On a family of algebraic vector bundles.

By

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Introduction. In the theory of algebraic vector bundles, it seems to the writer that it is very important to have a nice answer to the problem to "construct a lot of vector bundles on high dimensional algebraic variety." It is, of course, desirable that the structure of a vector bundle is easily known from its construction. A main purpose of this paper is to look for an answer to the above problem.

As for our problem, two answers are known:

(1) Schwarzenberger - Hironaka - Kleiman ( Empresa ).

For a vector bundle $E$ on a smooth quasi-projective variety $X$ over an infinite field there is a monoidal transformation $f : X' \to X$ with smooth center such that $f^*(E)$ contains a sublinebundle. This was proved by Schwarzenberger in the case where $X$ is a surface, by Hironaka in the characteristic 0 case and by Kleiman in the general case. The above fact implies that every vector bundle on a smooth
quasi-projective variety is a successive extension of line bundles

if one performs monoidal transformations on the base variety. Thus

every vector bundle on a smooth quasi-projective variety is obtained by

"extension + descent ". But the descent problem is very difficult.

In fact the answer to the descent problem is known only in the case

where the base space is a surface (Schwarzenberger [8]).

Schwarzenberger's answer was very powerful in his theory of almost
decomposable (i.e. non-simple) vector bundles. But he needed

another method in order to construct simple vector bundles on \( \mathbb{P}^2 \).

(2) Schwarzenberger - Oda ([7], [6]). If \( f : X' \to X \) is

a flat and finite morphism, then \( f^*(L) \) is a vector bundle (locally

free sheaf) on \( X \) for any linebundle \( L \) on \( X' \). Using this fact,

Schwarzenberger studied simple vector bundles on some algebraic

surfaces([19]). T. Oda studied \( f^*(L) \) in the case where \( f \) is

an isogeny of abelian varieties. This method faces the following problem:

What morphism and linebundle does a given vector bundle come from?

This is also difficult.
In general and in these treatment too, "base change" gives rise to some difficulties. Our starting point is to find a method to construct vector bundles without "base change". There is known a nice model, that is, the theory of elementary transformation of ruled surfaces, and we shall generalize elementary transformations of ruled surfaces to those of $\mathbb{P}^N$-bundles on a locally noetherian scheme. An interesting result is that every $\mathbb{P}^N$-bundle on a non-singular quasi-projective variety $S$ over an algebraically closed field $k$ with \( \dim S \leq 3 \) is obtained by an elementary transformation from the direct product $\mathbb{P}^N_k \times S$. This result leads us to the concept of regular vector bundles. Some big families of vector bundles are constructed in Chapter II. The family of regular vector bundles contains a large subfamily of simple vector bundles (see Chapter III, §1). This fact implies that if $S$ is a non-singular projective variety over $k$ and if $S \not\cong \mathbb{P}_k^1$, then there is a simple vector bundle on $S$ (Corollary 3.4.1).

In the rank 2 case, we have a very clear criterion that a regular vector
bundle is simple (Theorem 3.10). Using this criterion we can cover almost all results of Schwarzenberger without the theory of moduli of non-simple vector bundles and we get further result.

Notation and convention. Throughout this paper \( k \) denotes an algebraically closed field and all varieties are reduced and irreducible algebraic schemes over \( k \). We use the terms "vector bundles" and "locally free sheaves" interchangeably. For a monoidal transformation \( f : X' \rightarrow X \) with center \( Y \) and a subscheme \( Z \) of \( X \), \( f^{-1}(Z) \) denotes the total transform of \( Z \) and \( f^{-1}[Z] \) denotes the proper (strict) transform of \( Z \). If \( X \) and \( Y \) are smooth and if \( D = \sum n_i D_i \) (\( D_i \) irreducible) is a divisor on \( X' \), then \( f[D] \) denotes \( \sum n_i f'(D_i) \), where \( f' \) is the restriction of \( f \) to \( X' - f^{-1}(Y) \).

In the case where a birational map \( g : X_1 \rightarrow X_2 \) is a composition \( f_2 \circ f_1 \) of monoidal transformations with non-singular centers \( f_1 : X' \rightarrow X_1, f_2 : X' \rightarrow X_2 \), for a divisor \( D \) on \( X_1 \), \( g(D) \) denotes \( f_2[f_1^{-1}(D)] \) and \( g[D] \) denotes \( f_2[f_1^{-1}[D]] \). For a Cartier
divisor $D$ on a scheme $X$, $\mathcal{O}_X(D)$ denotes the invertible sheaf defined by $D$. If $L$ is a linebundle on a non-singular projective variety $X$, then $|L|$ denotes the complete linear system $|D|$ for a divisor $D$ on $X$ with $\mathcal{O}_X(D) \cong L$. For an algebraic $k$-scheme $X$, $X(k)$ denotes the set of $k$-rational points of $X$. If $E$ is a locally free $\mathcal{O}_S$-module (a vector bundle on $S$), then $P(E)$ denotes $\text{Proj}(\mathcal{S}_{\mathcal{O}_S}(E))$, where $\mathcal{S}_{\mathcal{O}_S}(E)$ is the $\mathcal{O}_S$-symmetric algebra of $E$.

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Chapter I. Elementary transformations of $\mathbb{P}^1$-bundles.

§ 1. Definition of elementary transformations.

Let $S$ be a locally noetherian scheme and let $\pi : X \to S$ be the projective bundle $\mathbb{P}(E)$ associated with a vector bundle $E$ of rank $N + 1$ ($N \geq 1$). Let $T$ and $Y$ be closed subschemes of $S$ and $X$ respectively, satisfying the following condition:

$(E_n^0)$ The ideal $I_T$ which defines $T$ is locally principal whose generator is non-zero divisor in every local ring of $S$, that is, $I_T$ is a Cartier divisor on $S$. $Y$ is a closed subscheme of $X_T$ and $\pi_T|_Y : Y \to T$ induces a $\mathbb{P}^n$-bundle on $T$ ($0 \leq n \leq N - 1$) such that $(\pi_T|_Y)^{-1}(t)$ is a linear subspace of $\pi_T^{-1}(t)$ for any $t \in T$. Roughly speaking $\pi_T|_Y : Y \to T$ is a subbundle of $\pi_T : X_T \to T$.

Now consider the monoidal transformation $f : \tilde{X} \to X$ with center $Y$ and put $f^{-1}[X_T] = \overline{X}_T$, $f^{-1}(Y) = E_Y$. In this situation we have the following theorem, whose proof will be given in the
Theorem 1.1. There exist a $P^N$-bundle $\pi' : X' \rightarrow S$ which is the projective bundle $P(E')$ associated with a vector bundle $E'$ and an $S$-morphism $g : \tilde{X} \rightarrow X'$ such that the closed subscheme $Y'$ of $X'$ defined by the ideal $g(I_{X_T}^T)$ with the defining ideal $I_{X_T}$ for $X_T$ in $X$ and $T$ satisfies the condition $(E_{N-n-1})$, and that $g^*(L) \cong f^*(O_X(1)) \otimes O_X(-E_Y)$ for some tautological line bundle $L$ on $X'$ and the tautological line bundle $O_X(1)$ on $X$ of $E^2$. $g$ is the monoidal transformation with center $Y'$. Moreover, such $(X', g)$ is unique, that is, if there exists another $(X'', g')$ satisfying the above conditions, then there is a unique bundle isomorphism $h : X' \rightarrow X''$ with $h \circ g = g'$.

The above theorem enables us to generalize elementary transformations of ruled surfaces to those of $P^N$-bundles. Namely:

Definition. Under the above notation the birational map $g \circ f^{-1}$ is called the elementary transformation of $X$ with center $Y$ and we denote it by $\text{elm}_Y^N$; we denote $X'$ by $\text{elm}_Y^n(X)$.

Corollary 1.1.1. $\text{elm}_Y^{N-n-1}(\text{elm}_Y^n(X)) = X$. 
We note that our treatment can be applied to $\mathbb{P}^n$-bundles if $S$ is factorial, that is, every local ring of $S$ is a unique factorization domain, because of the following fact:

Lemma 1.2. (A. Grothendieck [4]) If $\pi : X \to S$ is a $\mathbb{P}^n$-bundle (in Zariski topology) on a factorial scheme $S$, then there is a vector bundle $E$ on $S$ such that $\mathcal{P}(E) \cong X$.

Proof. Since $S$ is a direct sum of irreducible subschemes, we may assume that $S$ is irreducible. The exact sequence of group schemes on $S$

$$0 \to \mathbb{G}_m, S \to GL(n+1, S) \to PGL(N, S) \to 0$$

provides the exact sequence of cohomologies

$$H^0(S, GL(n+1, S)) \to H^0(S, PGL(N, S)) \to H^2(S, \mathcal{O}_S^*)$$

Thus we have only to prove $H^2(S, \mathcal{O}_S^*) = 0$. In order to see this, consider the exact sequence of sheaves

$$0 \to \mathcal{O}_S^* \to \mathcal{K}_S^* \to \mathcal{D}_S \to 0,$$
where $K_S^*$ is the sheaf of non-zero rational functions of $S$ and $D_S$ is the sheaf of Cartier divisors on $S$. Since every local ring of $S$ is a U.F.D., $D_S$ is isomorphic to the sheaf of Weil divisors on $S$. Thus $D_S$ is a flabby sheaf (because every Weil divisor on an open set is extensible to that on the whole space).

On the other hand, $K_S^*$ is also flabby because $K_S^*$ is a constant sheaf. Therefore the above sequence can be regarded as a part of a flabby resolution of $O_S^*$, whence $H^2(S, O_S^*) = 0$. q. e. d.

Sheaf theoretic interpretation of elementary transformations is stated as follows,

**Theorem 1.3.** Let $E$ be a locally free $O_S$-module of rank $N+1$ and let $T$, $Y$ be closed subschemes of $S$, $X = P(E)$ satisfying the condition $(E_n^*)$.

1. Denote by $I_Y$ the ideal defining $Y$ and by $O_X(1)$ a tautological line-bundle on $X$, then $E' = \mathbb{P}_X(I_Y \otimes O_X(1))$ is a locally free $O_S$-module, $P(E') \cong \text{elm}^n_Y(X)$ and $R^1\mathbb{P}_X(I_Y \otimes O_X(1))$
\[ i > 0 \text{, where } \pi : X \to S \text{ is the natural projection.} \]

(ii\(^3\)) Since \( (\pi|_\gamma)_{\ast}(\mathcal{O}_X \otimes \mathcal{O}_X(1)) \) is a locally free \( O_T \)-module of rank \( n+1 \), (i) can be said in other words; If \( F \) is a quotient bundle of \( E_T = E \otimes O_S O_T \) of rank \( n+1 \), then \( \ker \varphi = E' \) is a locally free \( O_S \)-module of rank \( n+1 \), where \( \varphi : E \to E_T \to F \) is the natural homomorphism. And we have the following exact commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \to & F' & \to & E_T & \to & F & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \downarrow & & \downarrow \\
0 & \to & E' & \to & E & \to & F & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \downarrow & & \downarrow \\
& & E \otimes I_T & = & E \otimes I_T & & & & \\
& & \uparrow & & \uparrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & & &
\end{array}
\]

Moreover, the locally free \( O_T \)-module \( F' \) of rank \( n+1 \) defines closed subscheme \( Y' \) of \( P(E') \) in Theorem 1.1 and the step obtaining \( E', F' \) corresponds to the inverse of \( \text{elm}_Y^{n} \) (see Corollary 1.1.1 and note that \( P(E) = P(E \otimes I_T) \)).
§ 2. Proof of Theorem 1.1 and Theorem 1.3.

In this section the notation of the preceding section is preserved.

The following is a key lemma.

Lemma 1.4. Assume that \( S = \text{Spec}(A) \), \( X = \text{Proj}(A \left[ t_0, \ldots, t_N \right]) \) and that the defining ideals \( I_T, I_Y \) for \( T \) and \( Y \) in \( A \) and
\[
A \left[ t_0, \ldots, t_N \right]
\]
respectively are generated by \( t \in A \) and \( t \),
\[
\gamma_{n+1}, \ldots, \gamma_N,
\]
respectively. Then \( \text{elm}_Y^n \) exists and \( \text{elm}_Y^n(x) = \text{Proj}(A \left[ \gamma_0, \ldots, \gamma_N \right]) \), where \( \gamma_i = \gamma_i (0 \leq i \leq n) \), \( \gamma_j = t \gamma_j \)
\((n+1 \leq j \leq N)\).

Proof. Put \( \gamma_0^0 = \gamma_0 \), \( \gamma_0^0 = \gamma_0 \) \((0 \leq \alpha, \beta \leq N)\) and put \( X' = \text{Proj}(A \left[ \gamma_0, \ldots, \gamma_N \right]) \). Let \( f : X' \to X \) be the monoidal transformation of \( X \) with center \( Y \), then
\[
X = \bigcup_{\alpha=0}^{N} X_{\alpha} \cup \bigcup_{0 \leq \alpha \leq N} X_{\alpha} \cup \bigcup_{0 \leq \alpha \leq N} X_{\alpha}
\]
where \( X_{\alpha} = \text{Spec}(A \left[ z_{\alpha}, \ldots, z_{\alpha} \right], z_{\alpha}/t, \ldots, z_{\alpha}/t) \) = \( \text{Spec}(A \left[ z_{\alpha}, \ldots, z_{\alpha}, z_{\alpha}/t, \ldots, z_{\alpha}/t \right]) \).
Moreover, we have

\[
U_\alpha \cap U_{\alpha'} = U_\alpha^\mu(\frac{3y}{x}, \frac{3y'}{x}) = U_{\alpha'}(\frac{3y}{x'}, \frac{3y'}{x'})
\]

\[
U_\alpha^\mu \cap U_{\alpha'}^\mu = U_\alpha^\mu(\frac{3y}{x}, \frac{3y'}{x'}) = U_{\alpha'}^\mu(\frac{3y}{x'}, \frac{3y'}{x'})
\]

\[
U_{\alpha'}^\mu \cap U_{\alpha'} = U_{\alpha'}(\frac{3y}{x}, \frac{3y'}{x'}) = U_{\alpha'}(\frac{3y}{x'}, \frac{3y'}{x'})
\]

\[
U_{\alpha}^\mu \cap U_{\alpha'}^\mu = U_{\alpha'}^\mu(\frac{3y}{x'}, \frac{3y'}{x'}) = U_{\alpha'}^\mu(\frac{3y}{x}, \frac{3y'}{x'})
\]

On the other hand, for the monoidal transformation \(g : X' \rightarrow X\)

with center \(Y' = \text{Proj}(A[\gamma_1, \ldots, \gamma_n]/(t, \gamma_1, \ldots, \gamma_n))\) we have an affine open covering \(\tilde{X}' = \left( \bigcup_{\gamma_n} V_{\gamma'} \right) \cup \left( \bigcup_{\alpha=1}^N U_{\alpha}^* \right) \cup \left( \bigcup_{\alpha=1}^N V_{\alpha}^* \right)\)

where

\[
V_{\gamma'} = \text{Spec}(A[\frac{3y}{x}, \ldots, \frac{3y}{x}, \frac{3y}{x}, \ldots, \frac{3y}{x}]) = \text{Spec}(A[\frac{3y}{x}, \ldots, \frac{3y}{x}, \frac{3y}{x}, \ldots, \frac{3y}{x}]) = U_{\gamma'}
\]
\[ V_\phi' = \text{Spec}(A \left[ \frac{\mathfrak{m}}{\mathfrak{m}^2}, \ldots, \frac{\mathfrak{m}^{n_1}}{\mathfrak{m}^{n_2}}, \frac{t}{\mathfrak{m}^0}, \frac{\mathfrak{p}}{\mathfrak{p}^0}, \frac{\mathfrak{s}}{\mathfrak{s}^0}, \ldots, \frac{\mathfrak{s}_n^{n_1}}{\mathfrak{s}_n^{n_2}} \right]) \]

\[ = \text{Spec}(A \left[ \frac{\mathfrak{m}^{n_1}}{\mathfrak{m}^{n_2}}, \ldots, \frac{\mathfrak{m}^{n_1}}{\mathfrak{m}^{n_2}}, \frac{t}{\mathfrak{m}^0}, \frac{\mathfrak{p}}{\mathfrak{p}^0}, \frac{\mathfrak{s}}{\mathfrak{s}^0}, \ldots, \frac{\mathfrak{s}_n^{n_1}}{\mathfrak{s}_n^{n_2}} \right]) \]

\[ = V_\theta \cdot V_\phi. \]

\[ V'_x' = \text{Spec}(A \left[ \frac{\mathfrak{m}}{\mathfrak{m}^2}, \ldots, \frac{\mathfrak{m}^{n_1}}{\mathfrak{m}^{n_2}} \right]) = \text{Spec}(A \left[ \frac{\mathfrak{m}}{\mathfrak{m}^2}, \ldots, \frac{\mathfrak{m}^{n_1}}{\mathfrak{m}^{n_2}} \right]) = V_\theta \cdot V_\phi. \]

Furthermore, we have

\[ V'_y \cap V'_y' = V'_y(\frac{\mathfrak{m}}{\mathfrak{m}^2}) = V_\theta(\frac{\mathfrak{m}}{\mathfrak{m}^2}) = V_\theta \cap V_\theta, \]

\[ V'_\phi \cap V_\phi' = V'_\phi(\frac{\mathfrak{m}}{\mathfrak{m}^2} / t \cdot \frac{\mathfrak{m}^{n_1}}{\mathfrak{m}^{n_2}}) = V_\theta(\frac{\mathfrak{m}}{\mathfrak{m}^2} \cdot \frac{t}{\mathfrak{m}^0}) = V_\theta \cap V_\theta. \]

\[ V'_\phi \cap V'_\phi' = V'_\phi(\frac{\mathfrak{m}}{\mathfrak{m}^2} \cdot \frac{\mathfrak{m}^{n_1}}{\mathfrak{m}^{n_2}}) = V_\theta(\frac{\mathfrak{m}}{\mathfrak{m}^2} \cdot \frac{\mathfrak{m}^{n_1}}{\mathfrak{m}^{n_2}}) = V_\theta \cap V_\theta. \]

\[ V'_\phi \cap V_\phi' = V'_\phi(\frac{\mathfrak{m}}{\mathfrak{m}^2} \cdot \frac{\mathfrak{m}^{n_1}}{\mathfrak{m}^{n_2}}) = V_\theta(\frac{\mathfrak{m}}{\mathfrak{m}^2} \cdot \frac{\mathfrak{m}^{n_1}}{\mathfrak{m}^{n_2}}) = V_\theta \cap V_\theta. \]

\[ V'_\phi \cap V'_\phi' = V'_\phi(\frac{\mathfrak{m}}{\mathfrak{m}^2} \cdot \frac{\mathfrak{m}^{n_1}}{\mathfrak{m}^{n_2}}) = V_\theta(\frac{\mathfrak{m}}{\mathfrak{m}^2} \cdot \frac{\mathfrak{m}^{n_1}}{\mathfrak{m}^{n_2}}) = V_\theta \cap V_\theta. \]

Thus we obtain \( \widetilde{X} = \widetilde{X}' \). It is easy to see that \( g^*(O_X) = O_{X'} \).

(see Lemma 1.5) Now let us prove \( g^*(I_{X'}) = I_{X'} \). In order to show this let us consider the affine covering \( X' = \bigcup_{i=0}^N W_i, W_i = \text{Spec}(A \left[ \frac{\mathfrak{m}}{\mathfrak{m}^2}, \ldots, \frac{\mathfrak{m}^{n_1}}{\mathfrak{m}^{n_2}} \right]) \) and put \( \widetilde{W}_i = g^{-1}(W_i) \). Then we have

\[ \widetilde{W}_i = V'_\alpha = V_\alpha \quad (0 \leq \alpha \leq n) \]
Since \( U_\alpha \cap \widetilde{X}_T = \{ t/t = 0 \} = \phi \), we know \( U_\alpha \cap g(\widetilde{X}_T) = \phi \), \( 0 \leq \alpha \leq n \).

Furthermore since the ideal of \( U_\alpha \cap \widetilde{X}_T \) (or, \( U_\alpha \cap \widetilde{X}_T \)) is generated by \( t \) (or, \( t/\gamma \), resp.). \( H^0(\widetilde{W}_T, \mathcal{O}_{\widetilde{X}_T}) \) is generated by \( t \).

\[ \frac{\chi}{\gamma}, \ldots, \frac{\chi}{\gamma} \text{ as } H^0(\widetilde{W}_T, \mathcal{O}_X) \text{-modules} \text{, whence } g_*(\mathcal{O}_{\widetilde{X}_T}) = \mathcal{O}_Y. \]

Finally we must show that for a tautological linebundle \( L \) on \( X \) there is a tautological linebundle \( L' \) on \( X' \) with \( g^*(L') \cong f^*(L) \otimes \mathcal{O}_X(-E_Y) \).

Assume that there are tautological linebundles \( L_1 \) on \( X \) and \( L'_1 \) on \( X' \) with \( g^*(L'_1) \cong f^*(L_1) \otimes \mathcal{O}_X(-E_Y) \), then \( L \cong L_1 \otimes \mathcal{O}_X(-E_Y) \) for some linebundle \( M \) on \( S \) and therefore \( g^*(L'_1 \otimes \mathcal{O}_X(M)) \cong g^*(L_1) \otimes g^*(\mathcal{O}_X(M)) \cong f^*(L_1) \otimes f^*(\mathcal{O}_X(M)) \cong f^*(L) \otimes \mathcal{O}_X(-E_Y) \), which implies that \( L'_1 \otimes \mathcal{O}_X(M) = L' \) is a desired linebundle.

Thus we may assume that \( L \) is an invertible sheaf with \( 1/\gamma^N \) as a generator in \( \mathcal{W}_X = \text{Spec}(A \left( \frac{\chi_0}{\gamma}, \ldots, \frac{\chi_N}{\gamma} \right)) \). Then a generator of \( f^*(L) \otimes \mathcal{O}_X(-E_Y) \) in \( \mathcal{W}_X = V^\alpha_n = U_\alpha (0 \leq \alpha \leq n) \) is \( t/\gamma^N = 1/\gamma^N \),

the one in \( V^\alpha_\gamma = U^\alpha_\gamma (0 \leq \alpha \leq n, \gamma \leq N) \) is \( 1/\gamma^N \) and the one in \( V^\alpha_\gamma = U^\alpha_\gamma (0 \leq \alpha \leq n, \gamma \leq N) \) is \( \frac{\chi}{\gamma} \gamma^N = 1/\gamma^N \), whence
the one in \( \mathcal{W}_1' = g^{-1}(\mathcal{W}_1) \) is \( 1/\mathcal{J}_1^N \). Thus if \( L' \) is an invertible
sheaf on \( X' \) whose generator in \( \mathcal{W}_1' \) is \( 1/\mathcal{J}_1^N \), then \( g^*(L') = f^*(L) \in \mathcal{O}(-\mathcal{E}_Y) \). It is clear that \( L' \) is a tautological linebundle.

q. e. d.

As a corollary to the above proof, we have

Lemma 1.5. If \( \mathcal{E}_Y \) is the Cartier divisor \( f^*(\mathcal{I}_Y) \), then

\[
f_* (\mathcal{O}_X(-r\mathcal{E}_Y)) = \mathcal{I}_Y, \quad f_* (\mathcal{O}_X(r\mathcal{E}_Y)) = \mathcal{O}_X \quad \text{for any} \quad r \geq 0.
\]

Proof. We have only to prove \( H^0(f^{-1}(U_{\alpha}), \mathcal{O}_{\mathcal{X}}(-r\mathcal{E}_Y)) = (t, \mathcal{J}_\alpha^{n+1}, \ldots, \mathcal{J}_\alpha^N) \) for \( U_{\alpha} = \text{Spec}(A[\mathcal{J}_\alpha^0, \ldots, \mathcal{J}_\alpha^N]) \), \( 0 \leq \alpha \leq n \) under the same
situation as in Lemma 1.4. If \( F \in \mathbb{Q}(\mathbb{A})(\mathcal{J}_\alpha^0, \ldots, \mathcal{J}_\alpha^N) \) is contained
in \( H^0(f^{-1}(U_{\alpha}), \mathcal{O}_{\mathcal{X}}(r\mathcal{E}_Y)) \), then \( t^\alpha F \in A[\mathcal{J}_\alpha^0, \ldots, \mathcal{J}_\alpha^N, \mathcal{J}_\alpha^{n+1}/t, \ldots, \mathcal{J}_\alpha^N/t], \ (\mathcal{J}_\alpha^0)^n F \in A[\mathcal{J}_\alpha^0, \ldots, \mathcal{J}_\alpha^N, t/\mathcal{J}_\alpha^n, \mathcal{J}_\alpha^{n+1}/t, \ldots, \mathcal{J}_\alpha^N/t], \)
for \( n+1 \leq \beta \leq N \). Thus \( F \in A[\mathcal{J}_\alpha^0, \ldots, \mathcal{J}_\alpha^N] \cap A[\mathcal{J}_\alpha^0, \ldots, \mathcal{J}_\alpha^N] \frac{1}{\mathcal{J}_\alpha^\beta} \)
for all \( \beta \leq N \). Conversely it is clear that \( A[\mathcal{J}_\alpha^0, \ldots, \mathcal{J}_\alpha^N] \)
\( \subseteq H^0(f^{-1}(U_{\alpha}), \mathcal{O}_{\mathcal{X}}(r\mathcal{E}_Y)) \). Since \( \mathcal{O}_{\mathcal{X}}(-r\mathcal{E}_Y) \) is a ideal of \( \mathcal{O}_{\mathcal{X}} \), and
since \( f_* (\mathcal{O}_X) = \mathcal{O}_X \) by the above proof, \( f_* (\mathcal{O}_{\mathcal{X}}(-r\mathcal{E}_Y)) \subseteq \mathcal{O}_{\mathcal{X}} \).
A \left[ \frac{3^0_\alpha}{3^0_\alpha}, \ldots, \frac{3^N_\alpha}{3^N_\alpha} \right] \text{ is contained in } H^0(f^{-1}(U_\alpha), O_X(-r E)) \iff \text{if and only if }

\frac{F/\alpha}{t/\alpha} \in A \left[ \frac{3^0_\alpha}{3^0_\alpha}, \ldots, \frac{3^{n+1}_\alpha}{3^{n+1}_\alpha}, \ldots, \frac{3^N_\alpha}{3^N_\alpha} \right], \quad r/(-\text{r } E) \in A \left[ \frac{3^0_\alpha}{3^0_\alpha}, \ldots, \frac{3^N_\alpha}{3^N_\alpha} \right] \quad (n+1 \leq \beta \leq N). \text{ Thus we obtain }

H^0(f^{-1}(U_\alpha), O_X(-r E)) = (t, \frac{3^{n+1}_\alpha}{3^{n+1}_\alpha}, \ldots, \frac{3^N_\alpha}{3^N_\alpha})^\alpha A \left[ \frac{3^0_\alpha}{3^0_\alpha}, \ldots, \frac{3^N_\alpha}{3^N_\alpha} \right].

Lemma 1.6. If \( P \)-bundle \( \pi_i : X_i \longrightarrow S \) \( i = 1, 2 \) and morphisms

\( g_1 : X_1 \longrightarrow X_2 \) satisfy the conditions stated in Theorem 1.1, then

there exists a unique isomorphism \( h : X_1 \longrightarrow X_2 \) such that \( h \cdot g_1 = g_2. \)

Proof. Let \( L_i \) \( i = 1, 2 \) be a tautological linebundle on \( X_i \)

with \( g^*_1(L_1) \cong f^*(O_X(1)) \otimes O_X(-E) \). Then since \( g^*_1(L_1) \cong g^*_2(L_2) \), we have \( L_2 \cong (g_2)(g^*_1(L_1))^* \). \( \pi_1 \) \( \pi_2 \) get, \( (\pi_1)_*(L_1) = (\pi_2)_*(L_1) = E_1 \) (cf. Lemma 1.5). Since \( P(E_2) = X_2 \), \( P(E_1) = X_1 \), this isomorphism yields an isomorphism \( h : X_1 \longrightarrow X_2 \) with \( h^*(L_2) = L_1 \). Thus we get an isomorphism \( h : X_1 \longrightarrow X_2 \) with \( h \cdot g_1 = g_2 \) (E, G, A, Chap. II, 4.2.3). Uniqueness clearly follows from the construction.

q. e. d.
Lemma 1.7. If $U \supseteq V$ are open subschemes of $S$ and if
\[ g_U : X_U \rightarrow \text{elm}^n_X(X_U) \] exists, then $g_V : X_V \rightarrow \text{elm}^n_X(X_V)$ exists
and there is a unique isomorphism $h^U_V : (\text{elm}^n_X(X_U))_V \rightarrow \text{elm}^n_X(X_V)$
with $h^U_V \cdot g_U \cdot g_U^{-1} = g_V$.

Proof. This is an immediate consequence of the definition of

elementary transformation and Lemma 1.6.

Now we proceed with the proofs of Theorem 1.1 and Theorem 1.3.

Proof of Theorem 1.1. Uniqueness has been proved in Lemma 1.6.

Let us cover $S$ by affine open subsets $\{ U_\lambda \}_{\lambda \in \Lambda}$ satisfying the
conditions in Lemma 1.4. By virtue of Lemma 1.4 there exists
\[ g_\lambda : \tilde{X}_{U_\lambda} \rightarrow X'_{\lambda, \mu} = \text{elm}^n_{X_{U_\lambda}}(X_{U_\lambda}), \text{hence } g_{\lambda, \mu} : \tilde{X}_{U_\lambda, \mu} \rightarrow X'_{\lambda, \mu} = \text{elm}^n_{X_{U_\lambda, \mu}}(X_{U_\lambda, \mu}), \]
exist (Lemma 1.7), where $U_{\lambda, \mu} = U_\lambda \cap U_\mu$, $U_{\lambda, \mu, \nu} = U_\lambda \cap U_\mu \cap U_\nu$.

By virtue of Lemma 1.7 there is a unique isomorphism $h^\lambda_{\mu} : X'_{\lambda, U_{\lambda, \mu}} \rightarrow X'_{\lambda, \mu}$
and the commutative diagram (+) is obtained. Thus

if $X'_{\lambda, U_{\lambda, \mu}}$ is identified with $X'_{\lambda, U_{\lambda, \mu}}$ by the isomorphism $\phi^\lambda_{\mu} = (h^\lambda_{\mu})^{-1} \cdot h^\lambda_{\mu}$, then we obtain a $P^N$-bundle $X'$ on $S$ because $\phi^\lambda_{\mu} \cdot \phi^\lambda_{\nu} = \phi^\lambda_{\lambda, U_{\lambda, \nu}}$ by virtue of the diagram (+). Moreover, since
In order to show that $X' \subseteq P(E')$ for some vector bundle $E'$ on $S$, we have only to prove that there exists a tautological linebundle $L'$ on $X'$ such that $g^*(L') \cong f^*(\mathcal{O}_X(1)) \otimes \mathcal{O}_X(-E_Y)$ for a tautological linebundle $\mathcal{O}_X(1)$ on $X$, which completes our proof. By virtue of Lemma 1.4 $(\xi_{U_X})_*$

$((\xi_{U_X})_*(\mathcal{O}_X(1)|_{U_\lambda}) \otimes \mathcal{O}_X(-E_Y|_{U_\lambda})) = I'_{U_\lambda}$ is an inversible sheaf on $X'_{U_\lambda}$ such that $(\xi_{U_X})^*(I'_{U_\lambda}) \cong (\xi_{U_X})^*(\mathcal{O}_X(1)|_{U_\lambda}) \otimes \mathcal{O}_X(-E_Y|_{U_\lambda})$. 

$h^\mu_{\mu'} \cdot g_{U_{r_{\mu'}}} = g_{U_{r_{\mu'}}} = h^\mu_{\mu'} \cdot g_{U_{r_{\mu'}}}$ we get a morphism $\bar{z} : X \rightarrow X'$. 

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Thus we know that \( L' = g_*(f^*(O_X(1)) \otimes O_Y(-E_Y)) \) is an invertible sheaf on \( X' \) such that \( g^*(L') = f^*(O_X(1)) \otimes O_Y(-E_Y) \). On the other hand,

\[ L'|_{U_\lambda} \text{ is a tautological linebundle on } X'_{U_\lambda} \text{ by virtue of Lemma 1.4.} \]

Thus \( \varphi'_*(L') = \mathcal{E}' \) is a locally free sheaf on \( S \) and \( P(E') = X' \), whence \( L' \) is a tautological linebundle.

\[ \text{q. e. d.} \]

Proof of Theorem 1.3. (i) Let \( O_{X'}(1) \) be a tautological linebundle on \( X' = \text{elm}_X(S) \). Then by virtue of Theorem 1.1 \( g^*(O_{X'}(1)) \cong f^*(O_X(1)) \otimes O_Y(-E_Y) \) for a tautological linebundle \( O_X(1) \) on \( X \). Thus

\[ E' = \varphi'_*(O_{X'}(1)) \cong \varphi'_*g^*(O_{X'}(1)) \cong \varphi'_*f^*(O_X(1)) \otimes O_Y(-E_Y) \]

\[ \cong \varphi'_*(O_X(1) \otimes f^*(O_Y(-E_Y))) \cong \varphi'_*(O_X(1) \otimes I_Y) \text{(see Lemma 1.5).} \]

Since \( X' = P(E') \), we know that \( \varphi'_*(O_X(1) \otimes I_Y) \) is a locally free \( O_S \)-module and \( P(\varphi'_*(O_X(1) \otimes I_Y)) \cong \text{elm}_Y(X) \). Let \( I_X(T) \) (or, \( J_Y \)) be the ideal of \( X_T \) in \( X \) (or, \( Y \) in \( X_T \), resp.) Then we have an exact sequence
0 \rightarrow I_{XT} \otimes O_X(1) \rightarrow I_Y \otimes O_X(1) \rightarrow J_Y \otimes O_X(1) \rightarrow 0.

Since $I_{XT} = \tau^*(I_T)$ and since $I_T$ is a Cartier divisor on $S$,

$I_{XT} \otimes O_X(1)$ is also a tautological linebundle on $X$, whence

$R^i \tau_* (I_{XT} \otimes O_X(1)) = 0, \quad i > 0$. On the other hand, the following

exact sequence

$$0 \rightarrow J_Y \otimes O_X(1) \rightarrow O_{XT}(1) \rightarrow O_Y \otimes O_X(1) \rightarrow 0$$

gives rise to an exact sequence

$$E_T \rightarrow F \rightarrow R^1 \tau_* (J_Y \otimes O_X(1)) \rightarrow R^1 \tau_* (O_{XT}(1)) \rightarrow$$

$$R^1 \tau_* (O_Y \otimes O_X(1)) \rightarrow \ldots \rightarrow R^{i-1} \tau_* (O_Y \otimes O_X(1)) \rightarrow$$

$$R^i \tau_* (J_Y \otimes O_X(1)) \rightarrow R^i \tau_* (O_{XT}(1)) \rightarrow \ldots,$$

where $E = \tau_* (O_X(1))$, $F = \tau_* (O_Y \otimes O_X(1))$. Since $P(E_T) = X_T$. 

$P(F) = Y$ and since $Y \hookrightarrow X_T$ is a closed immersion, $E_T \rightarrow F$ is a surjective map. $O_{X_T}(1)$ is a tautological linebundle on $X_T(\mathcal{O}, Y)$, hence $\mathcal{O}_{X_T}(1) = \mathcal{O}_{Y} \otimes \mathcal{O}_{X}(1)$ is a surjective map. Thus $R^i\mathcal{T}_*(\mathcal{O}_Y \otimes \mathcal{O}_X(1)) = 0$ ($\forall i > 0$). Hence the first exact sequence implies $R^i\mathcal{T}_*(\mathcal{O}_Y \otimes \mathcal{O}_X(1)) = 0$ ($\forall i > 0$).

(ii) Every assertion is clear except for $Y' = P(F')$. Let $s$ be a point of $S$ and let $U = \text{Spec}(A)$ be an affine neighborhood of $s$ such that $E, E', F$ are free and that $I_T$ is principal in $U$. If $e_0, \ldots, e_N(\mathcal{O}, \mathcal{O}_F, \ldots, \mathcal{O}_N)$ form a basis of $E_U(\mathcal{O}, E_U, \ldots, E_U)$, we may assume that $\varphi(e_0), \ldots, \varphi(e_N)$ form a basis of $F$ and the map $\varphi_U' : E_U' \rightarrow E_U$ is given by $\varphi_U'(e_i') = te_i$ ($0 \leq i \leq n$), $\varphi_U'(e_{n+1}) = e_{n+1}$. Then $\varphi_U'(e_{n+1})$, $\psi_U'(e_N)$ form a basis of $F'$, where $\varphi : E' \rightarrow F'$ is the natural map. This and Lemma 1.4 imply $P(F') = Y'$.

q. e. d.

§ 3. Some properties of elementary transformations.
Elementary transformations are compatible with base changes.

In fact.

Proposition 1.8. Let $\varphi : S' \to S$ be a morphism of locally noetherian schemes, let $\pi : X \to S$ be the projective bundle associated with a vector bundle $E$ of rank $N+1$, and let $T, Y$ be closed subschemes of $S, X$ satisfying the condition $(E_1^0)$. Assume that $\varphi^*(I_T)$ is also a Cartier divisor in $S'$ with the defining ideal for $T$ in $S$. Then $\varphi^{-1}(T), Y_S$ satisfy the condition $(E_n^0)$ for $\mathbb{P}^N$-bundle $\pi_S : X_S \to S'$ and $(\text{elm}_Y^0(X))_S \cong \text{elm}_Y^0(X_S)$.

Proof. It is clear that $\varphi^{-1}(T), Y_S$ satisfy the condition $(E_n^0)$. Note that if a $\mathbb{P}^N$-bundle $\pi' : X' \to S$ and a morphism $g : X \to X'$ exist and if there is an open covering $\bigcup U_\lambda = S$ such that $g_{U_\lambda} : X_{U_\lambda} \to X'_{U_\lambda}$ satisfies the conditions stated in Theorem 1.1, then $X' \cong \text{elm}_X^0(X)$ and $g \cdot f^{-1} = \text{elm}_Y^0$. (see Theorem 1.1 and its proof). Thus we may assume that $S = \text{Spec}(A), S' = \text{Spec}(B)$ and that $X, Y$ satisfy the condition in Lemma 1.4. Then our assertion is obvious by virtue of Lemma 1.4.

q. e. d.
Next, we assume that $S$ is a regular scheme. Let us consider the following condition for a $P^N$-bundle $\pi : X \to S$ and a closed subscheme $Y$ of $X$:

$$(E_n) \text{ } Y \text{ is a regular subscheme of pure dimension } n + \dim S - 1 \text{ \ (0 \leq n \leq N-1)} \text{ and } \pi^{-1}(s) \text{ is a } n\text{-dimensional linear subvariety}$$

$$(\mathcal{E}_n) \text{ of } P^N_k(s) = \pi^{-1}(s) \text{ for any } s \in T = \pi(Y), \text{ where } T \text{ has the unique reduced structure and where } \overline{\pi} : Y \to T \text{ is the restriction of } \pi \text{ to } Y.$$

Then we know that $Y$ is a $P^N$-bundle on $T$ ([7] Lemma 1.7, Theorem 1.8) and $T$ is a regular subscheme of $S$. Hence if $Y$ satisfies the condition $(E_n)$, then $Y, T$ satisfy the condition $(E^0_n)$.

The remaining part of this section will be devoted to prove that every $P^N$-bundle on a smooth quasi-projective $k$-variety with dimension smaller than 4 is obtained by an elementary transformation with center satisfying the condition $(E_{N-1})$ from the trivial bundle.

Proposition 1.9. Let $\pi : X \to S$ be a $P^N$-bundle on a
smooth k-variety $S$ and let $H_0, \ldots, H_N$ be positive divisors on $X$ such that $O_X(H_i)$ is a tautological linebundle for every $i$.

Assume that $H_0, \ldots, H_N$ are transversal to each other at any point of $\bigcap_{i=0}^N H_i$ and that $\dim \left( \bigcap_{i=0}^N H_i \cap \pi^{-1}(s) \right) \leq 0$ for every $s \in S$.

Then $Y = H_0 \cdot \ldots \cdot H_N$ satisfies the condition (E$_0$) and

\[ \text{elm}_Y(X) \subseteq \mathbb{P}(L_0 \oplus \ldots \oplus L_N), \text{ where } \pi^*(L_i) \cong O_X(H_0) \oplus O_X(-H_i). \]

In particular if $O_X(H_0) \cong O_X(H_i) (1 \leq i \leq N)$, then $\text{elm}_Y(X) \cong \mathbb{P}^N \times S$.

**Proof.** Since $H_0, \ldots, H_N$ are transversal to each other at any point of $\bigcap_{i=0}^N H_i$, $Y$ is k-smooth and pure dimension $(\dim S - 1)$.

Moreover, since $O_X(H_i)$ is a tautological linebundle and since

\[ \dim \left( Y \cap \pi^{-1}(s) \right) \leq 0, \text{ we see that } \pi^{-1}(s) = L_s^0 \text{ for every } s \in S. \]

Thus we know that $Y$ satisfies the condition (E$_0$). Next let $I_Y$ be the ideal sheaf of $Y$ in $O_X$. Let us consider the Koszul complex $K$, defined by $H_0, \ldots, H_N$:

\[ K_0 = O_X \]

\[ K_i = \bigoplus_{0 \leq x_1, \ldots, x_i \leq N} O_X(-H_{x_1} + \ldots + H_{x_i}), \quad 1 \leq i \leq N+1 \]

\[ K_j = 0 \quad j > N+1 \]
and the derivation \( d_i : K_i \to K_{i-1} \) is defined by

\[
(d_i)_x(e_{\alpha_1}, \ldots, e_{\alpha_1}; \ldots, e_{\alpha_i}) = \sum_{e_{\alpha_1} \leq \cdots \leq e_{\alpha_i} \leq e} (-1)^{k-1} a_{\alpha_1} \ldots a_{\alpha_i},
\]

where \( x \in X \), \( a_{\alpha_1}, \ldots, a_{\alpha_i} \in O_X(-H_1 + \ldots + H_i) \) and

\((-1)^{k-1} a_{\alpha_1} \ldots a_{\alpha_i} \) of the left hand side is regarded as an element of \( O_X(-H_{d_1} + \ldots + H_{d_{k-1}} + H_{d_{k+1}} + \ldots + H_{d_i}) \) by the natural inclusion

\[O_X(-H_{d_1} + \ldots + H_{d_{k-1}} + H_{d_{k+1}} + \ldots + H_{d_i}) \subseteq O_X(-H_{d_1} + \ldots + H_{d_{k-1}} + H_{d_{k+1}} + \ldots + H_{d_i}) \]

Then since \( H_0, \ldots, H_N \) are transversal to each other at any point of \( Y \),

\[0 \to K_{N+1} \to K_N \to \ldots \to K_1 \to I_Y \to 0\]

is an exact sequence (\( \Xi, \xi, \chi \), Chap. III, 1.1.4). Hence,

\[0 \to K_{N+1} \otimes_{O_X} O_X(H_0) \to K_N \otimes_{O_X} O_X(H_0) \to \cdots \]

\[\to K_1 \otimes_{O_X} O_X(H_0) \to I_Y \otimes_{O_X} O_X(H_0) \to 0\]

is also an exact sequence. Put \( M_i = \ker(d_i \otimes_{O_X} O_X(H_0)) = I_X(d_{i+1} \otimes_{O_X} O_X(H_0)) \). then we have the following exact sequences ;
\[ a_N : 0 \rightarrow O_X(-H_1 + \ldots + H_N) \otimes K_{N+1} \otimes O_X(H_0) \rightarrow K_N \otimes O_X(H_0) \rightarrow M_{N-1} \rightarrow 0 \]

\[ a_{N-1} : 0 \rightarrow M_{N-1} \rightarrow K_{N-1} \otimes O_X(H_0) \rightarrow M_{N-2} \rightarrow 0 \]

\[ a_1 : 0 \rightarrow M_1 \rightarrow K_1 \otimes O_X(H_0) \rightarrow I_Y \otimes O_X(H_0) \rightarrow 0 \]

Since \( O_X(-H_{d_1} + \ldots + H_{d_1}) \otimes O_X(H_0) \cong O_X(-iH_0) \otimes O_X(L_{d_1} \otimes \ldots \otimes L_{d_1}) \otimes O_X \)

\[ O_X(H_0) \cong O_X(-(i-1)H_0) \otimes O_X(L_{d_1} \otimes \ldots \otimes L_{d_1}), \]

we obtain

\[ \tau_*(K_i \otimes O_X(H_0)) \cong \bigotimes_{0 \leq d_1 < \ldots < d_i \leq N} \tau_*(O_X(-(i-1)H_0)) \otimes (L_{d_1} \otimes \ldots \otimes L_{d_1}) = 0, \quad 2 \leq i \leq N+1, \]

Thus the exact sequence \((a_N)\) implies that \( \tau_*(M_{N-1}) = R^1\tau_* (M_{N-1}) = 0 \).

Assume that \( \tau_*(M_i) = R^i\tau_* (M_i) = 0 (i > 1) \), then the exact sequence \((a_1)\) implies that \( \tau_*(M_{i-1}) = R^i\tau_* (M_{i-1}) = 0 \). By induction on \( i \),

therefore, we see that \( \tau_*(M_1) = R^2\tau_* (M_1) = 0 \). Hence by virtue of the exact sequence \((a_1)\) we obtain that \( L_0 \oplus \ldots \oplus I_N \cong \ldots \).
This and Theorem 1.3 assert that \( \ell_m^0(X) \cong P(L_0 \oplus \ldots \oplus L_N) \).

q. e. d.

By virtue of the above proposition we have only to find

\( H_0, \ldots, H_N \) satisfying the conditions in Proposition 1.9. in order to prove what we have been aiming.

**Lemma 1.10.** Let \( \tau : X \to S \) be a \( P^N \)-bundle on a quasi-projective \( k \)-variety \( S \) and let \( i : X \subset P_k \) be an immersion such that \( i^* (O_{P_k}(1)) \) is a tautological linebundle on \( X \). If \( H_0, \ldots, H_N \) are general hyperplanes of \( P_k \) and if \( \dim S \leq 3 \), then \( \dim \left( \bigcap_{i=0}^{N} H_i \right) \cap \tau^{-1}(s) \leq 0 \) for every \( s \in S \).

**Proof.** Since \( i^* (O_{P_k}(1)) \) is a tautological linebundle,

\( \left( \bigcap_{i=0}^{N} H_i \right) \cap \tau^{-1}(s) \) is a linear subspace of \( \tau^{-1}(s) \) for every \( s \in S \) and hyperplanes \( H_0, \ldots, H_N \) of \( P_k \). Thus we have only to prove that no line in \( \tau^{-1}(s) \) is contained in \( \bigcap_{i=0}^{N} H_i \) for general hyperplanes \( H_0, \ldots, H_N \) of \( P_k \). Let \( \text{Grass}^N_0 \) be the Grassmannian of the
\( \rho \)-dimensional linear subvariety of \( P_k^\rho \). Put \( \Gamma = \{ (L_1, L_2) \in \text{Grass}_t^1 \times \text{Grass}_{t-N-1} \mid L_1 \subset L_2 \} \) and let \( p_1 : \Gamma \to \text{Grass}_t^1, \quad p_2 : \Gamma \to \text{Grass}_{t-N-1} \), be natural projections, then \( \Gamma \) is an algebraic variety and \( p_1, p_2 \) are morphisms. We have \( \dim(p_1^{-1}(x)) = (N + 1)(t - N - 2) \) for any \( x \in \text{Grass}_t^1 \). On the other hand, \( E = \psi_*(\iota^*(\mathcal{O}_{P_t^1}(1))) \) is generated by a finite subset \( \{ u_0, \ldots, u_t \} \) of its global sections because \( \psi \) is \( \iota^*(\mathcal{O}_{P_t^1}(1)) \). The surjective homomorphism \( \varphi : \mathcal{O}_{P_k^{t+1}} \to E \) determined by \( u_0, \ldots, u_t \) defines a morphism \( \alpha : S \to \text{Grass}_N^t \). \( \alpha \) is nothing but the map defined by \( s \mapsto \iota(\varphi^{-1}(s)) \in \text{Grass}_N^t \). Let \( F = \{ L_1 \in \text{Grass}_t^1 \mid L_1 \subset L_2 \} \) for some \( L_2 \in \alpha(S) \), then \( F \) is a locally closed subset of \( \text{Grass}_t^1 \) and there is a natural morphism \( q : F \to \alpha(S) \). Since \( q^{-1}(s) = \text{Grass}_1^N \) for any \( s \in S, \dim F = \dim S + \dim(\text{Grass}_1^N) \).
= \dim S + 2(N - 1). \text{ Thus if } \dim S \leq 3, \text{ then } \dim(p_2^{-1}(F)) = \dim S + 2(N - 1) + (N + 1)(t - N - 2) = \dim S + (N + 1)(t - N) - 4 = \dim S + \dim(\text{Grass}^t_{t-N-1}) - 4 < \dim(\text{Grass}^t_{t-N-1}) \text{, whence } p_2(p_2^{-1}(F)) \subseteq \text{Grass}^t_{t-N-1}. \text{ Therefore if } \bigcap_{i=0}^{N} H_i \in (\text{Grass}^t_{t-N-1} - p_2(p_2^{-1}(F))) \text{ for } H_0, \ldots, H_N \in \text{Grass}^t_{t-1}, \text{ then } \bigcap_{i=0}^{N} H_i \text{ contains no line of } \pi^{-1}(s) \text{ for any } s \in S.

q. e. d.

Lemma 1.11. If \( \pi: X \to S \) be a \( \mathbb{P}^N \)-bundle on a quasi-projective smooth \( k \)-variety \( S \), then there is a tautological linebundle on \( X \) which is very ample over \( \text{Spec}(k) \).

Proof. By virtue of Lemma 1.2 there is a tautological linebundle \( \mathcal{O}_X(1) \) on \( X \) and the assumption implies that there is a very ample invertible sheaf \( L \) on \( S \). Since \( \mathcal{O}_X(1) \) is \( \mathbb{K} \)-very ample, \( \mathcal{O}_X(1) \otimes \pi^*(L^{\otimes n}) \) is very ample over \( \text{Spec}(k) \) for any \( n \geq n_0 \) (E.G.A. Chap.II, 4.4.10, (ii)).

q. e. d.
Now we come to the following theorem which extends a well-known theorem: Every \( P^1 \)-bundle on a complete non-singular curve \( C \) (that is, a geometrically ruled surface) is obtained from the direct product \( P^1 \times C \) by successive elementary transformations.

**Theorem 1.12.** Let \( \gamma : X \rightarrow S \) be a \( P^N \)-bundle on a smooth quasi-projective \( k \)-variety \( S \) with \( \dim S \leq 3 \). Then there is a \( k \)-subscheme \( Y \) of \( P^N_k \times S \) satisfying the condition \( (E_{N-1}) \) such that \( X \cong \text{elm}^{-1}_0(P^N_k \times S) \). Moreover, if \( \dim S = 2 \) or 3, we can choose such a \( Y \) as an irreducible subscheme.

**Proof.** By virtue of Lemma 1.11 there is an immersion \( i : X \hookrightarrow P^N_k \) such that \( i^*(\mathcal{O}(1)) \) is a tautological linebundle on \( X \) (see [E.G.A., (U), 4.4.7]). If \( H_0, \ldots, H_N \) are sufficiently general hyperplane sections of \( X \) in \( P^N_k \), then \( Y' = H_0 \cdot \ldots \cdot H_N \) satisfies the condition \( (E_0) \) by virtue of Proposition 1.9 and Lemma 1.10. By virtue of Proposition 1.9 we have that \( \text{elm}^{-1}_0(X) \cong P^N_k \times S \). Let \( Y \) be the center of \( (\text{elm}^{-1}_0)^{-1} \), then \( Y \) is a desired subscheme (see Theorem 1.1, Corollary 1.1.1). If
dim $S = 2$ or $3$, then $\dim Y' \geq 1$. Thus we can choose such a $Y'$ as an irreducible subscheme. Then the subscheme $Y$ determined by the $Y'$ as above is irreducible.

$q. e. d.$

Remark. 1.13. It seems that Theorem 1.12 is false in the case where $\dim S$ is greater than 3 (see Theorem 2.19). But we may present the following problem: Is every $P^N$-bundle on a smooth quasi-projective $k$-variety $S$ obtained from the direct product $P^N_k \times S$ by successive elementary transformations?
Chapter II. Regular vector bundles.

From now on we shall use the following notation unless otherwise stated:

$S$: a smooth projective variety over $k$ with dimension greater than 1

$P^N_S$: the direct product $P^N_k \times_k S$;

$\pi: \text{the projection } P^N_S \to S$;

$Z$: a hyperplane of $P^N_k$;

$H_0$: the subvariety $Z \times S$ of $P^N_S$;

$Y$: an irreducible subscheme of $P^N_S$ satisfying the condition $(E_{N-1})$;

$T$: the subscheme $\pi(Y)$ of $S$ with reduced structure;

$P^N_T$: the direct product $P^N_k \times T$ which is regarded as a subscheme of $P^N_S$.

$H_Y$: the divisor $H_0 + P^N_T$ on $P^N_S$;

$I_Y$: the ideal sheaf of $Y$ in $P^N_S$;

$f_Y: \tilde{\chi}(Y) \to P^N_S$: monoidal transformation with center $Y$;
\[ \overline{x}_T = f^{-1}_Y [\mathbb{P}^N_T], \quad E_Y = f^{-1}_Y (Y) \quad (\text{i.e. exceptional variety of } f_Y) ; \]

\[ \mathfrak{E}_Y : \overline{X}(Y) \rightarrow X(Y) : \text{the contraction with center } \overline{x}_T \text{ whose contractability is guaranteed by Theorem 1.1} \ ; \]

\[ \mathfrak{T}_Y : X(Y) \rightarrow S : \text{the projection of } \mathbb{P}^N \text{-bundle } X(Y) ; \]

\[ H'_Y : \text{the transform of } H_0 \text{ by } \text{elm}_{Y}^{N-1} \left( = g_Y f_Y^{-1} \right) \]

In the above situation we may assume that \( H_0 \) does not contain \( Y \).

\[ \begin{array}{ccc}
\overline{X}(Y) & \xrightarrow{f_Y} & \mathbb{P}^N_S \\
\downarrow \mathfrak{E}_Y & & \downarrow \pi \\
X(Y) & \xrightarrow{\pi_X} & S \\
\end{array} \]

\[ \begin{array}{ccc}
\mathbb{P}^N_T & \xleftarrow{\pi_T} & Y \\
\downarrow \pi & & \downarrow \pi |_Y \\
T & & T \\
\end{array} \]

\( \S \ 1. \quad \text{Definition of regular vector bundles.} \)

By virtue of Theorem 1.3 we know that \( E(Y) = \pi_* (I_Y \otimes \mathcal{O}_{\mathbb{P}^N_S} (H_Y)) \) is a locally free \( \mathcal{O}_S \)-module of rank \( N + 1 \). Thus it seems that the following definition is adequate.

Definition. A locally free \( \mathcal{O}_S \)-module which is isomorphic to \( E(Y) \) is called a regular vector bundle (defined by \( Y \)).

Of course a subscheme which defines a regular vector bundle may not be unique (see \( \S \ 2 \) of this chapter).
Lemma 2.1. Let \( P_i : X_i \to S \) (\( i = 1, 2 \)) be \( \mathbb{P}^N \)-bundles on 
\( S \), let \( T, Y_1 \) (or, \( T, Y_2 \)) be subvarieties of \( S \), \( X_1 \) (or, \( S \), 
\( X_2 \), resp.) satisfying the condition \((E_n)\) (or, \((E_{N-n-1})\), resp.)

with \( X_2 = \mathcal{E}_1 \gamma Y_1(X_1), \mathcal{E}_1 Y_2(X_1) \), \( \mathcal{E}_1 Y_2(X_1) \), and let \( f_i : \tilde{X} \to X_1 \) be the monoidal transformations of \( X_1 \) with center \( Y_1 \). Assume

that \( C_1 \) is a positive divisor on \( X_1 \) such that \( \mathcal{O}_{X_1}(C_1) \) is a

tautological linebundle on \( X_1 \). Put \( C_2 = \mathcal{E}_1 Y_2(C_1) \). 

(i) \( C_1 \not\supset Y_1 \) if and only if \( C_2 \not\supset Y_2 \). In this case

\[
f_2^{-1}(C_2) = f_2^{-1}(C_1) + f_2^{-1}(Y_2). 
\]

(ii) \( f_1^{-1}(P_1^{-1}(T)) = f_1^{-1}(P_1^{-1}(T)) + f_1^{-1}(Y_1). \)

Proof. Let \( x \) be a point of \( T \) and let \( U = \text{Spec}(A) \) be 
an affine open neighborhood of \( x \) in \( S \) such that \( X_1 \cup U = \text{Proj} \left( A \left[ \gamma_0, \ldots, \gamma_N \right] \right) \) and that the ideal of \( T \cap U \) (or, \( X_1 \cup U \)
\( \cap Y_1 \)) is generated by \( t \in A \) (or, \( t, \gamma_{n+1}, \ldots, \gamma_N \), resp.)

Then \( X_2 \cup U = \text{Proj} \left( A \left[ \gamma_0', \ldots, \gamma_N' \right] \right), \gamma_i' = \gamma_i \) (\( 0 \leq i \leq n \)),

\( t \gamma_i' = \gamma_i \) (\( n+1 \leq i \leq N \)) and the defining ideal for \( Y \) is generated

by \( t, \gamma_1', \ldots, \gamma_n' \) by virtue of Lemma 1.4. We may assume that
C_1 \cap X_1, U \text{ is defined by } \sum_{i=0}^{N} a_i \gamma_i = 0, a_i \in A. \quad C_1 \cap X_1, U

\iff \forall Y_1 \cap X_1, U \text{ if and only if } a_i \notin t A \text{ for some } 0 \leq i \leq n. \quad \text{Thus}

if \ C_1 \cap X_1, U \nsubseteq Y_1 \cap X_1, U, \text{ then } C_2 \cap X_2, U \text{ is defined by }

\sum_{i=0}^{n} a_i \gamma_i' + t \sum_{j=n+1}^{N} a_j \gamma_j'. \quad \text{Hence } C_2 \cap X_2, U \nsubseteq Y_2, U. \quad \text{Conversely assume }

that \ C_2 \cap X_2, U \nsubseteq Y_2 \cap X_2, U. \quad \text{We may assume that } C_2 \cap X_2, U \text{ is defined}

by \ \sum_{i=0}^{N} b_i \gamma_i = 0. \quad (b_i \in A, \text{ if } 0 \leq i \leq n, \quad b_i = t b_i', b_i' \notin A

if \ n+1 \leq i \leq N). \quad \text{Since } C_2 \text{ is the proper transform of } C_1 \text{ by }

\text{we may assume that } C_2 \cap X_1, U \text{ is defined by }

\sum_{i=0}^{N} b_i \gamma_i + \sum_{j=n+1}^{N} b_j \gamma_j = 0 \quad \text{and } b_i \notin t A \text{ for some } 0 \leq i \leq n. \quad \text{Then}

C_1 \cap X_1, U \text{ is defined by } \sum_{i=0}^{n} b_i \gamma_i + \sum_{j=n+1}^{N} b_j \gamma_j = 0 \quad \text{and } b_i \notin t A \text{ for some } 0 \leq i \leq n. \quad \text{Thus } C_1 \cap X_1, U \nsubseteq Y_1 \cap X_1, U, \quad C_1 \nsubseteq Y_1 \text{ if and only if } \ C_1 \cap X_1, U \nsubseteq Y_1 \cap X_1, U. \quad \text{Thus } C_1 \nsubseteq Y_1 \text{ if and only if } \ C_2 \cap X_2, U \nsubseteq Y_2 \cap X_2, U.

\text{We have } f_2^{-1}(C_2) = f_2^{-1} \{ C_2 \} + f_2^{-1}(Y_2) \text{ is clear because }

\sum_{i=0}^{N} b_i \gamma_i' \in I_{1,j}^{-1} \quad \text{for } I_j = (t, \gamma_0', \gamma_j', \ldots, \gamma_n', \gamma_j') \in A[\gamma_0', \gamma_j', \ldots, \gamma_n', \gamma_j'], \quad 0 \leq j \leq N \quad \text{(cf. Proof of Lemma 1.4). \quad Thus we get (i).}

Proof of (ii) is similar to the above. \quad q.e.d.
Lemma 2.2. If $E(Y)$ is the regular vector bundle defined by $Y$, then we have $E(Y) \cong \langle \nu_Y \rangle_*(\mathcal{O}_X(Y)(H'))$.

Proof. Put $f_Y^{-1}(H_0) = \sim H$, then $g_Y^{-1}(H_0') = \sim H + \sim H_T$ by virtue of the above lemma. Thus $f_Y^*(\mathcal{O}_Y(H)) \otimes_{\mathcal{O}_X} \mathcal{O}_Y(E_Y) \cong (g_Y)^*(\mathcal{O}_X(Y)(H')) \otimes \mathcal{O}_X(E_Y)$. We therefore obtain

$$\nu_Y^*(f_Y^*(\mathcal{O}_Y(H))) \otimes \mathcal{O}_X(-E_Y) \cong \nu_Y^*(I_Y \otimes \mathcal{O}_X(H_Y)) \cong E(Y) \quad \text{q. e. d.}$$

The following is a corollary to Theorem 1.12.

Proposition 2.3. Assume that the dimension of $S$ is equal to 2 or 3. Every very ample vector bundle $\mathcal{E}$ of rank $N + 1$ ($N \geq 1$) is regular and therefore, for any vector bundle $E$ of rank $N + 1$ ($N \geq 1$) on $S$, there exists a linebundle $L$ on $S$ such that $E \otimes L$ is a regular vector bundle.

Proof. Put $X = \mathbb{P}(E)$ and let $\mathcal{O}_X(1)$ be the tautological linebundle of $E$. Since $\mathcal{O}_X(1)$ is very ample by our assumption, the proof of
Theorem 1.12 shows that there is an isomorphism \( j : X \rightarrow \) 
\( \text{slm}^{N-1}_Y(P^n_X \times S) \) for the same \( Y \) obtained from \( O_X(1) \) as in the
proof. Moreover, \( (\text{slm}^{N-1}_Y)^{-1}(j(H_1)) = Z_1 \times S \) for a hyperplane
\( Z_1 \) of \( P^n_k \), where \( H_1 \) is the same as in the proof of Theorem 1.12.
Thus we obtain our assertion by virtue of Lemma 2.2 q. e. d.

§ 2. Families of regular vector bundles.

In this section we shall construct a moduli of a subfamily of
regular vector bundles.

Lemma 2.4. Let \( X \) be a factorial variety over \( k \) and let
\( W \) be a positive divisor on \( P^n_X \) such that 
\( O_X(W) \otimes O_X(k(x_0)) \)
\( O_{P^n_X}(r) \) for some \( x_0 \in X \). Then we have that \( r \geq 0 \) and
\( O_X(W) \otimes O_{P^n_X}(n(Z \times X)) \otimes O_X(D) \)
for some positive divisor \( D \) on \( X \), where \( p_2 : P^n_X \rightarrow X \) is the projection.

Proof. Invariance of Euler-Poincaré characteristic of a
proper flat family implies that 
\( O_X(W) \otimes O_X(k(x)) \cong O_X(r) \) for every
\( x \in X \). Then by virtue of the seesaw theorem (§4)
we know that \( \mathcal{O}_X(N(W)) \cong \mathcal{O}_X(r(Z \times X)) \otimes \mathcal{O}_Z(L) \) for some linebundle \( L \) on \( X \). On the other hand, the K"unneth formula implies that
\[
H^0(P^N_X, \mathcal{O}_X(N(W))) \cong H^0(P^N_k, \mathcal{O}_k(r)) \otimes H^0(X, L).
\]
Since \( W \) is a positive divisor, \( \dim_k H^0(P^N_X, \mathcal{O}_X(W)) \geq 0 \), whence
\[
\dim_k H^0(P^N_k, \mathcal{O}_k(r)) > 0, \quad \dim_k H^0(X, L) > 0.
\]
Thus we get that \( r \geq 0 \) and \( L \cong \mathcal{O}_X(D) \) for some positive divisor \( D \) on \( X \), q.e.d.

Now a regular vector bundle \( E \) of rank \( N + 1 \) \((N \geq 1)\) is completely determined by a subvariety \( Y \) of \( P^N_S \) satisfying the condition \((E_{N-1})\). Then \( T = \pi(Y) \) with reduced structure is a smooth subvariety of \( S \) of codimension 1 (\([7]\) Theorem 1.8, E.G.A. Chap. IV 6.8.3) and \( Y \) can be regarded as a positive divisor on \( P^N_T \).

Furthermore, since \( Y_t \) is a hyperplane of \( P^N_{\kappa(t)} \) for every \( t \in T \), we know by the above lemma that \( \mathcal{O}_S(Y) \cong \mathcal{O}_S(Z \times T) \otimes \mathcal{O}_T(D) \) for a positive divisor \( D \) of \( T \). Thus \( Y \) is a member of a complete linear system on \( P^N_T \) of type \( |Z \times T + (K_T)^{-1}(D)| \) which contains no fibre of \( P^N_T \). We have therefore the following principle.
Principle 2.5. To give a regular vector bundle of rank $N + 1$

($N \geq 1$) on $S$ is equivalent to give a member of a complete linear

system of the type $|Z \times T + \left(\mathcal{V}_T\right)^{-1}(D)|$ on $P^N_T$ which contains

no fibre of $P^N_T$, where $T$ is a suitable smooth subvariety of $S$

of codimension 1 and $D$ is a positive divisor on $T$.

Put $\text{Pic}^+(T) = \left\{ D \in \text{Pic}(T) \mid H^0(T, \mathcal{O}_T(D)) \not= 0 \right\}$. From

now on $R^r(S, T, D)^{10}$ denotes the set of isomorphism classes

of regular vector bundles of rank $r$ on $S$ which are determined by

members of $|Z \times T + \left(\mathcal{V}_T\right)^{-1}(D)|$ for $D \in \text{Pic}^+(T)$.

By virtue of K"unneth formula

$$H^0(P^N_T, \mathcal{O}_{P^N_T}(Z \times T + \left(\mathcal{V}_T\right)^{-1}(D))) \cong H^0(P^N_T, \mathcal{O}(1)) \otimes H^0(T, \mathcal{O}_T(D)) =$$

$$\otimes H^0(T, \mathcal{O}_T(D)) \otimes \cdots \otimes H^0(T, \mathcal{O}_T(D)).$$

Thus a member $Y$ of $|Z \times T + \left(\mathcal{V}_T\right)^{-1}(D)|$ is defined by

$s_0 + \cdots + s_N = 0$ for some $s_i \in H^0(T, \mathcal{O}_T(D))$. $Y$ contains

a fibre $\mathcal{V}_T(t)$ (for $t \in T$) if and only if $s_0(t) = \cdots = s_N(t) = 0$.

Hence Principles 2.5 can be said in other words as follows:
Principle 2.6. To give a regular vector bundle contained
in $\mathbb{H}^{N+1}(S,T,D) (N \geq 1)$ is equivalent to give an element
$(s_0, \ldots, s_N) \neq 0$ of $H^0(T, O_T(D)) \times \cdots \times H^0(T, O_T(D))$ such that
every $s_0(t), \ldots, s_N(t)$ is not zero for any $t \in T$.

Now let us construct a large family of regular vector bundles.

Lemma 2.7. The set $R^r(T)$ which consists of subschemes of
$\mathbb{P}^{r-1}_T$ satisfying the condition of Principle 2.5 forms an open subset
of $\text{Hilb}_{\mathbb{P}^{r-1}_T/k}$.

Proof. Since $\mathbb{P}^{r-1}_T$ is projective and non-singular, $\text{Div}_{\mathbb{P}^{r-1}_T/k}$
is open and closed in $\text{Hilb}_{\mathbb{P}^{r-1}_T/k}$ ([6] Proposition 4.1, Corollary 4.4,
[7] Theorem 2.1). Hence $\text{Div}_{\mathbb{P}^{r-1}_T/k}$ is a union of some connected
components of $\text{Hilb}_{\mathbb{P}^{r-1}_T/k}$. On the other hand,

$$D = \left\{ D \in \text{Div}_{\mathbb{P}^{r-1}_T/k} \mid O_{\mathbb{P}^{r-1}_T}(D)_t \equiv O_{\mathbb{P}^{r-1}_T}(1), \forall t \in T \right\}$$
is also a union of some connected components of $\text{Div}_{\mathbb{P}^{r-1}_T/k}$.
Moreover, $R^r(T)$ consists of the members of $D$ which contains no fibre of $\mathbb{P}^{r-1}_T$. Let

$W$ be the subscheme of $\mathbb{P}^{r-1}_T \times D$, induced from the universal family
of subschemes on $\mathbb{P}^{r-1}_T \times \text{Hilb}_{\mathbb{P}^{r-1}_T/k}$ by the natural inclusion.
Look at the following commutative diagram

\[
\begin{array}{ccc}
P^{-1} & \to & D \\
p \downarrow & & \downarrow q \\
T \times D & \to & D
\end{array}
\]

Since \( p' \) is proper, the set \( R' = \{ x \in T \times D \mid \dim p'^{-1}(x) = r-1 \} \), i.e. \( W \) contains the fibre \( p'^{-1}(x) \subseteq P^{-1}(x) \) is closed in \( T \times D \) (E.G.A. Chap. IV, 13.1.3). Since \( R^r(T) = D \cap q(R') \) and \( q \) is proper, \( R^r(T) \) is an open subset of \( D \). Thus \( R^r(T) \) is an open subset of \( \text{Hilb } P^{-1}/k \).

Lemma 2.8. Let \( \nu : X \rightarrow S, Y \rightarrow T \) be the same as in Theorem 1.1, and let \( j : S' \rightarrow S \) be a morphism such that \( j^{-1}(T) \) is also a Cartier divisor on \( S' \). Then canonically \( j^*(\nabla^* (I_Y \otimes 0_X(1))) \equiv (\nabla_{S'}^*) (I_Y \otimes i^* 0_X(1)) \) for a tautological line bundle \( 0_X(1) \) and the ideal \( I_Y \) of \( Y \) in \( X \), where \( i : X_S' \rightarrow X \) is the natural morphism induced by \( j \).
Proof. Put \( E = T_Y(0_X(1)) \), \( F = T_Y(0_Y \otimes 0_X(1)) \), \( E' = (T_Y S')_*(0_Y \otimes 0_X(1)) \), \( F' = (T_Y S')_*(0_Y \otimes 0_X(1)) \), Then \( \text{Ker} \phi = T_Y(0_Y \otimes 0_X(1)) \), \( \text{Ker} \phi' = (T_Y S')_*(0_Y \otimes 0_X(1)) \) for the canonical morphisms \( \phi : E \to F \), \( \phi' : E' \to F' \) (see Theorem 1.3, Proposition 1.8). Consider the following exact commutative diagram:

\[
\begin{array}{cccccc}
\alpha_1 & \psi & \alpha_2 & \alpha_3 & j^*F \\
\downarrow & \downarrow & \downarrow & \downarrow & j^*F \\
0 & \psi' & \alpha_2 & \alpha_3 & j^*F \\
0 & \downarrow & \downarrow & \downarrow & j^*F \\
(T_Y S')_*(0_Y \otimes 0_X(1)) & E' & F' & 0
\end{array}
\]

Since a local equation \( t \) for \( T \) at \( j(s') \in S \) is a non-zero divisor of \( 0_{S'} \), \( \text{Ker} \phi = 0 \) and therefore \( \phi \) is injective. On the other hand, \( (j^*F)_{s'} \to H^0(Y_{s'}, (0_Y \otimes 0_X(1))) \otimes k(s') \) is an isomorphism because \( j(s') = s \). Thus \( \alpha_3 \) is an isomorphism because \( j(s') = s \). Similarly \( \alpha_2 \) is an isomorphism. Therefore \( \alpha_1 \) is an isomorphism by virtue of the five lemma. q.e.d.

Theorem 2.9. Let \( S \) be a non-singular projective variety over \( k \), let \( T \) be a non-singular subvariety of \( S \) of codimension 1 and let \( R^r(T) \) be the open subscheme of \( \text{Hilb}_{r-1} T / k \) defined in Lemma 2.7.
Then there are a vector bundle $P^r(T)$ of rank $r$ on $S \times_k R^r(T)$

and a surjective map $\varphi_T^r : R^r(T)(k) \to \bigoplus_{D \in \text{Pic}(T)} R^r(S, T, D)$ such that $P(T, x) \cong \varphi_T^r(x)$ for any $k$-rational point $x$ of $R^r(T)$.

Proof. Let $W$ be the subscheme of $\mathbb{P}^{r-1} \times_k R^r(T) = \mathbb{P}^{r-1}_k (S \times_k R^r(T))$

induced from the universal family of subschemes on $\mathbb{P}^{r-1} \times_k \text{Hilb} \mathbb{P}^{r-1} / k$.

Since $T$ is a Cartier divisor on $S$, so is $T \times_k R^r(T)$ on $S \times_k R^r(T)$.

With the natural projection $p : \mathbb{P}^{r-1} \times_k R^r(T) \to S \times_k R^r(T)$, $(p_T \times \varphi_T^r(T)^{-1}(y))$

satisfy the condition (E.1-2) ([?], Theorem 1.8). Now put $P^r(T) = P^r(T) = P^r(T)$

$P^r(T) = P^r(T)$, where $I_W$ is the defining ideal for $W$ and $H_0$ is the

Cartier divisor $Z \times (S \times_k R^r(T))$ on $\mathbb{P}^{r-1} \times_k R^r(T)$. Then $P^r(T)$ is

a vector bundle of rank $r$ on $S \times_k R^r(T)$ by virtue of Theorem 1.3.

If $x$ is a $k$-rational point of $R^r(T)$, then $(x|_{\mathbb{P}^{r-1} \times_k R^r(T)})^{-1}(W) = W_x$

is contained in $| Z \times T + (\mathbb{P}_T^r)^{-1}(D) |$ for some $D \in \text{Pic}^+(T)$ and

contains no fibre of $P^{r-1}_T$, where $\alpha_x : P^{r-1}_T \to P^{r-1}_S \times_k R^r(T)$ is

the morphism induced by $x \to R^r(T)$. Thus for the natural morphism
Our next aim is to study conditions for two regular vector bundles to be isomorphic to each other. The following lemma is a key in the sequel.

**Lemma 2.10.** Let \( Z_0, \ldots, Z_N \) be linearly independent hyperplanes of \( \mathbb{P}^N_k \) and put \( H'_i = \text{elm} \mathcal{H}(R_i) \) for \( H_i = Z_i \times T \) and a subscheme \( Y \) of \( \mathbb{P}^N_S \) satisfying the condition \((E^0)\). Then \( Y' = \bigcap_{i=0}^{N} H'_i \) for the center \( Y' \) of \((\text{elm}_{\mathbb{P}^N})^{-1}\), that is, the ideal \( Y' \) is generated by those of \( H'_i \).

**Proof.** Since the property is local with respect to \( S \) and since \( H_0, \ldots, H_N \) form a basis of hyperplanes of \( \mathbb{P}^N_{S, s} \cong \mathbb{P}^N_{k(s)} \) for any \( s \in S \), we may assume that \( S = \text{Spec}(A) \), \( \mathbb{P}^N_S = \text{Proj}(A[\eta_0, \ldots, \eta_N]) \), the homogeneous ideal defining \( Y \) is generated by \( t(\epsilon A), \eta_{n+1}, \ldots, \eta_N \) and that \( H_i \) is defined by \( \eta_i = 0 \). Then by virtue of Lemma 1.4
\[ \text{elm}_{Y}^{N}(\mathcal{P}_{S}^{N}) = \text{Proj}(A \left[ \gamma_{0}', \ldots, \gamma_{N}' \right]) \quad \gamma_{i}' = \gamma_{i} \quad (0 \leq i \leq n), \quad t \gamma_{i}' = \gamma_{i} \quad (n+1 \leq i \leq N) \text{ and the homogeneous ideal defining } H_{1}' \text{ is generated by } \gamma_{i}' \quad (0 \leq i \leq n), \quad t \gamma_{i}' \quad (n+1 \leq i \leq N). \]

Thus in the affine open set \( U'_{1} = \{ t \gamma_{i}' = 0 \} \) the ideal defining \( \bigcap_{j=0}^{N} H_{j}' \) is generated by \( 1 \quad (0 \leq i \leq n); \quad t, \quad \gamma_{0}'/\gamma_{i}', \ldots, \gamma_{n}'/\gamma_{i}' \quad (n+1 \leq i \leq N). \) On the other hand, the ideal defining \( Y' \) in \( U_{1}' \) is generated by the same element because the homogeneous ideal of \( Y \) is generated by \( \gamma_{0}', \ldots, \gamma_{n}', t. \)

q.e.d.

For a non-singular subvariety \( T \) of codimension 1 of \( S \) put

\[ A_{T} = \left\{ D \mid D \in \text{Pic}(T), \quad H^{0}(T, \mathcal{O}_{T}(T^{2} - D)) = 0 \right\}. \]

Lemma 2.11. Let \( H_{0}, \ldots, H_{N} \) be as in the above lemma and put

\[ H_{i}' = \text{elm}_{Y}^{N-1}(H_{i}) \text{ for } Y \text{ satisfying the condition } (F_{N-1}). \text{ Let } T \text{ be the subvariety } \mathcal{T}_{L}(Y) \text{ (with reduced structure) of } S \text{ (then } T \text{ is non-singular and codimension 1). Assume that } Y \notin \left\{ \mathcal{X} T + (\mathcal{T}_{L})^{-1}(D) \right\}^{\text{num}} \text{ with } D \in A_{T}, \text{ then } H_{0}', \ldots, H_{N}' \text{ form a basis of the complete linear system } \left| H_{0}' \right| \text{ on } X(Y) = \text{elm}_{Y}^{N-1}(\mathcal{P}_{S}^{N}).
Proof. It is clear that \( H'_0, \ldots, H'_N \) are independent. Let \( L \) be the linear system spanned by \( H'_0, \ldots, H'_N \). Assume that \( L \subset H'_0 \) and we shall show a contradiction. Take a general member \( H' \) of \( H'_0 \) such that \( H' \) is irreducible and \( H' \not\in L \) (since at least one of \( H'_0, \ldots, H'_N \) is irreducible, such an \( H' \) exists). In the first place assume that \( H' \supset Y' \), then \( e^{-1}_Y[H'] + T_e \sim e^{-1}_Y[H'_0] + T_e \sim e^{-1}_Y[H_0 + P_T] \) by virtue of Lemma 2.1. Thus \( H = f_Y[e^{-1}_Y[H']] \sim H'_0 \). Since \( H'_0, \ldots, H'_N \) form a basis of \( H'_0 \), \( H = Z \times T \) for some hyperplane \( Z \) of \( P^N_k \) and \( H' \) is the total transform of \( H \). Thus \( H' \not\in L \), which is impossible. Next assume that \( H' \nsubseteq Y' \). By a similar argument as above we know that \( H \sim H'_0 + P_T^N \) and by virtue of Lemma 2.1 \( H \supset Y' \).

Thus \( H \sim H'_0 + P_T^N = Y + A, A > 0 \) and \( P_T^N(Y + A) \supseteq P_T^N(Z \times T + \overline{\nu^{-1}}(T^2)) \).

Thus \( P_T^N(Y + A) \supseteq P_T^N(Z \times T + \overline{\nu^{-1}}(T^2)) \).

Thus \( P_T^N(A) \supseteq \overline{\nu^{-1}}(P_T^N(Z \times T + \overline{\nu^{-1}}(T^2))) \), whence \( P_T^N(P_T^N(Z \times T + \overline{\nu^{-1}}(T^2))) \).

On the other hand, \( H^0(P_T^N, \overline{\nu^{-1}}(Q_T(T^2-D))) = H^0(T, \overline{\nu^{-1}}(Q_T(T^2-D))) \).

But this is contradictory to the fact that \( D \in A_T \).

\[ \text{q.e.d.} \]
Corollary 2.11.1. If $E \in \mathcal{R}(S, T, D)$, then $\dim_k \mathcal{H}^0(S, E) \geq r$.

Moreover if $D \in \mathcal{A}_T$, then $\dim_k \mathcal{H}^0(S, E) = r$.

Proof. Our assertion is clear if one notes $\mathcal{H}^0(S, E) = \mathcal{H}^0(P(E), \mathcal{O}_P(E)^{(1)})$. q.e.d.

Note that $\text{Aut}_S(P_{S}^{N-1}) = \text{PGL}(r-1)$ and that if a subscheme $Y$ of $P_{S}^{N}$ satisfying the condition $(E_{N-1})$, then so does $Y^\sigma$ for every $\sigma \in \text{PGL}(r-1)$. This enables us to show that the next proposition follows from the above two lemmas.

Proposition 2.12. Let $E_1 (i = 1, 2)$ be a regular vector bundle of rank $r$ on $S$ defined by $Y$; and let $Y_1 \in \mathcal{Z} \times T + (\mathcal{T}_Y)^{(1)}(D)$ for $D \in \mathcal{A}_T$ (notation is as above). Then $E_1$ is isomorphic to $E_2$ if and only if $Y_1 = Y_2^\sigma$ for some $\sigma \in \text{PGL}(r-1)$.

Proof. It is clear that if $Y_1 = Y_2^\sigma$, then $E_1 \cong E_2$. Conversely, assume that there is an isomorphism $i : E_2 \cong E_1$. $i$ induces an isomorphism $j : X_1 = P(E_1) \cong X_2 = P(E_2)$ such that $j^*(\mathcal{O}_{X_2}^{(1)}) \cong \mathcal{O}_{X_1}^{(1)}$ for the tautological linebundle $\mathcal{O}_{X_1}^{(1)}$ of $E_1$. Since $Y_1 \in \mathcal{Z} \times T + (\mathcal{T}_Y)^{(1)}(D)$ for $D \in \mathcal{A}_T$, $\dim_k \mathcal{H}^0(X_2, \mathcal{O}_{X_2}^{(1)}) = \dim_k \mathcal{H}^0(X_1, \mathcal{O}_{X_1}^{(1)})$. 


By virtue of Lemma 2.11 and Lemma 2.2, this and Lemma 2.10 imply that \( Y' = \bigcap_{H(1) \in \mathcal{O}_{Y_1}} H(1) \) is the center of \((\text{el}_r - 2)^{-1}\). We have therefore \( j(Y') = Y' \). Fix isomorphisms \( \varphi_1 : \mathbb{P}^{r-1}_S \to \text{el}_0(Y_1) \) and \( \varphi_2 : \mathbb{P}^{r-1}_S \to \text{el}_0(Y_2) \).

Put \( \varphi_1(Y_1) = Y'_1 \). Then it is easy to see that \( j \) induces an isomorphism \( \alpha : \text{el}_0(Y_1') \to \text{el}_0(Y_2') \) such that \( \alpha(Y'_1) = Y'_2 \). Hence we get a desired automorphism \( \tau_2^{-1} \alpha \tau_1 \) of \( \mathbb{P}^{r-1}_S \). q.e.d.

**Theorem 2.13** Let \( S \) be a non-singular projective variety over \( k \).

1. If \( E_i \in R^f(S, T_1, D_2) \) (i=1,2), \( D_2 \in A_T \), and \( T_1 \neq T_2 \), then \( E_1 \neq E_2 \).

2. If \( T \) is a non-singular subvariety of \( S \) of codimension 1, then \( \text{Aut}_S(\mathbb{P}^{r-1}_S) = \text{PGL}(r-1) \) acts on \( R^f(S, T) \). The set \( R^f_0(T) = \{ Y \in R^f(T) \mid Y \subseteq \{ z \in T \mid (\varphi_T^{-1}(D)) \text{ for some } D \} \text{ forms a PGL}(r-1)\)-stable open subset of \( R^f(S, T) \).

3. For the surjective map \( \varphi^f_T : R^f_0(T) \to R^f_0(S, T, D) \)

\[ \varphi^f_T(x) = \frac{z}{D \in A_T} R^f(S, T, D) \]

it holds that \( \varphi^f_T(x_1) = \varphi^f_T(x_2) \) if and only if \( x_1 = x_2 \).
Proof. (i) Assume that $E_1$ is defined by $Y_1$ and $E_1 \cong E_2$, then $Y_1 = Y_2^\sigma$ for some $\sigma \in \text{PGL}(r-1)$ by virtue of Proposition 2.12. Since $\sigma$ sends $P_{T_1}^{r-1}$ to itself and $\Gamma(Y_1) = T_1$, which is a contradiction. Thus $E_1 \ncong E_2$.

(ii) Each element $\sigma$ of $\text{PGL}(r-1)$ sends $P_{T_1}^{r-1}$ to itself and $Y^\sigma \sim Y$ in $P_{T}^{r-1}$. Thus if $Y \in R_0(T)$, then $Y^\sigma \in R_0(T)$, that is, $R_0(T)$ is $\text{PGL}(r-1)$-stable. Let $\overline{\Phi}$ be the canonical morphism $\text{Div}_{T/k} \rightarrow \text{Pic}(T)$ and let $\tau$ be the morphism of $\text{Pic}(T)$ to itself defined as follows: $\text{Pic}(T) \ni D \mapsto T^{-1}(D) \in \text{Pic}(T)$. Then $A_T = \text{Pic}(T) - \tau^{-1}(\overline{\Phi}(\text{Div}_{T/k}))$. Thus $A_T$ is an open subset of $\text{Pic}(T)$ because $\overline{\Phi}$ is projective ([6] Corollary 4.4). On the other hand, there is a canonical isomorphism $j : \text{Pic}(P_{T}^{r-1}) \rightarrow Z \times \text{Pic}(T)$ and $j \circ \overline{\Phi}$ sends $R^r(T)$ to $\{1\} \times \text{Pic}(T)$ for the canonical morphism $\overline{\Phi}$:

$$j \circ \overline{\Phi} : \text{Div}_{P_{T}^{r-1}/k} \rightarrow \text{Pic}(P_{T}^{r-1}) .$$

This map $j \circ \overline{\Phi} |_{R^r(T)}$ is defined as follows:

$$\forall D \in \text{Pic}(T), \quad | Z \times T + \langle \eta_T \rangle^{-1}(D) \Rightarrow Y \mapsto D \in \text{Pic}(T).$$
Thus $R^r_0(T) = \left( j_{T^r} \right)^{-1}(A_T) \cap R^r(T)$ which is an open subset of $R^r(T)$.

(iii) is a direct corollary of Proposition 2.12. q.e.d.

Theorem 2.14. Let $S$ be a non-singular projective variety over $k$, $T$ a non-singular subvariety of codimension 1 and let $D \in A_T$.

Then there is a subset $R^r(S,T,D)$ of $R^r(S,T,D)$ which carries the structure of an open set of Grass$_{r-1}(K)$, where $n + 1 = \dim_k H^0(T, O_T(D))$. Moreover, if $r = 2$, then $R^2(S,T,D) = R^2(S,T,D)$.

Proof. Fix a basis $a_0, \ldots, a_n$ of $H^0(T, O_T(D))$. If $(s_0, \ldots, s_{r-1}) = (s_0, \ldots, s_{r-1})$ is an element of $H^0(T, O_T(D)) \times \cdots \times H^0(T, O_T(D))$ and $s_i = \sum \alpha_{ij} a_j (\alpha_{ij} \in k)$, then $(s_0, \ldots, s_{r-1})$ or the $r \times (n+1)$-matrix $(\alpha_{ij})$ defines a member of $\left| Z \times T + (\nu_T)^{-1}(D) \right|$. For each $(\beta_{ij}) \in GL(r, k)$, the action $(\alpha_{ij}) \mapsto (\beta_{ij}) (\alpha_{ij})$ induces the action of $PGL(r-1)$ on $\left| Z \times T + (\nu_T)^{-1}(D) \right|$ which is the same action defined before Proposition 2.12. Let $U$ be the subset of $R^0(T, O_T(D)) \times \cdots \times R^0(T, O_T(D))$ which consists of element $(s_0, \ldots, s_{r-1})$.
such that \( s_0, \ldots, s_{r-1} \) are independent over \( k \) and let \( U' \) be the subset of \( \left\| \mathbb{Z} \times T + (\mathbb{N}_D)^{-1}(D) \right\| \) determined by \( U \) (\( U \) may be empty).

Then \( U \) (or, \( U' \)) is \( GL(r) \)-stable (or, \( PGL(r-1) \)-stable, resp.) and \( U/GL(r) \) is in bijective correspondence with \( U'/PGL(r-1) \). Furthermore it is clear that \( U/GL(r) = Grass_{r-1}^n \). Consider the following morphism

\[ \psi : \mathbb{A}^r \to TXU \]

\[ (t, s_0, \ldots, s_{r-1}) \mapsto (s_0(t), \ldots, s_{r-1}(t)) \in \mathbb{A}^r. \]

Then the set \( F = \left\{ (s_0, \ldots, s_{r-1}) \in U \mid s_0(t) = \ldots = s_{r-1}(t) = 0 \right\} \) is \( p(\mathbb{A}^r_0) \) for the projection \( p : TXU \to U \). Since \( T \) is projective, \( F \) is closed in \( U \) and it is \( GL(r) \)-stable. Thus \( (U-F)/GL(r) \) is an open set of \( Grass_{r-1}^n \).

By virtue of Principle 2.6 and Proposition 2.12 we see that \( ((U-F)/GL(r))(k) \)

\[ R^r(S,T,D) \]

is in bijective correspondence with a subset \( SR^r(S,T,D) \). Now, if \( r = 2 \) and \( s_0, s_1 \) are dependent (\( s_0 = 0 \)), then \( s_1 = \alpha s_0 \) for some \( \alpha \in k \), whence \( s_1(t) = 0 \) for any \( t \in T \) with \( s_0(t) = 0 \). Thus such a \( (s_0, s_1) \) defines no element of \( R^2(S,T,D) \). We know therefore

\[ SR^2(S,T,D) = R^2(S,T,D) \bigcap D \neq 0, \quad \text{q.e.d.} \]

Remark 2.15. \( SR^r(S,T,D) \) may be empty. We raise a problem :
Does there exist a $D$ for fixed $S$, $T$ such that $SR_r^r(S,T,D) \not= \emptyset$?

We know that if $r \geq \dim S$, then such a $D$ exists and that

$$\sup_{D \in A_T} (\dim SR_r^r(S,T,D)) = \infty.$$  

**Proof.** Take a very ample divisor $D$ on $T$ such that

$$\dim_k H^0(T, O_T(T^2-D)) = 0 \quad \text{and} \quad \dim_k H^0(T, O_T(D)) \geq r.$$  

Since $r \geq \dim S$ and $D$ is very ample, $s_0$, ..., $s_{r-1}$ are independent and each of $s_0(t)$, ..., $s_{r-1}(t)$ is not zero for any $t \in T$ if $s_0$, ..., $s_{r-1}$ are sufficiently general elements of $H^0(T, O_T(D))$. Then $(s_0, ..., s_{r-1})$ defines an element of $SR_r^r(S,T,D)$ and if $\dim_k H^0(T, O_T(D)) = n + 1,$

then $\dim SR_r^r(S,T,D) = \dim \text{Grass}_{r-1}^n = r(n + 1 - r).$ Thus $SR_r^r(S,T,D)$

$\not= \emptyset$ and $\sup_{D \in A_T} (\dim SR_r^r(S,T,D)) = \infty$.

**Remark 2.16.** i) $R^r(S,T, O) = \{ s_0, ..., s_r \}$

(ii) $R^2(S,T,D) \not= \emptyset$ for some $D \not= 0$ if and only if there exists a morphism $f$ of $T$ to a curve $C$.

**Proof.** (i) is a direct conclusion of Lemma 1.4 and Lemma 2.2.

(ii) If $R^2(S,T,D) \not= \emptyset$, then there exist two sections $s_0, s_1$ of $H^0(T, O_T(D))$ such that both $s_0(t), s_1(t)$ are not zero for any
Thus \( T \ni t \mapsto (s_0(t), s_1(t)) \in \mathbb{P}^1 \) is a morphism.

Conversely assume that there exists a morphism \( f : T \to C \) (we may assume that \( C \) is non-singular because so is \( T \)). Take a very ample divisor \( D \) on \( C \). Then \( H^0(T, f^*(\mathcal{O}_C(D))) \) contains two sections \( s_0, s_1 \) such that both \( s_0(t) \) and \( s_1(t) \) are not zero for any \( t \in T \). By virtue of Principle 2.6, we know therefore \( R^2(S, T, f^{-1}(D)) \neq 0 \). q.e.d.

The above proof show that if \( R^2(S, T, D) \neq 0 \) for some \( D \), then

\[
\sup_{D \in A_T} \dim R^2(S, T, D) = \infty \quad \text{and} \quad D^2 = 0.
\]

Example 2.17. i) \( R^2(P^3, T, D) = 0 \) for any \( D \neq 0 \) if \( T \) is a plane, \( R^2(P^3, Q, D) \neq 0 \) for some \( D \) if \( Q \) is a quadric surface because \( Q \cong P^1 \times P^1 \).

ii) \( R^2(F^r, T, D) = 0 \) (\( r \geq 0 \)) for any \( T \) and \( D \neq 0 \). For if there exists a morphism \( f \) of \( T \) to a curve \( C \), then \( \dim f^{-1}(p) = r - 2 \) for any \( p \in C \), which is a contradiction because \( \dim(f^{-1}(p) \cap f^{-1}(p')) \geq 0 \) and therefore \( f^{-1}(p) \cap f^{-1}(p') \neq \emptyset \). Thus every regular vector bundle of rank 2 on \( F^r \) (\( r \geq 4 \)) is isomorphic to \( \mathcal{O}_p \oplus \mathcal{O}_p(T) \) for some
non-singular subvariety \( T \) of codimension 1.

iii) If there exists a morphism of \( S \) to a curve, then \( R^2(S, T, D) \) is 0 for any \( T \) not contained in any fibre of the morphism and for some \( D \).

§ 3. Chern classes of regular vector bundles.

In this section we shall calculate Chern classes of regular vector bundles.

Lemma 2.18. Let \( E \) be a vector bundle of rank \( r (\geq 2) \) on \( S \) and let \( O_X(1) \) be the tautological line bundle of \( E \) for \( X = P(E) \).

If \( H_1, \ldots, H_r \) are divisors on \( X \) such that \( O_X(H_i) \cong O_X(1) \) for every \( i \) and that they intersect properly, then \( p_*(H_1 \cdots H_r) = c_1(E) \) for the natural projection \( p : X \to S \).

Proof. Consider the Chern polynomial

\[
H_1 \cdots H_r - p*(c_1(E)) - H_1 \cdot \cdots \cdot H_{r-1} + p*(c_2(E)) - H_1 \cdot \cdots \cdot H_{r-2} + \ldots + (-1)^{r-1}p*(c_{r-1}(E)) - H_1 + (-1)^r p*(c_r(E))
\]

= 0. Operating \( p_\ast \) on the polynomial, one gets \( p_*(H_1 \cdot \cdots \cdot H_r) = p_*(p*(c_1(E)) - H_1 \cdot \cdots \cdot H_{r-1}) = c_1(E) \) because \( p_*(H_1 \cdot \cdots \cdot H_{r-1}) = 1.\)
Lemma 2.2, Lemma 2.10 and Lemma 2.18 yield

Corollary 2.18.1. If $E \in R^r(S, T, D)$, then $\sigma_1(E) = T$.

However a more general result is given by the following theorem.

Theorem 2.19. If $E \in R^r(S, T, D)$, then

$$\text{ch}(E) = r + \sum_{i=1}^{\alpha} \frac{\gamma_i^i}{i!} + \sum_{m', n' = 1}^{\beta} \frac{(-1)^{m'-1} \cdot 1_a(D^n)}{m' \cdot n'}$$

where $\text{ch}(E)$ is the Chern character and $i : T \rightarrow S$ is the inclusion.

Proof Assume that $E$ is defined by $\mathcal{Y} \in 2 \times T + \frac{-1}{T}(D)$. The following exact sequence

$$0 \rightarrow \mathcal{Y} \otimes O_{p^{-1}(H, r)} \rightarrow O_{p^{-1}(H, r)} \rightarrow O_{p^{-1}(H, r)} \rightarrow 0$$

yields an exact sequence

$$0 \rightarrow E \rightarrow \tau_*(\mathcal{Y} \otimes O_{p^{-1}(H, r)}) \rightarrow 0$$

because $E \cong \tau_*(\mathcal{Y} \otimes O_{p^{-1}(H, r)})$, $\tau^1\tau_*(\mathcal{Y} \otimes O_{p^{-1}(H, r)}) = 0$ by virtue of the definition of regular vector bundle and Theorem 1.3. If one puts

$$F = (\tau_*)^*(-1)(\mathcal{Y} \otimes O_{p^{-1}(H, r)})$$

then by virtue of the Riemann-Roch theorem of Grothendieck for the morphism $i : T \rightarrow S$, we have
(1) \[ \text{ch}(E) = \text{ch}(\mathcal{O}_S(T) \otimes \mathcal{O}_{p-1}(H_Y)) \]

\[ = r \text{ch}(\mathcal{O}_S(T)) - \text{ch}(\mathcal{O}_S(T) \otimes \mathcal{O}_{p-1}(H_Y)) \]

\[ = r \text{ch}(\mathcal{O}_S(T)) - \text{ch}(\mathcal{O}_S(T)) \cdot \text{td}(N_{T/S})^{-1}, \]

where \( N_{T/S} \) is the normal bundle of \( T \) in \( S \) and \( \text{td} \) is the Todd class. On the other hand, for the ideal \( J_Y \) of \( Y \) in \( \mathbb{P}^{r-1}_T \), the following exact sequence

\[ 0 \rightarrow J_Y \otimes \mathcal{O}_{p-1}(H_Y) \rightarrow \mathcal{O}_{r-1}(H_Y) \rightarrow 0 \]

provides an exact sequence

\[ 0 \rightarrow (\mathcal{O}_{T^2}) \rightarrow \mathcal{O}_{r-1}(H_Y) \rightarrow 0 \rightarrow 0. \]

Since \( J_Y \otimes \mathcal{O}_{p-1}(H_Y) \cong \mathcal{O}_{r-1}(-\pi \times T + (\mathcal{O}_{T^2})^{-1}(T^2)) \)

\[ \cong (\mathcal{O}_{T^2}) \), we know \( R^1(\mathcal{O}_{T^2}) \rightarrow J_Y \otimes \mathcal{O}_{p-1}(H_Y) = 0. \) Thus

the above exact sequence implies

\[ \text{ch}(\mathcal{O}_T(\mathcal{O}_{T^2})) = r(\text{ch}(\mathcal{O}_T(\mathcal{O}_{T^2}) \otimes \mathcal{O}_T(-D))) \]

\[ = \text{ch}(\mathcal{O}_T(\mathcal{O}_{T^2})(r - \text{ch}(\mathcal{O}_T(-D))) \]

\[ = \left( \sum_{\alpha=0}^{\infty} \frac{T^\alpha}{\alpha!} \right) \left( r - 1 - \sum_{n=1}^{\infty} \frac{(-1)^n D^n}{n!} \right) \]

where \( T' = T^2 \) in \( T \). As to \( \text{td}(N_{T/S})^{-1} \) we get
\[
(3) \quad \text{td}(N_{T/S})^{-1} = \left(\frac{T'}{1 - e^{-T'}}\right)^{-1} = \sum_{\beta=1}^{\infty} \frac{(-1)^{\beta-1}T'}{\beta!}.
\]

The above (2), (3) yield

\[
(4) \quad \text{ch}(F) \cdot \text{td}(N_{T/S})^{-1} = (r-1)\left(\frac{\infty}{\alpha=0} \frac{(-1)^{\beta-1}T'_{\alpha+\beta-1}}{\beta!} \right) - \sum_{\alpha=0}^{\infty} \frac{(-1)^{\beta+n-1}T'_{\alpha+\beta-1}D^n}{\alpha! \beta! n!}
\]

\[
= (r-1) \sum_{\beta=1}^{\infty} T'_{\beta-1} \left(\sum_{\alpha=0}^{\infty} \frac{(-1)^{\beta-1}}{\alpha! \beta!} \right) \left(\sum_{n=1}^{\infty} \frac{(-1)^n D^n}{n!} \right)
\]

\[
\left(\sum_{\alpha+\beta=m}^{\infty} \frac{(-1)^{\beta-1}}{\alpha! \beta!} \right)
\]

Since \(\sum_{a+b=c} \frac{(-1)^{b-1}}{a! b!} = \frac{1}{c!}\), (4) reduces to the following:

\[
(4)' \quad \text{ch}(F) \cdot \text{td}(N_{T/S})^{-1} = (r-1) \sum_{\beta=1}^{\infty} T'_{\beta-1} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^n T'_{m-1} D^n}{m! n!}
\]

By virtue of (1), (4)'

\[
(5) \quad \text{ch}(E) = r \sum_{l=0}^{\infty} \frac{T'_{l}}{l!} - i_*((r-1) \sum_{l=1}^{\infty} \frac{T'_{l-1}}{l!}) - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^n T'_{m-1} D^n}{m! n!}
\]

Since \(i_*(T'^{l}) = T'^{l+1}\), and \(i_*(T'^{m-1}D^n) = T'^{m-1} i_*(D^n)\)
Corollary 2.19.1. If \( E \in R^r(S,T,A) \), then \( c_1(E) = T \), \( c_2(E) = D \), \( c_3(E) = i_*(D^2) \) in \( A(S) \otimes \mathbb{Q} \), where \( A(S) \) is the Chow ring of \( S \).

Proof. Note that \( \text{ch}(E) = r + c_1(E) + \frac{1}{2}(c_1(E)^2 - c_2(E)) + \frac{1}{6}(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)) + \text{higher term} \). Then our assertion is an immediate corollary of Theorem 2.19.

If \( r = 2 \), then \( c_3(E) = 0 \) and so the above corollary implies that \( i_*(D^2) = 0 \), but fortunately Remark 2.16 implies that if \( R^2(S,T,D) \neq \emptyset \) then \( D^2 = 0 \).

Remark 2.20. Corollary 2.18.1 asserts that if \( E \in R^r(S,T,D) \), then \( c_1(E) = T \) in \( A(S) \). Thus we have the following problem:

For \( E \in R^r(S,T,D) \), \( c_2(E) = D \), \( c_3(E) = i_*(D^2) \) in \( A(S) \)?
Chapter III. Simple vector bundles.

In this chapter we maintain the notation in the preceding chapter.

§ 1. Simple regular vector bundles.

Let $E$ be a vector bundle on a scheme $X$, then $\text{End}(E) = \text{Hom}_X(E, E)$ contains $O_X$ as scalar multiplications. Thus

$$\text{End}(E) = \text{Hom}_X(E, E) \text{ naturally contains } \Gamma(X, O_X).$$

**Definition.** A vector bundle $E$ on a scheme $X$ is called simple if $\text{End}(E) = \Gamma(X, O_X)$.

Our aim of this section is to show that $\text{SR}(S, T, D)$ in Theorem 2.14 consists of all simple vector bundles in $\text{R}(S, T, D)$.

**Lemma 3.1.** Let $X$ be a complete variety over $k$ and let $E$ be a vector bundle of rank $r$ on $X$.

(i) $\text{Aut}_X(E)$ is a connected linear group and $\dim \text{Aut}_X(E) = \dim_k \text{End}(E)$.

(ii) $E$ is indecomposable if and only if rank (i.e. dimension of a maximal torus) of $\text{Aut}_X(E) = 1$. 
Proof. If $X = \bigcup_{i \in I} U_i$ is a sufficiently fine open covering of $X$, then $E|U_i$ is free for any $i$, and an element $\sigma \in \text{End}(E)$ is represented by $\{\sigma_i\}_{i \in I}, \sigma_i \in \text{End}(r, E_i)$ such that

$$\sigma_i A_{ij} = A_{ij} \sigma_j$$

for the transition matrix $A_{ij}$ of $E$ in $U_i \cap U_j$.

Since $(\sigma_i - xI) A_{ij} = A_{ij} (\sigma_j - xI)$ in $U_i \cap U_j$ with an indeterminate $x$ and the unit matrix $I$, we see that $\det(\sigma_i - xI) = \det(\sigma_j - xI)$ in $U_i \cap U_j$ for any $i, j$. Thus there exists a polynomial $F(x) \in k[x]$ with $F(x) = \det(\sigma_i - xI)$ for any $i$ because $X$ is a complete variety over $k$. Hence every eigenvalue of $\sigma_i$ is independent of $i$, contained in $k$ and $\det \sigma = \det \sigma_i$ is an element of $k$. Take a free basis $e_1, \ldots, e_n$ of $\text{End}(E)$ with $e_1 = \text{id}_E$. $\sigma = \sigma_1 e_1 + \ldots + \sigma_n e_n$ is contained in $\text{Aut}_X(E)$ if and only if $\det \sigma \neq 0$. The above argument implies that $\det \sigma$ is a polynomial of $\sigma_1, \ldots, \sigma_n$ over $k$, and if $\sigma$ is not an eigenvalue of $\sigma$, then $\sigma - \sigma_1 e_1 = (\sigma_1 - \sigma') e_1 + \sigma_2 a_2 + \ldots + \sigma_n a_n$ is contained in $\text{Aut}_X(E)$. Thus $\text{Aut}_X(E)$ is an open dense subset in $\text{End}(E)$, which implies that $\text{Aut}_X(E)$ is a connected linear
group (ii) is easy, if one takes Lemma 6, Lemma 7 of \([\text{I}]\) and the
above argument into account.

q.e.d.

**Corollary 3.1.1.** A vector bundle \(E\) on a complete variety
over \(k\) is simple if and only if \(\text{Aut}_x(E) = G_m = k^*\).

**Proof.** If \(E\) is simple, then \(\text{End}(E) = \Gamma(X, O_X) = k\) which
acts on \(E\) as scalar multiplications. Thus \(\text{Aut}_x(E) = G_m\).

Conversely assume \(\text{Aut}_x(E) = G_m\). Then by virtue of Lemma 3.1
\(\dim_k \text{End}(E) = 1\), whence \(\text{End}(E) = k = \Gamma(X, O_X)\).

For a vector bundle \(E\) on a scheme put \(\Delta(E) = \left\{ L \mid L \text{ is the}
\text{isomorphism class of a linebundle } L \text{ with } E \cong E \oplus L \right\}\). Then we
get the following exact sequence of groups ([5] Corollary to
Proposition 2);

\[
e \to \text{Aut}_x(E) / \Gamma(X, O_X^*) \to \text{Aut}_x(P(E)) \to \Delta(E) \to e
\]

If \(X\) is a complete variety over \(k\), then \(\Gamma(X, O_X^*) = G_m\). If \(X\)
is complete and normal, then \(\Delta(E)\) is a finite group, because
\( \mathfrak{E} \subseteq \mathfrak{E} \otimes \mathfrak{L} \) implies \( L^{\mathfrak{G}(\text{rank}\mathfrak{E})} \cong \mathcal{O}_X \) and therefore \( \Delta(\mathfrak{E}) \) is contained in \( (\text{rank}\mathfrak{E}) - \text{torsion part of Pic}^{0}(X) \) which is an abelian variety.

Thus under these assumptions \( \text{Aut}_X(\mathfrak{E})/\mathbb{G}_{\text{m}} = \text{Aut}_X^0(\mathbb{P}(\mathfrak{E})) \), where \( \text{Aut}_X^0(\mathbb{P}(\mathfrak{E})) \) is the connected component of \( \text{Aut}_X(\mathbb{P}(\mathfrak{E})) \). Therefore we get

Corollary 3.1.2. A vector bundle \( \mathfrak{E} \) on a complete normal variety \( X \) over \( k \) is simple if and only if \( \text{Aut}_X^0(\mathbb{P}(\mathfrak{E})) = e \).

In order to investigate whether a regular vector bundle \( \mathfrak{E} \) on \( S \) is simple or not, let us study \( \text{Aut}_S^0(\mathbb{P}(\mathfrak{E})) \).

Lemma 3.2. If \( \mathfrak{E} \) is a regular vector bundle on \( S \) of rank \( r \) defined by \( Y \) and if \( \dim_k H^0(S, \mathfrak{E}) = r \) (cf. Corollary 2.11.1), then \( \text{Aut}_S^0(\mathbb{P}(\mathfrak{E})) \cong \{ \sigma : \sigma \in \text{PGL}(r - 1) = \text{Aut}_{\mathbb{P}^{r-1}}(S), \quad Y^\sigma = Y \} \).

Proof. The assumption \( \dim_k H^0(S, \mathfrak{E}) = r \) implies that \( H'_1, \ldots, H'_r \) form a basis of \( |H'_1| \), where \( H'_1 = \text{elm}_k(H_1), \quad H_i = \mathbb{P}_{\mathbb{F}_k}^1 + \mathbb{F}_k \) for independent hyperplanes \( Z_1, \ldots, Z_r \) of \( \mathbb{F}_k^r \). Since \( \sigma \in \text{Aut}_S(\mathbb{P}(\mathfrak{E})) \) is contained in \( \text{Aut}_S^0(\mathbb{P}(\mathfrak{E})) = \text{Aut}_S(\mathbb{P}(\mathfrak{E}))/\mathbb{G}_{\text{m}} \) if and only if \( \sigma^*(\mathcal{O}_S(1)) \cong \mathcal{O}_S(1) \) for the tautological linebundle \( \mathcal{O}_S(1) \) of \( E \) and since \( \mathcal{O}_S(H'_1) \cong \mathcal{O}_S(1) \), we have
(\bigwedge_{i=1}^{n} H_{1}^{i})^{*} \cong \bigwedge_{i=1}^{n} H_{1}^{i}. On the other hand, \bigwedge_{i=1}^{n} H_{1}^{i} coincides with the center \( Y' \) of \( (\text{elm}_{Y}^{n-2})^{-1} \) by virtue of Lemma 2.10. Thus \( Y' \sigma = Y' \).

Now we claim

**Lemma 3.3.** Let \( \mathcal{W} : X \rightarrow S \) be a \( \mathbb{P}^{N} \)-bundle and let \( T, Y \) be subschemes \( S, X \) satisfying the condition \( (E_{0}^{x}) \). If \( \sigma \in \text{Aut}_{S}(X) \) satisfies \( Y \sigma = Y \), then \( \mathcal{W} \) induces a unique element \( \sigma' \) of \( \text{Aut}_{S}(X') \) with \( X' = \text{elm}_{Y}^{n}(X) \) such that \( Y' \sigma' = Y' \) with the center \( Y' \) of \( (\text{elm}_{Y}^{n-1})^{-1} \) and \( \sigma'_{1} | X'(S-T) = \sigma'_{1} X(S-T) \) by the natural identification \( X'_{(S-T)} = X(S-T) \).

**Proof.** Cover \( X \) by a system of affine open sets \( \{ \mathcal{U}_{a} \} \) such that \( X_{\mathcal{U}_{a}} = \text{Proj}(A[\gamma_{0}, \ldots, \gamma_{N}]) \) and \( Y_{\mathcal{U}_{a}} \) is defined by the ideal \( (t \in A, \gamma_{n+1}, \ldots, \gamma_{N}) \). Let \( \gamma_{1}^{\sigma} = \sum_{j=0}^{N} a_{ij} \gamma_{j}^{'} \), \( a_{ij} \in A \), then the condition \( Y \sigma = Y \) implies \( a_{ij} = t a_{ij}^{'} \), \( a_{ij} \in A \) for \( n+1 \leq i \leq N, 0 \leq j \leq n \). By virtue of Lemma 1.4 \( \text{elm}_{Y}^{n}(X_{\mathcal{U}_{a}}) = \text{Proj}(A[\gamma_{0}^{'} , \ldots, \gamma_{N}^{'}]) \), \( \gamma_{1} = \gamma_{1}^{'}(0 \leq i \leq n), \quad \gamma_{1} = t \gamma_{1}^{'}(n+1 \leq i \leq N). \)

Thus \( \gamma_{1}^{\sigma} = \sum_{j=0}^{N} a_{ij} \gamma_{j}^{'} + \sum_{j=n+1}^{N} t a_{ij} \gamma_{j}^{'} \) \( (0 \leq i \leq n), \quad \gamma_{1}^{\sigma} = \sum_{j=0}^{N} a_{ij} \gamma_{j}^{'} \) \( (n+1 \leq i \leq N). \) Hence \( \sigma \) induces a morphism \( \sigma' | \mathcal{U}_{a} \).
of \( X'_{U_\alpha} \) to itself. \( \sigma'_{U_\alpha} \) is an automorphism of \( X'_U \) because if 

\( T \) is the inverse of \( \sigma' \), then 

\[
\tau'_{U_\alpha} \cdot \sigma'_{U_\alpha} = \sigma'_{U_\alpha} \cdot \tau'_{U_\alpha} = \text{id}_{X'_U}.
\]

Moreover \( \sigma'_{U_\alpha} \) coincides with \( \sigma'_{U_\beta} \) in an open dense subset \( X'(S-T) \cap X'_{\alpha \beta} = X'_{U_\alpha} \cap U_\beta \), which implies 

\[
\sigma'_{U_\alpha} \mid X'_{\alpha \beta} = \sigma'_{U_\beta} \mid X'_{\alpha \beta}
\]

because the set \( \{ x \in X'_{\alpha \beta} \mid x \sigma'_{U_\alpha} = x \sigma'_{U_\beta} \} \) is closed in \( X'_{\alpha \beta} \).

Thus \( \sigma \) induces an element \( \sigma' \) of \( \text{Aut}_S(X') \). It is obvious that 

\( \sigma' \) is a desired automorphism. If \( \sigma_1', \sigma_2' \) are automorphisms of \( X' \) induced by \( \sigma \), then \( \sigma_1' = \sigma_2' \) in an open dense subset, 

whence \( \sigma_1' = \sigma_2' \).

Now we shall come back to the proof of Lemma 3.2. By virtue of Lemma 3.3, \( \sigma \) induces an element of the group \( G(Y) = \{ \tau \in \text{PGL}(r-1) \mid Y^\tau = Y \} \). Thus we have a homomorphism \( \phi : \text{Aut}_S^0(P(E)) \rightarrow G(Y) \). We get also a homomorphism \( \psi : G(Y) \rightarrow \text{Aut}_S^0(P(E)) \) because \( \tau \in \text{Aut}_S^0(P(E)) \) induced by \( \tau \in G(Y) \) sends \( H_1 \) to an element of \( H_1 \), which means \( \tau^*(O_P(E)(1)) \cong O_P(E)(1) \). Clearly

\[
\phi \cdot \psi = \text{id}, \quad \psi \cdot \phi = \text{id},
\]

Thus \( \text{Aut}_S^0(P(E)) \cong G(Y) \).

q. e. d.
Now we come to a main theorem of this section.

Theorem 3.4. Let $S$ be a non-singular projective variety over $k$, let $T$ be a non-singular subvariety of $S$ of codimension one and let $D \in A_T$.

(i) $\overline{SR}(S, T, D)$ in Theorem 2.14 consists of all simple vector bundles in $R^*(S, T, D)$.

(ii) If $E \in R^*(S, T, D)$ is defined by $(s_1, \ldots, s_r) \in H^0(S, T, D) \times \cdots \times H^0(T, O_T(D))$ (cf. Principle 2.6) and if the dimension of the vector subspace of $H^0(T, O_T(D))$ generated by $s_1, \ldots, s_r$ is $r'$, then $E \cong O_S^{\oplus(r-r')} \oplus E'$ for some $E' \in \overline{SR}^{r'}(S, T, D)$.

Proof. Assume that $E$ is defined by $(s_1, \ldots, s_r) \in H^0(T, O_T(D)) \times \cdots \times H^0(T, O_T(D))$. In the first place note that $D \in A_T$ implies $\dim_k H^0(S, E) = r$ by virtue of Lemma 2.11, and therefore by virtue of Lemma 3.2 $\text{Aut}_S^0(P(E)) \cong G(Y) = \{ \sigma \mid \sigma \in \text{PGL}(r-1) \}$ for the subscheme $Y$ of $P^{r-1}_T$. 
whose ideal in $\mathbb{P}^{r-1}_T$ is generated by $s_1\gamma_1 + \ldots + s_r\gamma_r$, where

\(\gamma_1, \ldots, \gamma_r\) form a system of homogeneous coordinates of \(\mathbb{P}^{r-1}_S\) (cf. Principle 2.6 and Principle 2.6). Since every $s_1(t), \ldots, s_r(t)$ is not zero for any $t \in T$, the rational map \(\varphi : T \ni t \mapsto (s_1(t), \ldots, s_r(t)) \in \mathbb{P}^{r-1}\) is a morphism. On the other hand, since $Y_t \in \mathbb{P}$, where $\mathbb{P}$ is regarded as the dual space of \(\mathbb{P}^{r-1}_k\).

This map is nothing but \(\varphi\). Moreover the action of $\text{PGL}(r-1) = Aut_\mathbb{P}(\mathbb{P}^{r-1})$ on $\mathbb{P}^{r-1}_S = \mathbb{P}^{r-1}_k \times S$ induces that on the dual space $\mathbb{P}$ of $\mathbb{P}^{r-1}_k$ through contragradient linear transformations.

Thus the condition $Y^g = Y$ for $g \in \text{PGL}(r-1)$ is equivalent to $x^g = x$ for any $x \in \varphi(T)$ by the above action. This implies $G(Y) = \{ g \circ \varphi(T) \mid g \in \text{PGL}(r-1), x^g = x \text{ for any } x \in \varphi(T) \}$. Assume $\mathbb{E} \in \mathcal{SR}^r(S, T, D)$, that is, $s_1, \ldots, s_r$ are linearly independent in $H^0(T, \mathcal{O}_T(D))$, then $\varphi(T)$ is contained in no hyperplane of $\mathbb{P}$, whence there exist linearly independent $k$-rational points $x_1, \ldots, x_{r-1}, x_{i+1}, \ldots, x_r$, then $\bigcup_{i=1}^{r} L_i \nsubseteq \varphi(T)$ because $L_i \nsubseteq \varphi(T), 1 \leq i \leq r$. Thus there exists a $k$-rational point $x_{r+1}$ in $\varphi(T)$, let $L_i$ be the linear subspace in $\mathbb{P}$ generated by $\gamma_i$. 

\begin{align*}
\{ x_1, \ldots, x_r \text{ in } \varphi(T) \}, \text{ let } L_i \text{ be the linear subspace in } \mathbb{P} \text{ generated by } \gamma_i.
\end{align*}
in \( \Phi(T) - \bigcup_{i=1}^{r} L_i \). Then any \( r \) points in \( \{ x_1, \ldots, x_{r+1} \} \) are linearly independent in \( P \). We know therefore \( G(Y) = \{ e \} \) because \( \sigma \in G(Y) \) fixes every \( x_1, \ldots, x_{r+1} \). Hence if \( E \in \mathcal{S}R^r(S, T, D) \), then \( E \) is simple. Now we have only to prove (ii) because a simple vector bundle is indecomposable (see Lemma 3.1, (ii)). It is easy, however, that every vector bundle in \( \mathcal{R}^r(S, T, D) - \mathcal{S}R^r(S, T, D) \) is decomposable. In fact if \( E \in \mathcal{R}^r(S, T, D) - \mathcal{S}R^r(S, T, D) \), then \( s_1, \ldots, s_r \) are dependent, which means \( \Phi(T) \subseteq \mathcal{N} \) for some hyperplane \( \mathcal{N} \) of \( P \). Thus \( G(Y) \supseteq \{ \sigma \in \text{PGL}(r - 1) \mid x^\sigma = x \text{ for any } x \in \mathcal{N} \} \). Hence rank of \( \text{Aut}_S(E) \) = rank of \( \text{Aut}_S^0(P(E)) + 1 \) \( \geq 2 \), which asserts that \( E \) is decomposable by virtue of Lemma 3.1, (i) is therefore proved. Next let us proceed to the proof of (ii).

We may assume that \( s_{r+1}, \ldots, s_r \) are linearly independent for \( r'' = r - r' \). Since \( \Phi(T) \) is contained in a linear subspace of dimension \( r' - 1 \) of \( P \) and none of those of dimension \( r' - 2 \), there are \( k \)-rational points \( x_1, \ldots, x_{r'} \) such that \( \Phi(x_1), \ldots, \Phi(x_{r'}) \) are linearly independent in \( P \). Put
are linearly independent over \( k \) because \( \text{rank}(s_j(x_j)) = r' \) and

\[ s_1, \ldots, s_{r''} \text{ depend on } s_{r'+1}, \ldots, s_r. \]

Thus we can adopt

\[ \gamma_1, \ldots, \gamma_{r''}, \zeta_1, \ldots, \zeta_{r''} \]

as a homogeneous coordinate of \( \mathbb{P}^r_{S} \). Moreover \( Y \) is defined by \( s_1 \zeta_1 + \ldots + s'_{r''} \zeta_{r''} = 0 \) for some linearly independent \( s_1', \ldots, s_{r''}' \in H^0(T, \mathcal{O}_T(\mathcal{D})) \) because

\[ s_{r'+1}, \ldots, s_r \text{ are linearly independent.} \]

There are therefore an affine open covering \( \{ U_{\lambda} = \text{Spec}(A_{\lambda}) \}_{\lambda \in \Lambda} \) of \( S \) and a correspondence

\[ \Lambda \ni \lambda \longrightarrow \{ \lambda(1), \ldots, \lambda(\text{r}^* - 1) \} \subset \{ 1, \ldots, \text{r}' \} \]

such that \( \mathbb{P}^r_{S, U_{\lambda}} := \text{Proj}(A_{\lambda}[\gamma_1, \ldots, \gamma_{r''}, \zeta_1, \ldots, \zeta_{r''}, \lambda(1), \ldots, \lambda(\text{r}' - 1)], \zeta_{\lambda}) \). \( T \cap U_{\lambda} = \text{Spec}(A_{\lambda}/t_\lambda A_{\lambda}) \) for some \( t_\lambda \in A_{\lambda} \).

\[ \zeta_{\lambda} = a_1^{(\lambda)} \zeta_1 + \ldots + a_{r''}^{(\lambda)} \zeta_{r''} \text{ for some } a_1^{(\lambda)}, \ldots, a_{r''}^{(\lambda)} \in A_{\lambda} \]

and that the ideal of \( Y_{U_{\lambda}} \) in \( \mathbb{P}^r_{S, U_{\lambda}} \) is generated by \( t_\lambda, \zeta_{\lambda} \). Then

\[ X_{U_{\lambda}} = \text{elm}_{Y_{U_{\lambda}}}^r \mathcal{I}_{S, U_{\lambda}} = \text{Proj}(A_{\lambda}[\gamma_1, \ldots, \gamma_{r''}, \zeta_1(1), \ldots, \zeta_{r''}(1), \ldots, \zeta_{\lambda}, \ldots, \zeta_{\lambda}(r' - 1)], \zeta_{\lambda}) \]

for \( t_\lambda \zeta_{\lambda} = \zeta_{\lambda} \) by virtue of Lemma 1.4. By the construction the ideals \( I_{U_{\lambda}}, J_{U_{\lambda}} \) generated by \( \{ \gamma_1, \ldots, \gamma_{r''} \} \)

\[ \{ \zeta_1(1), \ldots, \zeta_{\lambda}(r' - 1), \zeta_{\lambda} \} \]

respectively define global
ideals, that is, there are ideals $I, J$ in $O_X$, for $X' = \text{elm}^{r-2}$ $P_{S}$ with $IO_{X_{\lambda}} = I_{U_{\lambda}}, JO_{X_{\lambda}} = J_{U_{\lambda}}$ for any $\lambda \in \Lambda$. $I, J$ define projective subbundle $P_1, P_2$ of $X' = P(E)$ such that $P_1 \cap P_2 = \emptyset, \dim P_{1,s} + \dim P_{2,s} = r-2$, for any $s \in S$. Thus $E$ is isomorphic to $E_1 \otimes E_2$ for $E_1 = \pi'_* (O_{p_1} \otimes O_{X', (1)}), E_2 = \pi'_* (O_{p_2} \otimes O_{X', (1)}), \text{where } \pi': X' \rightarrow S$ is the structure morphism and $O_{X', (1)}$ is the tautological linebundle of $E$. Since $\zeta'_1, ..., \zeta'_r$ form a basis of $E_2$ on $U_{\lambda}$ for any $\lambda \in \Lambda$, $E_2$ is isomorphic to $O_S^{\oplus r}$. On the other hand, since $\zeta_1, ..., \zeta_r$ form a local basis of $E_1$ on $U_{\lambda}$, $E_1$ is a regular vector bundle defined by $(s'^1_1, ..., s'^r_1) \in H^0(T, O_T(D)) \times \cdots \times H^0(T, O_T(D))$ by virtue of Lemma 1.4 and Lemma 2.2. $E_1$ is contained in $	ext{SR}^r(S, T, D)$ because $s'^1_1, ..., s'^r_1$, are linearly independent.

q. e. d.

In [3] A. Grothendieck proved that every vector bundle on $P^r_k$ is the direct sum of linebundles. In the same paper he posed a question whether this property characterizes $P^r_k$ in the category
of projective variety over \( k \). Van de Ven and J. Simonis solved this problem in the non-singular case (see \([17]\)). The above theorem provides an answer of this problem in a stronger form.

**Corollary 3.4.1.** Let \( S \) be a non-singular projective variety over \( k \) of dimension \( n \). If \( r \) is an integer greater than \( \max(n - 1, 1) \) and if \( S \cong \mathbb{P}^1_k \), then there is a simple vector bundle of rank \( r \) on \( S \).

**Proof.** If \( n \geq 2 \), then this is a direct corollary to Theorem 3.4 and Remark 2.15. It is well known that there is a stable vector bundle on \( S \) if \( n = 1 \) and \( S \cong \mathbb{P}^1_k \) (for example a nontrivial extension \( E \) of \( L \) by \( O_S \) is stable for a linebundle \( L \) of degree \( 1 \)). And every stable vector bundle is simple (\([17]\)).

q. e. d.

**Remark 3.5.** Our proof of Theorem 3.4 shows that without the assumption \( D \in A_T \) (ii) is true if one defines \( SR^p(S, T, D) \) as the set which consists of all elements in \( R^p(S, T, D) \) defined
by linearly independent \((s_1, \ldots, s_r)\). (i) is not necessarily
true without the condition \(D \in A_T\). But it would not be best to
assume the condition because there is a simple regular vector
bundle not satisfying the condition (see next section).

Example 3.6. For an ineducible conic \(C^2\) in \(P^2\) and a point
\(P \in C^2\) the unique element of \(SR^2(P^2, C, P) = R^2(P^2, C^2, P)\) is
\(O_{P^2}(1) \oplus O_{P^2}(1)\). Every element of \(SR^2(P^2, C^2, 2P) = R^2(P^2, C^2, 2P)\)
is indecomposable but not simple.

Proof. Assume that \(E \in R^2(P^2, C^2, P)\) is defined by \((s_1, s_2)\)
\(K^0(C^2, O_{C^2}(P)) \times K^0(C^2, O_{C^2}(P)). \ s_1\) corresponds to one point
divisor \(P_i\) on \(C^2\). Take a point \(Q \in C^2\) which is different
from \(P_1, P_2\) and two lines \(\sum_{j=0}^{2} a_j x_j = 0 \ (i = 1, 2)\). Then the
subvariety \(V\) of \(P_{P^2}^1 = X\) defined by \(\sum_{j=0}^{2} a_j x_j = 0 \ (i = 1, 2)\)
is non-singular and contains the subvariety \(Y\) of \(P_{C^2}^1\) defined by
\(s_1' y_1 + s_2' y_2 = 0\). It is easy to check that proper transform of
\(V\) by \(elm_{Y}^{0}\) is a section of \(elm_{Y}^{0}(X) = P(E)\). Thus \(E\) is an
extension of two linebundles. On the other hand, every extension of
two linebundles is trivial on $P^2$ because $H^1(P^2, L) = 0$ for any
linebundle $L$ on $P^2$. $E$ is therefore decomposable. Moreover
$c_1(E) = 2$, $c_2(E) = 1$ by virtue of Corollary 2.19.1. Thus $E \cong
O_{P^2}(1) \oplus O_{P^2}(1)$. Since $c_1(E) = 2$, $c_2(E) = 2$ for $E \in R^2(P^2,$
class $2$, $2P)$ by virtue of Corollary 2.19.1, $E$ is indecomposable.

That $E$ is not simple will be proved in the next section (see Example 3.11).

q. e. d.

§ 2. Simple regular vector bundles of rank 2.

In the rank 2 case we can study more fully simple regular vector
bundles. A distinguished fact on a vector bundle $E$ of rank 2 is
$P(E)^\vee \cong P(E)$ with the dual vector bundle $E^\vee$ of $E$. In fact

Lemma 3.7. If $E$ is a vector bundle of rank 2 on a scheme

then $E^\vee \cong E \otimes \det E^\vee$.

Proof. Let $A_{ij}$ be transition matrices of $E$, then those

...
The following lemma is due to Schwarzenberger ([9] Theorem 1).

Lemma 3.8. Let $E$ be a vector bundle of rank 2 on a non-singular projective variety $X$ over $k$. Then the following two conditions are equivalent to each other

(i) $E$ is not simple.

(ii) There is a linebundle $L$ on $X$ such that for $E' = E \otimes L$

$$\dim_k H^0(X, E') > 0, \dim_k H^0(X, E') > 0.$$ 

Now assume that $E \in \mathcal{R}^2(S, T, D)$ is defined by $Y$. Then the tautological linebundle of $E$ on $P(E) = \text{elm}^{-1}_Y(P_X^2) = X(Y)$ is $O_X(Y)(H_Y^1)$ in the notation of chapter II (see Lemma 2.2), and

$\det E^* \cong O_3(-T)$ by virtue of Corollary 2.18.1. Applying the above lemma $E$ is not simple if and only if
for some divisor $D_0$ on $S$ because $H^0(S, E \otimes O_S(D_0)) = H^0(X(Y), O_X(Y)(H_Y + T(D_0)))$ by virtue of Lemma 3.7. Thus $E$ is not simple if and only if there are positive divisors $A_1', A_2'$ with $A_1' - \overline{\tau}_Y(T)$ \notdivides 0, $A_2' - \overline{\tau}_Y(T)$ \notdivides 0 and non-negative integers $r_1, r_2$ such that

$$A_1' + r_1 \overline{\tau}_Y(T) \sim H_Y^1 + \overline{\tau}_Y(D_0), \quad A_2' + r_2 \overline{\tau}_Y(T) \sim H_Y^1 - \overline{\tau}_Y(T + D_0).$$

We may assume that $D_0 \notdivides T$ and replace $T$ by a suitable $T'$ such that $T' \notdivides T$, $T \sim T'$, because $S$ is projective. Put

$$A_1 = f_Y^{-1}[A_1'] \quad \text{and} \quad \tilde{H} = f_Y^{-1}[H_0].$$

(a) Assume that $A_1'$ contains the center $Y'$ of $(\text{EL}_Y)^{-1}$:

Since $g_Y^{-1}[A_1'] + \overline{x} + r_1(\overline{\tau}_Y - g_Y)^{-1}(T') \sim \tilde{H} + \overline{x} + (\overline{\tau}_Y - g_Y)^{-1}(D_0)$

and $f_Y^{-1}(H_0 + P_T^{-1}) = \tilde{H} + \overline{x} + E_Y$ by virtue of Lemma 2.1, we get

$f_Y^{-1}(H_0 + P_T^{-1}) \sim g_Y^{-1}[A_1'] + \overline{x} + r_1(\overline{\tau}_Y - f_Y)^{-1}(T') - (\overline{\tau}_Y - f_Y)^{-1}(D_0) + E_Y$
(note $\gamma \cdot f_y = \gamma Y \cdot \gamma Y$). On the other hand, $f_y^{-1}(A_1 + P_T^1 + r_1 T)^{-1} (T') = P_T^1 (D_0)$ by virtue of Lemma 2.1. Thus $f_y^{-1}(H_0 + P_T^1) = f_y^{-1}(A_1 + P_T^1 + r_1 T)^{-1} (T') - P_T^1 (D_0)$, which implies $A_1 \sim H_0 - r_1 P_T^1 + T^{-1} (D_0)$. Since $A_1 > 0$, there is a positive divisor $D_1$ with $D_1 \sim D_0 - r_1 T$.

(b) Assume that $A_2 \ni Y'$: By a similar argument as above we have $A_2 \sim H_0 - r_2 P_T^1 - P_T^1 (T + D_0) = H_0 - (r_2 + 1) P_T^1 - T^{-1} (D_0)$, whence there is a positive divisor $D_2$ with $D_2 \sim -(r_2 + 1) T - D_0$.

(a') Assume that $A_1 \ni Y'$: Since $g_T^{-1}[A_1] + r_1 (\gamma Y \cdot \gamma Y)^{-1} (T') \sim \gamma Y + f_T^{-1} (H_0 + P_T^1) = f_T^{-1} (A_1 + r_1 T^{-1} (T') - T^{-1} (D_0))$ by virtue of Lemma 2.1, we have $f_T^{-1} (H_0 + P_T^1) = f_T^{-1} (A_1 + r_1 T^{-1} (T') - T^{-1} (D_0))$. Thus $A_1 \sim H_0 - (r_1 - 1) P_T^1 + T^{-1} (D_0)$. Since $A_1 > 0$, there is a positive divisor $D_1'$ with $D_1' \sim D_0 - (r_1 - 1) T$.

(b') Assume $A_2 \ni Y'$: By a similar argument as above we have $A_2 \sim H_0 - r_2 P_T^1 - P_T^1 (T + D_0) = H_0 - r_2 P_T^1 - T^{-1} (D_0)$, whence there is a positive divisor $D_2'$ with $D_2' \sim -r_2 T - D_0$. 

We should therefore consider the following cases.

(1) The case where (a) and (b) are satisfied: Since
\[(r_1 + r_2 + 1)T \sim D_1 + D_2 > 0, \ r_1 + r_2 + 1 > 0, \ T \not\asymp 0 \text{ and since } S \text{ is projective, we have a contradiction.}\]

(2) The case where (a) and (b') are satisfied: Since
\[-(r_1 + r_2)T \sim D_1 + D_2' \geq 0, \text{ and } r_1, r_2 \text{ are non-negative integers, we get } r_1 = r_2 = 0, \text{ whence } D_0 \not\asymp 0. \text{ Thus } A_2 \not\sim H_0, \text{ and by virtue of Lemma 2.1, } A_2 \not\supset Y. \text{ Hence } Y = \mathbb{P} \times T \subseteq F^1_T \text{ for some point } \mathbb{P} \in F^1, \text{ which implies } E \in \mathbb{R}^2(S, T, 0). \text{ Therefore } E \cong O_S \oplus O_S(T) \text{ and this is not simple.}\]

(3) The case where (a') and (b) are satisfied: Since
\[-(r_1 + r_2)T \sim D_1 + D_2' > 0, \text{ and } r_1, r_2 \text{ are non-negative integers, we get } r_1 = r_2 = 0, \text{ whence } D_0 + T \sim D_1 > 0, \text{ and } -(D_0 + T) \sim D_2 > 0. \text{ Thus } D_0 \not\asymp -T. \text{ Hence } A_1 \not\sim H_0 \text{ and by virtue of Lemma 2.1, } A_1 \not\supset Y. \text{ Hence } E \cong O_S \oplus O_S(T) \text{ and } E \text{ is not simple.}\]

(4) The case where (a') and (b') are satisfied: Since
\[(1 - r_1 - r_2)T \sim D_1' + D_2' > 0, \text{ we get } 1 - r_1 - r_2 \geq 0. \text{ If either }
$r_1$ or $r_2$ is equal to 1, then by a similar argument as above we have $E \cong O_S \oplus O_S(T)$. Suppose $r_1 = r_2 = 0$, then $D'_1 \sim -D_0$, whence $D'_1 \sim T - D'_2$. Thus there are positive divisors $D'_1$, $D'_2$ on $S$ and $A_1$, $A_2$ on $P^1_S$ such that $D'_1 + D'_2 \sim T$, $A_1 \sim H_0 + \nu^{-1}(D'_1)$, $A_2 \sim H_0 + \nu^{-1}(D'_2)$ and both $A_1$ and $A_2$ contain $Y$. Conversely if these conditions are satisfied, then the calculation in $(a')$, $(b')$ shows that $A'_1 \sim H'_0 + \nu^{-1}(D'_0)$, $A'_2 \sim H'_0 - \nu^{-1}(T + D'_0)$ for $A'_1 = \mathfrak{elm}_Y[A'_1]$ and $D'_0 = D'_1 - T$.

Consequently we have

**Lemma 3.9.** Let $E \in R^2(S, T, D)$ be defined by $Y$. $E$ is not simple if and only if there are positive divisors $D_1$, $D_2$ on $S$ and $A_1$, $A_2$ on $P^1_S$ such that $D_1 + D_2 \sim T$, $A_1 \sim H_0 + \nu^{-1}(D_1)$, $A_2 \sim H_0 + \nu^{-1}(D_2)$ and that both $A_1$ and $A_2$ contain $Y$.

**Proof.** Note that if $D = 0$, that is, $E \cong O_S \oplus O_S(T)$, then the above conditions are satisfied by $D_1 = 0$, $D_2 = T$, $A_1 = H_0$, $A_2 = H_0 + \nu^{-1}(T)$. Then our assertion is clear by virtue of the argument before this lemma.

$q. e. d.$
Theorem 3.10. Assume that \( E \in \mathbb{R}(S, T, D) \) is defined by 
\[(s, s') \in H^0(T, O_T(D)) \times H^0(T, O_T(D)) \text{ (cf. Principle 2.6)}. \] \( E \) is not simple if and only if there are positive divisors \( C_1, C_1', C_2, C_2' \) such that \( C_1 + C_2 \sim T, C_1 \sim C_1'(i = 1, 2) \) and that \( C_1 \cdot T = |s| + B_1, C_1' \cdot T = |s'| + B_1 \) (1 = 1, 2) for positive divisors \( B_1 \) on \( T \), where \( |s| \) (or, \( |s'| \)) is the divisor defined by \( s = 0 \) (or, \( s' = 0 \), resp.).

Proof. \( E \) is defined by \( Y : s \bar{\gamma}_0 + s' \bar{\gamma}_1 = 0 \), where \( \bar{\gamma}_0, \bar{\gamma}_1 \) are a homogeneous coordinate of \( P^1_T \) induced from a homogeneous coordinate \( \gamma_0, \gamma_1 \) of \( P^1_S \). Assume that \( E \) is not simple, then there are \( D_1, D_2, A_1, A_2 \) satisfying the conditions in Lemma 3.9 with the same \( Y \) as above. \( A_1 \) is defined by \( s_{10} \bar{\gamma}_0 + s_{11} \bar{\gamma}_1 = 0 \) for \( s_{1j} \in H^0(S, O_S(D_j)) \). Let \( \bar{s}_{1j} \) be the element of \( H^0(T, O_T(D_1 \cdot T)) \) induced from \( s_{1j} \). Then \( A_1 \supset Y \) implies \( \bar{s}_{10} \bar{\gamma}_0 + \bar{s}_{11} \bar{\gamma}_1 = a_1(s \bar{\gamma}_0 + s' \bar{\gamma}_1) \) for some \( a_1 \in H^0(T, O_T(D_1 \cdot T - D)) \). Thus if one puts \( |s_{10}| = C_1, |s_{11}| = C_1', \) then \( C_1 \cdot T = |s| + B_1, C_1' \cdot T = |s'| + B_1 \) (\( B_1 = |a_1| \)). Since \( C_1 \sim C_1' \sim D_1 \), we know \( T \sim C_1 + C_2 \).

By virtue of Remark 2.16 \( E \) is not simple if \( p = 0 \). Thus we may assume that \( D \neq 0 \).
Conversely assume that $C_1, C_1', C_2, C_2'$ exist. The conditions

$$C_1 \cdot T = |s| + B_1, \quad C_1' \cdot T = |s'| + B_1$$

assert that there are

$$a_{ij} \in H^0(S, O_S(C_i)) \quad (j = 0, 1) \quad \text{and} \quad a_i \in H^0(T, O_T(B_i))$$

such that

$$|s_{i0}| = C_i, \quad |s_{i1}| = C_i', \quad |a_i| = B_i \quad \text{and that} \quad \overline{s_{i0}}_0 + \overline{s_{i1}}_1 = a_i(s_{i0} + s_{i1})$$

Let $A_i$ be the positive divisor on $P^1_S$ defined by $s_{i0} + s_{i1} = 0$, then $A_i \geq X$. Thus $D_1 = C_1, A_1$ satisfy the conditions in Lemma 3.9, which implies that $E$ is not simple.

$q. \ e. \ d.$

(D.5)

Corollary 3.10.1. $E \in R^2(S, T, D)$ is simple if $H^0(T, O_T(T^2 - 2D)) = 0$. 

Proof. If $E$ is not simple, then there are positive divisors $C_1, C_2$ such that $C_1 \cdot T \sim D + B_1, \ B_1 > 0$ on $T$ and $C_1 + C_2 \sim T$ (see Theorem 3.10). Thus $T^2 - 2D \sim ((C_1 + C_2) \cdot T) - 2D \sim B_1 + B_2 > 0$. We have therefore $H^0(T, O_T(T^2 - 2D)) \neq 0$.

$q. \ e. \ d.$

Example 3.11. Let $C^n$ be a non-singular curve of degree $n$ in $P^2$. Every element of $R^2(P^2, C^2, 2p)$ is indecomposable
and not simple. Every element of $R^2(P^2, C^3, P_1 + P_2 + P_3)$ is not simple if and only if $P_1, P_2, P_3$ are collinear.

Proof. Assume that $E \in R^2(P^2, C^2, 2P)$ is defined by $(s_1, s_2)$ of $H^0(C^2, O_{C^2}(2P)) \times H^0(C^2, O_{C^2}(2P))$. Put $s_1 = P_{11} + P_{12}$. Take the line $l_1$ going through $P_{11}$ and $P_{12}$ (if $P_{11} = P_{12}$, $l_1$ touches to $C^2$ at $P_{11}$). Then $l_1 = C_1 = C_2$, $l_2 = C'_1 = C'_2$ satisfy the conditions for $s_1, s_2$ in Theorem 3.10. Let us show the latter part. Note that $P_1, P_2, P_3$ are collinear if and only if $Q_1, Q_2, Q_3$ are collinear for any element $Q_1 + Q_2 + Q_3$ of $|P_1 + P_2 + P_3|$. Assume that $E \in R^2(P^2, C^2, P_1 + P_2 + P_3)$ is defined by $(s_1, s_2) \in H^0(C^3, O_{C^3}(P_1 + P_2 + P_3)) \times H^0(C^3, O_{C^3}(P_1 + P_2 + P_3))$ with $s_1 = \sum_{j} Q_{ij}$. If $E$ is not simple, then one of $Q_i$ in Theorem 3.10 is a line, whence $Q_{11}, Q_{12}, Q_{13}$ are collinear. Assume $P_1, P_2, P_3$ collinear, then $Q_{11}, Q_{12}, Q_{13}$ are collinear. Then $l_1 = C_1$, $l_2 = C'_1$, $2l_2 = C_2$, $2l_2 = C'_2$ satisfy the conditions of Theorem 3.10, whence $E$ is not simple.

q. e. d.

Take a line $l_1$ going through $Q_{11}, Q_{12}, Q_{13}$. 
As a matter of fact if \( P_1, P_2, P_3 \) are not collinear, then every element of \( R^2(p^2, C^3, P_1 + P_2 + P_3) \) is isomorphic to the tangent bundle of \( p^2 \) (see Example 4.8).

Example 3.12. Let \( T \) be a non-singular surfaces of degree 4 of \( p^3 \) which contains a line \( l \) and let \( \{ H_\lambda \}_{\lambda \in P^1} \) be the linear pencil which consist of hyperplanes of \( p^3 \) containing \( l \). Then 

\[
H_\lambda \cdot T = l + C_\lambda, \quad C_\lambda \cdot C_\mu = 0 \quad \text{because} \quad p_a(C_\lambda) = 1, \quad K_T \sim 0 \quad \text{with a canonical divisor} \quad K_T \quad \text{of} \quad T. \quad \text{Thus} \quad R^2(p^3, T, 2C_\lambda) \neq \emptyset \quad \text{by virtue of Remark 2.16. Let us show that every element of} \quad R^2(p^3, T, 2C_\lambda) \quad \text{is indecomposable and not simple.}

Proof. Since \( C_\lambda^2 = 0, \quad |2C_\lambda| = \{ D_\lambda^\mu = \alpha C_\lambda + \beta C_\mu \}_{\lambda, \mu \in P^1} \). Assume that

\( E \in R^2(p^3, T, 2C_\lambda) \) is defined by \((s_1, s_2) \in H^0(T, O_T(2C_\lambda)) \) 

\( H^0(T, O_T(2C_\lambda)) \) with \((s_1) = D_\lambda^\mu = \alpha C_\lambda + \beta C_\mu \). Then \( C_j = H_{2j} + H_{j_1}, \quad C_j = H_{2j} + H_{j_2} \) \( (j = 1, 2) \) satisfy the conditions in Theorem 3.10. Thus \( E \) is not simple. Since \( c_1(E) = T, \quad c_2(E) = 2C_\lambda, \quad \deg c_1(E) = 4 \) and \( \deg c_2(E) = 6 \), whence \( E \) is indecomposable.

\[ \text{q. e. d.} \]
As an application of the above theorem we shall give another proof of a theorem of Schwarzenberger (\cite{S7} Theorem 8).

Theorem 3.13. Let \( S \) be a non-singular projective surface over \( k \), \( c_1 \) a divisor on \( S \) and let \( c_2 \) be an integer. For \( r > 1 \), there exists a non-simple vector bundle of rank \( r \) on \( S \) with Chern classes \( c_1, c_2 \).

In order to prove the theorem we need a lemma.

Lemma 3.14. Let \( H \) be a very ample divisor on a non-singular projective surface \( S \) and let \( x_1, \ldots, x_n \) be mutually distinct points on \( S \). There exists a positive integer \( a_0 \) such that for any \( a \geq a_0 \) there is a non-singular irreducible member of \( |aH| \) going through all of \( x_1, \ldots, x_n \).

Proof. Let \( f : S' \rightarrow S \) be the blowing up with centers \( x_1, \ldots, x_n \) and let \( E_1 \) be the exceptional curve \( f^{-1}(x_1) \). Then \( \mathcal{O}_{S'}(-E_1 + \ldots + E_n) \) is \( f \)-very ample (\( E, G, A, \) Chap. II, 8.1.7). Since \( H \) is very ample, there exists a positive integer \( a_0 \) such
that $\mathcal{O}_S(-(E_1 + \ldots + E_n)) \otimes f^*(\mathcal{O}_S(H)^\otimes a) = \mathcal{O}_S((a f^{-1}(H) - (E_1 + \ldots + E_n))$

is very ample for any $a \geq a_0$ (E. G. A. Chap. II, 4.4.10). Then

a general member $H'$ of $\{ af^{-1}(H) - (E_1 + \ldots + E_n) \}$ is non-singular irreducible. Since $(H', E_1) = (af^{-1}(H) - (E_1 + \ldots + E_n), E_1) = 1$,

$f(H')$ goes through all $x_i$ with multiplicity 1. Thus $f(H')$ is non-singular irreducible because $S' = \bigcup_{i=1}^{n} E_i$ is isomorphic to $S - \{ x_1, \ldots, x_n \}$. Clearly $f(H') \sim aH$, $f(H')$ is therefore a desired member of $|aH|$.

q. e. d.

Proof of Theorem 3.13. For $r \geq 2$ and vector bundle $E$ of rank 2, $c_1(E) = c_1(\mathcal{O}_S^{(r-2)} \oplus E)$, $c_2(E) = c_2(\mathcal{O}_S^{(r-2)} \oplus E)$. Thus we have only to prove the theorem in the case of $r = 2$. Let $H$ be a very ample divisor on $S$ with $(H, H) = h$ and let $n$ be a non-negative integer. Take integers $\alpha, \beta$ such that $\alpha \geq 0$, $0 \leq \beta < h$, $n = \alpha h + \beta$. Let $H_1(1 \leq i \leq 4)$ be general members in $|H|$ such that $H_1 \cdot H_2 = \sum_{i=1}^{h} x_i$, $H_3 \cdot H_4 = \sum_{j=1}^{h} y_j$ with mutually distinct
points \( x_1, \ldots, x_h, y_1, \ldots, y_h \). Then by virtue of Lemma 3.14 there exists a positive integer such that for any \( a \geq a_0 \) there is a non-singular irreducible member of \( |aH| \) going through all of \( x_1, \ldots, x_h, y_1, \ldots, y_{h-\beta} \). Take such a member \( H' \) for \( a = 2r-1 \geq 2\max(a_0, 4\alpha + 3) \) with an even integer \( r \). We may assume that \( H' \) goes through none of \( y_{h-\beta+1}, \ldots, y_h \). If \( H_1 \cdot H' = \sum_{i=1}^{h} x_i + \sum_{k=1}^{h} y_k \) 

\[
(i = 1, 2), \quad H_1 \cdot H' = \sum_{j=1}^{h-\beta} y_j + \sum_{k=1}^{h-\beta} w_{k} \quad (i = 3, 4), \text{ then } A_1 = \]

\[
(2r-2)r \sum_{k=1}^{h} z_{1k} \sim A_2 = (2r-2)r \sum_{k=1}^{h} z_{2k}, \quad B_1 = (2r-2)r \sum_{k=1}^{h} w_{k} \sim B_2 = (2r-2)r \sum_{k=1}^{h} w_{2k} 
\]

and \( z_{1k} \sim z_{2k} \) (1 \( \leq k_1, k_2 \leq (2r-2)r \)), \( w_{1k} \sim w_{2k} \) (1 \( \leq k_1, k_2 \leq (2r-2)r \)). Take another general members \( H_5, H_6 \) in \( |H| \) such that \( A_1' = H_5 \cdot H' \) and \( A_2' = H_6 \cdot H' \) contain no common point, and put \( D_1 = \alpha A_1' + (\frac{r}{2} - \alpha - 1) A_1 + B_1 \) (i = 1, 2). Then \( D_1 \) and \( D_2 \) contain no common point and \( D_1 \sim D_2 \), whence \( R^2(S, H', D_1) \neq \emptyset \). The element \( E' \) of \( R^2(S, H', D_1) \) defined by 

\[
(s_1, s_2) \in H^0(H', O_{H'}(D_1)) \times H^0(H', O_{H'}(D_2)) \text{ with } |E'| = D_1
\]

(i = 1, 2) is not simple because 

\[
(\alpha) = \alpha H_5 + (\frac{r}{2} - \alpha - 1) H_1 + H_3,
\]

\[
C_1 = \alpha H_6 + (\frac{r}{2} - \alpha - 1) H_2 + H_4, \quad C_2 = (\alpha + r - 1) H_5 + (\frac{r}{2} - \alpha - 1) H_1 + H_3.
\]
\( G' = \alpha H_6 + (r - 1)H_5 + \left( \frac{r}{2} - \alpha - 1 \right) H_2 + H_4 \) obviously satisfy the conditions of Theorem 3.10. On the other hand, \( c_1(E') = (2r - 1)H \), \( c_2(E') = \alpha(2r - 1)h + (\frac{r}{2} - \alpha - 1)(2r - 2)h + (2r - 2)h + \beta = \)

\( r^2h - rh + \alpha h + \beta \) by virtue of Corollary 2.18.1 and Corollary 2.19.1.

Thus we obtain \( c_1(E' \otimes O_S(-(r - 1)H)) = H \), \( c_2(E' \otimes O_S(-(r - 1)H)) = \alpha h + \beta = n \). Therefore if \( H \) is a very ample divisor on \( S \) and if \( n \) is a non-negative integer, then there is a non-simple vector bundle \( E \) of rank 2 on \( S \) with \( c_1(E) = H \), \( c_2(E) = n \). For a given \( c_1 \), \( c_2 \), take a very ample divisor \( H'' \) and a positive integer \( t \) such that \( H = c_1 + 2rH'' \) is very ample and \( c_2 + r^2(H'', H') + r(c_1, H'') = n > 0 \) (these conditions are satisfied if one takes sufficiently large \( r \) for a very ample divisor \( H'' \)). As for these \( H, n \) there is a non-simple vector bundle \( E'' \) of rank 2 with \( c_1(E'') = H \), \( c_2(E'') = n \) by virtue of the above argument. Then \( c_1(E'' \otimes O_S(-rH'')) = c_1 \), \( c_2(E'' \otimes O_S(-rH'')) = c_2 \). Thus \( E = E'' \otimes O_S(-rH'') \) gives a desired vector bundle.

q. e. d.
Chapter IV. Some special cases.

In this chapter we shall study some special vector bundles on some special algebraic varieties, along the line developed in the preceding three chapters.

§ 1. Tangent bundle of $\mathbb{P}_k$.

Let $T_X$ be the tangent bundle of a non-singular variety $X$ over $k$. Then we have

Theorem 4.1. Let $H$ be an arbitrary hyperplane of $\mathbb{P}_k$. Then $\mathbb{P}^n(\mathbb{P}_k^n, H, \mathbb{H}^2)$ consists only of one element $T_{\mathbb{P}_k^n}(-1)$.

Proof. Let $\mathbb{P}$ be the dual space of $\mathbb{P}_k^n$, then $\mathbb{P} : X = \mathbb{F}_k \mathbb{P}_k^n \to \mathbb{P}_k^n$ is the trivial $\mathbb{P}^n$-bundle on $\mathbb{P}_k^n$. On the other hand, the $\mathbb{P}^{n-1}$-bundle $\mathbb{P} : Y = \mathbb{P}(\mathbb{T}_{\mathbb{P}_k^n}) \to \mathbb{P}_k^n$ may be regarded as the bundle whose fibre $\mathbb{P}^{n-1}(x)$ at $x$ is $(n-1)$-dimensional projective space consisting of all hyperplanes in $\mathbb{P}_k^n$ going through $x$. Thus $Y$ is naturally a subbundle of $X$. Take linearly independent points $x_1, \ldots, x_n$ in $H$. The set consisting of all hyperplanes
going through $x_i$ (for each fixed $i$) forms a hyperplane $Z_i$ in $P$.

Put $H_i = Z_i \times_k P_k^n$. Let us consider $H_1 \cdots \cdots H_n \cdot Y$ in $Y$. Since

$$\gamma^{-1}(y) \cap H_i = \left\{\text{hyperplanes of } P_k^n \text{ going through } x_i \text{ and } y\right\},$$

$$\gamma_i^{-1}(y) \cap (\bigcap_{i=1}^n H_i) \text{ is not empty if and only if } y \text{ is contained in } H; \text{ and if } y \in H, \text{ then } \gamma_i^{-1}(y) \cap (\bigcap_{i=1}^n H_i) \text{ is the point corresponding to } H.$$

Now let $X_0, \ldots, X_n$ be a homogeneous coordinate of $P_k^n$ such that $H$ is defined by $X_0 = 0$ and let $\gamma_0, \ldots, \gamma_n$ be the homogeneous coordinate of $P$ induced from $X_0, \ldots, X_n$. Let $U_i$ be the affine open set of $P_k^n$ defined by $X_i \neq 0$ and put $Z_i = X_i/X_j$. We may assume that $Z_i$ is defined by $\gamma_i = 0$. On the other hand, $Y_{U_i}$ is defined by $\sum_{j=0}^n \gamma_j Z_j = 0$ in $X_{U_i}$. Thus $H_1, \ldots, H_n, Y$ are transversal to each other at any point of $\left(\bigcap_{i=0}^n H_i\right)$. $Y$ and $H' = H_1 \cdots \cdots H_n \cdot Y$ is defined by the ideal generated by $Z_1, \gamma_1, \ldots, \gamma_n$. By virtue of Proposition 1.9 we know therefore $elm_{H'}(Y)/P_k^n \times_k P_k^n$. Let $H''$ be the center of $(elm_{H'}(Y))_{-1}$, then $H'' \subset P_k^n \times_k H$. Since the regular vector bundle $E$ defined by $H''$ is isomorphic to $T_{P_k^n}(r)$ and since $c_1(E) = H$, we have $E \cong T_{P_k^n}(-1)$. 
Moreover, since \( c_2(E) = c_2(T_{\mathbb{P}^n_k}(-1)) = R^2 \), \( T_{\mathbb{P}^n_k}(-1) \) is contained in \( R^n(\mathbb{P}^n_k, H, H^2) \). Conversely, if \( E \in R^n(\mathbb{P}^n_k, H, H^2) \) is defined by

\[(s_1, \ldots, s_n) \in H^0(H, O_H(H^2)) \times \cdots \times H^0(H, O_H(H^2)), \]

then \( s_1, \ldots, s_n \) are linearly independent because if \( s_1, \ldots, s_n \) are linearly dependent,

\[s_1(x) = \cdots = s_n(x) = 0 \text{ for some } x \in H.\]

Thus \( SR^n(\mathbb{P}^n_k, H, H^2) = R^n(\mathbb{P}^n_k, H, H^2) \) when one defines \( SR^n(\mathbb{P}^n_k, H, H^2) \) as in Remark 3.5.

On the other hand, there is a surjective map \( \text{Grass}^n_{n-1}(k) \rightarrow SR^n(\mathbb{P}^n_k, H, H^2) \) (see the proof of Theorem 2.14). Since \( \text{Grass}^n_{n-1}(k) \) has only one point, \( SR^n(\mathbb{P}^n_k, H, H^2) = R^n(\mathbb{P}^n_k, H, H^2) \) consists of one element \( T_{\mathbb{P}^n_k}(-1) \) only.

\[\text{q. e. d.}\]

It goes without saying that the above result has something to do with the fact that \( T_{\mathbb{P}^n_k}(-1) \) is a homogeneous vector bundle on the homogeneous space \( \mathbb{P}^n_k \). Furthermore, this theorem shows that the sufficient condition for a regular vector bundle to be simple stated in Theorem 3.4 is not best possible (Note \( H^2 \neq A_H \)).

As a corollary to the above proof we have the following, which
is well known.

Corollary 4.1.1. There is an exact sequence

\[ 0 \longrightarrow O_{P_k}(-1) \longrightarrow O_{P_k}(n+1) \longrightarrow T_{P_k} \longrightarrow 0. \]

Proof. Since \( O_X(H_1) \) is the tautological linebundle of \( O_{P_k}(n+1) \) on \( X \cong P(O_{P_k}(n+1)) \), \( O_X(H_1) \otimes O_Y \) is the tautological line bundle of \( T_{P_k}(-1) \) on \( Y = P(T_{P_k}) \) by virtue of Lemma 2.2 and the above proof and since \( Y \) is a subbundle of \( X \), we have a surjective homomorphism \( \varphi : \pi_*(O_X(H_1)) \otimes O_Y \longrightarrow T_{P_k}(-1) \). On the other hand, \( \text{Ker } \varphi \cong (n+1) \otimes (n-1) \cong O_{P_k}(-1) \).

q.e.d.

§ 2. Vector bundles on \( P_k^2 \).

We shall begin with an easy lemma.

Lemma 4.2. If \( E \) is a simple vector bundle of rank 2 on \( P_k^2 \) and if \( \deg E \geq -3 \), then \( H^2(P_k^2, E) = 0 \).
Proof. Assume that $H^2(P^2_k, E) \neq 0$. By the Serre duality,

$$\dim_k H^2(P^2_k, E) = \dim_k H^0(P^2_k, E \oplus O_{P^2_k}(-3)) > 0.$$ Since $\dim_k H^0(P^2_k, E) \geq \dim_k H^0(P^2_k, E \oplus O_{P^2_k}(-3))$, we have $H^0(P^2_k, E) \neq 0$. On the other hand, since $H^0(P^2_k, E \oplus O_{P^2_k}(-3)) = H^0(P^2_k, E \oplus (\det E) \otimes O_{P^2_k}(-3)) \neq 0$ and since $\deg((\det E) \otimes O_{P^2_k}(-3)) \geq 0$ by our assumption $\deg E \geq -3$, we have $H^0(P^2_k, E) \neq 0$. Thus $E$ is not simple by virtue of Lemma 3.8. This is contrary to the assumption that $E$ is simple.

q.e.d.

Let $E$ be a vector bundle of rank 2 with Chern classes $c_1(E), c_2(E)$ on a non-singular projective surface $S$. Define an integer $\Delta(E)$ to be $c_1(E)^2 - 4c_2(E)$. $\Delta(E)$ is the second Chern class of $\text{End}(E)$, hence it plays an important role in the theory of simple vector bundles. The following lemma is essentially due to Schwarzenberger ([3], Theorem 10).

Lemma 4.3. Let $E, S$ be as above and let $K$ be a canonical divisor of $S$. If $|-K| \not\equiv 0$ and $\Delta(E) \geq -(4p_a(S) + 1)$, then $E$
is not simple.

Proof. Since $\text{End}(E)$ is self-dual, $\dim_k H^2(S, \text{End}(E)) = \dim_k H^0(S, \text{End}(E) \otimes \mathcal{O}(K))$ by the Serre duality. On the other hand, the assumption $|K| = 0$ implies $\dim_k H^0(S, \text{End}(E) \otimes \mathcal{O}_S(K)) \leq \dim_k H^0(S, \text{End}(E))$. Thus $\chi(\text{End}(E)) = \Sigma(-1)^i \dim_k H^i(S, \text{End}(E)) \leq 2 \dim_k H^0(S, \text{End}(E))$. Besides the Riemann-Roch theorem provides equalities $\chi(\text{End}(E)) = \Delta(E) + \frac{1}{3}(K^2 + c_2(S))$ and $p_a(S) + 1 = \frac{1}{12}(K^2 + c_2(S))$

($c_2(S)$ is the second Chern class of $S$). We obtain therefore $2 \dim_k H^0(S, \text{End}(E)) \leq \Delta(E) + 4(1 + p_a(S)) > 2$, which is our assertion.

q. e. d.

Let us consider some special cases.

Corollary 4.3.1. Let $E, S$ be as above.

(i) If $S$ is a rational ruled surface or $P^2_k$ and if $\Delta(E) \geq -1$ then $E$ is not simple.

(ii) If $S$ is an abelian surface and if $\Delta(E) \geq 3$, then $E$ is not simple.
(ii)' If $S$ is an abelian surface, the characteristic of $k \neq 2$ and if $\Delta(E) \geq 1$, then $E$ is not simple.

(iii) If $k \sim 0$, $\dim_{k} H^{1}(S, O_{S}) = 0$ (for example $K3$ surfaces over $C$, a non-singular surface of degree $4$ in $P_{k}^{3}$) and if $\Delta(E) \geq -5$, then $E$ is not simple.

Proof. (i) Let $S$ be a rational ruled surface with minimal section $D$. Assume $(D, D) = -n$, then $-2D - (n+2)\ell$ is a canonical divisor on $S$, where $\ell$ is a fibre (= a generator of $S$). Thus we have $|K| \not\equiv \phi$. Let $S = P_{k}^{2}$ and let $C$ be a cubic curve, then $-C$ is a canonical divisor, whence $|K| \not\equiv \phi$. In any case $p_{a}(S) = 0$. Then (i) follows from the above lemma.

(ii) If $S$ is an abelian surface, then $K \sim 0$, $p_{a}(S) = -1$.

Thus we obtain (ii). As for (ii)' see [4] Corollary to Theorem 2.

(iii) In this case $K \sim 0$, $p_{a}(S) = 1$, whence our assertion is obvious by virtue of Lemma 4.3.

q. e. d.
Example 4.4. (i) If $S$ is a general non-singular surface of degree 4 in $\mathbb{P}^3_k$ which contains a line $l$ in $\mathbb{P}^2_k$, then there is a simple vector bundle $E$ of rank 2 on $S$ with $\Delta(E) = -2r$ for any $r \geq 3$.

(ii) If $S$ is a general surface of degree 4 in $\mathbb{P}^3_k$, then $\Delta(E) = 0$ for any vector bundle $E$ of rank 2 on $S$ and there is a simple vector bundle $E$ of rank 2 on $S$ with $\Delta(E) = -4r$ for any $r$.

Proof. (i) Take $H_\lambda, C_\lambda, \lambda \in \mathbb{P}^1$ as in Example 3.12. Then $C_\lambda$ is an elliptic curve for a general $\lambda \in \mathbb{P}^1$ and $(C_\lambda, C_\lambda) = 0, (l, l) = -2$. If one takes points $P_1, \ldots, P_r$ on $C_\lambda$ and $Q_1, \ldots, Q_s$ on $l$ for arbitrary $r (\geq 2)$, and $s (\geq 1)$, then $\sum_{i=1}^r P_i$ and $\sum_{j=1}^s Q_j$ are divisors free from base points. Thus $R^2(S, C_\lambda, \sum_{i=1}^r P_i), R^2(S, l, \sum_{j=1}^s Q_j)$ and every element in $R^2(S, C_\lambda, \sum_{i=1}^r P_i), R^2(S, l, \sum_{j=1}^s Q_j)$ is simple by virtue of Corollary 3.10.1. Since $\Delta(E_\lambda) = -4r, \Delta(E_s) = -2 - 4s$ for $E_\lambda \in R^2(S, C_\lambda, \sum_{i=1}^r P_i), E_s \in R^2(S, l, \sum_{j=1}^s Q_j)$, our assertion is proved.
(ii) Note that if \( S \) is a general surface of degree 4 in \( \mathbb{P}^3_C \), then \( \text{Pic}(S) \cong \mathbb{Z} \) whose generator is the class of hyperplane section \( (\text{LAA. LQ 13}) \). Thus \( D^2 \equiv 0 \) for any divisor \( D \) on \( S \), which shows the former assertion. In order to prove the latter, take a general hyperplane section \( C \). Then \( C \) is a non-singular plane curve of degree 4. Hence there is a positive divisor \( A_r \) of degree \( r \) free from base point on \( C \) for any \( r \geq 3 \). Thus \( R^2(S, C, A_r) \) \( \neq 0 \). Every element \( E_r \) of \( R^2(S, C, A_r) \) is simple by virtue of Corollary 3.10.1 and \( \triangle(E_r) = -4(r - 1) \) because of \((C, C) = 4\).

q. e. d.

Now let us come back to vector bundles on \( \mathbb{P}^3 \). The following lemma is very interest when one takes Corollary 4.3.1, (i) into account.

**Lemma 4.5.** If \( E \) is a simple vector bundle of rank 2 on \( \mathbb{P}^2 \), then \( \triangle(E) \neq -4 \).

**Proof.** We assume that \( E \) is a simple vector bundle of rank 2 on \( \mathbb{P}^2 \) with \( \triangle(E) = -4 \) and shall show a contradiction. By the
assumption $\Delta(E) = -4$ where is a linebundle $L$ such that $c_1(E \otimes L) = 0$, $c_2(E \otimes L) = -1$, whence we may assume $c_1(E) = 0$, $c_2(E) = 1$.

The Riemann-Roch theorem asserts for a vector bundle $E'$ of rank $2$ on $\mathbb{P}^2_k$:

$$\chi(E') = \sum_{i=0}^{2} (-1)^i \dim_k \mathcal{H}^i(\mathbb{P}^2_k, E') = 2 + \frac{3c_1(E')}{2} + \frac{c_1(E')^2 - 2c_2(E')}{2}.$$

Applying this to $E$ we have $\chi(E) = 1$. On the other hand, Lemma 4.2 implies $\mathcal{H}^2(\mathbb{P}^2_k, E) = 0$. Thus we obtain $\mathcal{H}^0(\mathbb{P}^2_k, E) \neq 0$. Moreover, since $E' \cong E \otimes (\det E) \cong E$, we know $\mathcal{H}^0(\mathbb{P}^2_k, E') \neq 0$. By virtue of Lemma 3.8 this is a contradiction.

q. e. d.

We have an interesting corollary.

**Corollary 4.5.1.** Let $C$ be a non-singular curve of degree $n$ in $\mathbb{P}^2_k$ and $D = \sum_{i=1}^{r} P_i$ be a positive divisor on $C$ such that $|D|$ is free from base point.

1) If $n (= 2m)$ is even and $r \leq m^2 + 1$, then there is a positive divisor $C'$ of degree $n$ in $\mathbb{P}^2_k$ such that $C \cdot C' - D > 0$. 
ii) If \( n (=2m + 1) \) is odd and \( r \leq m^2 + m \), then there is a positive divisor \( C' \) of degree \( m \) in \( \mathbb{P}^2_k \) such that \( C \cdot C' - D > 0 \).

Proof i) Since \( |D| \) is free from base point, there is \( D' \in |D| \) which contains none of \( \mathbb{P}_1 \). Let \( E \in \mathcal{H}^2(\mathbb{P}^2_k, C, D) \) be defined by

\[
(s, s') \in H^0(C, O_C(D)) \times H^0(C, O_C(D)) \text{ with } |s| = D, |s'| = D'.
\]

On the other hand, since \( \Delta(E) = C^2 - 4r \geq 4m^2 - 4(m^2 + 1) = -4, \)

\( 4|\Delta(E)| \), we know that \( E \) is not simple by virtue of Corollary 4.3.1 and Lemma 4.5. Applying Theorem 3.10 to \( E \), we obtain positive divisors \( C_1, C_2 \) such that \( C_1 + C_2 \sim C, C_1 \cdot C - D > 0, C_2 \cdot C - D > 0 \). Since either \( C_1 \) or \( C_2 \) has a degree not greater than \( n \), (i) is proved.

ii) Similar argument as above is available in this case too.

q. e. d.

Now we come to a theorem of Schwarzenberger ([19], Theorem 8)

Theorem 4.6. Let \( n, m \) be integers. There is a vector bundle \( E \) of rank 2 on \( \mathbb{P}^2_k \) with \( c_1(E) = n, c_2(E) = m \) if and
only if \( n^2 - 4m < 0, \frac{1}{4} - 4 \).

Proof. We have proved that if \( \Delta_2(E) \geq 0 \) or \( \Delta_2(E) = -4 \), then 

\( E \) is not simple (Corollary 4.3.1, Lemma 4.5). Let us show the "if" part of the theorem. Take a point \( P \) on a line \( C_1 \) and a point \( P' \) on an irreducible conic \( C_2 \). Since \( C_1 \) and \( C_2 \) are rational curves,

\[
r^2(p^2_k, C_1, rP) \equiv \phi, \quad r^2(p^2_k, C_2, sP') \equiv \phi
\]

for any \( r > 0, s > 0 \).

If \( E_r \in R^2(p^2_k, C_1, rP), \ E'_s \in R^2(p^2_k, C_2, sP') \), then \( E_r, E'_s \) are simple for any \( r \geq 1, s \geq 3 \) by virtue of Corollary 3.10.1. Put

\[
r = m + \left( \frac{1 - n^2}{4} \right) \text{ if } n \text{ is odd and put } s = m + 1 - \frac{n^2}{4} \text{ if } n \text{ is even.}
\]

The condition \( n^2 - 4m < 0, \frac{1}{4} - 4 \) implies \( r \geq 1, s \geq 3 \). Take 

\( E_r \) or \( E'_s \) and put \( E = E_r \otimes O_{p^2_k}(\frac{n - 1}{2}) \) or \( E'_s \otimes O_{p^2_k}(\frac{n - 1}{2}) \) according as \( n \) is odd or even. Then \( c_1(E) = n, \ c_2(E) = m. \)

q. e. d.

The following theorem is due to F. Takemoto [20], which can be proved along our line.
Theorem 4.7. If $E$ is a simple vector bundle of rank 2 on $\mathbb{P}_k^2$ with $\Delta(E) = -3$, then $E \cong \mathcal{T}_{\mathbb{P}_k^2}(n)$.

Proof. By the assumption $\Delta(E) = -3$ there is a line bundle $L$ such that $c_1(E \otimes L) = 1$, $c_2(E \otimes L) = 1$. Hence we may assume that $c_1(E) = 1$ and $c_2(E) = 1$. We know by virtue of the Riemann-Roch theorem and Lemma 4.2 that $\chi(E) = 3$, $\chi(E(-1)) = 0$, $E^2(\mathbb{P}_k^2, E) = H^2(\mathbb{P}_k^2, E(-1)) = 0$. Thus $\dim_k H^0(\mathbb{P}_k^2, E) \geq 3$ and $H^0(\mathbb{P}_k^2, E(-1)) = 0$ because $E \cong E(-1)$. Consequently we have $H^1(\mathbb{P}_k^2, E(-1)) = 0$. Let $O_{\chi}(1)$ be the tautological line bundle of $E$ on the $\mathbb{P}_1^3$-bundle

$\eta : X = P(E) \longrightarrow \mathbb{P}_k^2$. Leray's spectral sequence $E_2^{pq} = H^p(\mathbb{P}_k^2, R^q \eta_* O_{\chi}(1)) \Rightarrow H^{p+q}(X, O_{\chi}(1) \otimes \eta^* O_{\mathbb{P}_k^2}(-1))$ provides $E^1(X, O_{\chi}(1) \otimes \eta^* O_{\mathbb{P}_k^2}(-1)) = 0$ because $E_2^{1,0} = E_2^{0,1} = 0$. Let $\ell$ be a line in $\mathbb{P}_k^2$.

$0 \longrightarrow O_{\chi}(1) \otimes \eta^* O_{\mathbb{P}_k^2}(-1) \longrightarrow O_{\chi}(1) \longrightarrow O_{\chi}(1) \otimes O_{\eta^* \mathbb{P}_k^2(-1)} \longrightarrow 0$
is an exact sequence, which yields another exact sequence

\[ 0 \longrightarrow H^0(X, O_X(1) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) \longrightarrow H^0(X, O_X(1)) \longrightarrow 0 \]

\[ 0 = H^0(\mathbb{P}^2, \mathcal{E}(-1)) \longrightarrow H^0(\mathbb{P}^2, \mathcal{E}) \]

\[ H^0(\mathcal{W}^{-1}(\mathcal{I}), O_X(1) \otimes \mathcal{O}_{\mathcal{W}^{-1}(\mathcal{Q})}) \longrightarrow H^1(X, O_X(1) \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) = 0. \]

Therefore \( H^0(X, O_X(1)) \cong H^0(\mathcal{W}^{-1}(\mathcal{I}), O_X(1) \otimes \mathcal{O}_{\mathcal{W}^{-1}(\mathcal{Q})}) \). Since \( \mathcal{W}^{-1}(\mathcal{I}) \) is a rational ruled surface, this isomorphism implies that

\[ |O_X(1)| \text{ has no fixed fibre (a fixed fibre of } |O_X(1)| \text{ is that of} \]

\[ |O_X(1) \otimes \mathcal{O}_{\mathcal{W}^{-1}(\mathcal{Q})}|, \text{ which can not occur). If } D \text{ is a fixed component} \]

of \( |O_X(1)| \), then \( O_X(D) \) is a tautological linebundle and \( D \)

contains no fibre, whence \( D \) is a section of \( \mathcal{W} : X \longrightarrow \mathbb{P}^2 \). Then

\( E \) is decomposable, which is impossible because \( E \) is simple.

Thus \( |O_X(1)| \) has no fixed component. Hence if \( D_1, D_2 \) are

general members of \( |O_X(1)| \), they are irreducible and have no

common fibre (note that \( O_X(D_1) \) is a tautological linebundle on \( X \)).

Let \( I_{D_1} \) be the defining ideal of \( D_1 \) in \( X \) and let \( J \) be the

ideal generated by \( I_{D_1} \) and \( I_{D_2} \). Then the exact sequence
0 \rightarrow I_{D_1} \rightarrow J \rightarrow J/I_{D_1} \rightarrow 0.

yields an isomorphism \( \mathcal{T}_*(J) \cong \mathcal{T}_*(J/I_{D_1}) \) of ideals of \( O_{\mathbb{P}^2_k} \)
because \( \mathcal{T}_*(I_{D_1}) \cong \mathcal{T}_*(O_{\mathbb{X}}(-1)) = 0 \), \( R^1\mathcal{T}_*(I_{D_1}) \cong R^1\mathcal{T}_*(O_{\mathbb{X}}(-1)) = 0. \)

Since \( J/I_{D_1} \) are locally principal \( O_{D_1} \)-ideal and \( D_1 \cup U_i \cong U_i \) for some open covering \( U_1 \cup U_2 \) of \( \mathbb{P}^2_k \), \( \mathcal{T}_*(J) \) is locally principal.

This and \( c_1(E) = 1 \) imply that \( \mathcal{T}_*(J) \) defines a line. Thus \( D_1 \cdot D_2 = C \) is irreducible and \( \mathcal{T}_*(C) \) is non-singular because
\( D_1 \cup U_i \cong U_i \), \( U_1 \cup U_2 = \mathbb{P}^2_k \), \( C \) therefore satisfies the condition
\( (E_0) \) and \( \text{elm}^0_{C}(X) \cong \mathbb{P}^1_k \times \mathbb{P}^2_k \) by virtue of Proposition 1.9. Thus \( E \) is regular and \( E \in R^2(\mathbb{P}^2_k, Q) \) for a point \( P \in Q \) because
\( c_2(E) = 1 \). Then \( E \cong \mathcal{T}_{\mathbb{P}^2_k}(-1) \) by virtue of Theorem 4.1.

\[ \text{q. e. d.} \]

Example 4.8. As was shown in Example 3.11, \( E \in R^2(\mathbb{P}^2_k, C^3, Q_1+Q_2+Q_3) \) is simple if and only if \( Q_1, Q_2, Q_3 \) are not collinear.

On the other hand, if \( E \in R^2(\mathbb{P}^2_k, C^3, Q_1+Q_2+Q_3) \), then \( c_1(E) = 3 \),
\( c_2(E) = 3 \) and therefore \( \Delta(E) = -3. \) Thus \( R^2(\mathbb{P}^2_k, C^3, Q_1+Q_2+Q_3) \)
consists only of one element \( T_{p^2} \) if \( Q_1, Q_2, Q_3 \) are not collinear.

Let \( E \) be a vector bundle of rank \( r \) on \( P_k^n \) and let \( j : P_k^1 \to P_k^n \) be an embedding such that \( j(P_k^1) \) is a line of \( P_k^n \). By a famous theorem of Grothendieck we get \( j^*(E) \cong O_{P_k^n}(a_1) \oplus \cdots \oplus O_{P_k^n}(a_r) \).

\((a_1 \geq a_2 \geq \cdots \geq a_r)\). Let us consider the map \( \alpha_E : j(P_k^1) \to \mathbb{F}^{Gr^r} \).

**Lemma 4.9.** There is a non-empty open set \( U(E) \) of \( Grass^1_1 \) such that \( \alpha_E(x) \) is a constant for every \( x \in U(E)(k) \) and if \( \alpha_E(y) = \alpha_E(x_0) \) for an \( x_0 \in U(E)(k) \), then \( y \in U(E)(k) \).

**Proof.** Let \( G \) be the universal quotient bundle on \( X = Grass^1_1 \) and let \( p : O_{X}^{\Theta(n+1)} \to G \) be the canonical surjective homomorphism.

Then we have the following diagram:

\[
\begin{array}{ccc}
P_k^n \times X & \xrightarrow{i} & P(G) \\
\downarrow f' \downarrow f & & \\
\downarrow f & & \\
P_k^n & \xrightarrow{j} & X
\end{array}
\]

Then \( P(G), f, g = f \circ i \) are nothing but the graph of the incidence correspondence between \( P_k^n \) and \( Grass^1_1 \) and the natural projections.
respectively. Put \( E'(m) = g^*E(m) \). Since \( f \) is flat and \( E'(m) \) is locally free \( O_p(E) \)-module, \( x \mapsto \dim_k(x)H^0(f^{-1}(x), E'(m)_x) \) is upper semi-continuous on \( X \). Since \( X \) is a noetherian space,

\[
\dim_k(x)H^0(f^{-1}(x), E'(m)_x)
\]

is bounded. Thus \( b_1 = \inf_{x \in X} \) (the first term of \( \dim_k(x)H^0(f^{-1}(x), E'(m)_x) \)) is \( -\infty \). Put \( U_1 = \{ x \in X \mid \dim_k(x)H^0(f^{-1}(x), E'(m)_x) = 0 \} \). Then \( U_1 \) is a non-empty open set of \( X \) by virtue of above argument. Similarly \( b_2 = \inf_{x \in U_1} \) (the second term of \( \dim_k(x)H^0(f^{-1}(x), E'(m)_x) \)) and \( U_2 = \{ x \in U_1 \mid \dim_k(x)H^0(f^{-1}(x), E'(m)_x) = b_1 - b_2 \} \) is a non-empty open set of \( U_2 \). Inductively we get \( U_x \) and \( U_x \) is the desired open set of \( \text{Grass}_1^n \).

q.e.d.

Definition (Schwarzenberger). A line contained in \( \text{Grass}_1^n - U(E) \) is called an exceptional line of \( E \).

Van de Ven showed that if \( U(E) = \text{Grass}_1^n \), \( E \) is rank 2 and if the characteristic of \( k \) is 0, then \( E \cong O_{P_k}(s_1) \oplus O_{P_k}(s_2) \) or \( T_{P_k}(a) \) (see [2i]).
Theorem 4.10 (Schwarzengerger [8]). Let \( E \) be a non-simple vector bundle of rank 2 on \( \mathbb{P}^2_k \). The exceptional lines of \( E \) form a finite number of linear pencils. If \( E \) has no exceptional line, then \( E \) is decomposable.

Proof. Since the set of exceptional lines of \( E \otimes L \) is nothing but that of \( E \), we may assume that \( E \) is regular (Proposition 2.3). If \( E \) is defined by \( (s, s') \) of \( \mathcal{H}^0(C^n, O_{C^n(D)}) \times \mathcal{H}^0(C^n, O_{C^n(D)}) \) \((C^n : \text{non-singular curve of degree } n \text{ in } \mathbb{P}^2_k)\), there are \( u, v \in \mathcal{H}^0(p^2_k, O_{p^2_k(m)}) \) \((m < \frac{n}{1+2})\) such that \( u, v \) induce \( s, s' \) on \( C^n(a \in \mathcal{H}^0(p^2_k, O_{p^2_k(m)} \otimes O_{C^n(-D)}) \) because \( E \) is not simple (Theorem 3.10). Let \( \gamma, \gamma' \) be homogeneous coordinate of \( p^1_{C^n} \) induced from homogeneous coordinate \( \gamma, \gamma' \) of \( p^1_{p^2_k} \). Then \( E \) is defined by \( Y : s \gamma + s' \gamma' = 0 \). Let \( A \) be the positive divisor on \( p^1_{p^2_k} \) defined by \( u \gamma + v \gamma' = 0 \), then \( A > Y \). If \( A \) is reducible, then there are an irreducible component \( A_1 \) and a positive divisor \( C \) with \( \deg C < n \) such that \( A = A_1 + p^1_C \).

Since \( \deg C < n \) and \( Y \) is irreducible, \( Y \subset A_1 \). Thus we may
assume that $A$ is irreducible. Hence $\text{elm}_Y^0(A) = A'$ contains only a finite number of fibres $\pi_Y^{-1}(x_1), \ldots, \pi_Y^{-1}(x_r)$ of

$\pi: X = \mathbb{P}(E) = \text{elm}_Y^0(\mathbb{P}_k^1 \times \mathbb{P}_k^2) \rightarrow \mathbb{P}^2_k$. Take a general line $l_0$ in $\mathbb{P}^2_k$, then $P^1_{l_0} \cdot A = B_{l_0}$ is a section with $(B_{l_0}, B_{l_0}) = 2m \leq n$ and $P^1_{l_0} \cdot Y = P^1 + \ldots + P^n$. By virtue of Proposition 1.8 we have

$\pi^{-1}(l_0) \cong \text{elm}_{\{P_1, \ldots, P_n\}}(P^1_{l_0})$. Put $\text{elm}_{\{P_1, \ldots, P_n\}}[B_{l_0}] = B'_{l_0}$,

then $(B'_{l_0}, B'_{l_0}) = 2m - n = -b < 0$. Thus $B'_{l_0}$ is a minimal section of the choice of a line in $\mathbb{P}^2_k$, $A' \cdot \pi^{-1}(l_0) = B'_{l_0}$ and since $B'_{l_0} = B'_{l_0} \times \text{fibres (B'_{l_0} : section)},$ we get $(B''_{l_0}, B''_{l_0}) = -b \leq -b$, whence

$B''_{l_0}$ is a minimal section of $\pi^{-1}(l)$ and $\pi^{-1}(l) \cong F_{b_1} = \text{Proj}(O_{P^1_k} \oplus O_{P^2_k}(b_1))$. Since if $\pi_E(l) = (a_1, a_2)$, then $a_1 + a_2 = n, a_1 - a_2 = b_1$ and since $B''_{l_0} \not\subseteq B''_{l_0}$ if and only if $l$ contains one of $x_1, \ldots, x_r$, we have the set of exceptional lines of $E = \bigcup_{i=1}^{r} \{ \text{lines containing } x_i \}$. Therefore the first assertion is proved.

If $E$ has no exceptional line, then $r = 0$ and therefore $A'$ is a section of $\mathbb{P}(E)$. Thus $E$ is an extension of line bundles.

Since $H^1(\mathbb{P}^2_k, L) = 0$ for any line bundle $L$, $E$ is decomposable.

\[\text{q. e. d.}\]
We shall finish this section with some examples of exceptional lines.

Example 4.11. Schwarzenberger conjectured in [1] that if a vector bundle $E$ of rank 2 on $\mathbb{P}^2_k$ is simple, then the set of exceptional lines of $E$ does not form a finite number of linear pencils. But his conjecture is disproved. In fact let $C^3$ be a non-singular cubic in $\mathbb{P}^2_k$, let $\ell_0, \ell_1$ be lines in $\mathbb{P}^2_k$ whose intersection is not on $C^3$ and let $\ell_i \cdot C^3 = p_{11} + p_{12} + p_{13}$. Let $s_1$ be an element of $H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2}(1))$ with $|s_1| = \ell_i$ and let $\overline{s}_1$ be the element of $H^0(C^3, \mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{C^3})$ induced from $s_1$.

Then the regular vector bundle $E$ defined by $(\overline{s}_1^2, \overline{s}_2^2)$ is simple and the set of exceptional lines forms a linear pencil.

Proof. Take $\ell_0, \ell_1, \ell_0', \ell_1'$ as in the proof of Theorem 4.10. Then the positive divisor $A$ defined by $s_0^2 \ell_0 + s_1^2 \ell_1 = 0$ contains $Y : \overline{s}_0^2 \ell_0 + \overline{s}_1^2 \ell_1 = 0$, and $A' = \text{elm}_Y[A]$ contains only one fibre $\nabla^{-1}(x)$ with $x = \ell_0 \cdot \ell_1$ and $\nabla : \mathbb{P}(E) \rightarrow \mathbb{P}^2_k$. If $\ell$ is a line not containing $x$, it is easy to see that $A' \cdot (\nabla^{-1}(\ell))$
is a section of \( \tau^{-1}(l) \) and \((B_t, B_t') = 1\). Since \( n \) is the minimum of self-intersection numbers of sections of \( F_n = \text{Proj} (O_{F_k} \oplus O_{F_k}(n)) \) with non-negative self-intersection number, we know \( \tau^{-1}(l) \cong F_1 \). On the other hand, if \( l' \) is a line containing \( x \), then
\[
A' \cdot \tau^{-1}(l') = B_{l'} = 2 \tau^{-1}(x), \quad B_{l'} \quad \text{is a section of} \quad \tau^{-1}(l') \quad \text{and} \quad \quad (B_{l'}, B_{l'}) = (B_t, B_t') = 1, \quad \text{Thus} \quad \tau^{-1}(l') \quad \geq \quad F_2. \quad \text{We see therefore that the set of exceptional lines of} \quad E \quad \text{is the linear pencil formed by lines containing} \quad x. \quad \text{Since} \quad E \subset R^2(P_k^2, C^3, 2(P_{01} + P_{02} + P_{03})), \quad \text{E is simple by virtue of Corollary 3.10.1.}

Proofs of the following examples are similar as above.

Example 4.12. i) Let \( \ell \) be a line in \( P_k^2 \), let \( P \) be a point on \( \ell \) and let \( E \subset R^2(P_k^2, Q, nP) \). If \( n = 1 \), then \( E \cong T_{P_k^2}(-1) \) and therefore \( E \) has no exceptional line. If \( n > 1 \), then \( E \) has only one exceptional line \( \ell \).

ii) Let \( C^n \) be a non-singular curve in \( P_k^2 \) of degree \( n \).

Let \( D_0, D_1 \) be general conics in \( P_k^2 \) and let \( D_1 \cdot C^2 = \sum_{j=1}^{4} P_{ij} \).
If $s_0, s_1$ are element of $H^0(C^2, O_{P_2} \otimes O_{C^2})$ with $|s_1| = \sum_{j=1}^4 \sum_{i=1}^3 P_{ij}$, then the set of exceptional lines of the regular vector bundle of rank 2 defined by $(s_0, s_1)$ can not form a finite number of linear pencils.

iii) Let $E \in R^2(P^2_k, C^3, P_1 + P_2 + P_3)$. If $P_1, P_2, P_3$ is not collinear, then $E \cong T_{P_2} (E)$ (Example 4.8) and therefore $E$ has no exceptional line. If $P_1, P_2, P_3$ is collinear and if $E$ is defined by $(s_0, s_1) \in H^0(C^3, O_{P_2} \otimes O_{C^3}) \times H^0(C^3, O_{P_2} \otimes O_{C^3})$ with $|s_1| = \sum_{j=1}^3 Q_{ij}$, then the set of exceptional lines of $E$ is the linear pencil formed by lines containing the point $P_E$ which is the common point of lines $Q_i (i = 0, 1)$ going through $Q_{i1}, Q_{i2}, Q_{i3}$. Thus if $P_E \neq P_{E'}$, then $E \not\cong E'$. Conversely it is easy to see that if $P_E = P_{E'}$, then $E \cong E'$. Thus $R^2(P^2_k, C^3, P_1 + P_2 + P_3)$ is in bijective correspondence with $P^2_k - C^3$ if $P_1, P_2, P_3$ are collinear.

§ 3. Vector bundles on rational ruled surfaces.

A rational ruled surface over $k$ is isomorphic to $\varpi_n : F_n = \text{Proj}(O_{P^1_k} \oplus O_{P^1_k}(n)) \hookrightarrow P^1_k$ for some non-negative integer $n$. There
is a section $M$ on $F_n$ with $(M, M) = -n$. If $n > 0$, then $M$ is the unique irreducible curve with negative self-intersection number (see [13]). $M$ is called a minimal section of $F_n$. Let $N$ be a fibre (= a generator) of $F_n$. By virtue of the seesaw theorem, every divisor $D$ on $F_n$ is linearly equivalent to $aM + bN$, where $a = (D, N)$, $b = (D, M) + an$. On the other hand, $-2M - (n+2)N$ is a canonical divisor of $F_n$.

**Lemma 4.13.** Let $E$ be a vector bundle of rank 2 on $F_n$. If $c_1(E) = aM + bN$ for $a > -2$, $b \geq -(n+2)$ and if $E$ is simple, then $H^2(F_n, E) = 0$.

**Proof.** If one notes that $-2M - (n+2)N$ is a canonical divisor on $F_n$, the proof is similar to that of Lemma 4.2.

**Lemma 4.14.** If $E$ is a simple vector bundle of rank 2 on $F_n$ with $c_1(E) = aM + bN$, $c_2(E) = c$, then one of the following conditions is satisfied:

1. Both $a$ and $b$ are even and $2ab - a^2n - 4c = -4r$ $(r \geq 2)$.

2. Both $a$ and $b$ are odd and $2ab - a^2n - 4c = -n + 2 - 4r$.
\( a \) is odd, \( b \) is even \( a \) and \( 2ab - a^2n - 4c = -4r \) \((r \geq 1)\)

\( r \geq 1 \) if \( n = 0; \ r \geq 2 \) if \( n = 0 \)

(3) \( a \) is even, \( b \) is odd and \( 2ab - a^2n - 4c = -4r \) \((r \geq 1)\)

Proof. In the first place, note that \( c_1(E) = aM + bN, c_2(E) = c \) imply \( \Delta(E) = 2ab - a^2n - 4c \). The Riemann-Roch theorem asserts

the following equality for a vector bundle of \( E' \) of rank 2 on \( \mathbb{P}_n \):

\[ \chi(E') = 2 + (2M + (n+2)N) c_1(E') + (c_1(E')^2 - 2c_2(E'))/2 \]

1) Assume that both \( a \) and \( b \) are even, then \( c_1(E) = 0 \) and \( c_2(E) = c \). For such an \( E \)
we have \( \chi(E) = 2 - c \). On the other hand, \( H^2(\mathbb{P}_n, E) = 0 \) by
virtue of Lemma 4.13. Thus \( \dim_k H^0(\mathbb{P}_n, E) \geq 2 - c \). Since \( E' \cong E \) and \( E \) is simple, we have \( H^0(\mathbb{P}_n, E) = 0 \). Hence \( c \geq 2 \), which
implies \( \Delta(E) = -4c = -4r \) \((r \geq 2)\).

2) Assume that both \( a \) and \( b \) are odd. By a similar reason
as above we may assume that \( c_1(E) = M + N, c_2(E) = c \). Since
\( c_1(E') = -(M + N), c_2(E') = c \), we have \( \chi(E') = 1 - c \), \( H^2(\mathbb{P}_n, E) \).
$E^V = 0$, whence $\dim_k H^0(F_n, E^V) \geq 1 - c$. On the other hand, since

$H^0(F_n, E^V) = H^0(F_n, E \otimes_{\mathcal{O}_F} (-M-N))$ is a linear subspace of $H^0(F_n, E)$

and since $E$ is simple, we have $H^0(F_n, E^V) = 0$. Thus we get $c \geq 1$,

which implies $\Delta(E) = (M+N)^2 - 4c = -n^2 + 2 - 4c = -n + 2 - 4r (r \geq 1)$. Moreover, in the case where $n = 0$ and $\Delta(E) = -2$,

consider $E_1 = E \otimes_{\mathcal{O}_F} (-M)$ and $E_2 = E \otimes_{\mathcal{O}_F} (-N)$. Then $E_1 = E_2$,

$c_1(E_1) = N - M$, $c_1(E_2) = M - N$ and $c_2(E_1) = c_2(E_2) = 0$. Thus

$\chi(E_1) = \chi(E_2) = 1$ and $H^2(F_n, E_1) = H^2(F_n, E_2) = 0$. Therefore

$\dim_k H^0(F_n, E_1) > 0$, $\dim_k H^0(F_n, E_2) > 0$, which is impossible. Hence if $n = 0$, then $\Delta(E) = -4r + 2 (r \geq 2)$.

3) Assume that $a$ is even and $b$ is odd. Then we may assume

that $c_1(E) = N$, $c_2(E) = c$. Since $\Delta(E) = -4c$ and since

$\Delta(E) \leq -1$ by virtue of Corollary 4.3.1, (i), we have $\Delta(E) = -4r (r \geq 1)$.

4) Finally assume that $a$ is odd and $b$ is even. Then we may assume

that $c_1(E) = M$, $c_2(E) = c$. Since $c_1(E^V) = -M$, $c_2(E^V) = c$, we have $\chi(E^V) = 1 - c$, $H^2(F_n, E^V) = 0$, whence
\[
\dim_k H^0(F_n, E^\vee) \geq 1 - c. \quad \text{On the other hand, since } \dim_k H^0(F_n, E^\vee) \\
\leq \dim_k H^0(F_n, E) \quad \text{and } E \text{ is simple, we have } H^0(F_n, E^\vee) = 0.
\]

Thus \( c \geq 1 \), which implies that \( \Delta(E) = M^2 - 4c = -n - 4r \) \((r \geq 1)\).

\[ q.e.d. \]

Each of the conditions of the above lemma is sufficient for the existence of a vector bundle \( E \) of rank 2 on \( F_n \) with \( c_1(E) = aM + bN \), \( c_2(E) = c \). In fact,

\[ (i) \]

Theorem 4.15. There is a vector bundle \( E \) of rank 2 on \( F_n \) with \( c_1(E) = aM + bN \), \( c_2(E) = c \) if and only if one of the conditions (1), (2), (3), (4) of Lemma 4.14 is satisfied.

Proof. By virtue of Lemma 4.14 we have only to prove the "if" part.

1) Assume that the condition (1) is satisfied. Take a general member \( C \) of \( |2M + (2n + 2)N| \), then \( C \) is a non-singular curve because \( 2M + (2n + 2)N \) is very ample. Let \( P_1, \ldots, P_{n+2+r} \) \((r \geq 2)\) be sufficiently general points on \( C \) and put \( D_r = \sum_{i=1}^{n+2+r} P_i. \) Since
the genus of $C$ is $n+1$ and $D_r$ is general, $\dim |D_r| = r-1 \geq 1$

and $|D_r|$ is free from base point. Let $E_r \in R^2(F_n, C, D_r)$ be defined by $(s_0, s_1) \in H^0(C, O_C(D_r)) \times H^0(C, O_C(D_r))$ with $|s_0| = D_r$. Assume that $E_r$ is not simple, then there are positive divisors $C_1, C_2$ on $F_n$ such that $C_1 + C_2 \sim C \sim 2M + 2(n+1)N$ and $C_1 \cdot C - D_r > 0$ (see Theorem 3.10). $C_1$ is linearly equivalent to $a_1M + b_1N$ with $a_1 \geq 0$, $b_1 \geq 0$, $a_1 + a_2 = 2$, $b_1 + b_2 = 2n + 2$. If one of $a_1$, for instance $a_1$, is $0$, then $b_1 \geq n+2+r$ because $C_1 = N_1 + \ldots + N_{b_1}$ for some fibres $N_1, \ldots, N_{b_1}$ of $F_n$. $C_1$ goes through $P_1, \ldots, P_{n+2+r}$. Thus $C_2 = 2M + N_1' + \ldots + N_{b_2}'$ for some fibres $N_1', \ldots, N_{b_2}'$ because $2M + b_2N = 2M$ if $b_2 < n$. Then $C_2$ cannot go through $P_1, \ldots, P_{n+2+r}$. We may assume therefore that $C_1 \sim M + b_1N$, $b_1 \leq n+1$. Since $\deg(C_1 \cdot C) = 2b_1 + 2$, we have $\dim |C_1 \cdot C| \leq n+3$ by virtue of the Riemann-Roch theorem on $C$ and Clifford's theorem (which asserts that if $D$ is a special divisor of degree $n$ on a curve, then $2\dim |D| \leq n$). Thus $C_1$ cannot go through $P_1, \ldots, P_{n+2+r}$ if $P_1, \ldots, P_{n+2+r}$ are.
sufficiently general with \( r \geq 2 \). This is a contradiction.

Therefore \( E_r \) is simple. On the other hand, \( \triangle(E_r) = -4r \). Thus

\[
E = E_r \otimes O_{\mathbb{P}^n}((a-2/2)M + (b/2 - n-1)N)
\]

is the desired vector bundle.

2) Assume that the condition (2) is satisfied. Take a general member \( C \) of \( |M + mN| \), where \( m = n \) or \( n+1 \) according as \( n \) is odd or not. Then \( C \) is a non-singular curve because \( C \) is a section of \( F_n \). For a general positive divisor \( D_r \) of degree \( r + (m-1)/2 \) (\( r \geq 1 \)), construct a vector bundle \( E_r \in R^2(F_n, C, D_r) \) as in the proof(1) above. Assume that \( E_r \) is not simple, then there are positive divisors \( C_1, C_2 \) on \( F_n \) such that \( C_1 + C_2 \sim C \sim M + mN \) and \( C_1 \cdot C - D_r > 0 \). We may assume that \( C_1 \sim M + b_1N \), \( C_2 \sim b_2N \) with \( b_1 + b_2 = m \), \( b_1, b_2 > 0 \). We have \( b_2 \geq r + (m-1)/2 \) because \( C_2 \) goes through every point of \( \text{Supp}(D_r) \). Thus \( b_1 \leq (m+1)/2 - r \), whence \( M + b_1N \sim M + \text{fibres if } n \not\equiv 0 \). Thus if \( n \not\equiv 0 \), \( C_1 \) cannot go through every points of \( \text{Supp}(D_r) \) because \( b_1 < r + (m-1)/2 = \text{deg } D_r \). This is a contradiction. Therefore \( E_r \) is simple if \( n \not\equiv 0 \), \( r \geq 1 \). On the other hand, if \( n = 0 \), then
(C, C) = 2 and therefore $E_r$ is simple for any $r \geq 2$ by virtue of Corollary 3.10.1. Since $\Delta(E_r) = -n + 2 - 4r$, $E_r \otimes \mathcal{O}_{P_n}((a-1/2)M + (b-1/2)N) = E$ is the desired vector bundle.

3) Assume that the condition (3) is satisfied. Let $P$ be a point on $N$, then $R^2(F_n, N, rP) \neq \emptyset$ for any $r \geq 1$ because $N$ is a non-singular rational curve. Since $(N, N) = 0$, an element $E_r$ of $R^2(F_n, N, rP)$ is simple by virtue of Corollary 3.10.1. Thus $E = E_r \otimes \mathcal{O}_{P_n}((a/2)M + (b-1/2)N)$ is the desired vector bundle.

4) Assume that the condition (4) is satisfied. Let $P$ be a point of $M$, then $R^2(F_n, M, rP) \neq \emptyset$ for any $r \geq 1$ because $M$ is a non-singular rational curve. Since $(M, M) = -n \leq 0$, and element $E_r$ of $R^2(F_n, M, rP)$ is simple by virtue of Corollary 3.10.1. Thus $E = E_r \otimes \mathcal{O}_{P_n}((a-1/2)M + (b/2)N)$ is the desired vector bundle.

q. e. d.

As an example let us consider the family of simple vector bundles of rank 2 with $\Delta(E) = -4$ on $P_n$. 
Theorem 4.16. Let $S(n, a, b)$ be the set of isomorphism classes of simple vector bundles $E$ of rank 2 on $F_n$ with $c_1(E) = aM + bN$, $\Delta(E) = -4$. If (1) $a$ is even, $b$ is odd and $n \neq 0$ or (2) one of $a$ and $b$ is odd, the other is even and $n = 0$, then there is a bijective map $\varphi_{n,a,b} : P^1(k) \rightarrow S(n, a, b)$. Moreover, there is a vector bundle $S(a, b)$ on $F_0 \times P^1$ such that $S(a, b)_x = \varphi_{0,a,b}(x)$ for any $x \in P^1(k)$.

Proof. First of all, note that if $n, a, b$ satisfy the above conditions then $S(n, a, b)$ is empty by virtue of Theorem 4.15. Since $F_0 = P^1 \times P^1$ and since $M = P^1 \times Q$, $N = R \times P^1$ for some $Q, R \in P^1$, we may assume that $a$ is even and $b$ is odd even if $n = 0$. Take an $E' \in S(n, a, b)$ and let us consider $E = E' \otimes O_{F_n}(-(a/2)M - (b-1/2)N)$. Then $c_1(E) = N$ and $c_2(E) = 1$. Since $\chi(E) = 2$, $\chi(E^\vee) = 0$, $H^2(F_n, E) = H^2(F_n, E^\vee) = 0$, we have $\dim_k H^0(F_n, E) \geq 2$ and therefore $H^0(F_n, E \otimes O_{F_n}(-N)) = H^0(F_n, E^\vee) = 0$, $H^1(F_n, E \otimes O_{F_n}(-N)) = H^1(F_n, E^\vee) = 0$. By a similar argument as in the proof of Theorem 4.7 we have $H^0(X, O_X(1)) \cong H^0(\nu^{-1}(N), O_X(1) \otimes$
with the tautological linebundle \( \mathcal{O}_X(1) \) of \( E \) on 
\[ \pi: X = \mathbb{P}(E) \to F_n. \]
Since \( \mathbb{P}^{-1}(N) \) is a rational ruled surface
and since the fibre \( N \) is chosen arbitrarily in the above argument,
the above isomorphism implies that \( \mathcal{O}_X(1) \) has no fixed fibre.
Thus if \( D \) is the fixed component of \( \mathcal{O}_X(1) \), then \( D \) is a section
of \( \pi: X \to F_n \), whence \( E \) is an extension of linebundles;
\[
0 \to \mathcal{O}_{F_n}(aM + bN) \to E \to \mathcal{O}_{F_n}(a'M + b'N) \to 0.
\]
Since \( c_1(E) = (a + a')M + (b + b')N = N \), \( c_2(E) = -aa'n + ab' + b'b = -1 \), we obtain \( a = 1, \ a' = -1, \ b = n/2, \ b' = 1 - n/2 \) or \( a = -1 \)
\( a' = 1, \ b = 1 - n/2, \ b' = n/2. \) Thus \( n \) is even. If \( a = 1 \), then
\( n = 0 \) because \( H^0(F_n, \mathcal{O}_{F_n}(aM + (b - 1)N)) \subseteq H^0(F_n, E \otimes \mathcal{O}_{F_n}(-N)) = 0. \)
Since \( H^1(F_0, \mathcal{O}_{F_0}(2M - N)) = 0 \) by virtue of the Riemann-Roch theorem,
the above extension splits in this case, and we obtained a contradiction.
If \( a = -1 \), then \( n = 0 \) also because \( H^0(F_n, \mathcal{O}_{F_n}(a'M + (b' - 1)N)) \subseteq \)
\( H^4(F_n, \mathcal{O}_{F_n}(aM + (b-1)N)) \) and because \( \dim_k H^0(F_n, \mathcal{O}_{F_n}(a'M + (b' - 1)N)) = n/2, \ dim_k H^1(F_n, \mathcal{O}_{F_n}(aM + (b-1)N)) = 0. \) On the other hand, since
the exact sequence

\[ 0 \longrightarrow O_{F_0}(N - M) \longrightarrow E \longrightarrow O_{F_0}(M) \longrightarrow 0 \]

provides \( \dim_k H^0(X, O_X(1)) = \dim_k H^0(F_0, E) = 2 \) and since \( O_X(1) \cong O_X(D) \otimes \pi_*(L) \) for some linebundle \( L \) on \( F_0 \), we have \( O_X(1) \cong O_X(D + \pi^{-1}(M)) \) or \( O_X(D + \pi^{-1}(N)) \). But, in any case, \( \pi_*(D_1 \cdot D_2) \sim N \) for \( D_1, D_2 \) with \( O_X(1) \cong O_X(D_1) \) because \( \pi_*(D_1 \cdot D) \sim M \).

This contradicts the fact that \( c_1(E) = N \). We see therefore that \( |O_X(1)| \) has no fixed component. Then by a similar argument as

in the proof of Theorem 4.7, for any general members \( D_1, D_2 \) in \( |O_X(1)| \), we see that \( D_1 \cdot D_2 = Y \) is a non-singular curve

satisfying the condition \( (E_0) \) such that \( \pi_*(Y) = (a \text{ fibre } N_1 \text{ of } F_n) \). Thus \( \operatorname{elm}_2^0(\pi(E)) \cong \mathbb{P}^1_k \times F_n \) and \( E \in R^2(F_n, N_1, P) \) for a point \( P \in N_1 \). Conversely every element of \( R^2(F_n, N_1, P) \) is simple because \( (N_1, N_1) = 0 \). Since \( \dim_k H^0(N_1, O_{N_1}(P)) = 2 \), \( R^2(F_n, N_1, P) \) consists only of one element by virtue of Theorem 2.14. Moreover, if \( N_1, N_2 \) are mutually distinct fibres of \( F_n \),
then $E_1 \neq E_2$ for $E_i \in \mathcal{R}^2(F_n, N_i, \mathcal{P}_i)$ $(i = 1, 2)$ by virtue of Theorem 2.13. Thus there is a bijective map $\varphi : \{\text{fibres of } F_n\} \rightarrow S(n, 0, 1)$. Since $F_n$ is a rational ruled surface, there is a canonical bijective map $\varphi' : P^1(k) \rightarrow \{\text{fibres of } F_n\}$.

Therefore we obtain a bijective map $\varphi_{n,0,1} = \varphi \cdot \varphi' : P^1(k) \rightarrow S(n, 0, 1)$. Since $S(n, a, b) = \mathcal{S} \otimes O_{F_n}((a/2)M + (b-1/2)N)|E \in \mathcal{S}(n, 0, 1)$, we obtain a bijective map $\varphi_{n,a,b} : P^1(k) \rightarrow S(n, a, b)$.

In order to prove the last assertion, consider $Z = P_k^1 \times P_k^1 \times P_k^1$ whose system coordinates is $(z_0^{(1)}, z_1^{(1)}; z_0^{(2)}, z_1^{(2)}; z_0^{(3)}, z_1^{(3)})$.

Let $Y$ be the subvariety defined by $z_0^{(1)}z_0^{(2)} + z_1^{(1)}z_1^{(2)} = 0$

and let $j : Z \rightarrow P_k^1 \times F_0 \times P_k^1 = P_k^1 \times P_k^1 \times P_k^1 \times P_k^1$ be the closed immersion defined by $j((x, y, z)) = (x, y, z, z)$. Then $j(Y) = Y'$ is a subvariety of $P^1$-bundle $P_k^1 \times F_0 \times P_k^1 \rightarrow F_0 \times P_k^1$ satisfying the condition $(E_0)$. Let $S(0, 1)$ be the regular vector bundle on $F_0 \times P^1$ defined by $Y'$, then it is clear that for any even integer $a$ and odd integer $b$, $S(a, b) = S(0, 1) \otimes p^*O_{F_0}((a/2)M + (b-1/2)N)$ is the desired vector bundle with the natural projection.
\[ p : \mathcal{F}_0 \times P^1_K \longrightarrow \mathcal{F}_0. \]

\[ \text{q. e. d.} \]
References


Footnotes.

1) In fact $g_*(O_X) = O_X$, and so $g_*(I_{\mathcal{T}})$ is an ideal of $O_X$ (see Lemma 1.5).

2) Let $\pi: X \to S$ be the projective bundle $P(E)$ associated with a vector bundle $E$ of rank $N+1$ ($N \geq 1$). A linebundle $L$ on $X$ is, by abuse of language, called a tautological linebundle when $L$ is the tautological linebundle of a vector bundle $E'$ with $P(E') = X$. In the case where $S$ is reduced, $L$ is a tautological linebundle if and only if $L_s = L \otimes_{O_S} k(s)$ is the linebundle associated with the hyperplane $\pi^{-1}(s) = P_k(s)$ for any $s \in S$. If $L_1$, $L_2$ are tautological linebundles on $X$, then there is a linebundle $M$ on $S$ such that $L_1 = L_2 \otimes_{O_S} \pi^{-1}(M)$.

3) The direct proof of this fact is easy. But geometric interpretation of this (i.e., the relation between Theorem 1.1 and Theorem 1.3) is very important.

4) For an affine scheme $Z = \text{Spec}(B)$ and $b \in B$, $Z(b)$
denotes \( \text{Spec}(R_0) \).

5) Note that locally this complex \( K_* \) is isomorphic to the usual Koszul complex defined by \( h_0, \ldots, h_N \) with a local equation \( h_1 \) of \( H_1 \). Note also that if \( K_* (f_1, \ldots, f_r) \) is the Koszul complex defined by elements \( f_1, \ldots, f_r \) of \( A \) and if \( \{ f_1, \ldots, f_r \} \) contains a unit element, then \( H_i(K_* (f_1, \ldots, f_r) \otimes M) = 0 \) \( \forall i > 0 \) for every \( A \)-module \( M \).

6) As a matter of fact \( \mathcal{Y} \) is an immersion.

7) If \( \dim S = 1 \), then the theory in the sequel is trivial because we assume that \( Y \) is irreducible (cf. Remark 2.16).

8) Of course, \( C_1 \supset Y_1 \) means that the support of \( C_1 \) contains \( Y_1 \).

9) Very ample in the sense of Sumihiro: A vector bundle \( E \) on \( S \) is called very ample if the tautological linebundle of \( E \) on \( P(E) \) is very ample in the sense of Grothendieck. H. Sumihiro proved the following; (i) For any vector bundle \( E \) there is a linebundle \( L \) such that \( E \otimes L \) is very ample if \( S \) is projective.
(see Lemma 1.11). (ii) $E = E_1 \oplus E_2$ is very ample if and only if both $E_1$ and $E_2$ are very ample. (iii) If $f : E \rightarrow E'$ is a surjective homomorphism of vector bundles and if $E$ is very ample, then $E'$ is very ample. (iv) If $E$ is ample in the sense of Hartshorne, then there is an integer $n_0$ such that $S^n(E)$ is very ample for any $n \geq n_0$ ($S^n(E)$ is the symmetric tensor product of grade $n$). (v) If $E$ is very ample, then $E$ is generated by its global sections and the morphism $g : S \rightarrow \text{Grass}$ defined by $E$ is a closed immersion.

10) In the next section we shall show that $c_1(E) = T$, $c_2(E) = D$ for $E \in R^r(S, T, D)$.

11) This means that $f_0, \ldots, f_N$ form a basis of $H^0(X(Y), O_X(Y)(H^i))$ if $(f_1) = H^1 - H^0$.

12) In the next chapter we shall show that $SR^r(S, T, D)$ consists of all simple vector bundles in $R^r(S, T, D)$.

13) Note

$$\sum_{a+b=c, a \geq 0, b \geq 0} (-1)^{b-1/a} b = (-1/c') \sum_{b=0}^{c} (-1)^{b} \begin{pmatrix} c \\ b \end{pmatrix} = (-1/c)(1-1)^c = 0.$$
14) For divisors $D_1, D_2$ on a non-singular surface, $(D_1, D_2)$
denotes the intersection number of $D_1, D_2$.

15) In [11] Schwarzenberger says that there is a simple vector
bundle $E$ of rank 2 with $c_1(E) = n$, $c_2(E) = m$ if $n^2 - 4m < 0$.
But this is not true as we have shown. His error comes from an
incorrect statement (b) in the proof of his Theorem 7.

16) In [11] Schwarzenberger says without proof that for any
$a, b, c$ with $ab - 2c < 0$ there is a simple vector bundle $E$ of
rank 2 on $F_0$ with $c_1(E) = aM + bN$, $c_2(E) = c$. But this is not
true (see the above conditions (1), (2)).

17) A simple vector bundle $E$ with $\Delta(E) = -4$ which does
not satisfy these conditions exists only on $F_2$ (see Theorem 4.15).