a trace on $C^*(U_hP(i,k)\mid h\in H)\simeq M_{n_k}(\mathbb{C})$. If $f(\phi)=f(\psi)$, then the above calculations imply $\phi=\psi$ on $\mathcal{O}_\Gamma^{\mathbb{T}}$. By the KMS condition, $\phi(b)=0=\psi(b)$ for $b\notin\mathcal{O}_\Gamma^{\mathbb{T}}$. Thus $\phi=\psi$ and f is injective. Therefore ϕ_ν is the unique β-KMS state for α^ω. □

Remarks and Examples Let ν be the corresponding probability measure with the gauge action α. Under the identification $L^\infty(\Omega,\nu)\rtimes_w\Gamma\simeq\pi_\nu(\mathcal{O}_\Gamma)''$, we can determine the type of the factor by essentially the same arguments as in [EFW2]. If H is trivial, then $\mathcal{O}_\Gamma$ is a Cuntz-Krieger algebra for some irreducible matrix with 0-1 entries. In this case, we can always apply the result in [EFW2]. This fact generalizes [RR]. If H is not trivial, then by using the condition of simplicity of $\mathcal{O}_\Gamma$ in Corollary 6.4 to check the irreducibility of the matrix A_Γ, we can apply Theorem 8.1. In the special case where $G_i=G$ for all $i\in I$, we can easily determine the type of the factor $\pi_\nu(\mathcal{O}_\Gamma)''$ for the gauge action. The factor $\pi_\nu(\mathcal{O}_\Gamma)''$ is of type III_λ where $\lambda=1/([G:H]-1)^2$ if $|I|=2$ and $\lambda=1/(|I|-1)([G:H]-1)$ if $|I|>2$. For instance, let $\Gamma=\mathfrak{S}_4 *_{\mathfrak{S}_3}\mathfrak{S}_4$. We have already obtained the matrix A_Γ in section 7, but we can determine that the factor $L^\infty(\Omega,\nu)\rtimes_w\Gamma$ is of type $\mathrm{III}_{1/9}$ without using A_Γ.

We next discuss the converse. Namely any $\mathbb{R}$-actions that have KMS states induced by a probability measure μ on Γ with some conditions is, in fact, a generalized gauge action.

Let μ be a given probability measure on Γ with $\text{supp}(\mu) = \bigcup_{i\in I} G_i \setminus H$. By [W1], there exists an unique probability measure ν on Ω such that $\mu * \nu = \nu$. Let (π_ν, H_ν, x_ν) be the GNS-representation of $\mathcal{O}_\Gamma$ with respect to the state ϕ_ν. We also denote a vector state of x_ν by ϕ_ν.

$$\phi_\nu(a) = \langle a x_\nu, x_\nu\rangle \qquad \text{for} \quad a \in \pi_\nu(\mathcal{O}_\Gamma)''$$

Let σ_t^ν be the modular automorphism group of ϕ_ν.

Theorem 8.4 *Suppose that μ is a probability measure on Γ such that* $\text{supp}(\mu) = \bigcup_{i\in I} G_i \setminus H$ *and $\mu(g) = \mu(hg)$ for any $g \in \bigcup_{i\in I} G_i \setminus H$, $h \in H$. If ν is the corresponding stationary measure with respect to μ, then there exists $\omega_g \in \mathbb{R}_+$ such that*

$$\sigma_t^\nu(\pi_\nu(S_g)) = e^{\sqrt{-1}\omega_g t}\pi_\nu(S_g) \quad \textit{for} \quad g \in G_i \setminus H, i \in I,$$

and

$$\sigma_t^\nu(\pi_\nu(U_h)) = \pi_\nu(U_h) \quad \textit{for} \quad h \in H.$$

Proof To prove that $\sigma_t^\nu(\pi_\nu(S_g)) = e^{\sqrt{-1}\omega_g t}\pi_\nu(S_g)$, it suffices to show that there exists $\zeta_g \in \mathbb{R}_+$ such that

$$(*) \qquad \phi_\nu(\pi_\nu(S_g)a) = \zeta_g \phi_\nu(a\pi_\nu(S_g)) \quad \text{for} \quad g \in G_i\setminus, a \in \pi_\nu(\mathcal{O}_\Gamma)''$$

In fact, Let Δ_ν be the modular operator and J_ν be the modular conjugate of ϕ_ν.

$$\begin{aligned}
&(\text{left hand side of } (*)) = \langle \pi_\nu(S_g) a x_\nu, x_\nu\rangle \\
&= \langle a x_\nu, \pi_\nu(S_g)^* x_\nu\rangle \\
&= \langle a x_\nu, J_\nu \Delta_\nu^{1/2}\pi_\nu(S_g) x_\nu\rangle \\
&= \langle \Delta_\nu^{1/2}\pi_\nu(S_g) x_\nu, J_\nu a x_\nu\rangle \\
&= \langle \Delta_\nu^{1/2}\pi_\nu(S_g) x_\nu, \Delta_\nu^{1/2} a^* x_\nu\rangle.
\end{aligned}$$

and

$$\begin{aligned}
&(\text{right hand side of } (*)) = \zeta_g\langle a\pi_\nu(S_g) x_\nu, x_\nu\rangle \\
&= \zeta_g\langle \pi_\nu(S_g) x_\nu, a^* x_\nu\rangle.
\end{aligned}$$

Therefore for $a \in \pi_\nu(\mathcal{O}_\Gamma)''$,

$$\langle \Delta_\nu^{1/2}\pi_\nu(S_g) x_\nu, \Delta_\nu^{1/2} a^* x_\nu\rangle = \zeta_g\langle \pi_\nu(S_g) x_\nu, a^* x_\nu\rangle.$$

and hence for $y \in \text{dom}(\Delta_\nu^{1/2})$, we have

$$\langle \Delta_\nu^{1/2}\pi_\nu(S_g) x_\nu, \Delta_\nu^{1/2} y\rangle = \zeta_g\langle \pi_\nu(S_g) x_\nu, y\rangle.$$

Thus $\Delta_\nu^{1/2}\pi_\nu(S_g)x_\nu \in \mathrm{dom}(\Delta_\nu^{1/2})$ and we obtain

$$\Delta_\nu \pi_\nu(S_g)x_\nu = \zeta_g \pi_\nu(S_g)x_\nu.$$

Therefore

$$\Delta_\nu^{\sqrt{-1}t}\pi_\nu(S_g)x_\nu = \zeta_g^{\sqrt{-1}t}\pi_\nu(S_g)x_\nu,$$

and then

$$(\sigma_t^\nu(\pi_\nu(S_g)) - \zeta_g^{\sqrt{-1}t}\pi_\nu(S_g))x_\nu = 0,$$

where σ_t^ν is the modular automorphism group of ϕ_ν. Since x_ν is a separating vector,

$$\sigma_t^\nu(\pi_\nu(S_g)) = \zeta_g^{\sqrt{-1}t}\pi_\nu(S_g).$$

Now we will show that

$$\phi_\nu(\pi_\nu(S_g)a) = \zeta_g \phi_\nu(a\pi_\nu(S_g)) \quad \text{for} \quad g \in G_i \setminus H, a \in \pi_\nu(\mathcal{O}_\Gamma)''$$

We may assume that $a = f\lambda_{g^{-1}}$ for $f \in C(\Omega)$. Recall that $S_g = \lambda_g \chi_{\Omega\setminus Y_i} \in C(\Omega) \rtimes_r \Gamma$. Since

$$\phi_\nu(\pi_\nu(S_g a)) = \int_{\Omega\setminus Y_i} f(g^{-1}\omega)d\nu(\omega) = \int_{\Omega\setminus Y_i} f(\omega)\frac{dg^{-1}\nu}{d\nu}(\omega)d\nu(\omega),$$

we claim that

$$\frac{dg^{-1}\nu}{d\nu}(\omega) = \zeta_g \quad \text{on} \quad \Omega \setminus Y_i.$$

This is the Martin kernel $K(g^{-1}, \omega)$, (See [W1]). Hence it suffices to show that $K(g^{-1}, x)$ is constant for any $x = x_1 \cdots x_n \in \Gamma$ such that $x_1 \notin G_i$. By [W1], we have

$$K(g^{-1}, x) = \frac{G(g^{-1}, x)}{G(e, x)},$$

where $G(y, z) = \sum_{k=1}^{\infty} p^{(k)}(y, z)$ is the Green kernel. Since any probability from g^{-1} to x must be through elements of H at least once, we have

$$G(g^{-1}, x) = \sum_{h\in H} F(g^{-1}, h)G(h, x),$$

where $s^x = \inf\{n \geq 0 \mid Z_n = x\}$ and $F(g, x) = \sum_{n=0}^{\infty} \mathrm{Pr}_g[s^x = n]$ in [W2]. By hypothesis $\mu(g) = \mu(hg)$ for any $g \in \bigcup_{i\in I} G_i \setminus H$ and $h \in H$, we have

$$G(h, x) = G(e, x) \quad \text{for any} \quad h \in H.$$

Therefore we have $\omega_g = \log(\sum_{h\in H} F(g^{-1}, h))$. $\sigma_t^\nu(\pi_\nu(U_h)) = \pi_\nu(U_h)$ can be proved in the same way. Hence we are done. □

9 Appendix

Trees We first review trees based on [FN]. A *graph* is a pair (V, E) consisting of a set of vertices V and a family E of two-element subsets of V, called edges. A *path* is a finite sequence $\{x_1, \ldots, x_n\} \subseteq V$ such that $\{x_i, x_{i+1}\} \in E$. (V, E) is said to be *connected* if for $x, y \in V$ there exists a path $\{x_1, \ldots, x_n\}$ with $x_1 = x, x_n = y$. If (V, E) is a tree, then for $x, y \in V$ there exists a unique path $\{x_1, \ldots, x_n\}$ joining x to y such that $x_i \neq x_{i+2}$. We denote this path by $[x, y]$. A tree is said to be *locally finite* if every vertex belongs to finitely many edges. The number of edges to which a vertex of a locally finite tree belongs is called a *degree*. If the degree is independent of the choice of vertices, then the tree is called *homogeneous*.

We introduce trees for amalgamated free product groups based on [Ser]. Let $(G_i)_{i\in I}$ be a family of groups with an index set I. When H is a group and every G_i contains H as a subgroup, then we denote $*_H G_i$ by Γ, which is the amalgamated free product of the groups. If we choose sets Ω_i of left representatives of G_i/H with $e \in \Omega_i$ for any $i \in I$, then each $\gamma \in \Gamma$ can be written uniquely as

$$\gamma = g_1 g_2 \cdots g_n h,$$

where $h \in H, g_1 \in \Omega_{i_1} \setminus \{e\}, \ldots, g_n \in \Omega_{i_n} \setminus \{e\}$ and $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n$.

Now we construct the corresponding tree. At first, we assume that $I = \{1, 2\}$. Let

$$V = \Gamma/G_1 \coprod \Gamma/G_2 \text{ and } E = \Gamma/H,$$

and the original and terminal maps $o : \Gamma/H \to \Gamma/G_1$ and $t : \Gamma/H \to \Gamma/G_2$ are natural surjections. It is easy to see that $G_T = (V, E)$ is a tree. In general, we assume that the element 0 does not belong to I. Let $G_0 = H$ and $H_i = H$ for $i \in I$. Then we define

$$V = \coprod_{i\in I\cup\{0\}} \Gamma/G_i \text{ and } E = \coprod_{i\in I} \Gamma/H_i.$$

Now we define two maps $o, t : E \to V$ For $H_i \in E$, let

$$o(H_i) = G_0 \text{ and } t(H_i) = G_i.$$

For any $\gamma H_i \in E$, we may assume that $\gamma H = g_1 \cdots g_n H_i$ such that $g_k \in \Omega_{i_k}$ with $i_1 \neq \cdots \neq i_n$. If $i = i_n$ we define

$$o(\gamma H_i) = \gamma G_{i_n} \text{ and } t(\gamma H_i) = \gamma G_0.$$

If $i \neq i_n$ we define

$$o(\gamma H_i) = \gamma G_0 \text{ and } t(\gamma H_i) = \gamma G_i.$$

Then we have a tree $G_T = (V, E)$.

For a tree (V, E), the set V is naturally a metric space. The distance $d(x, y)$ is defined by the number of edges in the unique path $[x, y]$. An *infinite chain* is an infinite path $\{x_1, x_2, \dots\}$ such that $x_i \neq x_{i+2}$. We define an equivalence relation on the set of infinite chains. Two infinite chains $\{x_1, x_2, \dots\}, \{y_1, y_2, \dots\}$ are equivalent if there exists an integer k such that $x_n = y_{n+k}$ for a sufficiently large n. The boundary Ω of a tree is the set of the equivalence classes of infinite chains. The boundary may be thought of as a point at infinity. Next we introduce the topology into the space $V \cup \Omega$ such that $V \cup \Omega$ is compact, the points of V are open and V is dense in $V \cup \Omega$. It suffices to define a basis of neighborhoods for each $\omega \in \Omega$. Let x be a vertex. Let $\{x, x_1, x_2, \dots\}$ be an infinite chain representing ω. For each $y = x_n$, the neighborhood of ω is defined to consist of all vertices and all boundary points of the infinite chains which include $[x, y]$.

Hyperbolic groups We introduce hyperbolic groups defined by Gromov. See [GH] for details. Suppose that (X, d) is a metric space. We define a product by

$$\langle x|y\rangle_z = \frac{1}{2}\{d(x,z) + d(y,z) - d(x,y)\},$$

for $x, y, z \in X$. This is called the Gromov product. Let $\delta \geq 0$ and $w \in X$. A metric space X is said to be δ-hyperbolic with respect to w if For $x, y, z \in X$,

$$\langle x|y\rangle_w \geq \min\{\langle x|z\rangle_w, \langle y|z\rangle_w\} - \delta. \tag{‡}$$

Note that if X is δ-hyperbolic with respect to w, then X is 2δ-hyperbolic with respect to any $w' \in X$.

Definition 9.1 *The space X is said to be hyperbolic if X is δ-hyperbolic with respect to some $w \in X$ and some $\delta \geq 0$.*

Suppose that Γ is a group generated by a finite subset S such that $S^{-1} = S$. Let $G(\Gamma, S)$ be the Cayley graph. The graph $G(\Gamma, S)$ has a natural word metric. Hence $G(\Gamma, S)$ is a metric space.

Definition 9.2 *A finitely generated group Γ is said to be hyperbolic with respect to a finite generator system S if the corresponding Cayley graph $G(\Gamma, S)$ is hyperbolic with respect to the word metric.*

In fact, hyperbolicity is independent of the choice of S. Therefore we say that Γ is a hyperbolic group, for short.

We define the hyperbolic boundary of a hyperbolic space X. Let $w \in X$ be a point. A sequence (x_n) in X is said to *converge to infinity* if $\langle x_n|x_m\rangle_w \to \infty$, $(n, m \to \infty)$. Note that this is independent of the choice of w. The set X_∞ is the set of all sequences converging to infinity in X. Then we define an equivalence relation in X_∞. Two sequences $(x_n), (y_n)$ are equivalent if $\langle x_n|y_n\rangle_w \to \infty$, $(n \to \infty)$. Although this is not an equivalence relation in general, the hyperbolicity assures that it is indeed an equivalence relation. The set of all equivalent classes of X_∞ is called the *hyperbolic boundary (at infinity)* and denoted by ∂X. Next we define the Gromov product on $X \cup \partial X$. For $x, y \in X \cup \partial X$, we choose sequences $(x_n), (y_n)$ converging to x, y, respectively. Then we define $\langle x|y\rangle = \liminf_{n\to\infty}\langle x_n|y_n\rangle_w$. Note that this is well-defined and if $x, y \in X$ then the above product coincides with the Gromov product on X.

Definition 9.3 *The topology of $X \cup \partial X$ is defined by the following neighborhood basis:*

$$\{y \in X \mid d(x,y) < r\} \qquad \textit{for } x \in X, r > 0,$$
$$\{y \in X \cup \partial X \mid \langle x|y\rangle > r\} \qquad \textit{for } x \in \partial X, r > 0.$$

We remark that if X is a tree, then the hyperbolic boundary ∂X coincides with the natural boundary Ω in the sense of [Fre].

Finally we prove that an amalgamated free product $\Gamma = *_H G_i$, considered in this paper, is a hyperbolic group.

Lemma 9.4 *The group $\Gamma = *_H G_i$ is a hyperbolic group.*

Proof. Let $S = \{g \in \bigcup_i G_i \mid |g| \le 1\}$. Let $G(\Gamma, S)$ be the corresponding Cayley graph. It suffices to show (‡) for $w = e$. For $x, y, z \in \Gamma$, we can write uniquely as follows:

$$\begin{aligned} x &= x_1 \cdots x_n h_x, \\ y &= y_1 \cdots y_m h_y, \\ z &= z_1 \cdots z_k h_z, \end{aligned}$$

where

$$\begin{array}{llll} x_1 \in \Omega_{i(x_1)}, & \dots, & x_n \in \Omega_{i(x_n)}, & h_x \in H, \\ y_1 \in \Omega_{i(y_1)}, & \dots, & y_m \in \Omega_{i(y_m)}, & h_y \in H, \\ z_1 \in \Omega_{i(z_1)}, & \dots, & z_k \in \Omega_{i(z_k)}, & h_z \in H. \end{array}$$

such that each element has length one. Then $d(x,e) = n$, $d(y,e) = m$ and $d(z,e) = k$. If $i(x_1) = i(y_1), \cdots, i(x_{l(x,y)}) = i(y_{l(x,y)})$ and $i(x_{l(x,y)+1}) \ne i(y_{l(x,y)+1})$, then $\langle x|y\rangle_e = l(x,y)$. Similarly, we obtain the positive integers $l(x,z), l(y,x)$ such that $\langle x|z\rangle_e = l(x,z), \langle y|z\rangle_e = l(y,z)$. We can have (‡) with $\delta = 0$. □

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TYPE III FACTORS ARISING FROM CUNTZ-KRIEGER ALGEBRAS

RUI OKAYASU

ABSTRACT. We determine the types of factors arising from GNS-representations of quasi-free KMS states on Cuntz-Krieger algebras. Applying our result to the Cuntz-Krieger algebras arising from the boundary actions of some amalgamated free product groups, we also determine the types of the harmonic measures on the boundaries.

1. INTRODUCTION

The Cuntz algebra $\mathcal{O}_n$ [Cun] and the Cuntz-Krieger algebra $\mathcal{O}_A$ [CK], a generalization of $\mathcal{O}_n$, are important examples of C^*-algebras. The Cuntz-Krieger algebra $\mathcal{O}_A$, associated with a 0-1 matrix A, is the universal C^*-algebra generated by the family of partial isometries $\{S_i\}_{i=1}^N$ satisfying the Cuntz-Krieger relations. The universal property of $\mathcal{O}_A$ allows us to define the so-called gauge action on $\mathcal{O}_A$. The existence of KMS states for one-parameter automorphisms is one of the natural questions. The KMS states for the gauge actions on $\mathcal{O}_n$ and $\mathcal{O}_A$ were obtained by D. Olesen and G. K. Pedersen [OP], and M. Enomoto, M. Fujii and Y. Watatani [EFW], respectively. More generally, D. E. Evans determined the KMS states on $\mathcal{O}_n$ for the quasi-free actions in [Eva]. In order to construct examples of subfactors, M. Izumi determined the types of factors obtained by the GNS-representations of quasi-free KMS states in [Izu]. One of the purposes in this paper is to generalize his result to Cuntz-Krieger algebras. The existence and the uniqueness of quasi-free KMS states on Cuntz-Krieger algebras were proved by R. Exel and M. Laca in [EL]. It implies that the von Neumann algebras arising from their GNS-representations are factors. We will compute the Connes spectrum of the modular automorphism group and determine the types of quasi-free KMS states.

As an application, we can give a construction of type III factors from geometric objects. J. Spielberg proved in [Spi] that some Cuntz-Krieger algebras can be obtained by the crossed product construction of the boundary action $(\partial\Gamma, \Gamma)$, where Γ is the free product of cyclic groups and $\partial\Gamma$ is the hyperbolic boundary as a hyperbolic group. This construction was generalized to amalgamated free product groups in [Oka]. Under this identification, it was shown that there is one-to-one correspondence between quasi-free KMS states and some class of random walks on Γ. Namely, by identifying $\partial\Gamma$ with the Poisson boundary, harmonic measures on $\partial\Gamma$ induce quasi-free KMS states. We will apply the main result to the harmonic measures and determine the types of them. It turns out that the resulting factors are either of type III_1 or of type III_λ $(0 < \lambda < 1)$, where λ is some algebraic number. Therefore, by combining these results, we can make type III factors from boundary actions and harmonic measures on the boundary, which generalizes J. Ramagge and G. Robertson's result in [RR].

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2. Preliminaries

2.1. Perron-Frobenius theorem. Let $A = [A(i,j)]_{i,j=1}^N$ be an $N \times N$ matrix with non-negative entries. We denote the (i,j)-entry of A^m by $A^m(i,j)$. A matrix A is *irreducible* if for every pair of indices i and j there is an $m > 0$ with $A^m(i,j) > 0$. For $1 \le i,j \le N$, put $E(i,j) = \{m \in \mathbf{N} \mid A^m(i,j) > 0\}$ and $p(i) = \text{g.c.d.}\{m \in \mathbf{N} \mid A^m(i,i) \neq 0\}$. Note that if A is irreducible, then $p \equiv p(i)$ for any i and we call it the *period* of A. An irreducible matrix A is said to be *periodic of period* p if $p > 1$ and *aperiodic* if $p = 1$. Set $I_k = \{i \mid 1 \le i \le N, E(i,1) = k - 1 \pmod p\}$ for $k = 1,\dots,p$. If A is periodic, then the index set $\{1,\dots,N\}$ can be decomposed into distinct subsets $I_1,\dots,I_p$ such that the matrix A translates from I_k into I_{k+1}, (I_p into I_1), and the restriction of A^p to I_k is aperiodic. If A is irreducible, the Perron-Frobenius theorem guarantees the existence of the strictly positive eigenvector with respect to the simple root α of the characteristic polynomial such that $\alpha \ge |\beta|$ for any other eigenvalue β. Moreover, the following theorem is known.

Theorem 2.1 ([Kit, Theorem 1.3.8]). *Let A be an irreducible matrix with non-negative entries and p the period of A. If $x = {}^T(x_1,\dots,x_N)$ and $y = (y_1,\dots,y_N)$ are the right and left Perron eigenvectors of the Perron eigenvalue α such that $\sum_{i=1}^N x_i y_i = p$, then*

$$\lim_{n\to\infty} A^{pn}(i,j)/\alpha^{pn} = x_i y_j,$$

for any $i,j = 1,\dots,N$.

2.2. Cuntz-Krieger algebras. Let A be an $N \times N$ 0-1 matrix without zero rows. Then the Cuntz-Krieger algebra $\mathcal{O}_A$ is the universal C^*-algebra generated by the family of partial isometries $S_1,\dots,S_N$ satisfying:

$$S_i^* S_i = \sum_{j=1}^N A(i,j) S_j S_j^*, \quad \text{and} \quad 1 = \sum_{i=1}^N S_i S_i^*.$$

For $i = 1,\dots,N$, let us denote the initial projection of S_i by Q_i and the range projection by P_i. We say that $\xi = (\xi_1,\dots,\xi_n) \in \prod_{i=1}^n \{1,\dots,N\}$ with $A(\xi_i,\xi_{i+1}) \neq 0$ is an *admissible word* and denote the set of all admissible words by $\mathcal{W}_A$. We define two maps s and r by $s(\xi) = \xi_1$ and $r(\xi) = \xi_n$. For $\xi = (\xi_1,\dots,\xi_n), \eta = (\eta_1,\dots,\eta_m) \in \mathcal{W}_A$ with $A(\xi_n,\eta_1) = 1$, we define the concatenation $\xi \cdot \eta$ in $\mathcal{W}_A$ by $(\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_m)$. Let us say that an admissible word $\xi = (\xi_1,\dots,\xi_n)$ is a *loop* if $A(\xi_n,\xi_1) = 1$. We say that a loop ξ is a *circle* if $\xi_k \neq \xi_l$ for any $1 \le k,l \le n$, $(k \neq l)$.

Let $\omega = (\omega_1,\dots,\omega_N) \in \mathbf{R}_+^N$. We define the action α^ω of $\mathbf{R}$ on $\mathcal{O}_A$ by $\alpha_t^\omega(S_i) = e^{\sqrt{-1}\omega_i t} S_i$ for $t \in \mathbf{R}$ and $i = 1,\dots,N$. If $\omega = (1,\dots,1)$, then α^ω is the gauge action. We define two word-length functions. For $\xi = (\xi_1,\dots,\xi_n) \in \mathcal{W}_A$, we denote the canonical one by $|\xi| = n$ and the other by $\omega_\xi = \omega_{\xi_1} + \cdots + \omega_{\xi_n}$. Let $\Omega_A = \{(a_k)_{i=k}^\infty \mid A(a_k,a_{k+1}) = 1\}$ be the set of all one-sided infinite admissible words. Note that there is the faithful conditional expectation Φ from $\mathcal{O}_A$ onto

$\overline{\text{span}}\{S_\xi S_\xi^* \mid \xi \in \mathcal{W}_A\} \simeq C(\Omega_A)$ (see [CK]). We assume that there is $\beta \in \mathbf{R}_+$ and $x_i > 0$ that satisfies:

$$x_i = \sum_{j=1}^{N} e^{-\beta\omega_i} A(i,j) x_j, \quad \text{and} \quad 1 = x_1 + \cdots + x_N.$$

We can define a probability measure ν on Ω_A by

$$\nu(\Omega_A(\xi_1, \ldots, \xi_{n-1}, \xi_n)) = e^{-\beta\omega_{\xi_1}} \cdots e^{-\beta\omega_{\xi_{n-1}}} x_{\xi_n},$$

where $\Omega_A(\xi_1, \ldots, \xi_n)$ is the cylinder set $\{(a_k)_{k=1}^\infty \in \Omega_A \mid a_1 = \xi_1, \ldots, a_n = \xi_n\}$. This probability measure induces a β-KMS state for α^ω on $\mathcal{O}_A$ by $\phi^\omega = \nu \circ \Phi$. Set $A_\omega(i,j) = e^{-\beta\omega_i} A(i,j)$. Note that the vector $x = {}^T(x_1, \ldots, x_N)$ is the right Perron eigenvector of the matrix A_ω with respect to the Perron eigenvalue 1. R. Exel and M. Laca, in fact, showed the following in [EL].

Theorem 2.2 ([EL, Theorem 18.5]). *If A is irreducible, then there exists the unique β-KMS state ϕ^ω of the Cuntz-Krieger algebra $\mathcal{O}_A$ for the action α^ω and the inverse temperature β is also unique.*

Throughout this paper, we assume that A is irreducible and not a permutation matrix. Let $(\pi_{\phi^\omega}, H_{\phi^\omega}, \xi_{\phi^\omega})$ be the GNS-triple of ϕ^ω The above theorem, in particular, says that the von Neumann algebra $M = \pi_{\phi^\omega}(\mathcal{O}_A)''$ becomes a factor.

2.3. AF-algebras. The following results are based on [SV, Theorem I.3.12.]. Consider an AF-algebra $B = \overline{\bigcup_{n\geq 0} B_n}$, where $\{B_n\}_{n=0}^\infty$ is an increasing family of finite C^*-subalgebras. We assume that $B_0 = \mathbf{C}1$. We define a maximal abelian subalgebra C of B as follows. Let $C_0 = B_0$ and C_{n+1} the C^*-subalgebra generated by C_n and D_{n+1}, where D_{n+1} is a masa of B_{n+1}, containing C_n. We define $C = \overline{\bigcup_{n\geq 0} C_n}$. One can check that C is a masa of B. There is a conditional expectation Ψ from B onto C, and there is a topological dynamical system (Ω, Γ) such that $C \simeq C(\Omega)$, $B = \overline{\text{span}}\{fu \mid f \in C(\Omega), u \in \Gamma\}$ and $\Gamma = \bigcup_{n\geq 0} \Gamma_n$, where Γ_n consists of all unitaries u in B_n with $uC_n u^* = C_n$. Let ν be a Γ-quasi-invariant probability measure on Ω. It induces a state $\psi = \nu \circ \Psi$ of B. Let $(\pi_\psi, H_\psi, \xi_\psi)$ be the GNS-triple of ψ. Then we obtain the following:

(1) $\pi_\psi(C)''$ is a masa in $\pi_\psi(B)''$.
(2) $\pi_\psi(C)'' \simeq L^\infty(\Omega, \nu)$.
(3) The conditional expectation Ψ can extend to $\pi_\psi(B)''$ whose image is $\pi_\psi(C)''$

3. Lemmata

We denote by $\mathcal{O}_A^{\alpha^\omega}$ the fixed-point algebra under α^ω. We first introduce an equivalence relation on the index set $I = \{1, \ldots, N\}$. We say that i is equivalent to j if there are $\xi, \eta \in \mathcal{W}_A$ such that $s(\xi) = i, s(\eta) = j, r(\xi) = r(\eta)$ and $\omega_\xi = \omega_\eta$. It is easy to check that this is an equivalence relation. We obtain the corresponding disjoint union $I = I_1^\omega \cup \cdots \cup I_{n_\omega}^\omega$. Note that if α^ω is the gauge action, then this decomposition coincides with the one with respect to the period of A. Set $P_{I_k^\omega} = \sum_{i\in I_k^\omega} P_i$. Our goal in this section is to prove the following lemma.

Lemma 3.1.

$$Z(\pi_{\phi^\omega}(\mathcal{O}_A^{\alpha^\omega})'') = \pi_{\phi^\omega}(\mathcal{O}_A^{\alpha^\omega})'' \cap \pi_{\phi^\omega}(\mathcal{O}_A^{\alpha^\omega})' = \bigoplus_{k=1}^{n_\omega} \mathbf{C}\pi_{\phi^\omega}(P_{I_k^\omega}).$$

We need some lemmata to show the above.

Lemma 3.2. *The fixed-point algebra $\mathcal{O}_A^{\alpha^\omega}$ is an AF-algebra.*

Proof. Set $F_t^i = \operatorname{span}\{S_\xi P_i S_\eta^* \mid \omega_\xi = \omega_\eta = t\}$ for $t \in \{\omega_\xi \mid \xi \in \mathcal{W}_A\}$. Since $\{S_\xi P_i S_\eta^*\}$ gives the matrix units, F_t^i is a simple finite-dimensional C^*-algebra. We can define finite-dimensional C^*-algebras F_n as follows:

$$\begin{aligned} F_{-1} &= \mathbf{C}1, \\ F_n &= \bigvee_{i\in I}\bigvee_{t\in K_n^i} F_t^i = \bigoplus_{i\in I}\bigoplus_{t\in K_n^i} F_t^i \quad \text{for } n \geq 0, \end{aligned}$$

where $\omega_{\min} = \min\{\omega_i \mid i \in I\}$ and

$$K_n^i = \{\omega_\xi \mid \xi \in \mathcal{W}_A, A(r(\xi), i) = 1, n\omega_{\min} - \omega_i < \omega_\xi \leq n\omega_{\min}\}.$$

Indeed, let $S_{\xi_1} P_i S_{\eta_1}^* \in F_t^i$ for $t \in K_n^i$ and $S_{\xi_2} P_j S_{\eta_2}^* \in F_s^j$ for $s \in K_n^j$. We assume that $S_{\xi_1} P_i S_{\eta_1}^* S_{\xi_2} P_j S_{\eta_2}^* \neq 0$. If $|\eta_1| = |\xi_2|$, then $\eta_1 = \xi_2$ and thus $s = t$ and $i = j$. We now suppose that $|\eta_1| \neq |\xi_2|$. Without loss of generality, we may assume that $|\eta_1| < |\xi_2|$. Since $P_i S_{\eta_1}^* S_{\xi_2} P_j \neq 0$, we have $P_i S_{\eta_1}^* S_{\xi_2} P_j = S_\xi P_j$ for some ξ with $\xi_2 = \eta_1 \cdot \xi$ and $s(\xi) = i$. Hence we obtain $\omega_i \leq \omega_\xi$. However,

$$\omega_{\xi_2} = \omega_{\eta_1} + \omega_\xi > n\omega_{\min} - \omega_i + \omega_i = n\omega_{\min}.$$

Thus $n\omega_{\min} < \omega_{\xi_2} \leq n\omega_{\min}$ and this is a contradiction.

We next show that F_n is a C^*-subalgebra of F_{n+1}. Let $S_\xi P_i S_\eta^* \in F_t^i$ with $t \in K_n^i$. If $(n+1)\omega_{\min} - \omega_i < t$, then we have $S_\xi P_i S_\eta^* \in F_{n+1}$. If $t \leq (n+1)\omega_{\min} - \omega_i$, then we have

$$S_\xi P_i S_\eta^* = \sum_{j\in I} A(i,j) S_\xi S_i P_j S_i^* S_\eta^* \in F_{n+1}.$$

We can define an AF-algbera $F = \overline{\bigcup_n F_n}$.

We claim that $F = \mathcal{O}_A^{\alpha^\omega}$ It is clear that $F \subseteq \mathcal{O}_A^{\alpha^\omega}$ To show the converse, we need the conditional expectation. If $\omega_i/\omega_j \in \mathbf{Q}$ for all $i, j \in I$, then we can define the faithful conditional expectation from $\mathcal{O}_A$ onto $\mathcal{O}_A^{\alpha^\omega}$ by the integration on $\mathbf{T}$. If not, we consider an action $\tilde{\alpha}$ of $\mathbf{T}^N$ such that $\tilde{\alpha}_z(S_i) = z_i S_i$ for $z \in (z_1, \ldots, z_N) \in \mathbf{T}^N$ Since there is the embedding of $\mathbf{R}$ into $\mathbf{T}^N$, $t \mapsto (e^{\sqrt{-1}\omega_1 t}, \ldots, e^{\sqrt{-1}\omega_N t})$, we can consider the closure of $\mathbf{R}$ in $\mathbf{T}^N$ via this embedding. Therefore the conditional expectation is given by the integration on the compact group $\overline{\mathbf{R}}$. One can easily check that the fixed-point algebra under $\tilde{\alpha}|_{\overline{\mathbf{R}}}$ coinsides with $\mathcal{O}_A^{\alpha^\omega}$ and thus we can show that $\mathcal{O}_A^{\alpha^\omega} = F$ by using this conditional expectation. □

We will need one more lemma. Let p be the period of the matrix A. We define partial isometries for $m \in \mathbf{N}$, $i \in I$ by

$$\theta_m^{(i)} = \sum_{\xi,\eta\in L_i(mp)} S_\xi S_\eta P_i S_\xi^* S_\eta^*,$$

where $L_i(n) = \{\xi \in \mathcal{W}_A \mid s(\xi) = i, A(r(\xi), i) = 1, |\xi| = n\}$ is the set of all loops of i with length n. Note that $\theta_m^{(i)}$ is self-adjoint. We define the tracial state by $\psi^\omega = \phi^\omega|_{\mathcal{O}_A^{\alpha^\omega}}$ on $\mathcal{O}_A^{\alpha^\omega}$, and use the same symbol ψ^ω for its normal extension to $\pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$ for simplicity.

Lemma 3.3. *Let $f \in \pi_{\psi^\omega}(C(\Omega_A))''$ and $a \in \pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$. Then for any $i \in I$,*

$$\lim_{m\to\infty} \psi^\omega(\theta_m^{(i)} f \theta_m^{(i)} a) = \psi^\omega(P_i f)\psi^\omega(P_i a) x_i y_i^2,$$

where $y = (y_1, \ldots, y_N)$ is the left Perron eigenvector of A_ω with $\sum_{i\in I} x_i y_i = p$.

Proof. Note that $C(\Omega_A) \simeq \overline{\mathrm{span}}\{S_\xi S_\xi^* \mid \xi \in \mathcal{W}_A\}$ is a masa in the AF-algebra $\mathcal{O}_A^{\alpha^\omega}$. We denote by Ψ the conditional expectation from $\pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$ onto $\pi_{\psi^\omega}(C(\Omega_A))'' \simeq L^\infty(\Omega_A, \nu)$. We first prove the lemma for $f \in C(\Omega_A)$ and $a \in \mathcal{O}_A^{\alpha^\omega}$. Remark that $\psi^\omega = \nu \circ \Psi$. We may assume that $a \in C(\Omega_A)$. Indeed, if the statement holds for $\Psi(a)$ instead of a, then since $\theta_m^{(i)} f \theta_m^{(i)} \in C(\Omega_A)$, we will have

$$\begin{aligned}
\lim_{m\to\infty} \psi^\omega(\theta_m^{(i)} f \theta_m^{(i)} a) &= \lim_{m\to\infty} \psi^\omega(\theta_m^{(i)} f \theta_m^{(i)} \Psi(a)) \\
&= \psi^\omega(P_i f)\psi^\omega(P_i \Psi(a)) x_i y_i^2 \\
&= \psi^\omega(P_i f)\psi^\omega(P_i a) x_i y_i^2.
\end{aligned}$$

It suffices to check the statement for $f = S_{\zeta_1} P_{j_1} S_{\zeta_1}^*$, $a = S_{\zeta_2} P_{j_2} S_{\zeta_2}^*$ with $|\zeta_1| = kp, |\zeta_2| = lp$ and $s(\zeta_1) = s(\zeta_2) = i$. In this case, for sufficiently large m we have

$$\theta_m^{(i)} f \theta_m^{(i)} a = \sum_{\xi', \eta'} S_{\zeta_2} S_{\xi'} S_{\zeta_1} S_{\eta'} P_i S_{\eta'}^* S_{\zeta_1}^* S_{\xi'}^* S_{\zeta_2}^*,$$

where ξ' and η' run over all admissible words from j_2, j_1 to i with $|\xi'| = (m-l)p, |\eta'| = (m-k)p$. Therefore

$$\begin{aligned}
\psi^\omega(\theta_m^{(i)} f \theta_m^{(i)} a) &= e^{-\beta\omega_{\zeta_2}} A_\omega^{(m-l)p}(j_2, i) e^{-\beta\omega_{\zeta_1}} A_\omega^{(m-k)p}(j_1, i) x_i \\
&\longrightarrow e^{-\beta\omega_{\zeta_2}} x_{j_2} y_i e^{-\beta\omega_{\zeta_1}} x_{j_1} y_i x_i \quad (m \to \infty) \\
&= \psi^\omega(f)\psi^\omega(a) x_i y_i^2.
\end{aligned}$$

Next let $f \in L^\infty(\Omega_A, \nu)$. We choose $g \in C(\Omega_A)$ with $\|(f-g)\xi_{\psi^\omega}\| < \varepsilon$. Then

$$\begin{aligned}
|\psi^\omega(\theta_m^{(i)} f \theta_m^{(i)} a) - \psi^\omega(P_i f)\psi^\omega(P_i a) x_i y_i^2| &\le |\psi^\omega(\theta_m^{(i)} (f-g) \theta_m^{(i)} a)| \\
&+ |\psi^\omega(\theta_m^{(i)} g \theta_m^{(i)} a) - \psi^\omega(P_i g)\psi^\omega(P_i a) x_i y_i^2| \\
&+ |\psi^\omega(P_i(f-g))\psi^\omega(P_i a) x_i y_i^2|,
\end{aligned}$$

and we get the following estimate of the first term:

$$\begin{aligned}
|\psi^\omega(\theta_m^{(i)} (f-g) \theta_m^{(i)} a)| &= |\psi^\omega(\theta_m^{(i)} a \theta_m^{(i)} (f-g))| \\
&\le \psi^\omega(\theta_m^{(i)} a \theta_m^{(i)} \theta_m^{(i)} a^* \theta_m^{(i)})^{1/2} \psi^\omega((f-g)^*(f-g))^{1/2} \\
&\le \|a\| \|(f-g)\xi_{\psi^\omega}\|,
\end{aligned}$$

because ψ^ω is tracial. In a similar way, we can show the statement for $a \in \pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$ □

We will use the following folklore among specialists, (e.g. see [Izu]).

Lemma 3.4 ([Izu, Lemma 4.1]). *Let B be a unital C^*-algebra, ϕ a state of B and $(\pi_\phi, H_\phi, \xi_\phi)$ the GNS-triple of ϕ. We assume that the cyclic vector ξ_ϕ is a separating vector for $\pi_\phi(B)''$ Let C be a unital C^*-sublagebra of B and ψ the restriction of ϕ to C. Then $(\pi_\phi|_C, H_\phi)$ is quasi-equivalent to the GNS-representation (π_ψ, H_ψ) of ψ.*

Now we have the necessary ingredients for the proof of Lemma 3.1.

Proof of Lemma 3.1. It is easy to show that $\pi_{\psi^\omega}(P_{I_k^\omega}) \in Z(\pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})'')$ for $k = 1, \ldots, n_\omega$. By Lemma 3.4, $\pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$ is isomorphic to $\pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$. It therefore suffices to show that $Z(\pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})'') = \bigoplus_{k=1}^{n_\omega} \mathbb{C}P_{I_k^\omega}$. Let $z \in Z(\pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})'')$ be a non-trivial projection. Since $L^\infty(\Omega_A)$ is a masa in $\pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$, we have $z \in L^\infty(\Omega_A)$. We can apply Lemma 3.3 to $f = P_i z$:

$$\lim_{m\to\infty} \psi^\omega(\theta_m^{(i)} z \theta_m^{(i)} a) = \psi^\omega(P_i z)\psi^\omega(P_i a) x_i y_i^2.$$

On the other hand, since z is centered, we get

$$\lim_{m\to\infty} \psi^\omega(\theta_m^{(i)} z \theta_m^{(i)} a) = \lim_{m\to\infty} \psi^\omega(\theta_m^{(i)} \theta_m^{(i)} z a) = \psi^\omega(P_i)\psi^\omega(P_i z a) x_i y_i^2.$$

Therefore

$$\psi^\omega(P_i)\psi^\omega(P_i z a) = \psi^\omega(P_i z)\psi^\omega(P_i a).$$

Since ψ^ω is faithful on $\pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$ and a is arbitrary, we get

$$P_i z = \frac{\psi^\omega(P_i z)}{\psi^\omega(P_i)} P_i.$$

Therefore we obtain $z = \sum_{i\in I} c_i P_i$ for $c_i \in \{0,1\}$. If $c_i \neq 0$ for $i \in I_k^\omega$, then there are admissible words ξ, η with $s(\xi) = i, s(\eta) = j, r(\xi) = r(\eta)$ and $\omega_\xi = \omega_\eta$ for $j \in I_k^\omega$, and z must commute with $S_\xi S_\eta^* \in \mathcal{O}_A^{\alpha^\omega}$ Hence we have $z = \sum_{k=1}^{n_\omega} c_k P_{I_k^\omega}$ for $c_k \in \{0,1\}$. □

4. Main theorem

We first review some notations in [Con]. Let $(M, \mathbb{R}, \sigma)$ be a W^*-dynamical system. For $f \in L^1(\mathbb{R})$, we define a σ-weakly continuous linear map on M by

$$\sigma_f(x) = \int f(t)\sigma_t(x)dt \quad \text{for } x \in M.$$

The *Arveson spectrum* of σ is defined by

$$\mathrm{Sp}(\sigma) = \bigcap \{Z(f) \mid f \in L^1(\mathbb{R}),\ \sigma_f = 0\},$$

where $Z(f) = \{r \in \hat{\mathbb{R}} \mid \hat{f}(r) = 0\}$ and $\hat{\mathbb{R}}$ is the dual group of $\mathbb{R}$. Then the *Connes spectrum* of σ is defined by

$$\Gamma(\sigma) = \bigcap_p \mathrm{Sp}(\sigma|_{pMp}),$$

where p runs over all non-zero projections in $Z(M^\sigma) = M^\sigma \cap (M^\sigma)'$ Note that $\Gamma(\sigma) \subseteq \mathrm{Sp}(\sigma|_{pMp})$ holds for any non-zero projection p in M^σ

For each $i \in I$, let G_i be the closed additive subgroup of $\mathbb{R}$ generated by $\beta\omega_\xi$ for all loops ξ with $s(\xi) = i$ and G the closed additive subgroup generated by $\beta\omega_\xi$ for all circles ξ.

Lemma 4.1. *For any $i \in I$, $G = G_i$.*

Proof. It is clear that $G \subseteq G_i$. Conversely, let ξ be a loop with $s(\xi) = i$. Then there are circles $\xi(1), \ldots, \xi(n)$ such that $\omega_\xi = \omega_{\xi(1)} + \cdots + \omega_{\xi(n)}$. Thus $G_i \subseteq G$. □

We will prove the following main theorem.

Theorem 4.2. (1) *If $\omega_\xi/\omega_\eta \in \mathbb{Q}$ for all circles ξ, η, then $M = \pi_{\psi^\omega}(\mathcal{O}_A)''$ is the AFD type III_λ factor for $\lambda = e^{-r}$, where $G = r\mathbb{Z}$ for some $r \in \mathbb{R}_+$.*

(2) *If $\omega_\xi/\omega_\eta \notin \mathbf{Q}$ for some circles ξ, η, then $M = \pi_{\phi^\omega}(\mathcal{O}_A)''$ is the AFD type III_1 factor.*

Proof. Since ϕ^ω is α^ω-invariant, α^ω can be extended to an action on M. We use the same symbol ϕ^ω for its normal extension. Let σ^{ϕ^ω} be the modular automorphism group for ϕ^ω , which satisfies $\sigma_t^{\phi^\omega} = \alpha^\omega_{-\beta t}$ for $t \in \mathbf{R}$. We first claim that $M^\sigma = \pi_{\phi^\omega}(\mathcal{O}_A^\sigma)''$ One can check that the conditional expectation from $\mathcal{O}_A$ onto $\mathcal{O}_A^{\alpha^\omega}$ in the proof of Lemma 3.2 can extend to the one on $\pi_{\phi^\omega}(\mathcal{O}_A)''$ Thus by the approximation arguments, we can obtain our claim.

By Lemma 3.1, we obtain $\Gamma(\sigma^{\phi^\omega}) = \bigcap_k \mathrm{Sp}(\sigma^{\phi^\omega}|_{P_{I_k^\omega} M P_{I_k^\omega}})$. Since $\mathrm{Sp}(\sigma^{\phi^\omega}|_{P_i M P_i}) \subseteq \mathrm{Sp}(\sigma^{\phi^\omega}|_{P_{I_k^\omega} M P_{I_k^\omega}})$ for $i \in I_k^\omega$, we have $\Gamma(\sigma^{\phi^\omega}) = \bigcap_{i \in I} \mathrm{Sp}(\sigma^{\phi^\omega}|_{P_i M P_i})$.

We now claim that $\mathrm{Sp}(\sigma^{\phi^\omega}|_{P_i M P_i}) = G_i$ for each $i \in I$. Let ξ, η be loops with $s(\xi) = s(\eta) = i$. If $f \in \mathrm{Ker}\sigma^{\phi_\omega}|_{P_i M P_i}$, then

$$0 = \sigma_f^{\phi_\omega}(P_i S_\xi S_\eta^* P_i) = \hat{f}(\beta(\omega_\xi - \omega_\eta))P_i S_\xi S_\eta^* P_i.$$

Since $P_i S_\xi S_\eta^* P_i \neq 0$, we have $\beta(\omega_\xi - \omega_\eta) \in \mathrm{Sp}(\sigma^{\phi^\omega}|_{P_i M P_i})$. Thus a group generated by $\beta\omega_\xi$ for all loops ξ with $s(\xi) = i$ is contained in $\mathrm{Sp}(\sigma^{\phi^\omega}|_{P_i M P_i})$. Since $\mathrm{Sp}(\sigma^{\phi^\omega}|_{P_i M P_i})$ is closed, $G_i \subseteq \mathrm{Sp}(\sigma^{\phi^\omega}|_{P_i M P_i})$ holds. Conversely let $r \in \mathbf{R} \setminus G_i$. Choose a function $f \in L^1(\mathbf{R})$ with $\hat{f}(r) \neq 0$ and $\hat{f}|_{G_i} = 0$. We have

$$\sigma_f^{\phi_\omega}|_{P_i M P_i}(P_i S_\xi S_\eta^* P_i) = \hat{f}(\beta(\omega_\xi - \omega_\eta))P_i S_\xi S_\eta^* P_i.$$

If $P_i S_\xi S_\eta^* P_i \neq 0$, then we have $s(\xi) = s(\eta) = i$ and $A(r(\xi), j) = A(r(\eta), j) = 1$ for some $j \in I$. Since A is irreducible, there is an admissible word ζ with $s(\zeta) = j$ and $A(r(\zeta), i) = 1$. Two admissible words $\xi \cdot \zeta$, $\eta \cdot \zeta$ are loops with $s(\xi \cdot \zeta) = s(\eta \cdot \zeta) = i$. Hence

$$\beta(\omega_\xi - \omega_\eta) = \beta(\omega_\xi + \omega_\zeta - \omega_\eta - \omega_\zeta) = \beta(\omega_{\xi\cdot\zeta} - \omega_{\eta\cdot\zeta}) \in G_i.$$

We therefore obtain $f \in \mathrm{Ker}\sigma^{\phi_\omega}|_{P_i M P_i}$. It follows from Lemma 4.1 that $\Gamma(\sigma^{\phi^\omega}) = G$. In the case (1), we have $G = r\mathbf{Z}$ for some $r \in \mathbf{R}_+$ and λ is determined by e^{-r} □

Example 4.3. Let $\mathbf{F}_n$ be the free group with the canonical generators $a_1, \ldots, a_n$ and $S = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$ the generating set. The corresponding Cayley graph is the homogeneous tree with degree $2n$. We define a compact space by

$$\Omega = \{(x_i)_{i=1}^\infty \mid x_i \neq x_{i+1}^{-1}\} \subseteq \prod_{i=1}^\infty S.$$

Note that Ω is compact and Γ acts on Ω by left multiplications. We remark that Ω coinsides with the hyperbolic boundary $\partial\mathbf{F}_n$ of $\mathbf{F}_n$. In [Spi], Spielberg showed the identification $\mathcal{O}_A \simeq C(\Omega) \rtimes \mathbf{F}_n$, where

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 1 \end{pmatrix} \quad (2n \times 2n\text{-matrix}).$$

We now apply Theorem 4.2 to $\mathcal{O}_A \simeq C(\Omega) \rtimes \mathbb{F}_n$. Note that the canonical masa $C(\Omega_A)$ of $\mathcal{O}_A$ coinsides with $C(\Omega)$. Let $\omega = (\omega_x)_{x\in S} \in \mathbb{R}_+^{2n}$ and ν the corresponding probability measure on Ω, which induces the KMS state for α^ω By Theorem 4.2, we have the following:

(1) *If $\omega_x/\omega_y \in \mathbb{Q}$ for all $x, y \in S$, then $L^\infty(\Omega,\nu) \rtimes \mathbb{F}_n$ is the AFD type* III_λ *factor for some* $0 < \lambda < 1$.

(2) *If $\omega_x/\omega_y \notin \mathbb{Q}$ for some $x, y \in S$, then $L^\infty(\Omega,\nu) \rtimes \mathbb{F}_n$ is the AFD type* III_1 *factor.*

Let μ be a probability measure on $\mathbb{F}_n$ with $\mathrm{supp}\mu = S$. By [Oka], the random walk with law μ induces the harmonic measure ν on Ω such that the modular automorphism group of the state $\nu\circ\Phi$ has the form α^ω for some $\omega = (\omega_x)_{x\in S} \in \mathbb{R}_+^{2n}$ Therefore the above result also means that we determine the type of harmonic measures on Ω (cf. [RR]).

Remark 4.4. We can also prove the same results for $\mathcal{O}_\Gamma$ in [Oka] in the same way, where Γ is an amalgamated free product group $*_H G_i$. Here, we will give a sketch of the proof.

Let I be a finite index set and G_i a group containing a copy of a group H as a subgroup for $i \in I$. We assume that G_i is finite for simplicity. $\mathcal{O}_\Gamma$ is the universal C^*-algebra generated by partial isometries S_g, $g \in \bigcup_{i\in I} G_i \setminus H$ and unitaries U_h, $h \in H$ satisfying certain conditions (see [Oka]). We use some symbols in [Oka]. For $\omega = (\omega_i)_{i\in I} \in \mathbb{R}_+^{|I|}$, we consider the action α^ω of $\mathbb{R}$ given by

$$\begin{array}{rcll} \alpha_t^\omega(S_g) &=& e^{\sqrt{-1}\omega_i t} S_g & \text{for} \quad i \in I,\ g \in G_i \setminus H, \\ \alpha_t^\omega(U_h) &=& U_h & \text{for} \quad h \in H, \end{array}$$

where $|I|$ is the cardinality of I. Remark that there is an identification $\mathcal{O}_\Gamma \simeq C(\Omega) \rtimes \Gamma$ for some compact space Ω ([Oka, Theorem 5.3]). Let Φ be the canonical conditional expectation from $C(\Omega) \rtimes \Gamma$ onto $C(\Omega)$. It was shown that there is the unique β-KMS state $\phi = \nu \circ \Phi$ for α^ω, where ν is the corresponding probability measure on Ω. However the difference from the above example is that $C(\Omega)$ may be not a masa of the fixed-point algebra under α^ω. Therefore we need some arguments to obtain the similar result for $\mathcal{O}_\Gamma$. Choose a masa C, containing $C(\Omega)$. We assume that $\Gamma = *_H G_i$ satisfies the following condition:

For any $i \in I$, there is an element $\gamma_i = g_1 \cdots g_n \in \Gamma$ such that $h\gamma_i H \neq \gamma_i H$ for any $(e \neq) h \in H$, where $g_k \in G_{i_k} \setminus H$ with $i = i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n$.

We remark that the above assumption holds if $\Gamma = *_H G_i$ satisfies the condition of [Oka, Corollary 6.4]. Fix γ_i satisfying the above. Let ψ be the restriction of ϕ on the fixed-point algebra under $\mathcal{O}_\Gamma^{\alpha^\omega}$ For $g \in G_i \setminus H$, we set

$$\theta_m^{(g)} = \sum_{\xi,\eta} S_\xi S_{\gamma_i} S_\eta P_g S_\xi^* S_{\gamma_i}^* S_\eta^*,$$

where ξ, η run over all words from g to an element, which is not in G_i, with length m if $|I| > 2$ and length $2m$ if $|I| = 2$. Let π_ψ be the GNS-representation of ψ. Then we will get the similar result of Lemma 3.3.

Lemma 4.5. *For $f \in \pi_\psi(C)''$ and $a \in \pi_\psi(\mathcal{O}_\Gamma^{\alpha^\omega})''$, we have*

$$\lim_{m\to\infty} \psi(\theta_m^{(g)} f \theta_m^{(g)} a) = \psi(P_g f)\psi(P_g a) x_g y_g^2 z_{\gamma_i},$$

where x_g, y_g, z_{γ_i} are some constants.

Using this lemma, we can prove the following similarly.

Proposition 4.6.

$$Z(\pi_\phi(\mathcal{O}_\Gamma^{\alpha^\omega})'') = \begin{cases} \bigoplus_{i\in I} \mathbb{C}P_i & \text{if } |I| = 2, \\ \mathbb{C}1 & \text{if } |I| > 2. \end{cases}$$

Hence we can compute the Connes spectrum of the modular automorphism group in the similar way. This gives a generalization on [RR].

Corollary 4.7. *Let $\mathcal{O}_\Gamma, \omega, \phi, \nu$ be the above and π_ϕ the GNS-representation of ϕ. Then*

(1) *If $\omega_i/\omega_j \in \mathbb{Q}$ for any $i, j \in I$, then $\pi_\phi(\mathcal{O}_\Gamma)'' \simeq L^\infty(\Omega, \nu) \rtimes \Gamma$ is the AFD type III_λ factor for some $0 < \lambda < 1$.*

(2) *If $\omega_i/\omega_j \notin \mathbb{Q}$ for some $i, j \in I$, then $\pi_\phi(\mathcal{O}_\Gamma)'' \simeq L^\infty(\Omega, \nu) \rtimes \Gamma$ is the AFD type III_1 factor.*

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