学位申請論文

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主論文

On a family of algebraic vector bundles.

By

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Introduction. In the theory of algebraic vector bundles, it seems to the writer that it is very important to have a nice answer to the problem to "construct a lot of vector bundles on high dimensional algebraic variety." It is, of course, desirable that the structure of a vector bundle is easily known from its construction. A main purpose of this paper is to look for an answer to the above problem.

As for our problem, two answers are known:

(1) Schwarzenberger - Hironaka - Kleiman ([18], [8], [10]).

For a vector bundle E on a smooth quasi-projective variety X over an infinite field there is a monoidal transformation $f : X' \longrightarrow X$ with smooth center such that $f^*(E)$ contains a sublinebundle. This was proved by Schwarzenberger in the case where X is a surface, by Hironaka in the characteristic 0 case and by Kleiman in the general case. The above fact implies that every vector bundle on a smooth quasi-projective variety is a successive extension of linebundles if one performs monoidal transformations on the base variety. Thus every vector bundle on a smooth quasi-projective variety is obtained by "extension + descent ". But the descent problem is very difficult. In fact the answer to the descent problem is known only in the case where the base space is a surface (Schwarzenberger [18]). Schwarzenberger's answer was very powerful in his theory of almost decomposable (i.e. non-simple) vector bundles. But he needed another method in order to construct simple vector bundles on P^2.

(2) Schwarzenberger - Oda ([19] , [15]). If $f : X' \longrightarrow X$ is a flat and finite morphism, then $f_*(L)$ is a vector bundle (locally free sheaf) on X for any linebundle L on X'. Using this fact, Schwarzenberger studied simple vector bundles on some algebraic surfaces([19]). T. Oda studied $f_*(L)$ in the case where f is an isogeny of abelian varieties. This method faces the following problem:

What morphism and linebundle does a given vector bundle come from ?

This is also difficult.

In general and in these treatment too, "base change" gives rise to some difficulties . Our starting point is to find a method to construct vector bundles without "base change". There is known a nice model, that is, the theory of elementary transformation of ruled surfaces, and we shall generalize elementary transformations of ruled surfaces to those of P^N-bundles on a locally noetherian scheme. An interesting result is that every P^N-bundle on a non-singular quasi-projective variety S over an algebraically closed field k with dim $S \leq 3$ is obtained by an elementary transformation from the direct product $P_k^N \times S$. This result leads us to the concept of regular vector bundles. Some big families of vector bunles are constructed in Chapter II. The family of regular vector bundles contains a large subfamily of simple vector bundles (see Chapter III, § 1) This fact implies that if S is a non-singular projective variety over k and if $S \not\cong P_k^1$, then there is a simple vector bundle on S (Corollary 3. 4. 1) In the rank 2 case, we have a very clear criterion that a regular vector bundle is simple (Theorem 3.10). Using this criterion we can cover almost all results of Schwarzenberger without the theory of moduli of non-simple vector bundles and we get further result.

Notation and convention. Throughout this paper k denotes an algebraically closed field and all varieties are reduced and irreducible algebraic schemes over k. We use the terms "vector bundles" and "locally free sheaves" interchangeablly. For a monoidal transformation $f : X' \to X$ with center Y and a subscheme Z of X, $f^{-1}(Z)$ denotes the total transform of Z and $f^{-1}[Z]$ denotes the proper (strict) transform of Z. If X and Y are smooth and if $D = \sum n_i D_i$ (D_i: irreducible) is a divisor on X', then $f[D]$ denotes $\sum_{D_i \not\subseteq f^{-1}(Y)} n_i \overline{f'(D_i)}$, where f' is the restriction of f to $X' - f^{-1}(Y)$. In the case where a birational map $g : X_1 \to X_2$ is a composition $f_2 \cdot f_1^{-1}$ of monoidal transformations with non-singular centers $f_1 : X' \to X_1$, $f_2 : X' \to X_2$, for a divisor D on X_1, $g(D)$ denotes $f_2[f_1^{-1}(D)]$ and $g[D]$ denotes $f_2[f_1^{-1}[D]]$. For a Cartier divisor D on a scheme X, $O_X(D)$ denotes the invertible sheaf defined by D. If L is a linebundle on a non-singular projective variety X, then $|L|$ denotes the complete linear system $|D|$ for a divisor D on X with $O_X(D) \cong L$. For an algebraic k-scheme X, X(k) denotes the set of k-rational points of X. If E is a locally free O_S-module (=a vector bundle on S), then P(E) denotes $\mathrm{Proj}(S_{O_S}(E))$, where $S_{O_S}(E)$ is the O_S-symetric algebra of E.

The author wishes to thank Professors M. Nagata, H. Hironaka, S. Nakano and T. Oda for their encouragement and valuable suggestions.

Contents.

Chapter I. Elementary transformations of P^N-bundles.

§ 1. Definition of elementary transformations.

Let S be a locally noetherian scheme and let $\pi : X \longrightarrow S$ be the projective bundle $P(E)$ associated with a vector bundle E of rank $N + 1$ $(N \geq 1)$. Let T and Y be closed subschemes of S and X respectively, satisfying the following condition;

(E_n^0) The ideal I_T which defines T is locally principal whose generator is non-zero divisor in every local ring of S, that is, I_T is a Cartier divisor on S. Y is a closed subscheme of X_T and $\pi_T|_Y : Y \longrightarrow T$ induces a P^n-bundle on T $(0 \leq n \leq N-1)$ such that $(\pi_T|_Y)^{-1}(t)$ is a linear subspace of $\pi^{-1}(t)$ for any $t \in T$. Roughly speaking $\pi_T|_Y : Y \longrightarrow T$ is a subbundle of $\pi_T : X_T \longrightarrow T$.

Now consider the monoidal transformation $f : \tilde{X} \longrightarrow X$ with center Y and put $f^{-1}[X_T] = \overline{X}_T$, $f^{-1}(Y) = E_Y$. In this situation we have the following theorem, whose proof will be given in the next section.

Theorem 1.1. There exist a P^N-bundle $\pi' : X' \longrightarrow S$ which is the projective bundle $P(E')$ associated with a vector bundle E' and an S-morphism $g : \widetilde{X} \longrightarrow X'$ such that the closed subscheme Y' of X' defined by the ideal $g(I_{X_T})$ 1) with the defining ideal $I_{\overline{X}_T}$ for $\overline{X}_T$ in X and T satisfies the condition (E^0_{N-n-1}), and that $g^*(L) \cong f^*(O_X(1)) \otimes O_X(-E_Y)$ for some tautological linebundle L on X' and the tautological line bundle $O_X(1)$ on X of E. 2) g is the monoidal transformation with center Y'. Moreover, such (X', g) is unique, that is, if there exists another (X'', g') satisfying the above conditions, then there is a unique bundle isomorphism $h : X' \longrightarrow X''$ with $h \cdot g = g'$.

The above theorem enables us to generalize elementary transformations of ruled surfaces to those of P^N-bundles. Namely :

Definition. Under the above notation the birational map $g \cdot f^{-1}$ is called the elementary transformation of X with center Y and we denote it by elm^n_Y ; we denote X' by $elm^n_Y(X)$.

Corollary 1.1.1. $elm^{N-n-1}_{Y'}(elm^n_Y(X)) = X$.

We note that our treatment can be applied to P^N-bundles if S is factorial, that is, every local ring of S is a unique factorization domain, because of the following fact :

Lemma 1.2. (A. Grothendieck [4]) If $\pi : X \longrightarrow S$ is a P^N-bundle (in Zariski topology) on a factorial scheme S, then there is a vector bundle E on S such that $P(E) \cong X$.

Proof. Since S is a direct sum of irreducible subschemes, we may assume that S is irreducible. The exact sequence of group schemes on S

$$e \longrightarrow G_{m,S} \longrightarrow GL(n+1, S) \longrightarrow PGL(N, S) \longrightarrow e$$

provides the exact sequence of cohomologies

$$H^1(S, GL(N+1, S)) \longrightarrow H^1(S, PGL(N, S)) \longrightarrow H^2(S, O_S^*)$$

Thus we have only to prove $H^2(S, O_S^*) = 0$. In order to see this, consider the exact sequence of sheaves

$$0 \longrightarrow O_S^* \longrightarrow K_S^* \longrightarrow D_S \longrightarrow 0,$$

where K_S^* is the sheaf of non-zero rational functions of S and D_S is the sheaf of Cartier divisors on S. Since every local ring of S is a U.F.D., D_S is isomorphic to the sheaf of Weil divisors on S. Thus D_S is a flabby sheaf (because every Weil divisor on an open set is extensible to that on the whole space). On the other hand, K_S^* is also flabby because K_S^* is a constant sheaf. Therefore the above sequence can be regarded as a part of a flabby resolution of O_S^*, whence $H^2(S, O_S^*) = 0$. q. e. d.

Sheaf theoretic interpretation of elementary transformations is stated as follows.

Theorem 1.3. Let E be a locally free $\mathcal{O}_S$-module of rank $N+1$ and let T, Y be closed subschemes of S, $X = P(E)$ satisfying the condition (E_n^*).

(i) Denote by I_Y the ideal defining Y and by $O_X(1)$ a tautological line-bundle on X , then $E' = \pi_*(I_Y \otimes O_X(1))$ is a locally free O_S-module, $P(E') \cong elm_Y^n(X)$ and $R^1\pi_*(I_Y \otimes O_X(1))$

$= 0 \quad (i > 0)$, where $\pi : X \longrightarrow S$ is the natural projection.

(ii)[3] Since $(\pi_T|_Y)_*(O_Y \otimes O_X(1))$ is a locally free O_T-module of rank n+1, (1) can be said in other words ; If F is a quotient bundle of $E_T = E \otimes_{O_S} O_T$ of rank n+1, then $\mathrm{Ker}\,\varphi = E'$ is a locally free O_S-module of rank N+1, where $\varphi : E \longrightarrow E_T \longrightarrow F$ is the natural homomorphism. And we have the following exact commutative diagram ;

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & F' & \longrightarrow & E_T & \longrightarrow & F & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & E \otimes I_T & = & E \otimes I_T & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Moreover, the locally free O_T-module F' of rank N-n defines closed subscheme Y' of $P(E')$ in Theorem 1.1 and the step obtaining E', F' corresponds to the inverse of elm_Y^n (see Corollary 1.1.1 and note that $P(E) = P(E \otimes I_T)$).

§ 2. Proof of Theorem 1.1 and Theorem 1.3.

In this section the notation of the preceding section is preserved.

The following is a key lemma.

Lemma 1.4. Assume that $S = \mathrm{Spec}(A)$, $X = \mathrm{Proj}(A[\eta_0, \dots, \eta_N])$ and that the defining ideals I_T, I_Y for T and Y in A and $A[\eta_0, \dots, \eta_N]$ respectively are generated by $t \in A$ and t, $\eta_{n+1}, \dots, \eta_N$, respectively. Then elm^n_Y exists and $\mathrm{elm}^n_Y(X) = \mathrm{Proj}(A[\eta'_0, \dots, \eta'_N])$, where $\eta_i = \eta'_i$ $(0 \leq i \leq n)$, $\eta_j = t\,\eta'_j$ $(n+1 \leq j \leq N)$.

Proof. Put $\xi^\beta_\alpha = \eta_\beta/\eta_\alpha$, $\xi'^\beta_\alpha = \eta'_\beta/\eta'_\alpha$ $(0 \leq \alpha, \beta \leq N)$ and put $X' = \mathrm{Proj}(A[\eta'_0, \dots, \eta'_N])$. Let $f : \tilde{X} \longrightarrow X$ be the monoidal transformation of X with center Y, then

$$X = \Big(\bigcup_{\alpha=0}^{n} U'_\alpha\Big) \cup \Big(\bigcup_{\substack{0 \leq \alpha \leq n \\ n+1 \leq \beta \leq N}} U^\beta_\alpha\Big) \cup \Big(\bigcup_{\gamma=n+1}^{N} U''_\gamma\Big)$$

where $U'_\alpha = \mathrm{Spec}(A[\xi'^0_\alpha, \dots, \xi^N_\alpha, \xi^{n+1}_\alpha/t, \dots, \xi^N_\alpha/t]) =$ $\mathrm{Spec}(A[\xi^0_\alpha, \dots, \xi^n_\alpha, \xi^{n+1}_\alpha/t, \dots, \xi^N_\alpha/t])$,

$U_\alpha^\beta = \mathrm{Spec}(A\left[\zeta_\alpha^0, \ldots, \zeta_\alpha^n, t/\zeta_\alpha^\beta, \zeta_\alpha^{n+1}/\zeta_\alpha^\beta, \ldots, \zeta_\alpha^N/\zeta_\alpha^\beta\right]) =$

$\mathrm{Spec}(A\left[\zeta_\alpha^0, \ldots, \zeta_\alpha^n, \zeta_\alpha^\beta, t/\zeta_\alpha^\beta, \zeta_\beta^{n+1}, \ldots, \zeta_\beta^N\right]),$

$U''_\gamma = \mathrm{Spec}(A\left[\zeta_\gamma^0, \ldots, \zeta_\gamma^N\right])$ (Everything is considered in

$Q(A)(\zeta_0^1, \ldots, \zeta_0^N))$.

Moreover, we have

$U'_\alpha \cap U'_{\alpha'} = U'_\alpha(\zeta_\alpha^{\alpha'}) = U'_{\alpha'}(\zeta_{\alpha'}^\alpha)$ 4)

$U'_\alpha \cap U_{\alpha'}^\beta = U'_\alpha(\zeta_\alpha^\beta/t \cdot \zeta_\alpha^{\alpha'}) = U_{\alpha'}^\beta(t/\zeta_{\alpha'}^\beta \cdot \zeta_\alpha^{\alpha'})$

$U_\alpha^\beta \cap U_{\alpha'}^{\beta'} = U_\alpha^\beta(\zeta_\alpha^{\alpha'} \cdot \zeta_\beta^{\beta'}) = U_{\alpha'}^{\beta'}(\zeta_{\alpha'}^\alpha \cdot \zeta_{\beta'}^\beta)$

$U'_\alpha \cap U''_\gamma = U'_\alpha(\zeta_\alpha^\gamma) = U''_\gamma(\zeta_\gamma^\alpha t)$

$U_\alpha^\beta \cap U''_\gamma = U_\alpha^\beta(\zeta_\alpha^\gamma \cdot \zeta_\beta^\gamma) = U''_\gamma(\zeta_\gamma^\alpha \cdot \zeta_\gamma^\beta)$

$U''_\gamma \cap U''_{\gamma'} = U''_\gamma(\zeta_\gamma^{\gamma'}) = U''_{\gamma'}(\zeta_{\gamma'}^\gamma)$

On the other hand, for the monoidal transformation $g : \tilde{X}' \longrightarrow X'$

with center $Y' = \mathrm{Proj}(A[\eta'_0, \ldots, \eta'_N]/(t, \eta'_0, \ldots, \eta'_n))$ we

have an affine open covering $\tilde{X}' = \left(\bigcup_{\gamma=n+1}^{N} V'_\gamma\right) \cup \left(\bigcup_{\substack{0\le\alpha\le n \\ n+1\le\beta\le N}} V_\beta^\alpha\right) \cup \left(\bigcup_{\alpha=0}^{n} V''_\alpha\right)$

where

$V'_\gamma = \mathrm{Spec}(A[\zeta'^{n+1}_\gamma, \ldots, \zeta'^N_\gamma, \zeta'^0_\gamma/t, \ldots, \zeta'^n_\gamma/t]) =$

$\mathrm{Spec}(A\left[\zeta_\gamma^{n+1}, \ldots, \zeta_\gamma^N, \zeta_\gamma^0, \ldots, \zeta_\gamma^n\right]) = U''_\gamma,$

$$V_\beta^\alpha = \mathrm{Spec}(A[\xi'^{n+1}_\beta, \ldots, \xi'^N_\beta, \xi'^\alpha_\beta, t/\xi'^\alpha_\beta, \xi'^0_\alpha, \ldots, \xi'^n_\alpha])$$

$$= \mathrm{Spec}(A[\xi^{n+1}_\beta, \ldots, \xi^N_\beta, t/\xi^\beta_\alpha, \xi^\beta_\alpha, \xi^0_\alpha, \ldots, \xi^n_\alpha])$$

$$= U^\beta_\alpha .$$

$$V''_\alpha = \mathrm{Spec}(A[\xi'^0_\alpha, \ldots, \xi'^N_\alpha]) = \mathrm{Spec}(A[\xi^0_\alpha, \ldots, \xi^n_\alpha, \xi^{n+1}_\alpha/t, \ldots, \xi^N_\alpha/t]) = U'_\alpha .$$

Furthermore, we have

$$V'_\gamma \cap V'_{\gamma'} = V'_\gamma(\xi'^{\gamma'}_\gamma) = U''_\gamma(\xi^{\gamma'}_\gamma) = U''_\gamma \cap U''_{\gamma'}$$

$$V'_\gamma \cap V^\alpha_\beta = V'_\gamma(\xi'^\alpha_\gamma/t \cdot \xi'^\beta_\gamma) = U''_\gamma(\xi^\alpha_\gamma \cdot \xi^\beta_\gamma) = U''_\gamma \cap U^\beta_\alpha$$

$$V^\alpha_\beta \cap V^{\alpha'}_{\beta'} = V^\alpha_\beta(\xi'^{\beta'}_\beta \cdot \xi'^\alpha_\gamma) = U^\beta_\alpha(\xi^{\alpha'}_\alpha \cdot \xi^{\beta'}_\beta) = U^\beta_\alpha \cap U^{\beta'}_{\alpha'}$$

$$V'_\gamma \cap V''_\alpha = V'_\gamma(\xi'^\alpha_\gamma) = U''_\gamma(t\,\xi^\alpha_\gamma) = U'_\alpha \cap U''_\gamma$$

$$V^\alpha_\beta \cap V''_{\alpha'} = V^\alpha_\beta(\xi'^{\alpha'}_\beta \cdot \xi'^{\alpha'}_\alpha) = U^\beta_\alpha(t/\xi^\beta_{\alpha'} \cdot \xi^{\alpha'}_\alpha)$$

$$V''_\alpha \cap V''_{\alpha'} = V''_\alpha(\xi'^{\alpha'}_\alpha) = U'_\alpha(\xi^{\alpha'}_\alpha) = U'_\alpha \cap U'_{\alpha'}$$

Thus we obtain $\tilde{X} = \tilde{X}'$. It is easy to see that $g_*(O_X) = O_{X'}$. (see Lemma 1.5) Now let us prove $g_*(I_{\overline{X}_T}) = I_{Y'}$. In order to show this let us consider the affine open covering $X' = \bigcup_{i=0}^N W'_i$, $W'_i = \mathrm{Spec}(A[\xi'^0_i, \ldots, \xi'^N_i])$ and put $\tilde{W}'_i = g^{-1}(W'_i)$. Then we have

$$\tilde{W}'_\alpha = V''_\alpha = U'_\alpha \quad (0 \le \alpha \le n)$$

$$\widetilde{W}'_\gamma = V'_\gamma \cup (\bigcup_{\alpha=0}^{n} V^\alpha_\gamma) = U''_\gamma \cup (\bigcup_{\alpha=0}^{n} U^\gamma_\alpha) \quad (n+1 \le \gamma \le N)$$

Since $U'_\alpha \cap \overline{X}_T = \{t/t = 0\} = \phi$, we know $U'_\alpha \cap g(\overline{X}_T) = \phi$, $(0 \le \alpha \le n)$.

Furthermore since the ideal of $U''_\gamma \cap \overline{X}_T$ (or, $U^\gamma_\alpha \cap \overline{X}_T$) is generated by t (or, $t/\xi^\gamma_\alpha = \xi'^\alpha_\gamma$, resp.), $H^0(\widetilde{W}'_\gamma, I_{\overline{X}_T})$ is generated by t, $\xi'^0_\gamma, \ldots, \xi'^n_\gamma$ as $H^0(\widetilde{W}'_\gamma, O_{\widetilde{X}})$-module (see the proof of Lemma 1.5), whence $g_*(I_{\overline{X}_T}) = I_{Y'}$.

Finally we must show that for a tautological linebundle L on X there is a tautological linebundle L' on X' with $g^*(L') \cong f^*(L) \otimes O_{\widetilde{X}}(-E_Y)$.

Assume that there are tautological linebundles L_1 on X and L'_1 on X' with $g^*(L'_1) \cong f^*(L_1) \otimes O_{\widetilde{X}}(-E_Y)$, then $L \cong L_1 \otimes \pi^*(M)$ for some linebundle M on S and therefore $g^*(L'_1 \otimes \pi'^*(M)) \cong g^*(L'_1) \otimes g^* \pi'^*(M)$ $\cong f^*(L_1) \otimes O_{\widetilde{X}}(-E_Y) \otimes f^* \pi^*(M) \cong f^*(L_1 \otimes \pi^*(M)) \otimes O_{\widetilde{X}}(-E_Y) \cong f^*(L) \otimes$ $O_X(-E_Y)$, which implies that $L'_1 \otimes \pi'^*(M) = L'$ is a desired linebundle.

Thus we may assume that L is an invertible sheaf with $1/\xi^N_\alpha$ as a generator in $W_\alpha = \mathrm{Spec}(A[\xi^0_\alpha, \ldots, \xi^N_\alpha])$. Then a generator of $f^*(L) \otimes O_{\widetilde{X}}(-E_Y)$ in $\widetilde{W}'_\alpha = V''_\alpha = U'_\alpha\ (0 \le \alpha \le n)$ is $t/\xi^N_\alpha = 1/\xi'^N_\alpha$, the one in $V'_\gamma = U''_\gamma (n+1 \le \gamma \le N)$ is $1/\xi^N_\gamma = 1/\xi'^N_\gamma$ and the one in $V^\alpha_\gamma = U^\gamma_\alpha$ $(0 \le \alpha \le n,\ n+1 \le \gamma \le N)$ is $\xi^\gamma_\alpha/\xi^N_\alpha = 1/\xi'^N_\gamma$, whence the one in $\tilde{W}'_i = g^{-1}(W'_i)$ is $1/\xi'^N_i$. Thus if L' is an invertible sheaf on X' whose generator in W'_i is $1/\xi'^N_i$, then $g^*(L') = f^*(L) \otimes O_{\tilde{X}}(-E_Y)$. It is clear that L' is a tautological linebundle.

q. e. d.

As a corollary to the above proof, we have

Lemma 1.5. If E_Y is the Cartier divisor $f^*(I_Y)$, then $f_*(O_{\tilde{X}}(-rE_Y)) = I_Y^r$, $f_*(O_{\tilde{X}}(rE_Y)) = O_X$ for any $r \geq 0$.

Proof. We have only to prove $H^0(f^{-1}(U_\alpha), O_{\tilde{X}}(-rE_Y)) = (t, \xi_\alpha^{n+1}, \ldots, \xi_\alpha^N)^r A[\xi_\alpha^0, \ldots, \xi_\alpha^N]$ and $H^0(f^{-1}(U_\alpha), O_{\tilde{X}}(rE_Y)) = A[\xi_\alpha^0, \ldots, \xi_\alpha^N]$ for $U_\alpha = \mathrm{Spec}(A[\xi_\alpha^0, \ldots, \xi_\alpha^N])$, $0 \leq \alpha \leq n$ under the same situation as in Lemma 1.4. If $F \in Q(A)(\xi_\alpha^0, \ldots, \xi_\alpha^N)$ is contained in $H^0(f^{-1}(U_\alpha), O_{\tilde{X}}(rE_Y))$, then $t^r F \in A[\xi_\alpha^0, \ldots, \xi_\alpha^n, \xi_\alpha^{n+1}/t, \ldots, \xi_\alpha^N/t]$, $(\xi_\alpha^\beta)^r F \in A[\xi_\alpha^0, \ldots, \xi_\alpha^N, t/\xi_\alpha^\beta, \xi_\alpha^{n+1}/\xi_\alpha^\beta, \ldots, \xi_\alpha^N/\xi_\alpha^\beta]$ for $n+1 \leq \beta \leq N$. Thus $F \in A_t[\xi_\alpha^0, \ldots, \xi_\alpha^N] \cap A[\xi_\alpha^0, \ldots, \xi_\alpha^N]_{\xi_\alpha^\beta} = A[\xi_\alpha^0, \ldots, \xi_\alpha^N]$. Conversely it is clear that $A[\xi_\alpha^0, \ldots, \xi_\alpha^N] \subseteq H^0(f^{-1}(U_\alpha), O_{\tilde{X}}(r E_Y))$. Since $O_{\tilde{X}}(-r E_Y)$ is a ideal of $O_{\tilde{X}}$, and since $f_*(O_{\tilde{X}}) = O_X$ by the above proof, $f_*(O_{\tilde{X}}(-r E_Y)) \subseteq O_X$. $F \in$

$A[\xi_\alpha^0, \ldots, \xi_\alpha^N]$ is contained in $H^0(f^{-1}(U_\alpha), O_{\tilde{X}}(-r E_Y))$ if and only if $F/t^r \in A[\xi_\alpha^0, \ldots, \xi_\alpha^n, \xi_\alpha^{n+1}/t, \ldots, \xi_\alpha^N/t]$, $F/(\xi_\alpha^\beta)^r \in A[\xi_\alpha^0, \ldots, \xi_\alpha^N, t/\xi_\alpha^\beta, \xi_\alpha^{n+1}/\xi_\alpha^\beta, \ldots, \xi_\alpha^N/\xi_\alpha^\beta]$ $(n+1 \leq \beta \leq N)$. Thus we obtain $H^0(f^{-1}(U_\alpha), O_{\tilde{X}}(-r E_Y)) = (t, \xi_\alpha^{n+1}, \ldots, \xi_\alpha^N)^r A[\xi_\alpha^0, \ldots, \xi_\alpha^N]$.

Lemma 1.6. If P^N-bundle $\pi_i : X_i \longrightarrow S (i = 1, 2)$ and morphisms $g_i : \tilde{X} \longrightarrow X_i$ satisfy the conditions stated is Theorem 1.1, then there exists a unique isomorphism $h : X_1 \longrightarrow X_2$ such that $h \cdot g_1 = g_2$.

Proof. Let L_i $(i = 1, 2)$ be a tautological linebundle on X_i with $g_1^*(L_1) \cong f^*(O_X(1)) \otimes O_{\tilde{X}}(-E_Y)$. Then since $g_1^*(L_1) \cong g_2^*(L_2)$, we have $L_2 \cong (g_2)_*(g_1^*(L_1))$. Put $E_i = (\pi_i)_*(L_i)$, then $E_2 \cong (\pi_2)_*(g_2)_* g_1^*(L_1) \cong (\pi_1)_*(g_1)_* g_1^*(L_1) \cong (\pi_1)_*(L_1) = E_1$ (cf Lemma 1.5). Since $P(E_2) = X_2$, $P(E_1) = X_1$, this isomorphism yields an isomorphism $h : X_1 \longrightarrow X_2$ with $h^*(L_2) = L_1$. Thus we get an isomorphism $h : X_1 \longrightarrow X_2$ with $h \cdot g_1 = g_2$ (E. G. A. Chap. II, 4.2.3). Uniqueness clearly follows from the construction.

q. e. d.

Lemma 1.7. If $U \supseteq V$ are open subschemes of S and if $g_U : \widetilde{X}_U \longrightarrow \mathrm{elm}^n_{Y_U}(X_U)$ exists, then $g_V : \widetilde{X}_V \longrightarrow \mathrm{elm}^n_{Y_V}(X_V)$ exists and there is a unique isomorphism $h^U_V : (\mathrm{elm}^n_{Y_U}(X_U))_V \longrightarrow \mathrm{elm}^n_{Y_V}(X_V)$ with $h^U_V \cdot g_{U,V} = g_V$.

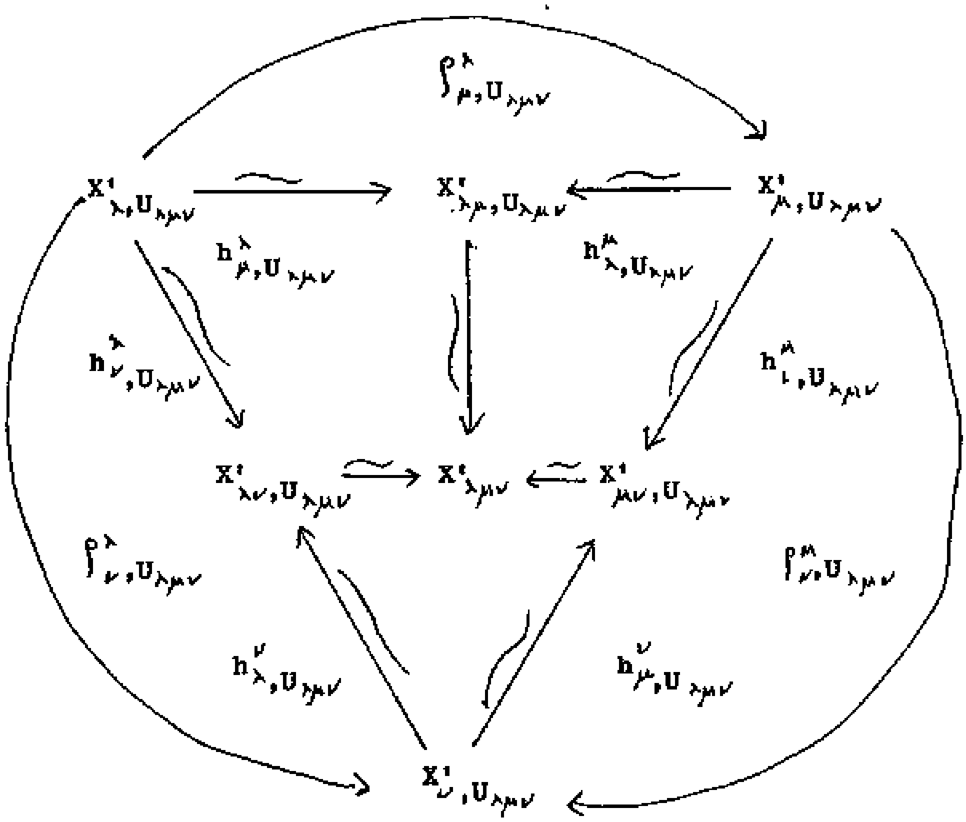
Proof. This is an immediate consequence of the definition of elementary transformation and Lemma 1.6.

Now we proceed with the proofs of Theorem 1.1 and Theorem 1.3.

Proof of Theorem 1.1. Uniqueness has been proved in Lemma 1.6. Let us cover S by affine open subsets $\{U_\lambda\}_{\lambda \in \Lambda}$ satisfying the conditions in Lemma 1.4. By virtue of Lemma 1.4 there exists $g_\lambda : \widetilde{X}_{U_\lambda} \longrightarrow X'_\lambda = \mathrm{elm}^n_{Y_U}(X_U)$, hence $g_{\lambda\mu} : \widetilde{X}_{U_{\lambda\mu}} \longrightarrow X'_{\lambda\mu} = \mathrm{elm}^n_{Y_{U_{\lambda\mu}}}(X_{U_{\lambda\mu}})$, $g_{\lambda\mu\nu} : X_{U_{\lambda\mu\nu}} \longrightarrow X'_{\lambda\mu\nu} = \mathrm{elm}^n_{Y_{U_{\lambda\mu\nu}}}(X_{U_{\lambda\mu\nu}})$ exist (Lemma 1.7), where $U_{\lambda\mu} = U_\lambda \cap U_\mu$, $U_{\lambda\mu\nu} = U_\lambda \cap U_\mu \cap U_\nu$. By virtue of Lemma 1.7 there is a unique isomorphism $h^\lambda_\mu : X'_{\lambda, U_{\lambda\mu}} \longrightarrow X'_{\lambda\mu}$ and the commutative diagram (+) is obtained. Thus if $X'_{\lambda, U_{\lambda\mu}}$ is identified with $X'_{\mu, U_{\lambda\mu}}$ by the isomorphism $\rho^\lambda_\mu = (h^\mu_\lambda)^{-1} \cdot h^\lambda_\mu$, then we obtain a P^N-bundle X' on S because $\rho^\mu_\nu \cdot \rho^\lambda_\mu = \rho^\lambda_\nu$ on $X_{\lambda, U_{\lambda\mu\nu}}$ by virtue of the diagram (+). Moreover, since

$h^{\lambda}_{\mu} \cdot g_{\lambda,U_{\lambda\mu}} = g_{\lambda\mu} = h^{\mu}_{\lambda} \cdot g_{\mu,U_{\lambda\mu}}$, we get a morphism $g : \tilde{X} \longrightarrow X'$. In order to show that $X' \cong P(E')$ for some vector bundle E' on S, we have only to prove

(+)



that there exists a tautological linebundle L' on X' such that $g^*(L') \cong f^*(O_X(1)) \otimes O_X(-E_Y)$ for a tautological linebundle $O_X(1)$ on X, which completes our proof. By virtue of Lemma 1.4 $(g_{U_\lambda})_* ((f_{U_\lambda})^*(O_X(1)|U_\lambda) \otimes O_{\tilde{X}_{U_\lambda}}(-E_Y|U_\lambda)) = L'_{U_\lambda}$ is an inversible sheaf on X'_{U_λ} such that $(g_{U_\lambda})^*(L'_{U_\lambda}) \cong (f_{U_\lambda})^*(O_X(1)|U_\lambda) \otimes O_{\tilde{X}_{U_\lambda}}(-E_Y|U_\lambda)$.

Thus we know that $L' = g_*(f^*(O_X(1)) \otimes O_{\tilde{X}}(-E_Y))$ is an invertible sheaf on X' such that $g^*(L') = f^*(O_X(1)) \otimes O_{\tilde{X}}(-E_Y)$. On the other hand, $L'|_{U_\lambda}$ is a tautological linebundle on X'_{U_λ} by virtue of Lemma 1.4. Thus $\pi'_*(L') = E'$ is a locally free sheaf on S and $P(E') = X'$, whence L' is a tautological linebundle.

q. e. d.

Proof of Theorem 1.3. (i) Let $O_{X'}(1)$ be a tautological linebundle on $X' = elm^n_Y(X)$. Then by virtue of Theorem 1.1 $g^*(O_{X'}(1)) \cong f^*(O_X(1)) \otimes O_{\tilde{X}}(-E_Y)$ for a tautological linebundle $O_X(1)$ on X. Thus

$$E' = \pi'_*(O_{X'}(1)) \cong \pi'_* g_* g^*(O_{X'}(1)) \cong \pi_* f_*(f^*(O_X(1)) \otimes O_{\tilde{X}}(-E_Y))$$

$$\pi_*(O_X(1) \otimes f_*(O_{\tilde{X}}(-E_Y))) \cong \pi_*(O_X(1) \otimes I_Y) \text{ (see Lemma 1.5) .}$$

Since $X' = P(E')$, we know that $\pi_*(O_X(1) \otimes I_Y)$ is a locally free O_S-module and $P(\pi_*(O_X(1) \otimes I_Y)) \cong elm^n_Y(X)$. Let I_{X_T}(or, J_Y) be the ideal of X_T in X(or, Y in X_T, resp.) Then we have an exact sequence

$$0 \longrightarrow I_{X_T} \otimes O_X(1) \longrightarrow I_Y \otimes O_X(1) \longrightarrow J_Y \otimes O_X(1) \longrightarrow 0 .$$

Since $I_{X_T} = \pi^*(I_T)$ and since I_T is a Cartier divisor on S, $I_{X_T} \otimes O_X(1)$ is also a tautological linebundle on X, whence $R^i \pi_*(I_{X_T} \otimes O_X(1)) = 0$, $i > 0$. On the other hand, the following exact sequence

$$0 \longrightarrow J_Y \otimes O_X(1) \longrightarrow O_{X_T}(1) \longrightarrow O_Y \otimes O_X(1) \longrightarrow 0$$

gives rise to an exact sequence

$$E_T \longrightarrow F \longrightarrow R^1\pi_*(J_Y \otimes O_X(1)) \longrightarrow R^1\pi_*(O_{X_T}(1)) \longrightarrow$$
$$R^1\pi_*(O_Y \otimes O_X(1)) \longrightarrow \dots \longrightarrow R^{i-1}\pi_*(O_Y \otimes O_X(1)) \longrightarrow$$
$$R^i\pi_*(J_Y \otimes O_X(1)) \longrightarrow R^i\pi_*(O_{X_T}(1)) \longrightarrow \dots,$$

where $E = \pi_*(O_X(1))$, $F = \pi_*(O_Y \otimes O_X(1))$. Since $P(E_T) = X_T$,

$P(F) = Y$ and since $Y \hookrightarrow X_T$ is a closed immersion, $E_T \longrightarrow F$ is a surjective map. $O_{X_T}(1)$ ($\sim$, $O_Y \otimes O_X(1)$) is a tautological linebundle on X_T($\sim$, Y, resp.), hence $R^i\pi_*(O_{X_T}(1)) = R^i\pi_*(O_Y \otimes O_X(1)) = 0 (\forall i > 0)$. Thus $R^i\pi_*(J_Y \otimes O_X(1)) = 0 \; (\forall i > 0)$. Hence the first exact sequence implies $R^i\pi_*(I_Y \otimes O_X(1)) = 0 \; (\forall i > 0)$.

(ii) Every assertion is clear except for $Y' = P(F')$. Let s be a point of S and let $U = \mathrm{Spec}(A)$ be an affine neighborhood of s such that E, E', F are free and that I_T is principal in U. If $e_0, \ldots, e_N$ ($\sim$, $e'_0, \ldots, e'_N$, resp.) form a basis of E_U ($\sim$, E'_U, resp.), we may assume that $\varphi(e_0), \ldots, \varphi(e\ )$ form a basis of F and the map $\alpha_U : E'_U \longrightarrow E_U$ is given by $\alpha_U(e'_i) = te_i (0 \leq i \leq n)$, $\alpha_U(e'_i) = e_i (n+1 \leq i \leq N)$. Then $\psi_U(e'_{n+1}), \ldots, \psi_U(e'_N)$ form a basis of F', where $\psi : E' \longrightarrow F'$ is the natural map. This and Lemma 1.4 imply $P(F') = Y'$.

q. e. d.

§ 3. Some properties of elementary transformations.

Elementary transformations are compatible with base changes.

In fact .

Proposition 1.8. Let $\varphi : S' \longrightarrow S$ be a morphism of locally noetherian schemes, let $\pi : X \longrightarrow S$ be the projective bundle associated with a vector bundle E of rank $N+1$, and let T, Y be closed subscheme of S, X satisfying the condition (E_n^0). Assume that $\varphi^*(I_T)$ is also a Cartier divisor in S' with the defining ideal for T in S. Then $\varphi^{-1}(T)$, $Y_{S'}$ satisfy the condition (E_n^0) for P^N-bundle $\pi_{S'} : X_{S'} \longrightarrow S'$ and $(elm_Y^n(X))_{S'} \cong elm_{Y_{S'}}^n(X_{S'})$.

Proof. It is clear that $\varphi^{-1}(T)$, $Y_{S'}$ satisfy the condition (E_n^0). Note that if a P^N-bundle π' ; $X' \longrightarrow S$ and a morphism g : $\tilde{X} \longrightarrow X'$ exist and if there is an open covering $\bigcup_{\lambda \in \Lambda} U_\lambda = S$ such that $g_{U_\lambda} : \tilde{X}_{U_\lambda} \longrightarrow X'_{U_\lambda}$ satisfies the conditions stated in Theorem 1.1, then $X' \cong elm_Y^n(X)$ and $g \cdot f^{-1} = elm_Y^n$. (see Theorem 1.1 and its proof). Thus we may assume that $S = \mathrm{Spec}(A)$, $S' = \mathrm{Spec}(B)$ and that X, Y satisfy the condition in Lemma 1.4. Then our assertion is obvious by virtue of Lemma 1.4.

q. e. d.

Next, we assume that S is a regular scheme. Let us consider the following condition for a P^N-bundle $\pi : X \longrightarrow S$ and a closed subscheme Y of X ;

(E_n) Y is a regular subscheme of pure dimension $n + \dim S-1$ $(0 \leq n \leq N-1)$ and $\bar{\pi}^{-1}(s)$ is a n-dimensional linear subvariety L^n_s of $P^N_{k(s)} = \pi^{-1}(s)$ for any $s \in T = \pi(Y)$, where T has the unique reduced structure and where $\bar{\pi} : Y \longrightarrow T$ is the restriction of π to Y.

Then we know that Y is a P^n-bundle on T ([9] Lemma 1.7, Theorem 1.8) and T is a regular subscheme of S. Hence if Y satisfies the condition (E_n), then Y, T satisfy the condition (E^0_n).

The remaining part of this section will be devoted to prove that every P^N-bundle on a smooth quasi-projective k-variety with dimension smaller than 4 is obtained by an elementary transformation with center satisfying the condition (E_{N-1}) from the trivial bundle.

Proposition 1.9. Let $\pi : X \longrightarrow S$ be a P^N-bundle on a smooth k-variety S and let $H_0, \ldots, H_N$ be positive divisors on X such that $O_X(H_i)$ is a tautological linebundle for every i. Assume that $H_0, \ldots, H_N$ are transversal to each other at any point of $\bigcap_{i=0}^{N} H_i$ and that $\dim ((\bigcap_{i=0}^{N} H_i) \cap \pi^{-1}(s)) \leq 0$ for every $s \in S$. Then $Y = H_0 \cdot \cdots \cdot H_N$ satisfies the condition (E_0) and $\mathrm{elm}_Y^0(X) \cong P(L_0 \oplus \ldots \oplus L_N)$, where $\pi^*(L_i) \cong O_X(H_0) \otimes O_X(-H_i)$. In paticular if $O_X(H_0) \cong O_X(H_i)$ $(1 \leq i \leq N)$, then $\mathrm{elm}_Y^0(X) \cong P_k^N \times S$.

Proof. Since $H_0, \ldots, H_N$ are transversal to each other at any point of $\bigcap_{i=0}^{N} H_i$, Y is k-smooth and pure dimension $(\dim S - 1)$. Moreover, since $O_X(H_i)$ is a tautological linebundle and since $\dim (Y \cap \pi^{-1}(s)) \leq 0$, we see that $\pi^{-1}(s) = L_s^0$ for every $s \in S$. Thus we know that Y satisfies the condition (E_0). Next let I_Y be the ideal sheaf of Y in O_X. Let us consider the Koszul complex $K.$ defined by $H_0, \ldots, H_N$;

$$K_0 = O_X$$

$$K_i = \bigoplus_{0 \leq \alpha_1 < \cdots < \alpha_i \leq N} O_X(-(H_{\alpha_1} + \ldots + H_{\alpha_i})), \qquad 1 \leq i \leq N+1$$

$$K_j = 0 \qquad j > N+1$$

.

and the derivation $d_i : K_i \longrightarrow K_{i-1}$ is defined by

$$(d_i)_x\Big(\sum_{0\le\alpha_1<\cdots<\alpha_i\le N} a_{\alpha_1,\ldots,\alpha_i}\Big) = \sum_{0\le\alpha_1<\cdots<\alpha_i\le N}\sum_{k=1}^{i}(-1)^{k-1}\, a_{\alpha_1,\ldots,\alpha_i},$$

where $x \in X$, $a_{\alpha_1,\ldots,\alpha_i} \in O_X(-(H_{\alpha_1} + \ldots + H_{\alpha_i}))_x$ and $(-1)^{k-1} a_{\alpha_1,\ldots,\alpha_i}$ of the left hand side is regarded as an element of $O_X(-(H_{\alpha_1} + \ldots + H_{\alpha_{k-1}} + H_{\alpha_{k+1}} + \ldots + H_{\alpha_i}))_x$ by the natural inclusion $O_X(-(H_{\alpha_1} + \ldots + H_{\alpha_i}))_x \hookrightarrow O_X(-(H_{\alpha_1} + \ldots + H_{\alpha_{k-1}} + H_{\alpha_{k+1}} + \ldots + H_{\alpha_i}))_x$. Then since $H_0, \ldots, H_N$ are transversal to each other at any point of Y,

$$0 \longrightarrow K_{N+1} \longrightarrow K_N \longrightarrow \cdots\cdots \longrightarrow K_1 \longrightarrow I_Y \longrightarrow 0$$

is an exact sequence (E.G.A. Chap.III, 1.1.4)[5]. Hence,

$$0 \longrightarrow K_{N+1} \otimes_{O_X} O_X(H_0) \longrightarrow K_N \otimes_{O_X} O_X(H_0) \longrightarrow \cdots$$
$$\longrightarrow K_1 \otimes_{O_X} O_X(H_0) \longrightarrow I_Y \otimes_{O_X} O_X(H_0) \longrightarrow 0$$

is also an exact sequence. Put $M_i = \mathrm{Ker}\,(d_i \otimes_{O_X} O_X(H_0)) = \mathrm{Im}(d_{i+1} \otimes_{O_X} O_X(H_0))$, then we have the following exact sequences ;

$$a_N)\quad 0 \to O_X(-(H_1+\dots+H_N)) = K_{N+1} \otimes_{O_X} O_X(H_0) \to K_N \otimes_{O_X} O_X(H_0) \to M_{N-1} \to 0$$

$$a_{N-1})\quad 0 \to M_{N-1} \to K_{N-1} \otimes_{O_X} O_X(H_0) \to M_{N-2} \to 0$$

$$\vdots$$

$$a_1)\quad 0 \to M_1 \to K_1 \otimes_{O_X} O_X(H_0) \to I_Y \otimes_{O_X} O_X(H_0) \to 0$$

Since $O_X(-(H_{\alpha_1}+\dots+H_{\alpha_i})) \otimes_{O_X} O_X(H_0) \cong O_X(-iH_0) \otimes_{O_X} \pi^*(L_{\alpha_1} \otimes \dots \otimes L_{\alpha_i}) \otimes_{O_X}$ $O_X(H_0) \cong O_X(-(i-1)H_0) \otimes_{O_X} \pi^*(L_{\alpha_1} \otimes \dots \otimes L_{\alpha_i})$, we obtain

$$\pi_*(K_i \otimes_{O_X} O_X(H_0)) \cong \bigoplus_{0 \le \alpha_1 < \dots < \alpha_i \le N} \pi_*(O_X(-(i-1)H_0)) \otimes_{O_X} (L_{\alpha_1} \otimes \dots \otimes L_{\alpha_i}) = 0,$$

$$2 \le i \le N+1,$$

$$\pi_*(K_1 \otimes_{O_X} O_X(H_0)) \cong L_0 \oplus \dots \oplus L_N.$$

$$R^j\pi_*(K_i \otimes_{O_X} O_X(H_0)) = \bigoplus_{0 \le \alpha_1 < \dots < \alpha_i \le N} R^j\pi_*(O_X(-i-1)H_0)) \otimes_{O_X} (L_{\alpha_1} \dots L_{\alpha_i}) = 0,$$

$$1 \le j, \quad 1 \le i \le N+1.$$

Thus the exact sequence (a_N) implies that $\pi_*(M_{N-1}) = R^j\pi_*(M_{N-1}) = 0$. Assume that $\pi_*(M_i) = R^j\pi_*(M_i) = 0$ $(i > 1)$, then the exact sequence (a_i) implies that $\pi_*(M_{i-1}) = R^j\pi_*(M_{i-1}) = 0$. By induction on i, therefore, we see that $\pi_*(M_1) = R^1\pi_*(M_1) = 0$. Hence by virtue of the exact sequence (a_1) we obtain that $L_0 \oplus \dots \oplus L_N \cong \cdot$

$\pi_*(I_Y \otimes O_X(H_0))$. This and Theorem 1.3 assert that $\mathrm{elm}_Y^0(X) \cong$ $P(L_0 \oplus \ldots \oplus L_N)$.

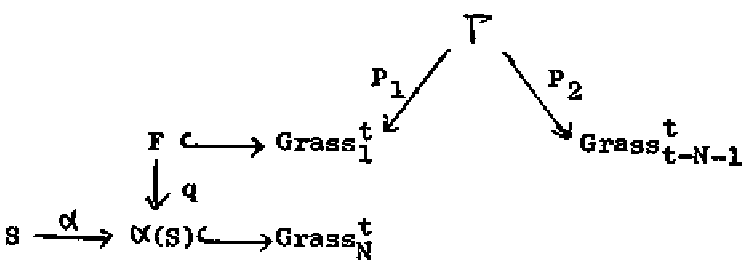
q. e. d.

By virtue of the above proposition we have only to find $H_0, \ldots, H_N$ satisfying the conditions in Proposition 1.9. in order to prove what we have been aiming.

Lemma 1.10. Let $\pi : X \longrightarrow S$ be a P^N-bundle on a quasi-projective k-variety S and let $i : X \hookrightarrow P_k^t$ be an immersion such that $i^*(O_{P_k^t}(1))$ is a tautological linebundle on X. If $H_0, \ldots, H_N$ are general hyperplanes of P_k^t and if $\dim S \leq 3$, then $\dim((\bigcap_{i=0}^{N} H_i) \cap \pi^{-1}(s))$ ≤ 0 for every $s \in S$.

Proof. Since $i^*(O_{P_k^t}(1))$ is a tautological linebundle, $(\bigcap_{i=0}^{N} H_i) \cap \pi^{-1}(s)$ is a linear subspace of $\pi^{-1}(s)$ for every $s \in S$ and hyperplanes $H_0, \ldots, H_N$ of P_k^t. Thus we have only to prove that no line in $\pi^{-1}(s)$ is contained in $\bigcap_{i=0}^{N} H_i$ for general hyperplanes $H_0, \ldots, H_N$ of P_k^t. Let $\mathrm{Grass}_{\beta}^{\alpha}$ be the Grassmannian of the

β-dimensional linear subvariety of P_k^α. Put $\Gamma = \{(L_1, L_2) \in \mathrm{Grass}_1^t \times \mathrm{Grass}_{t-N-1} \mid L_1 \subset L_2\}$ and let $p_1 : \Gamma \longrightarrow \mathrm{Grass}_1^t$, $p_2 : \Gamma \longrightarrow \mathrm{Grass}_{t-N-1}^t$, be natural projections, then Γ is an algebraic variety and p_1, p_2 are morphisms. We have $\dim(p_1^{-1}(x)) = (N + 1)(t - N - 2)$ for any $x \in \mathrm{Grass}_1^t$. On the other hand, $E = \pi_*(i^*(O_{P_k^t}(1))$ is generated by a finite subset $\{u_0, \ldots, u_t\}$ of its global sections because $\mathfrak{S}\mathfrak{Q}$ is $i^*(O_{P_k^t}(1))$. The surjective homomorphism $\varphi : O_S^{t+1} \longrightarrow E$ determined by $u_0, \ldots, u_t$ defines a morphism $\alpha : S \longrightarrow \mathrm{Grass}_N^t$.[6] α is nothing but the map defined by $s \longrightarrow i(\pi^{-1}(s)) \in \mathrm{Grass}_N^t$. Let $F = \{L_1 \in \mathrm{Grass}_1^t \mid L_1 \subset L_2$ for some $L_2 \in \alpha(S)\}$, then F is a locally closed subset of Grass_1^t and there is a natural morphism $q : F \longrightarrow \alpha(S)$. Since $q^{-1}(s) = \mathrm{Grass}_1^N$ for any $s \in S$, $\dim F = \dim S + \dim(\mathrm{Grass}_1^N)$



$= \dim S + 2(N-1)$. Thus if $\dim S \leq 3$, then $\dim(p_1^{-1}(F)) = \dim S + 2(N-1) + (N+1)(t-N-2) = \dim S + (N+1)(t-N) - 4 = \dim S + \dim(\mathrm{Grass}^t_{t-N-1}) - 4 < \dim(\mathrm{Grass}^t_{t-N-1})$, whence $p_2(p_1^{-1}(F)) \subsetneqq \mathrm{Grass}^t_{t-N-1}$. Therefore if $\bigcap_{i=0}^{N} H_i \in (\mathrm{Grass}^t_{t-N-1} - p_2(p_1^{-1}(F)))$ for $H_0, \ldots, H_N \in \mathrm{Grass}^t_{t-1}$, then $\bigcap_{i=0}^{N} H_i$ contains no line of $\pi^{-1}(s)$ for any $s \in S$.

q. e. d.

Lemma 1.11. If $\pi : X \longrightarrow S$ be a P^N-bundle on a quasi-projective smooth k-variety S, then there is a tautological linebundle on X which is very ample over $\mathrm{Spec}(k)$.

Proof. By virtue of Lemma 1.2 there is a tautological linebundle $O_X(1)$ on X and the assumption implies that there is a very ample invertible sheaf L on S. Since $O_X(1)$ is π-very ample, $O_X(1) \otimes \pi^*(L^{\otimes n})$ is very ample over $\mathrm{Spec}(k)$ for any $n \geq n_0$ (E.G.A. Chap.II, 4.4.10, (ii)).

q. e. d.

Now we come to the following theorem which extends a well known theorem : Every P^1-bundle on a complete non-singular curve C (that is, a geometrically ruled surface) is obtained from the direct product $P^1 \times C$ by succesive elementary transformations.

Theorem 1. 12. Let $\pi : X \longrightarrow S$ be a P^N-bundle on a smooth quasi-projective k-variety S with $\dim S \leq 3$. Then there is a k-subscheme Y of $P_k^N \times S$ satisfying the condition (E_{N-1}) such that $X \cong elm_Y^{N-1}(P_k^N \times S)$. Moreover, if $\dim S = 2$ or 3, we can choose such a Y as an irreducible subscheme.

Proof. By virtue of Lemma 1. 11 there is an immersion $i : X \hookrightarrow P_k^t$ such that $i^*(O_{P_k^t}(1))$ is a tautological linebundle on X (E.G.A. Chap.II, 4.4.7). If $H_0, \ldots, H_N$ are sufficiently general hyperplane sections of X in P_k^t, then $Y' = H_0 \cdot \ldots \cdot H_N$ satisfies the condition (E_0) by virtue of Proposition 1.9 and Lemma 1. 10. By virtue of Proposition 1.9 we have that $elm_{Y'}^0(X) \cong P_k^N \times S$. Let Y be the center of $(elm_{Y'}^0)^{-1}$, then Y is a desired subscheme (see Theorem 1.1, Corollary 1.1.1). If dim S = 2 or 3, then dim $Y' \geq 1$. Thus we can choose such a Y' as an irreducible subscheme. Then the subscheme Y determined by the Y' as above is irreducible.

q. e. d.

Remark. 1.13. It seems that Theorem 1.12 is false in the case where dim S is greater than 3 (see Theorem 2.19). But we may present the following problem : Is every P^N-bundle on a smooth quasi-projective k-variety S obtained from the direct product $P_k^N \times S$ by succesive elementary transformations?

Chapter II. Regular vector bundles.

From now on we shall use the following notation unless otherwise stated :

S : a smooth projective variety over k with dimension greater than 1[7)]

P^N_S : the direct product $P^N_k \times_k S$;

π : the projection $P^N_S \to S$;

Z : a hyperplane of P^N_k ;

H_0 : the subvariety $Z \times_k S$ of P^N_S ;

Y : an irreducible subscheme of P^N_S satisfying the condition (E_{N-1}) ;

T : the subscheme $\pi(Y)$ of S with reduced structure ;

P^N_T : the direct product $P^N_k \times_k T$ which is regarded as a subscheme of P^N_S.

H_Y : the divisor $H_0 + P^N_T$ on P^N_S

I_Y : the ideal sheaf of Y in P^N_S

f_Y : $\widetilde{X}(Y) \to P^N_S$: monoidal transformation with center Y ;

$\overline{X}_T = f_Y^{-1}[P_T^N]$, $E_Y = f_Y^{-1}(Y)$ (i.e. exceptional variety of f_Y) ;

$g_Y : \widetilde{X}(Y) \to X(Y)$: the contraction with center $\overline{X}_T$ whose contractability is garanteed by Theorem 1.1 ;

$\pi_Y : X(Y) \to S$: the projection of P^N-boundle $X(Y)$;

H'_Y : the transform of H_0 by elm_Y^{N-1} $(= g_Y f_Y^{-1})$

In the above situation we may assume that H_0 does not contain Y.

$$\begin{array}{ccccccc}
\widetilde{X}(Y) & \xrightarrow{f_Y} & P_S^N & \hookleftarrow & P_T^N & \hookleftarrow & Y \\
\downarrow{\scriptstyle g_Y} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_T} & \swarrow{\scriptstyle \pi|_Y} & \\
X(Y) & \xrightarrow{\pi_Y} & S & \hookleftarrow & T & &
\end{array}$$

§ 1. Definition of regular vector bundles.

By virtue of Theorem 1.3 we know that $E(Y) = \pi_*(I_Y \otimes O_{P_S^N}(H_Y))$ is a locally free O_S-module of rank $N + 1$. Thus it seems that the following definition is adequate.

Definition. A locally free O_S-module which is isomorphic to $E(Y)$ is called a regular vector bundle (defined by Y).

Of course a subscheme which defines a regular vector bundle may not be unique (see § 2 of this chapter).

Lemma 2.1. Let $p_i : X_i \rightarrow S$ $(i=1,2)$ be P^N-bundles on S, let T, Y_1 (or, T, Y_2) be subvarieties of S, X_1 (or, S, X_2, resp.) satisfying the condition (E_n) (or, (E_{N-n-1}), resp.) with $X_2 = \mathrm{elm}_{Y_1}^{n}(X_1)$, $\mathrm{elm}_{Y_2}^{N-n-1} = (\mathrm{elm}_{Y_1}^{n})^{-1}$ and let f_i ; $\tilde{X} \longrightarrow X_i$ be the monoidal transformations of X_i with center Y_i. Assume that C_1 is a positive divisor on X_1 such that $O_{X_1}(C_1)$ is a tautological linebundle on X_1. Put $C_2 = \mathrm{elm}_{Y_1}^{n}[C_1]$.

(i) $C_1 \not\supset Y_1$ if and only if $C_2 \supset Y_2$.[8] In this case $f_2^{-1}(C_2) = f_2^{-1}[C_2] + f_2^{-1}(Y_2)$.

(ii) $f_1^{-1}(P_1^{-1}(T)) = f_1^{-1}[P_1^{-1}(T)] + f_1^{-1}(Y_1)$.

Proof. Let x be a point of T and let $U = \mathrm{Spec}(A)$ be an affine open neighborhood of x in S such that $X_{1,U} = \mathrm{Proj}\,(A[\eta_0, \ldots, \eta_N])$ and that the ideal of $T \cap U$ (or, $X_{1,U} \cap Y_1$) is generated by $t \in A$ (or, $t, \eta_{n+1}, \ldots, \eta_N$, resp.) Then $X_{2,U} = \mathrm{Proj}\,(A[\eta'_0, \ldots, \eta'_N])$, $\eta'_i = \eta_i$ $(0 \leq i \leq n)$, $t\eta'_i = \eta_i$ $(n+1 \leq i \leq N)$ and the defining ideal for Y is generated by $t, \eta'_1, \ldots, \eta'_n$ by virtue of Lemma 1.4. We may assume that

$C_1 \cap X_{1,U}$ is defined by $\sum_{i=0}^{N} a_i \eta_i = 0, a_i \in A$. $C_1 \cap X_{1,U} \not\supset Y_1 \cap X_{1,U}$ if and only if $a_i \notin tA$ for some $0 \leq i \leq n$. Thus if $C_1 \cap X_{1,U} \not\supset Y_1 \cap X_{1,U}$, then $C_2 \cap X_{2,U}$ is defined by $\sum_{i=0}^{n} a_i \eta_i' + t \sum_{j=n+1}^{N} a_j \eta_j'$. Hence $C_2 \cap X_{2,U} \supset Y_{2,U}$. Conversely assume that $C_2 \cap X_{2,U} \supset Y_2 \cap X_{2,U}$. We may assume that $C_2 \cap X_{2,U}$ is defined by $\sum_{i=0}^{N} b_i \eta_i' = 0$. ($b_i \in A$, if $0 \leq i \leq n$, $b_i = t\, b_i'$, $b_i' \in A$ if $n+1 \leq i \leq N$). Since C_2 is the proper transform of C_1 by elm_Y^n, $C_2 - p_2^{-1}(T) \not\geq 0$, whence $b_i \notin tA$ for some $0 \leq i \leq n$. Then $C_1 \cap X_{1,U}$ is defined by $\sum_{i=0}^{n} b_i \eta_i + \sum_{j=n+1}^{N} b_j' \eta_j = 0$ and $b_i \notin tA$ for some $0 \leq i \leq n$. Thus $C_1 \cap X_{1,U} \not\supset Y_1 \cap X_{1,U}$. Since Y_i is irreducibe, $C_i \supset Y_i$ if and only if $C_i \cap X_{i,U} \supset Y_i \cap X_{i,U}$. Thus $C_1 \not\supset Y_1$ if and only if $C_2 \supset Y_2$. $f_2^{-1}(C_2) = f_2^{-1}[C_2] + f_2^{-1}(Y_2)$ is clear because $\sum_{i=0}^{N} b_i \eta_i'/\eta_j' \in I_j - I_j^2$ for $I_j = (t, \eta_0'/\eta_j', \dots, \eta_n'/\eta_j')\ A[\eta_0'/\eta_j' \cdots \eta_N'/\eta_j']$, $0 \leq j \leq N$ (cf. Proof of Lemma 1.4). Thus we get (i). Proof of (ii) is similar to the above. q. e. d.

Lemma 2.2. If $E(Y)$ is the regular vector bundle defined by Y, then we have $E(Y) \cong (\pi_Y)_*(O_{X(Y)}(H'_Y))$.

Proof. Put $f_Y^{-1}(H_0) = \tilde{H}$, then $g_Y^{-1}(H'_Y) = \tilde{H} + \bar{X}_T$ by virtue of the above lemma. Thus $f_Y^*(O_{P_S^N}(H_0) \otimes O_{P_S^N}(P_T^N)) \cong$ $O_{\tilde{X}}(\tilde{H}) \otimes O_{\tilde{X}}(\bar{X}_T) \otimes O_{\tilde{X}}(E_Y) \cong (g_Y)^*(O_{X(Y)}(H'_Y)) \otimes O_{\tilde{X}}(E_Y)$. We therefore obtain $(\pi_Y)_*(O_{X(Y)}(H'_Y)) \cong (\pi_Y)_*(g_Y)_*(g_Y)^*(O_{X(Y)}(H'_Y))$

$\pi_*(f_Y)_*(f_Y^*(O_{P_S^N}(H_Y)) \otimes O_{\tilde{X}}(-E_Y)) \cong \pi_*(I_Y \otimes O_{P_S^N}(H_Y)) \cong E(Y)$ (see Lemma 1.5).

q. e. d.

The following is a corollary to Theorem 1.12.

Proposition 2.3. Assume that the dimension of S is equal to 2 or 3. Every very ample vector bundle[9)] of rank $N + 1$ $(N \geq 1)$ is regular and therefore, for any vector bundle E of rank $N + 1$ $(N \geq 1)$ on S, there exists a linebundle L on S such that $E \otimes L$ is a regular vector bundle.

Proof. Put $X = P(E)$ and let $O_X(1)$ be the tautological linebundle of E. Since $O_X(1)$ is very ample by our assumption, the proof of

Theorem 1.12 shows that there is an isomorphism $j : X \longrightarrow \mathrm{elm}_Y^{N-1}(P_k^N \times S)$ for the same Y obtained from $O_X(1)$ as in the proof. Moreover, $(\mathrm{elm}_Y^{N-1})^{-1}\{j(H_i)\} = Z_i \times S$ for a hyperplane Z_i of P_k^n, where H_i is the same as in the proof of Theorem 1.12. Thus we obtain our assertion by virtue of Lemma 2.2 q. e. d.

§ 2. Families of regular vector bundles.

In this section we shall construct a moduli of a subfamily of regular vector bundles.

Lemma 2 4. Let X be a factorial variety over k and let W be a positive divisor on P_X^N such that $O_{P_X^N}(W) \otimes_{O_X} k(x_0)$ $O_{P^N_{k(x_0)}}(r)$ for some $x_0 \in X$. Then we have that $r \geq 0$ and $O_{P_X^N}(W) \cong O_{P_X^N}(n(Z \times X)) \otimes p_2^*(O_X(D))$ for some positive divisor D on X, where $p_2 : P_X^N \rightarrow X$ is the projection.

Proof Invariance of Euler-Poincaré characteristic of a proper flat family implies that $O_{P_X^N}(W) \otimes_{O_X} k(x) \cong O_{P^N_{k(x)}}(r)$ for every $x \in X$. Then by virtue of the seesaw theorem ([12] p54) we know that $O_{P_X^N}(W) \cong O_{P_X^N}(r(Z \times X)) \otimes p_2^*(L)$ for some linebundle L on X. On the other hand, the Künneth formula implies that $H^0(P_X^N, O_{P_X^N}(W)) \cong H^0(P_k^N, O_{P_k^N}(r)) \otimes_k H^0(X,L)$. Since W is a positive divisor, $\dim_k H^0(P_X^N, O_{P_X^N}(W)) > 0$, whence $\dim_k H^0(P_k^N, O_{P_k^N}(r)) > 0$, $\dim_k H^0(X,L) > 0$. Thus we get that $r \geq 0$ and $L \cong O_X(D)$ for some positive divisor D on X.

q.e.d.

Now a regular vector bundle E of rank $N + 1$ $(N \geq 1)$ is completely determined by a subvariety Y of P_S^N satisfying the condition (E_{N-1}). Then $T = \pi(Y)$ with reduced structure is a smooth subvariety of S of codimension 1 ([7] Theorem 1.8, E.G.A. Chap. IV 6.8.3) and Y can be regarded as a positive divisor on P_T^N. Furthermore, since Y_t is a hyperplane of $P_{k(t)}^N$ for every $t \in T$, we know by the above lemma that $O_{P_T^N}(Y) \cong O_{P_T^N}(Z \times T) \otimes (\pi_T^*(O_T(D))$ for a positive divisor D of T. Thus Y is a member of a complete linear system on P_T^N of type $| Z \times T + (\pi_T)^{-1}(D) |$ which contains no fibre of P_T^N. We have therefore the following principle

Principle 2.5. To give a regular vector bundle of rank $N + 1$ $(N \geq 1)$ on S is equivalent to give a member of a complete linear system of the type $|Z \times T + (\pi_T)^{-1}(D)|$ on P_T^N which contains no fibre of P_T^N, where T is a suitable smooth subvariety of S of codimension 1 and D is a positive divisor on T.

Put $\mathrm{Pic}^+(T) = \{ D \in \mathrm{Pic}(T) \mid H^0(T, O_T(D)) \neq 0 \}$. From now on $R^r(S, T, D)$[10)] denotes the set of isomorphism classes of regular vector bundles of rank r on S which are determined by members of $|Z \times T + (\pi_T)^{-1}(D)|$ for $D \in \mathrm{Pic}^+(T)$.

By virtue of Künneth formula

$$H^0(P_T^N, O_{P_T^N}(Z \times T + (\pi_T)^{-1}(D))) \cong H^0(P_k^N, O_{P_k^N}(1)) \otimes_k H^0(T, O_T(D)) =$$
$$\zeta_0 H^0(T, O_T(D)) \oplus \ldots \oplus \zeta_N H^0(T, O_T(D)).$$

Thus a member Y of $|Z \times T + (\pi_T)^{-1}(D)|$ is defined by $s_0\zeta_0 + \ldots + s_N\zeta_N = 0$ for some $s_i \in H^0(T, O_T(D))$. Y contains a fibre $\pi^{-1}(t)$ $(t \in T)$ if and only if $s_0(t) = \ldots = s_N(t) = 0$.

Hence Principles 2.5 can be said in other word as follows:

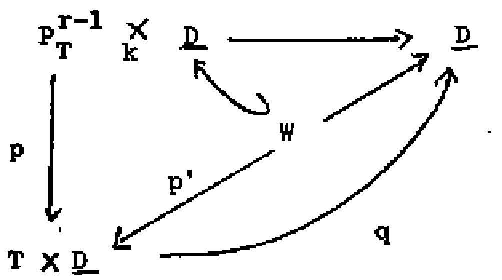
Principle 2.6. To give a regular vector bundle contained in $R^{N+1}(S,T,D)$ $(N \geq 1)$ is equivalent to give an element $(s_0, \ldots, s_N) \neq 0$ of $H^0(T, O_T(D)) \times \ldots \times H^0(T, O_T(D))$ such that every $s_0(t), \ldots, s_N(t)$ is not zero for any $t \in T$.

Now let us construct a large family of regular vector bundles.

Lemma 2.7. The set $R^r(T)$ which consists of subschemes of P_T^{r-1} satisfying the condition of Principle 2.5 forms an open subset of $\mathrm{Hilb}_{P_T^{r-1}/k}$.

Proof. Since P_T^{r-1} is projective and non-singular, $\mathrm{Div}_{P_T^{r-1}/k}$ is open and closed in $\mathrm{Hilb}_{P_T^{r-1}/k}$ ([6] Proposition 4.1, Corollary 4.4, [7] Theorem 2.1). Hence $\mathrm{Div}_{P_T^{r-1}/k}$ is a union of some connected components of $\mathrm{Hilb}_{P_T^{r-1}/k}$. On the other hand, $\underline{D} = \left\{ D \in \mathrm{Div}_{P_T^{r-1}/k} \;\middle|\; O_{P_T^{r-1}}(D)_t \cong O_{P_{k(t)}^{r-1}}(1), \quad \forall\, t \in T \right\}$ is also a union of some connected components of $\mathrm{Div}_{P_T^{r-1}/k}$. Moreover, $R^r(T)$ consists of the members of $\underline{D}$ which contains no fibre of P_T^{r-1}. Let W be the subscheme of $P_T^{r-1} \times \underline{D}$ induced from the universal family of subschemes on $P_T^{r-1} \times \mathrm{Hilb}_{P_T^{r-1}/k}$ by the natural inclusion

$P_T^{r-1} \times_k \underline{D} \subset P_T^{r-1} \times_k \text{Hilb}\, P_T^{r-1}/k$. Look at the following commutative diagram



Since p' is proper, the set $R' = \{x \in T \times \underline{D} \mid \dim p'^{-1}(x) = r-1$, i.e. W contains the fibre $p^{-1}(x) \cong P^{r-1}_{k(x)}\}$ is closed in $T \times \underline{D}$ (E.G.A. Chap. IV, 13.1.3). Since $R^r(T) = \underline{D} - q(R')$ and q is proper, $R^r(T)$ is an open subset of $\underline{D}$. Thus $R^r(T)$ is an open subset of $\text{Hilb}\, P_T^{r-1}/k$.

Lemma 2.8. Let $\pi : X \longrightarrow S$, Y, T be the same as in Theorem 1.1. and let $j : S' \longrightarrow S$ be a morphism such that $j^{-1}(T)$ is also a Cartier divisor on S'. Then canonically $j^*(\pi_*(I_Y \otimes O_X(1))) \cong (\pi_{S'})_*(I_Y O_{X_{S'}} \otimes i^* O_X(1))$ for a tautological linebundle $O_X(1)$ and the defining ideal I_Y of Y in X, where $i : X_{S'} \longrightarrow X$ is the natural morphism induced by j.

Proof. Put $E = \pi_*(O_X(1))$, $F = \pi_*(O_Y \otimes O_X(1))$, $E' = (\pi_{S'})_*(i^*(O_X(1)))$, $F' = (\pi_{S'})_*(O_{Y_{S'}} \otimes i^*(O_X(1)))$. Then $\mathrm{Ker}\,\varphi = \pi_*(I_Y \otimes O_X(1))$, $\mathrm{Ker}\,\varphi' = (\pi_{S'})_*(I_Y O_{X_{S'}} \otimes i^* O_X(1))$ for the canonical morphisms $\varphi : E \to F$, $\varphi' : E' \to F'$ (see Theorem 1.3, Proposition 1.8). Consider the following exact commutative diagram;

$$\begin{array}{ccccccc} j^*\pi_*(I_Y \otimes O_X(1)) & \xrightarrow{\psi} & j^*E & \xrightarrow{j^*\varphi} & j^*F & \longrightarrow & 0 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \\ 0 \to (\pi_{S'})_*(I_Y O_{X_{S'}} \otimes i^* O_X(1)) & \longrightarrow & E' & \longrightarrow & F' & \longrightarrow & 0 \end{array}$$

Since a local equation t for T at $j(s') \in S$ is a non-zero divisor of $O_{S',s'}$, $\mathrm{Tor}_1^{O_{S,s}}(O_{S,s}/tO_{S,s},\ O_{S',s'}) = 0$ and therefore ψ is injective. On the other hand, $(j^*F)_{s'} \xrightarrow{\beta} H^0(Y_s, (O_Y \otimes O_X(1))_s) \otimes_{k(s)} k(s') \xrightarrow[\sim]{\gamma} H^0(Y_{S',s'}, (O_{Y_{S'}} \otimes i^* O_X(1))_{s'}) \xleftarrow[\sim]{\delta} F'_{s'}$ for any $s' \in j^{-1}(T)$, $j(s') = s$. Thus α_3 is an isomorphism because $\delta \cdot (\alpha_3)_{s'} = \gamma \cdot \beta$. Similary α_2 is an isomorphism. Therefore α_1 is an isomorphism by virtue of the five lemma. q.e.d.

Theorem 2.9. Let S be a non-singular projective variety over k, let T be a non-singular subvariety of S of codimension 1 and let $R^r(T)$ be the open subscheme of $\mathrm{Hilb}_{P_T^{r-1}/k}$ defined in Lemma 2.7.

Then there are a vector bundle $P^r(T)$ of rank r on $S \underset{k}{\times} R^r(T)$ and a surjective map $\varphi^r_T : R^r(T)(k) \longrightarrow \coprod_{D \in Pic^+(T)} R^r(S,T,D)$ such that $P(T)_x \cong \varphi^r_T(x)$ for any k-rational point x of $R^r(T)$

Proof. Let W be the subscheme of $P^{r-1}_T \underset{k}{\times} R^r(T) = P^{r-1}_k \underset{k}{\times} (T \underset{k}{\times} R^r(T))$ induced from the universal family of subschemes on $P^{r-1}_T \underset{k}{\times}$ Hilb $_{P^{r-1}_T/k}$. Since T is a Cartier divisor on S, so is $T \underset{k}{\times} R^r(T)$ on $S \underset{k}{\times} R^r(T)$. With the natural projection $p : P^{r-1}_S \times_k R^r(T) \to S \underset{k}{\times} R^r(T)$, $(p_{T \times R^r(T)}|W)^{-1}(y) \cong P^{r-2}_{k(y)}$ ($\forall y \in T \underset{k}{\times} R^r(T)$). Thus the subscheme W in $P^{r-1}_k \underset{k}{\times} (S \underset{k}{\times} R^r(T))$ satisfy the condition (E^0_{r-2}) ([9] Theorem 1.8). Now put $P^r(T) = P_*(I_W \otimes O_X(H_0))$, where I_W is the defining ideal for W and H_0 is the Cartier divisor $Z \times (S \underset{k}{\times} R^r(T))$ on $P^{r-1}_S \underset{k}{\times} R^r(T)$. Then $P^r(T)$ is a vector bundle of rank r on $S \times R^r(T)$ by virtue of Theorem 1.3 If x is a k-rational point of $R^r(T)$, then $(\alpha_x|_{P^{r-1}_T})^{-1}(W) = W_x$ is contained in $|Z \times T + (\pi_T)^{-1}(D)|$ for some $D \in Pic^+(T)$ and contains no fibre of P^{r-1}_T, where $\alpha_x : P^{r-1}_T \to P^{r-1}_S \times R^r(T)$ is the morphism induced by $x \to R^r(T)$. Thus for the natural morphism

$\beta_x : S \longrightarrow S \times R^r(T)$, $(\beta_x)^*(P^r(T))$ is contained in $R^r(S,T,D)$ by virtue of Lemma 2.8. Hence if one defineds $\varphi^r_T(x)$ = the regular vector bundle defined by W_x for $x \in R^r(T)(k)$, then clearly φ^r_T, $P^r(T)$ fulfill our requirement. q.e.d.

Our next aim is to study conditions for two regular vector bundles to be isomorphic to each other. The following lemma is a key in the sequel.

Lemma 2.10. Let $Z_0, \ldots, Z_N$ be linearly independent hyperplanes of P^N_k and put $H'_i = \text{elm}^n_Y(H_i)$ for $H_i = Z_i \times T$ and a subscheme Y of P^N_S satisfying the condition (E^0_n). Then $Y' = \bigcap_{i=0}^N H'_i$ for the center Y' of $(\text{elm}^n_r)^{-1}$, that is, the ideal Y' is generated by those of H'_i.

Proof. Since the property is local with respect to S and since $H_{0,s}, \ldots, H_{N,s}$ form a basis of hyperplanes of $P^N_{S,s} \cong P^N_{k(s)}$ for any $s \in S$, we may assume that $S = \text{Spec}(A)$, $P^N_S = \text{Proj}(A[\eta_0, \ldots, \eta_N])$, the homogeneous ideal defining Y is generated by $t (\in A)$, $\eta_{n+1}, \ldots, \eta_N$ and that H_i is defined by $\eta_i = 0$. Then by virtue of Lemma 1.4

$\mathrm{elm}_Y^n(P_S^N) = \mathrm{Proj}(A\{\eta'_0, \ldots, \eta'_N\})$ $\eta'_i = \eta_i$ $(0 \leq i \leq n)$, $t\eta'_i = \eta_i$ $(n+1 \leq i \leq N)$ and the homogeneous ideal defining H'_i is generated by η'_i $(0 \leq i \leq n)$, $t\eta'_i$ $(n+1 \leq i \leq N)$. Thus in the affine open set $U'_i = \{\eta'_i = 0\}$ the ideal defining $\bigcap_{j=0}^N H'_j$ is generated by 1 $(0 \leq i \leq n)$; t, $\eta'_0/\eta'_i, \ldots, \eta'_n/\eta'_i$ $(n+1 \leq i \leq N)$. On the other hand, the ideal defining Y' in U'_i is generated by the same element because the homogeneous ideal of Y is generated by $\eta'_0, \ldots, \eta'_n$, t.

q.e.d.

For a non-singular subvariety T of codimension 1 of S put

$$A_T = \left\{ D \;\middle|\; D \in \mathrm{Pic}^+(T),\quad H^0(T, O_T(T^2-D)) = 0 \right\}.$$

Lemma 2.11. Let $H_0, \ldots, H_N$ be as in the above lemma and put $H'_i = \mathrm{elm}_Y^{N-1}(H_i)$ for Y satisfying the condition(E_{N-1}). Let T be the subvariety $\pi(Y)$ (with reduced structure) of S (then T is non-singular and codimension 1). Assume that $Y \in |Z \times T + (\pi_T)^{-1}(D)|$ with a $D \in A_T$, then $H'_0, \ldots, H'_N$ form a basis of the complete linear system $|H'_0|$ on $X(Y) = \mathrm{elm}_Y^{N-1}(P_S^N)$.

Proof. It is clear that $H'_0, \dots, H'_N$ are independent. Let L be the linear system spanned by $H'_0, \dots, H'_N$. Assume that $L \subsetneq H'_0$ and we shall show a contradiction. Take a general member H' of $|H'_0|$ such that H' is irreducible and $H' \notin L$ (since at least one of $H'_0, \dots, H'_N$ is irreducible, such an H' exists). In the first place assume that $H' \supset Y'$, then $g_Y^{-1}[H'] + \overline{X}_T \sim g_Y^{-1}[H'_0] + \overline{X}_T \sim f_Y^{-1}(H_0 + P^N_T) - E_Y$ by virtue of Lemma 2.1. Thus $H = f_Y[g_Y^{-1}[H']] \sim H_0$. Since $H_0, \dots, H_N$ form a basis of $|H_0|$, $H = Z \times T$ for some hyperplane Z of P^N_k and H' is the total transform of H. Thus $H' \in L$, which is impossible. Next assume that $H' \not\supset Y'$. By a similar argument as above we know that $H \sim H_0 + P^N_T$ and by virtue of Lemma 2.1 $H \supset Y$. Thus $H \cdot P^N_T = Y + A$, $A > 0$ and $O_{P^N_T}(Y + A) \cong O_{P^N_T}(Z \times T + \pi^{-1}(T^2))$. Thus $O_{P^N_T}(A) \cong \pi^*(O_T(T^2-D))$, whence $H^0(P^H_T, \pi^*(O_T(T^2-D))) = H^0(P^N_T, O_{P^N_T}(A)) \neq 0$ because $O_{P^N_T}(Y) \cong O_{P^N_T}(Z \times T) \otimes \pi^*(O_T(D))$. On the other hand, $H^0(P^N_T, \pi^*(O_T(T^2-D))) = H^0(T, \pi_* \pi^*(O_T(T^2-D))) \cong H^0(T, O_T(T^2-D))$. But this is contradictory to the fact that $D \in A_T$.

q.e.d.

Corallary 2.11.1. If $E \in R^r(S,T,D)$, then $\dim_k H^0(S, E) \geq r$. Moreover if $D \in A_T$, then $\dim_k H^0(S, E) = r$.

Proof. Our assertion is clear if one notes $H^0(S, E) = H^0(P(E), O_{P(E)}(1))$. q.e.d.

Note that $\mathrm{Aut}_S(P_S^{r-1}) \cong PGL(r-1)$ and that if a subscheme Y of P_S^N satisfying the condition (E_{N-1}), then so does Y^{σ} for every $\sigma \in PGL(r-1)$. This enables us to show that the next proposition follows from the above two lemmas.

Proposition 2.12. Let E_i $(i = 1, 2)$ be a regular vector bundle of rank r on S defined by Y : and let $Y_1 \in |Z \times T + (\pi_T)^{-1}(D)|$ for $D \in A_T$ (notation is as above). Then E_1 is isomorphic to E_2 if and only if $Y_1 = Y_2^{\sigma}$ for some $\sigma \in PGL(r-1)$.

Proof. It is clear that if $Y_1 = Y_2^{\sigma}$, then $E_1 \cong E_2$. Conversely, assume that there is an isomorphism $i : E_2 \xrightarrow{\sim} E_1$. i induces an isomorphism $j : X_1 = P(E_1) \xrightarrow{\sim} X_2 = P(E_2)$ such that $j^*(O_{X_2}(1)) \cong O_{X_1}(1)$ for the tautological linebundle $O_{X_i}(1)$ of E_i. Since $Y_1 \in |Z \times T + (\pi_T)^{-1}(D)|$ for $D \in A_T$, $\dim_k H^0(X_2, O_{X_2}(1)) = \dim_k H^0(X_1, O_{X_1}(1))$

$= r$ by virtue of Lemma 2.11 and Lemma 2.2. This and Lemma 2.10 imply that $Y'_i = \bigcap_{H^{(1)} \in |O_{X_i}(1)|} H^{(1)}$ is the center of $(\mathrm{elm}_{Y_i}^{r-2})^{-1}$. We have therefore $j(Y'_1) = Y'_2$. Fix isomorphisms $\tau_i : P_S^{r-1} \longrightarrow \mathrm{elm}^0_{Y'_1}(X_1)$ and put $\tau_i(Y_i) = Y''_i$. Then it is easy to see that j induces an isomorphism $\alpha : \mathrm{elm}^0_{Y'_1}(X_1) \to \mathrm{elm}^0_{Y'_2}(X_2)$ such that $\alpha(Y''_1) = Y''_2$. Hence we get a desired automorphism $\tau_2^{-1} \alpha \tau_1$ of P_S^{r-1}. q.e.d.

Theorem 2.13 Let S be a non-singular projective variety over k.

(i) If $E_i \in R^r(S, T_i, D_i)$ $(i=1,2)$, $D_1 \in A_{T_1}$ and $T_1 \neq T_2$, then $E_1 \not\cong E_2$.

(ii) If T is a non-singular subvariety of S of codimension 1, then $\mathrm{Aut}_S(P_S^{r-1}) = PGL(r-1)$ acts on $R^r(T)$. The set $R_0^r(T) = \{ Y \in R^r(T) \mid Y \in |Z \times T + (\pi_T)^{-1}(D)| \text{ for some } D \in A_T \}$ forms a $PGL(r-1)$-stable open subset of $R^r(T)$.

(iii) For the surjective map $\varphi_T^r|R_0^r(T) : R_0^r(T) \longrightarrow \coprod_{D \in A_T} R^r(S,T,D)$ it holds that $\left(\varphi_T^r|_{R_0^r(T)}\right)(x_1) = \left(\varphi_T^r|_{R_0^r(T)}\right)(x_2)$ if and only if $x_1 =$

x_2^{σ} for some $\sigma \in PGL(r-1)$.

Proof. (i) Assume that E_i is defined by Y_i and $E_1 \cong E_2$, then $Y_1 = Y_2^{\sigma}$ for some $\sigma \in PGL(r-1)$ by virtue of Proposition 2.12. Since σ sends $P_{T_1}^{r-1}$ to itself and $\pi(Y_i) = T_i$, which is a contradiction. Thus $E_1 \not\cong E_2$.

(ii) Each element σ of PGL(r-1) sends P_T^{r-1} to itself and $Y^{\sigma} \sim Y$ in P_T^{r-1}. Thus if $Y \in R_0^r(T)$, then $Y^{\sigma} \in R_0^r(T)$, that is, $R_0^r(T)$ is PGL(r-1)-stable. Let Φ be the canonical morphism $Div_{T/k} \longrightarrow Pic(T)$ and let τ be the morphism of Pic(T) to itself defined as follows ; $Pic(T) \ni D \longmapsto T^2 - D \in Pic(T)$. Then $A_T = Pic(T) - \tau^{-1}(\Phi(Div_{T/k}))$. Thus A_T is an open subset of Pic(T) because Φ is projective ([6] Corollary 4.4). On the other hand, there is a canonical isomorphism $j : Pic(P_T^{r-1}) \rightarrow \mathbb{Z} \times Pic(T)$ and $j \circ \Psi$ sends $R^r(T)$ to $\{1\} \times Pic(T)$ for the canonical morphism Ψ : $Div_{P_T^{r-1}/k} \rightarrow Pic(P_T^{r-1})$. This map $j \cdot \Psi|_{R^r(T)}$ is defined as follows ;

$$\forall D \in Pic(T), \quad |Z \times T + (\pi_T)^{-1}(D)| \ni Y \rightarrow D \in Pic(T).$$

Thus $R_0^r(T) = (j\cdot\Psi)^{-1}(A_T) \cap R^r(T)$ which is an open subset of $R^r(T)$.

(iii) is a direct corollary of Proposition 2.12. q.e.d.

Theorem 2.14. Let S be a non-singular projective variety over k, T be a non-singular subvariety of codimension 1 and let $D \in A_T$. Then there is a subset $SR^r(S,T,D)^{(2)}$ of $R^r(S,T,D)$ which carries the structure of an open set of $\mathrm{Grass}^n_{r-1}(K)$, where $n+1 = \dim_k H^0(T, O_T(D))$. Moreover, if $r = 2$ and $D \neq 0$ then $SR^2(S,T,D) = R^2(S,T,D)$.

Proof. Fix a basis $a_0, \ldots, a_n$ of $H^0(T, O_T(D))$. If $(s_0, \ldots, s_{r-1}) (\neq 0)$ is an element of $H^0(T, O_T(D)) \times \ldots \times H^0(T, O_T(D))$ and $s_i = \sum \alpha_{ij} a_j$ $(\alpha_{ij} \in k)$, then $(s_0, \ldots, s_{r-1})$ or the $r\times(n+1)$-matrix (α_{ij}) defines a member of $|Z \times T + (\pi_T)^{-1}(D)|$. For each $(\beta_{ij}) \in GL(r, k)$, the action $(\alpha_{ij}) \rightarrow (\beta_{ij})(\alpha_{ij})$ induces the action of $PGL(r-1)$ on $|Z \times T + (\pi_T)^{-1}(D)|$ which is the same action defined before Proposition 2.12. Let U be the subset of $H^0(T, O_T(D)) \times \ldots \times H^0(T, O_T(D))$ which consists of element $(s_0, \ldots, s_{r-1})$

such that $s_0, \ldots, s_{r-1}$ are independent over k and let U' be the subset of $|Z \times T + (\pi_T)^{-1}(D)|$ determined by U (U may be empty). Then U (or, U') is GL(r)-stable (or, PGL(r-1)-stable, resp.) and U/GL(r) is in bijective correspondence with U'/PGL(r-1). Furthermore it is clear that $U/GL(r) = Grass^n_{r-1}$. Consider the following morphism ψ of $T \times U$ to the r-dimensional affine space A^r over k ; $T \times U$ $(t, s_0, \ldots, s_{r-1}) \to (s_0(t), \ldots, s_{r-1}(t)) \in A^r$. Then the set $F = \{(s_0, \ldots, s_{r-1}) \in U \mid s_0(t) = \ldots = s_{r-1}(t) = 0\}$ is $p(\psi^{-1}(0))$ for the projection $p : T \times U \to U$. Since T is projective, F is closed in U and it is GL(r)-stable. Thus (U-F)/GL(r) is an open set of $Grass^n_{r-1}$. By virtue of Principle 2.6 and Proposition 2.12 we see that ((U-F)/GL(r))(k) is in bijective correspondence with a subset $SR^r(S,T,D)$ of $R^r(S,T,D)$. Now, if $r = 2$ and s_0, s_1 are dependent ($s_0 \neq 0$), then $s_1 = \alpha s_0$ for some $\alpha \in k$, whence $s_1(t) = 0$ for any $t \in T$ with $s_0(t) = 0$. Thus such a (s_0, s_1) defines no element of $R^2(S,T,D)$. We know therefore $SR^2(S,T,D) = R^2(S,T,D)$ if $D \neq 0$, q.e.d.

Remark 2.15. $SR^r(S,T,D)$ may be empty. We raise a problem :

Does there exist a D for fixed S, T such that $SR^r(S,T,D) \neq \emptyset$?

We know that if $r \geq \dim S$, then such a D exists and that

$$\sup_{D \in A_T} (\dim SR^r(S,T,D)) = \infty .$$

Proof. Take a very ample divisor D on T such that $\dim_k H^0(T, O_T(T^2-D)) = 0$ and $\dim_k H^0(T, O_T(D)) \geq r$. Since $r \geq \dim S$ and D is very ample, $s_0, \ldots, s_{r-1}$ are independent and each of $s_0(t), \ldots, s_{r-1}(t)$ is not zero for any $t \in T$ if $s_0, \ldots, s_{r-1}$ are sufficiently general elements of $H^0(T, O_T(D))$. Then $(s_0, \ldots, s_{r-1})$ defines an element of $SR^r(S,T,D)$ and if $\dim_k H^0(T, O_T(D)) = n + 1$, then $\dim SR^r(S,T,D) = \dim \mathrm{Grass}^n_{r-1} = r(n + 1 - r)$. Thus $SR^r(S,T,D) \neq \emptyset$ and $\sup_{D \in A_T} (\dim SR^r(S,T,D)) = \infty$.

Remark 2.16. i) $R^r(S,T,0) = \left\{ O_S \oplus \cdots \oplus O_S \oplus O_S(T) \right\}$

(ii) $R^2(S,T,D) \neq \emptyset$ for some $D \neq 0$ if and only if there exists a morphism f of T to a curve C.

Proof. (i) is a direct conclusion of Lemma 1.4 and Lemma 2.2.

(ii) If $R^2(S,T,D) \neq \emptyset$, then there exist two sections s_0, s_1 of $H^0(T, O_T(D))$ such that both $s_0(t)$, $s_1(t)$ are not zero for any

$t \in T$. Thus $T \ni t \longrightarrow (s_0(t), s_1(t)) \in P^1$ is a morphism.

Conversely assume that there exists a morphism $f : T \longrightarrow C$ (we may assume that C is non-singular because so is T). Take a very ample divisor A on C. Then $H^0(T, f^*(O_C(A)))$ contains two sections s_0, s_1 such that both $s_0(t)$ and $s_1(t)$ are not zero for any $t \in T$. By virtue of Principle 2.6, we know therefore $R^2(S,T,f^{-1}(A)) \neq \emptyset$. q.e.d.

The above proof show that if $R^2(S,T,D) \neq \emptyset$ for some D, then $\sup_{D \in A_T} (\dim SR^2(S,T,D)) = \infty$ and $D^2 = 0$.

Example 2.17. i) $R^2(P^3,T,D) = \emptyset$ for any $D \neq 0$ if T is a plane, $R^2(P^3,Q,D) \neq \emptyset$ for some D if Q is a quadratic surface because $Q \cong P^1 \times P^1$.

ii) $R^2(P^r,T,D) = \emptyset$ $(r \geq 4)$ for any T and $D \neq 0$. For if there exists a morphism f of T to a curve C, then $\dim f^{-1}(p) = r-2$ for any $p \in C$, which is a contradiction because $\dim(f^{-1}(p) \cap f^{-1}(p')) \geq 0$ and therefore $f^{-1}(p) \cap f^{-1}(p') \neq \emptyset$. Thus every regular vector bundle of rank 2 on P^r $(r \geq 4)$ is isomorphic to $O_{P^r} \oplus O_{P^r}(T)$ for some non-singular subvariety T of codimension 1.

iii) If there exists a morphism of S to a curve, then $R^2(S,T,D) \neq \emptyset$ for any T not contained in any fibre of the morphism and for some D.

§ 3. Chern classes of regular vector bundles.

In this section we shall calculate Chern classes of regular vector bundles.

Lemma 2.18. Let E be a vector bundle of rank r (≥ 2) on S and let $O_X(1)$ be the tautological line bundle of E for $X = P(E)$. If $H_1, \ldots, H_r$ are divisors on X such that $O_X(H_i) \cong O_X(1)$ for every i and that they intersect properly, then $p_*(H_1 \ldots H_r) = c_1(E)$ for the natural projection $p : X \to S$.

Proof. Consider the Chern polynomial $H_1 \cdot \ldots \cdot H_r - p^*(c_1(E)) \cdot H_1 \cdot \ldots \cdot H_{r-1} + p^*(c_2(E)) \cdot H_1 \cdot \ldots \cdot H_{r-2} + \ldots + (-1)^{r-1} p^*(c_{r-1}(E)) \cdot H_1 + (-1)^r p^*(c_r(E)) = 0$. Operating p_* on the polynomial, one gets $p_*(H_1 \cdot \ldots \cdot H_r) = p_*(p^*(c_1(E)) \cdot H_1 \cdot \ldots \cdot H_{r-1}) = c_1(E)$ because $p_*(H_1 \cdot \ldots \cdot H_{r-1}) = 1$,

$p_*(H_1 \cdot \ldots \cdot H_i) = 0$ for $i < r-1$. q.e.d.

Lemma 2.2, Lemma 2.10 and Lemma 2.18 yield

Corollary 2.18.1. If $E \in R^r(S,T,D)$, then $c_1(E) = T$.

However a more general result is given by the following theorem.

Theorem 2.19. If $E \in R^r(S,T,D)$, then

$$ch(E) = r + \sum_{l=1}^{\infty} \frac{T^l}{l!} + \sum_{m,n=1}^{\infty} \frac{(-1)^n T^{m-1} \cdot i_*(D^n)}{m!\, n!} ,$$

where $ch(E)$ is the Chern character of E ([2] p112) and $i : T \longrightarrow S$ is the inclusion.

Proof Assume that E is defined by $Y \in |Z \times T + \pi_T^{-1}(D)|$.

The following exact sequence

$$0 \longrightarrow I_Y \otimes O_{P_S^{r-1}}(H_Y) \longrightarrow O_{P_S^{r-1}}(H_Y) \longrightarrow O_Y \otimes O_{P^{r-1}}(H_Y) \longrightarrow 0$$

yields an exact sequence

$$0 \longleftarrow E \longrightarrow O_S(T)^{\oplus r} \longrightarrow \pi_*(O_Y \otimes O_{P_S^{r-1}}(H_Y)) \longrightarrow 0$$

because $E \cong \pi_*(I_Y \otimes O_{P_S^{r-1}}(H_Y))$, $R^1\pi_*(I_Y \otimes O_{P_S^{r-1}}(H_Y)) = 0$ by virtue of the definition of regular vector bundle and Theorem 1.3. If one puts $F = (\pi|_Y)_*(O_Y \otimes O_{P_S^{r-1}}(H_Y))$, then by virtue of the Riemann-Roch theorem of Grothendieck for the morphism $i : T \longrightarrow S$ we have

(1) $\quad ch(E) = ch(O_S(T)^{\otimes r}) - ch(\pi_*(O_Y \otimes O_{P_S^{r-1}}(H_Y)))$

$$= r\, ch(O_S(T)) - ch\,(i_*(\pi|Y)_*(O_Y \otimes O_S^{r-1}(H_Y))$$

$$= r\, ch(O_S(T)) - i_*(ch(F)\cdot td(N_{T/S})^{-1}),$$

where $N_{T/S}$ is the normal bundle of T in S and td is the Todd class. On the other hand, for the ideal J_Y of Y in P_T^{r-1} the following exact sequence

$$0 \to J_Y \otimes O_{P_S^{r-1}}(H_Y) \to O_{P_T^{r-1}} \otimes O_{P_S^{r-1}}(H_Y) \to O_Y \otimes O_{P_S^{r-1}}(H_Y) \to 0$$

provides an exact sequence

$$0 \to (\pi_T)_*(J_Y \otimes O_{P_S^{r-1}}(H_Y)) \to O_T(T^2)^{\oplus r} \to F \to R^1(\pi_T)_*(J_Y \otimes O_{P_S^{r-1}}(H_Y)).$$

Since $J_Y \otimes O_{P_S^{r-1}}(H_Y) \cong O_{P_T^{r-1}}(-Z \times T - (\pi_T)^{-1}(D)) \otimes O_{P_T^{r-1}}(Z \times T + (\pi_T)^{-1}(T^2))$ $\cong (\pi_T)^*(O_T(T^2-D))$, we know $R^1(\pi_T)_*(J_Y \otimes O_{P_S^{r-1}}(H_Y)) = 0$. Thus the above exact sequence implies

(2) $\quad ch(F) = r(ch(O_T(T^2)) - ch(O_T(T^2) \otimes O_T(-D))$

$$= ch(O_T(T^2))(r - ch(O_T(-D)))$$

$$= \left(\sum_{\alpha=0}^{\infty} \frac{T'^{\alpha}}{\alpha!}\right)\left(r - 1 - \sum_{n=1}^{\infty} \frac{(-1)^n D^n}{n!}\right),$$

where $T' = T^2$ in T. As to $td(N_{T/S})^{-1}$ we get

$$(3)\qquad td(N_{T/S})^{-1} = \left(\frac{T'}{1-e^{-T'}}\right)^{-1} = \sum_{\beta=1}^{\infty} \frac{(-1)^{\beta-1} T'^{\,\beta-1}}{\beta!}$$

The above (2),(3) yield

$$(4)\qquad ch(F)\cdot td(N_{T/S})^{-1}$$

$$= (r-1)\left(\sum_{\substack{\alpha=0\\ \beta=1}}^{\infty} \frac{(-1)^{\beta-1}\, T'^{\,\alpha+\beta-1}}{\alpha!\ \beta!}\right) - \sum_{\substack{\alpha=0\\ \beta=1\\ n=1}}^{\infty} \frac{(-1)^{\beta+n-1}\, T'^{\,\alpha+\beta-1}\, D^n}{\alpha!\ \beta!\ n!}$$

$$= (r-1) \sum_{l=1}^{\infty} T'^{\,l-1} \left(\sum_{\substack{\alpha+\beta=l\\ \alpha\geq 0,\ \beta\geq 1}} \frac{(-1)^{\beta-1}}{\alpha!\ \beta!}\right) - \sum_{n=1} \frac{(-1)^n D^n}{n!} \left(\sum_{m=1}^{\infty} T'^{\,m-1} \left(\sum_{\substack{\alpha+\beta=m\\ \alpha\geq 0,\ \beta\geq 1}} \frac{(-1)^{\beta-1}}{\alpha!\ \beta!}\right)\right)$$

Since $\sum_{\substack{a+b=c\\ a\geq 0,\ b\geq 1}} \frac{(-1)^{b-1}}{a!\ b!} = \frac{1}{c!}$ [3], (4) reduces to the following ;

$$(4)'\qquad ch(F)\cdot td(N_{T/S})^{-1} = (r-1)\sum_{l=1}^{\infty} \frac{T'^{\,l-1}}{l!} - \sum_{\substack{m=1\\ n=1}}^{\infty} \frac{(-1)^n T'^{\,m-1} D^n}{m!\ n!}$$

By virtue of (1),(4)'

$$(5)\qquad ch(E) = r\sum_{l=0}^{\infty} \frac{T^l}{l!} - i_*\left((r-1)\sum_{l=1}^{\infty} \frac{T'^{\,l-1}}{l!} - \sum_{\substack{m=1\\ n=1}}^{\infty} \frac{(-1)^n T'^{\,m-1} D^n}{m!\ n!}\right)$$

Since $i_*(T'^{\,l}) = T^{\,l+1}$, and $i_*(T'^{\,m-1}D^n) = T^{m-1}\, i_*(D^n)$

$$ch(E) = r + \sum_{\ell=1}^{\infty} \frac{T}{\ell!} + \sum_{\substack{m=1\\n=1}}^{\infty} \frac{(-1)^n T^{m-1} i_*(D^n)}{m!\,n!}$$ q.e.d.

Corollary 2.19.1. If $E \in R^r(S,T,A)$, then $c_1(E) = T$, $c_2(E) = D$, $c_3(E) = i_*(D^2)$ in $A(S) \otimes_Z Q$, where $A(S)$ is the Chow ring of S.

Proof. Note that $ch(E) = r + c_1(E) + \frac{1}{2}(c_1(E)^2 - c_2(E)) + \frac{1}{6}(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E))$ + higher term. Then our assertion is an immediate corollary of Theorem 2.19.

If $r = 2$, then $c_3(E) = 0$ and so the above corollary implies that $i_*(D^2) = 0$, but fortunately Remark 2.16 implies that if $R^2(S,T,D) \neq \phi$ then $D^2 = 0$.

Remark 2.20. Corollary 2.18.1 asserts that if $E \in R^r(S,T,D)$, then $c_1(E) = T$ in $A(S)$. Thus we have the following problem ; For $E \in R^r(S,T,D)$, $c_2(E) = D$, $c_3(E) = i_*(D^2)$ in $A(S)$?

Chapter III. Simple vector bundles.

In this chapter we maintain the notation in the preceding chapter.

§ 1. Simple regular vector bundles.

Let E be a vector bundle on a scheme X, then $\underline{End}(E) = \underline{Hom}_{O_X}(E, E)$ contains O_X as scalar multiplications. Thus $End(E) = Hom_{O_X}(E, E)$ naturally contains $\Gamma(X, O_X)$.

Definition. A vector bundle E on a scheme X is called simple if $End(E) = \Gamma(X, O_X)$.

Our aim of this section is to show that $SR^r(S, T, D)$ in Theorem 2. 14 consists of all simple vector bundles in $R^r(S, T, D)$.

Lemma 3.1. Let X be a complete variety over k and let E be a vector bundle of rank r on X.

(i) $Aut_X(E)$ is a connected linear group and $\dim Aut_X(E) = \dim_k End(E)$.

(ii) E is indecomposable if and only if rank (i.e. dimension of a maximal torus) of $Aut_X(E) = 1$.

Proof. If $X = \bigcup_{i \in I} U_i$ is a sufficiently fine open covering of X, then $E|U_i$ is free for any i, and an element $\sigma \in End(E)$ is represented by $\{\sigma_i | i \in I, \sigma_i \in M(r, \Gamma(U_i, O_X))$ such that $\sigma_i A_{ij} = A_{ij} \sigma_j$ for the transition matrix A_{ij} of E in $U_i \cap U_j$. Since $(\sigma_i - xI) A_{ij} = A_{ij}(\sigma_j - xI)$ in $U_i \cap U_j$ with an indeterminate x and the unit matrix I, we see that $\det(\sigma_i - xI) = \det(\sigma_j - xI)$ in $U_i \cap U_j$ for any i, j. Thus there exists a polynomial $F(x) \in k[x]$ with $F(x) = \det(\sigma_i - xI)$ for any i because X is a complete variety over k. Hence every eigenvalue of σ_i is independent of i, contained in k and $\det \sigma = \det \sigma_i$ is an element of k. Take a free basis $e_1, \ldots, e_n$ of End(E) with $e_1 = id_E$. $\sigma = \alpha_1 e_1 + \ldots + \alpha_n e_n$ is contained in $Aut_X(E)$ if and only if $\det \sigma \neq 0$. The above argument implies that $\det \sigma$ is a polynomial of $\alpha_1, \ldots, \alpha_n$ over k, and if α is not an eigenvalue of , then $\sigma - \alpha e_1 = (\alpha_1 - \alpha)e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n$ is contained in $Aut_X(E)$. Thus $Aut_X(E)$ is an open dense subset in End(E), which implies that $Aut_X(E)$ is a connected linear group (ii) is easy, if one takes Lemma 6, Lemma 7 of [1] and the above argument into account.

q. e. d.

Corollary 3.1.1. A vector bundle E on a complete variety over k is simple if and only if $Aut_X(E) = G_m = k^*$.

Proof. If E is simple, then $End(E) = \Gamma(X, O_X) = k$ which acts on E as scalar multiplications. Thus $Aut_X(E) = G_m$. Conversely assume $Aut_X(E) = G_m$. Then by virtue of Lemma 3.1 $\dim_k End(E) = 1$, whence $End(E) = k = \Gamma(X, O_X)$.

For a vector bundle E on a scheme put $\Delta(E) = \{\bar{L} \mid \bar{L}$ is the isomorphism class of a linebundle L with $E \cong E \otimes L\}$. Then we get the following exact sequence of groups ([5] Corollary to Proposition 2);

$$e \longrightarrow Aut_X(E)/\Gamma(X,O_X^*) \longrightarrow Aut_X(P(E)) \longrightarrow \Delta(E) \longrightarrow e$$

If X is a complete variety over k, then $\Gamma(X, O_X^*) = G_m$. If X is complete and normal, then $\Delta(E)$ is a finite group, because

$E \cong E \otimes L$ implies $L^{\otimes(\mathrm{rank}E)} \cong O_X$ and therefore $\Delta(E)$ is contained in (rankE) - torsion part of $\mathrm{Pic}^0(X)$ which is an abelian variety. Thus under these assumptions $\mathrm{Aut}_X(E)/G_m = \mathrm{Aut}_X^0(P(E))$, where $\mathrm{Aut}_X^0(P(E))$ is the connected component of $\mathrm{Aut}_X(P(E))$. Therefore we get

Corollary 3.1.2. A vector bundle E on a complete normal variety X over k is simple if and only if $\mathrm{Aut}_X^0(P(E)) = e$.

In order to investigate whether a regular vector bundle E on S is simple or not, let us study $\mathrm{Aut}_S^0(P(E))$.

Lemma 3.2. If E is a regular vector bundle on S of rank r defined by Y and if $\dim_k H^0(S, E) = r$ (cf. Cororally 2.11.1), then $\mathrm{Aut}_S^0(P(E)) \cong \{\sigma \mid \sigma \in PGL(r-1) = \mathrm{Aut}_S(P_S^{r-1}),\ Y^\sigma = Y\}$.

Proof. The assumption $\dim_k H^0(S, E) = r$ implies that $H_1', \ldots, H_r'$ form a basis of $|H_1'|$, where $H_i' = \mathrm{elm}_Y(H_i)$, $H_i = Z_i \times S$ for independent hyperplanes $Z_1, \ldots, Z_r$ of P_k^{r-1}. Since $\sigma \in \mathrm{Aut}_S(P(E))$ is contained in $\mathrm{Aut}_S^0(P(E)) = \mathrm{Aut}_S(E)/G_m$ if and only if $\sigma^*(O_{P(E)}(1)) \cong O_{P(E)}(1)$ for the tautological linebundle $O_{P(E)}(1)$ of E and since $O_{P(E)}(H_i') \cong O_{P(E)}(1)$, we have

$(\bigcap_{i=1}^{r} H'_i)^{\sigma} = \bigcap_{i=1}^{r} H'_i$ for any $\sigma \in \mathrm{Aut}^{\circ}_S(\mathbb{P}(E))$. On the other hand, $\bigcap_{i=1}^{r} H'_i$ coincides with the center Y' of $(\mathrm{elm}_Y^{r-2})^{-1}$ by virtue of Lemma 2.10. Thus $Y'^{\sigma} = Y'$. Now we claim

Lemma 3.3. Let $\pi : X \longrightarrow S$ be a P^N-bundle and let T, Y be subschemes S, X satisfying the condition (E_r^0). If $\sigma \in \mathrm{Aut}_S(X)$ satisfies $Y^{\sigma} = Y$, then σ induces a unique element σ' of $\mathrm{Aut}_S(X')$ with $X' = \mathrm{elm}_Y^n(X)$ such that $Y'^{\sigma'} = Y'$ with the center Y' of $(\mathrm{elm}_Y^n)^{-1}$ and $\sigma' \mid X'_{(S-T)} = \sigma \mid X_{(S-T)}$ by the natural identification $X'_{(S-T)} = X_{(S-T)}$.

Proof. Cover X by a system of affine open sets $\{U_\alpha\}$ such that $X_{U_\alpha} = \mathrm{Proj}(A[\eta_0, \ldots, \eta_N])$ and Y_{U_α} is defined by the ideal $(t \in A, \eta_{n+1}, \ldots, \eta_N)$. Let $\eta_i^{\sigma} = \sum_{j=0}^{N} a_{ij}\eta_j$, $a_{ij} \in A$, then the condition $Y^{\sigma} = Y$ implies $a_{ij} = t\, a'_{ij}$, $a'_{ij} \in A$ for $n+1 \leq i \leq N$, $0 \leq j \leq n$. By virtue of Lemma 1.4 $\mathrm{elm}^n_{Y_{U_\alpha}}(X_{U_\alpha}) = \mathrm{Proj}(A[\eta'_0, \ldots, \eta'_N])$, $\eta_i = \eta'_i\,(0 \leq i \leq n)$, $\eta_i = t\,\eta'_i\,(n+1 \leq i \leq N)$. Thus $\eta'^{\sigma}_i = \sum_{j=0}^{n} a_{ij}\eta'_j + \sum_{j=n+1}^{N} t\, a_{ij}\eta'_j$ $(0 \leq i \leq n)$, $\eta'^{\sigma}_i = \sum_{j=0}^{n} a'_{ij}\eta'_j + \sum_{j=n+1}^{N} a_{ij}\eta'_j$ $(n+1 \leq i \leq N)$. Hence σ induces a morphism σ'_{U_α}

of X'_{U_α} to itself. σ'_{U_α} is an automorphism of X'_U because if τ is the inverse of σ, then $\tau'_{U_\alpha} \cdot \sigma'_{U_\alpha} = \sigma'_{U_\alpha} \cdot \tau'_{U_\alpha} = id_{X'_{U_\alpha}}$. Moreover σ'_{U_α} coincides with σ'_{U_β} in an open dense subset $X'_{(S-T)} \cap X'_{\alpha\beta}$ of $X'_{\alpha\beta} = X'_{U_\alpha \cap U_\beta}$, which implies $\sigma'_{U_\alpha} | X'_{\alpha\beta} = \sigma'_{U_\beta} | X'_{\alpha\beta}$ because the set $\{x \in X'_{\alpha\beta} \mid x^{\sigma'_{U_\alpha}} = x^{\sigma'_{U_\beta}}\}$ is closed in $X'_{\alpha\beta}$. Thus σ induces an element σ' of $Aut_S(X')$. It is obvious that σ' is a desired automorphism. If σ'_1, σ'_2 are automorphisms of X' induced by σ, then $\sigma'_1 = \sigma'_2$ in an open dense subset, whence $\sigma'_1 = \sigma'_2$.

Now we shall come back to the proof of Lemma 3.2. By virtue of Lemma 3.3, σ induces an element of the group $G(Y) = \{\tau \in PGL(r-1) \mid Y^{\tau} = Y\}$. Thus we have a homomorphism $\varphi : Aut^0_S(P(E)) \longrightarrow G(Y)$. We get also a homomorphism $\psi : G(Y) \longrightarrow Aut^0_S(P(E))$ because $\sigma \in Aut^0_S(P(E))$ induced by $\tau \in G(Y)$ sends H'_i to an element of $|H'_1|$, which means $\sigma^*(O_{P(E)}(1)) \cong O_{P(E)}(1)$. Clearly $\varphi \cdot \psi = id$, $\psi \cdot \varphi = id$. Thus $Aut^0_S(P(E)) \cong G(Y)$.

q. e. d.

Now we come to a main theorem of this section.

Theorem 3.4. Let S be a non-singular projective variety over k, let T be a non-singular subvariety of S of codimension one and let $D \in A_T$.

(i) $SR^r(S, T, D)$ in Theorem 2.14 consists of all simple vector bundles in $R^r(S, T, D)$.

(ii) If $E \in R^r(S, T, D)$ is defined by $(s_1, \ldots, s_r) \in H^0(S, T, D) \times \ldots \times H^0(T, O_T(D))$ (cf. Principle 2.6) and if the dimension of the vector subspace of $H^0(T, O_T(D))$ generated by $s_1, \ldots, s_r$ is r', then $E \cong O_S^{\oplus (r-r')} \oplus E'$ for some $E' \in SR^{r'}(S, T, D)$.

Proof. Assume that E is defined by $(s_1, \ldots, s_r) \in H^0(T, O_T(D)) \times \ldots \times H^0(T, O_T(D))$. In the first place note that $D \in A_T$ implies $\dim_k H^0(S, E) = r$ by virtue of Lemma 2.11, and therefore by virtue of Lemma 3.2 $\mathrm{Aut}_S^0(P(E)) \cong G(Y) = \{ \sigma \mid \sigma \in PGL(r-1) = \mathrm{Aut}_S(P_S^{r-1}),\ Y^{\sigma} = Y \}$ for the subscheme Y of P_T^{r-1}

! Y_t is a hyperplane of $P_{k(t)}$ for any $t \in T$, we can define a map $T \ni t \longrightarrow$)

whose ideal in P_T^{r-1} is generated by $s_1 \eta_1 + \ldots + s_r \eta_r$, where $\eta_1, \ldots, \eta_r$ form a system of homogeneous coordinates of P_S^{r-1} (cf. Principle 2.5 and Principle 2.6). Since every $s_1(t), \ldots, s_r(t)$ is not zero for any $t \in T$, the rational map $\varphi : T \ni t \longrightarrow (s_1(t), \ldots, s_r(t)) \in P \cong P_k^{r-1}$ is a morphism. On the other hand, since $Y_t \in P$, where P is regarded as the dual space of P_k^{r-1}. This map is nothing but φ. Moreover the action of $PGL(r-1) = Aut_S(P_S^{r-1}) = Aut_k(P_k^{r-1})$ on $P_S^{r-1} = P_k^{r-1} \times_k S$ induces that on the dual space P of P_k^{r-1} through contragradient linear transformations. Thus the condition $Y^{\sigma} = Y$ for $\sigma \in PGL(r-1)$ is equivalent to $x^{\sigma} = x$ for any $x \in \varphi(T)$ by the above action. This implies $G(Y) \cong \{\sigma \mid \sigma \in PGL(r-1),\ x^{\sigma} = x \text{ for any } x \in \varphi(T)\}$. Assume $E \in SR^r(S, T, D)$, that is, $s_1, \ldots, s_r$ are linearly independent in $H^0(T, O_T(D))$, then $\varphi(T)$ is contained in no hyperplane of P, whence there exist linearly independent k-rational points

{$x_1, \ldots, x_r$ in $\varphi(T)$. Let L_i be the linear subspace in P generated by }

$x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r$, then $\bigcup_{i=1}^{r} L_i \not\supset \varphi(T)$ because $L_i \not\supset \varphi(T)$, $1 \leq {}^{\forall}i \leq r$. Thus there exists a k-rational point x_{r+1}

in $\varphi(T) - \bigcup_{i=1}^{r} L_i$. Then any r points in $\{x_1, \ldots, x_{r+1}\}$ are linearly independent in P. We know therefore $G(Y) = \{e\}$ because $\sigma \in G(Y)$ fixes every $x_1, \ldots, x_{r+1}$. Hence if $E \in SR^r(S, T, D)$, then E is simple. Now we have only to prove (ii) because a simple vector bundle is indecomposable (see Lemma 3.1, (ii)). It is easy, however, that every vector bundle in $R^r(S, T, D) - SR^r(S, T, D)$ is decomposable. In fact if $E \in R^r(S, T, D) - SR^r(S, T, D)$, then $s_1, \ldots, s_r$ are dependent, which means $\varphi(T) \subset H$ for some hyperplane H of P. Thus $G(Y) \supset \{\sigma \in PGL(r-1) \mid x^\sigma = x$ for any $x \in H\} \supset G_m$. Hence rank of $\mathrm{Aut}_S(E) =$ rank of $\mathrm{Aut}_S^0(P(E)) + 1 \geq 2$, which asserts that E is decomposable by virtue of Lemma 3.1. (i) is therefore proved. Next let us proceed to the proof of (ii). We may assume that $s_{r''+1}, \ldots, s_r$ are linearly independent for $r'' = r - r'$. Since $\varphi(T)$ is contained in a linear subspace of dimension $r' - 1$ of P and none of those of dimension $r' - 2$, there are k-rational points $x_1, \ldots, x_{r'}$ such that $\varphi(x_1), \ldots, \varphi(x_{r'})$ are linearly independent in P. Put

$\xi_i = \sum_{j=1}^{r} s_j(x_i)\eta_j$ $(1 \leq i \leq r')$, then $\eta_1, \ldots, \eta_{r''}, \xi_1, \ldots, \xi_{r'}$ are linearly independent over k because $\mathrm{rank}(s_i(x_j)) = r'$ and $s_1, \ldots, s_{r''}$ depend on $s_{r''+1}, \ldots, s_r$. Thus we can adopt $\eta_1, \ldots, \eta_{r''}, \xi_1, \ldots, \xi_{r'}$ as a homogeneous coordinate of P_S^{r-1}. Moreover Y is defined by $s_1' \xi_1 + \ldots + s_{r'}' \xi_{r'} = 0$ for some linearly independent $s_1', \ldots, s_{r'}' \in H^0(T, O_T(D))$ because $s_{r''+1}, \ldots, s_r$ are linearly independent. There are therefore an afine open covering $\{U_\lambda = \mathrm{Spec}(A_\lambda)\}_{\lambda \in \Lambda}$ of S and a correspondence

$$\Lambda \ni \lambda \longrightarrow \{\lambda(1), \ldots, \lambda(r'-1)\} \subset \{1, \ldots, r'\}$$

such that $P_{S,U_\lambda}^{r-1} = \mathrm{Proj}(A_\lambda[\eta_1, \ldots, \eta_{r''}, \xi_{\lambda(1)}, \ldots, \xi_{\lambda(r'-1)}, \zeta_\lambda])$, $T \cap U_\lambda = \mathrm{Spec}(A_\lambda/t_\lambda A_\lambda)$ for some $t_\lambda \in A_\lambda$, $\zeta_\lambda = a_1^{(\lambda)}\xi_1 + \ldots + a_{r'}^{(\lambda)}\xi_{r'}$ for some $a_1^{(\lambda)}, \ldots, a_{r'}^{(\lambda)} \in A_\lambda$ and that the ideal of Y_{U_λ} in P_{S,U_λ}^{r-1} is generated by t_λ, ζ_λ. Then $X'_{U_\lambda} = \mathrm{elm}_{Y_{U_\lambda}}^{r-2}(P_{S,U_\lambda}^{r-1}) = \mathrm{Proj}(A_\lambda[\eta_1, \ldots, \eta_{r''}, \xi_{\lambda(1)}, \ldots, \xi_{\lambda(r'-1)}, \zeta'_\lambda])$ for $t_\lambda \zeta'_\lambda = \zeta_\lambda$ by virtue of Lemma 1.4. By the construction the ideals I_{U_λ}, J_{U_λ} generated by $\{\eta_1, \ldots, \eta_{r''}\}$ $\{\xi_{\lambda(1)}, \ldots, \xi_{\lambda(r'-1)}, \zeta'_\lambda\}$ respectively define global ideals, that is, there are ideals I, J in $O_{X'}$ for $X' = elm_Y^{r-2} P_S^{r-1})$ with $IO_{X_{U_\lambda}} = I_{U_\lambda}$, $JO_{X_{U_\lambda}} = J_{U_\lambda}$ for any $\lambda \in \Lambda$. I, J define projective subbundle P_1, P_2 of $X' = P(E)$ such that $P_1 \cap P_2 = \phi$, $\dim P_{1,s} + \dim P_{2,s} = r-2$, for any $s \in S$. Thus E is isomorphic to $E_1 \oplus E_2$ for $E_1 = \pi'_*(O_{P_1} \otimes O_{X'}(1))$, $E_2 = \pi'_*(O_{P_2} \otimes O_{X'}(1))$, where $\pi' : X' \longrightarrow S$ is the structure morphism and $O_{X'}(1)$ is the tautological linebundle of E. Since $\eta_1, \ldots, \eta_{r''}$ form a basis of E_2 on U_λ for any $\lambda \in \Lambda$, E_2 is isomorphic to $O_S^{\oplus r''}$. On the other hand, since $\xi_{\lambda(1)}, \ldots, \xi_{\lambda(r'-1)}, \zeta'_\lambda$ form a local basis of E_1 on U_λ, E_1 is a regular vector bundle defined by $(s'_1, \ldots, s'_{r'}) \in H^0(T, O_T(D)) \times \cdots \times H^0(T, O_T(D))$ by virtue of Lemma 1.4 and Lemma 2.2. E_1 is contained in $SR^{r'}(S, T, D)$ because $s'_1, \ldots, s'_{r'}$ are linearly independent.

q. e. d.

In [3] A. Grothendieck proved that every vector bundle on P_k^1 is the direct sum of linebundles. In the same paper he posed a question whether this property characterizes P_k^1 in the category of projective variety over k. Van de Ven and J. Simonis solved this problem in the non-singular case (see [17]). The above theorem provides an answer of this problem in a stronger form.

Corollary 3.4.1. Let S be a non-singular projective variety over k of demension n. If r is an integer greater than max(n - 1, 1) and if $S \not\cong P_k^1$, then there is a simple vector bundle of rank r on S.

Proof. If $n \geq 2$, then this is a direct corollary to Theorem 3.4 and Remark 2.15. It is well known that there is a stable vector bundle on S if n = 1 and $S \not\cong P_k^1$ (for example a non-trivial extension E of L by O_S is stable for a linebundle L of degree 1). And every stable vector bundle is simple ([14]).

q. e. d.

Remark 3.5. Our proof of Theorem 3.4 shows that without the assumption $D \in A_T$ (ii) is true if one defines $SR^r(S, T, D)$ as the set which consists of all elements in $R^r(S, T, D)$ defined by linearly independent $(s_1, \ldots, s_r)$. (i) is not necessarily true without the condition $D \in A_T$. But it would not be best to assume the condition because there is a simple regular vector bundle not satisfying the condition (see next section).

Example 3.6. For an ineducible conic C^2 in P^2 and a point $P \in C^2$ the unique element of $SR^2(P^2, C, P) = R^2(P^2, C^2, P)$ is $O_{P^2}(1) \oplus O_{P^2}(1)$. Every element of $SR^2(P^2, C^2, 2P) = R^2(P^2, C^2, 2P)$ is indecomposable but not simple.

Proof. Assume that $E \in R^2(P^2, C^2, P)$ is defined by (s_1, s_2) $H^0(C^2, O_{C^2}(P)) \times H^0(C^2, O_{C^2}(P))$. s_i corresponds to one point divisor P_i on C^2. Take a point $Q \in C^2$ which is different from P_1, P_2 and two lines $\sum_{j=0}^{2} a_j^{(i)} X_j = 0$ $(i = 1, 2)$ (which go through Q, P_i). Then the subvariety V of $P^1_{P^2} = X$ defined by $(\sum_{j=0}^{2} a_j^{(1)} X_j)\eta_1 + (\sum_{j=0}^{2} a_j^{(2)} X_j)\eta_2 = 0$ with a system of homogeneous coordinates η_1, η_2 of X is non-singular and contains the subvariety Y of $P^1_{C^2}$ defined by $s_1\eta_1 + s_2\eta_2 = 0$. It is easy to check that proper transform of V by elm^0_Y is a section of $\mathrm{elm}^0_Y(X) = P(E)$. Thus E is an extension of two linebundles. On the other hand, every extension of two linebundles is trivial on P^2 because $H^1(P^2, L) = 0$ for any linebundle L on P^2. E is therefore decomposable. Moreover $c_1(E) = 2$, $c_2(E) = 1$ by virtue of Corollary 2.19.1. Thus $E \cong O_{P^2}(1) \oplus O_{P^2}(1)$. Since $c_1(E) = 2$, $c_2(E) = 2$ for $E \in R^2(P^2, c^2, 2\bar{P})$ by virtue of Corollary 2.19.1, E is indecomposable. That E is not simple will be proved in the next section (see Example 3.11).

q. e. d.

§ 2. Simple regular vector bundles of rank 2.

In the rank 2 case we can study more fully simple regular vector bundles. A distinguished fact on a vector bundle E of rank 2 is $P(E^\vee) \cong P(E)$ with the dual vector bundle $E^\vee$ of E. In fact

Lemma 3.7. If E is a vector bundle of rank 2 on a scheme X, then $E^\vee \cong E \otimes \det E^\vee$.

Proof. Let A_{ij} be transition matrices of E, then those of E are ${}^{t}A_{ij}^{-1}$ = the contragradient of A_{ij} . Then the isomorphism $f : E \otimes \det E^{\vee} \longrightarrow E^{\vee}$ can be given by the following matrix identity ;

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot A_{ij} \cdot (\det A_{ij}) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = {}^{t}A_{ij}^{-1} \ .$$

q. e. d.

The following lemma is due to Schwarzenberger ([19] Theorem 1).

Lemma 3.8. Let E be a vector bundle of rank 2 on a non-singular projective variety X over k. Then the following two conditions are equivalent to each other

(i) E is not simple.

(ii) There is a linebundle L on X such that for $E' = E \otimes L$

$\dim_k H^0(X, E') > 0, \ \dim_k H^0(X, E') > 0.$

Now assume that $E \in R^2(S, T, D)$ is defined by Y. Then the tautological linebundle of E on $P(E) = \mathrm{elm}_Y^0(P_S^1) = X(Y)$ is $O_{X(Y)}(H_Y')$ in the notation of chapter II (see Lemma 2.2), and $\det E^{\vee} \cong O_S(-T)$ by virtue of Corollary 2.18.1. Applying the above lemma E is not simple if and only if

$$\dim_k H^0(X(Y),\ O_{X(Y)}(H'_Y + \pi_Y^{-1}(D_0))) > 0,$$

$$\dim_k H^0(X(Y),\ O_{X(Y)}(H'_Y - \pi_Y^{-1}(T + D_0))) > 0$$

for some divisor D_0 on S because $H^0(S,\ E \otimes O_S(D_0)) = H^0(X(Y),\ O_{X(Y)}(H'_Y + \pi_Y^{-1}(D_0)))$, $H^0(S,\ (E \otimes O_S(D_0))^{\vee}) = H^0(X(Y),\ O_{X(Y)}(H'_Y - \pi_Y^{-1}(T + D_0)))$ by virtue of Lemma 3.7. Thus E is not simple if and only if there are positive divisors A'_1, A'_2 with $A'_1 - \pi_Y^{-1}(T) \not> 0$, $A'_2 - \pi_Y^{-1}(T) \not> 0$ and non-negative integers r_1, r_2 such that

$$A'_1 + r_1\pi_Y^{-1}(T) \sim H'_Y + \pi_Y^{-1}(D_0),\quad A'_2 + r_2\pi_Y^{-1}(T) \sim H'_Y - \pi_Y^{-1}(T+D_0).$$

We may assume that $D_0 \not\supset T$ and replace T by a suitable T' such that $T' \not\supset T$, $T \sim T'$, because S is projective. Put $A_i = f_Y\{g_Y^{-1}[A'_i]\}$ and $\widetilde{H} = f_Y^{-1}[H_0]$.

(a) Assume that A'_1 contains the center Y' of $(\mathrm{elm}_Y^0)^{-1}$: Since $g_Y^{-1}[A'_1] + \overline{X}_T + r_1(\pi_Y \cdot g_Y)^{-1}(T') \sim \widetilde{H} + \overline{X}_T + (\pi_Y \cdot g_Y)^{-1}(D_0)$ and $f_Y^{-1}(H_0 + P_T^1) = \widetilde{H} + \overline{X}_T + E_Y$ by virtue of Lemma 2.1, we get

$$f_Y^{-1}(H_0 + P_T^1) \sim g_Y^{-1}[A'_1] + \overline{X}_T + r_1(\pi f_Y)^{-1}(T') - (\pi \cdot f_Y)^{-1}(D_0) + E_Y$$

(note $\pi \cdot f_Y = \pi_Y \cdot g_Y$). On the other hand, $f_Y^{-1}(A_1 + P_T^1 + r_1\pi^{-1}(T') - \pi^{-1}(D_0)) = g_Y^{-1}[A_1'] + \bar{X}_T + E_Y + r_1(\pi f_Y)^{-1}(T') - (\pi \cdot f_Y)^{-1}(D_0)$ by virtue of Lemma 2.1. Thus $f_Y^{-1}(H_0 + P_T^1) \sim f_Y^{-1}(A_1 + P_T^1 + r_1\pi^{-1}(T') - \pi^{-1}(D_0))$, which implies $A_1 \sim H_0 - r_1P_T^1 + \pi^{-1}(D_0)$. Since $A_1 > 0$, there is a positive divisor D_1 with $D_1 \sim D_0 - r_1T$.

(b) Assume that $A_2' \supset Y'$: By a similar argument as above we have $A_2 \sim H_0 - r_2P_T^1 - \pi^{-1}(T + D_0) = H_0 - (r_2 + 1)P_T^1 - \pi^{-1}(D_0)$, whence there is a positive divisor D_2 with $D_2 \sim -(r_2 + 1)T - D_0$.

(a') Assume that $A_1' \not\supset Y'$: Since $g_Y^{-1}[A_1'] + r_1(\pi_Y \cdot g_Y)^{-1}(T') \sim \tilde{H} + \bar{X}_T + (\pi_Y \cdot g_Y)^{-1}(D_0)$ and $f_Y^{-1}(A_1 + r_1\pi^{-1}(T') - \pi^{-1}(D_0)) = g_Y^{-1}[A_1'] + E_Y + r_1(\pi \cdot f_Y)^{-1}(T') - (\pi \cdot f_Y)^{-1}(D_0)$ by virtue of Lemma 2.1, we have $f_Y^{-1}(H_0 + P_T^1) \sim f_Y^{-1}(A_1 + r_1\pi^{-1}(T') - \pi^{-1}(D_0))$. Thus $A_1 \sim H_0 - (r_1 - 1)P_T^1 + \pi^{-1}(D_0)$. Since $A_1 > 0$, there is a positive divisor D_1' with $D_1' \sim D_0 - (r_1 - 1)T$.

(b') Assume $A_2' \not\supset Y'$: By a similar argument as above we have $A_2 \sim H_0 - r_2P_T^1 + P_T^1 - \pi^{-1}(T + D_0) = H_0 - r_2P_T^1 - \pi^{-1}(D_0)$, whence there is a positive divisor D_2' with $D_2' \sim -r_2T - D_0$.

We should therefore consider the following cases.

(1) The case where (a) and (b) are satisfied : Since $-(r_1 + r_2 + 1)T \sim D_1 + D_2 > 0$, $r_1 + r_2 + 1 > 0$, $T \gneq 0$ and since S is projective, we have a contradiction.

(2) The case where (a) and (b') are satisfied : Since $-(r_1 + r_2)T \sim D_1 + D_2' > 0$, and r_1, r_2 are non-negative integers, we get $r_1 = r_2 = 0$, whence $D_0 \sim 0$. Thus $A_2 \sim H_0$, and by virtue of Lemma 2.1, $A_2 \supset Y$. Hence $Y = \mathfrak{p} \times T \subset P^1_T$ for some point $\mathfrak{p} \in P^1$, which implies $E \in R^2(S, T, 0)$. Therefore $E \cong O_S \oplus O_S(T)$ and this is not simple.

(3) The case where (a') and (b) are satisfied : Since $-(r_1 + r_2)T \sim D_1' + D_2 > 0$ and r_1, r_2 are non-negative integers, we get $r_1 = r_2 = 0$, $D_0 + T \sim D_1' > 0$, $-(D_0 + T) \sim D_2 > 0$, whence $D_0 \sim -T$. Thus $A_1 \sim H_0$ and by virtue of Lemma 2.1 $A_1 \supset Y$. Hence $E \cong O_S \oplus O_S(T)$ and E is not simple.

(4) The case where (a') and (b') are satisfied : Since $(1 - r_1 - r_2)T \sim D_1' + D_2' > 0$, we get $1 - r_1 - r_2 \geq 0$. If either

r_1 or r_2 is equal to 1, then by a similar argument as above we have $E \cong O_S \oplus O_S(T)$. Suppose $r_1 = r_2 = 0$, then $D_2' \sim -D_0$, whence $D_1' \sim T - D_2'$. Thus there are positive divisors D_1', D_2' on S and A_1, A_2 on P_S^1 such that $D_1' + D_2' \sim T$, $A_1 \sim H_0 + \pi^{-1}(D_1')$, $A_2 \sim H_0 + \pi^{-1}(D_2')$ and both A_1 and A_2 contain Y. Conversely if these conditions are satisfied, then the calculation in (a'), (b') shows that $A_1' \sim H_Y' + \pi_Y^{-1}(D_0)$, $A_2' \sim H_Y' - \pi_Y^{-1}(T + D_0)$ for $A_i' = \mathrm{elm}_Y^0[A_i]$ and $D_0 = D_1' - T$.

Consequently we have

Lemma 3.9. Let $E \in R^2(S, T, D)$ be defined by Y. E is not simple if and only if there are positive divisors D_1, D_2 on S and A_1, A_2 on P_S^1 such that $D_1 + D_2 \sim T$, $A_1 \sim H_0 + \pi^{-1}(D_1)$, $A_2 \sim H_0 + \pi^{-1}(D_2)$ and that both A_1 and A_2 contain Y.

Proof. Note that if $D = 0$, that is, $E \cong O_S \oplus O_S(T)$, then the above conditions are satisfied by $D_1 = 0$, $D_2 = T$, $A_1 = H_0$, $A_2 = H_0 + \pi^{-1}(T)$. Then our assertion is clear by virtue of the argument before this lemma.

q. e. d.

Theorem 3.10. Assume that $E \in R^2(S, T, D)$ is defined by $(s, s') \in H^0(T, O_T(D)) \times H^0(T, O_T(D))$ (cf. Principle 2.6). E is not simple if and only if $D = 0$ or there are positive divisors C_1, C_1', C_2, C_2' such that $C_1 + C_2 \sim T$, $C_i \sim C_i'$ $(i = 1, 2)$ and that $C_i \cdot T = |s| + B_i$, $C_i' \cdot T = |s'| + B_i$ $(i = 1, 2)$ for positive divisors B_i on T, where $|s|$ (or, $|s'|$) is the divisor defined by $s = 0$ (or, $s' = 0$, resp.).

Proof. By virtue of Remark 2.16 E is not simple if $D = 0$. Thus we may assume that $D \neq 0$. E is defined by $Y : s\bar{\eta}_0 + s'\bar{\eta}_1 = 0$, where $\bar{\eta}_0, \bar{\eta}_1$ are a system of homogeneous coordinate of P_T^1 induced from a system of homogeneous coordinate η_0, η_1 of P_S^1. Assume that E is not simple, then there are D_1, D_2, A_1, A_2 satisfying the conditions in Lemma 3.9 with the same Y as above. A_i is defined by $s_{i0}\eta_0 + s_{i1}\eta_1 = 0$ for $s_{ij} \in H^0(S, O_S(D_i))$. Let $\overline{s_{ij}}$ be the element of $H^0(T, O_T(D_i \cdot T))$ induced from s_{ij}. Then $A_i \supset Y$ implies $\bar{s}_{i0}\bar{\eta}_0 + \bar{s}_{i1}\bar{\eta}_1 = a_i(s\bar{\eta}_0 + s'\bar{\eta}_1)$ for some $a_i \in H^0(T, O_T(D_i \cdot T - D))$. Thus if one puts $|s_{i0}| = C_i$, $|s_{i1}| = C_i'$, then $C_i \cdot T = |s| + B_i$, $C_i' \cdot T = |s'| + B_i$ $(B_i = |a_i|)$. Since $C_i \sim C_i' \sim D_i$, we know $T \sim C_1 + C_2$.

Conversely assume that C_1, C'_1, C_2, C'_2 exist. The conditions $C_i \cdot T = |s| + B_i$, $C'_i \cdot T = |s'| + B_i$ assert that there are $s_{ij} \in H^0(S, O_S(C_i))$ $(j = 0, 1)$ and $a_i \in H^0(T, O_T(B_i))$ such that $|s_{i0}| = C_i$, $|s_{i1}| = C'_i$, $|a_i| = B_i$ and that $\bar{s}_{i0}\xi_0 + \bar{s}_{i1}\xi_1 = a_i(s\xi_0 + s'\xi_1)$. Let A_i be the positive divisor on P^1_S defined by $s_{i0}\xi_0 + s_{i1}\xi_1 = 0$, then $A_i \supset Y$. Thus $D_i = C_i$, A_i satisfy the conditions in Lemma 3.9, which implies that E is not simple.

q. e. d.

Corollary 3.10.1. $E \in R^2(S, T, D)$ $(D \neq 0)$ is simple if $H^0(T, O_T(T^2 - 2D)) = 0$.

Proof. If E is not simple and $D \neq 0$, then there are positive divisors C_1, C_2 such that $C_i \cdot T \sim D + B_i$, $B_i > 0$ on T and $C_1 + C_2 \sim T$ (see Theorem 3.10). Thus $T^2 - 2D \sim ((C_1 + C_2) \cdot T) - 2D \sim B_1 + B_2 > 0$. We have therefore $H^0(T, O_T(T^2 - 2D)) \neq 0$.

q. e. d.

Example 3.11. Let C^n be a non-singular curve of degree n in P^2. Every element of $R^2(P^2, C^2, 2p)$ is indecomposable and not simple. Every element of $R^2(P^2, C^3, P_1+P_2+P_3)$ is not simple if and only if P_1, P_2, P_3 are collinear.

Proof. Asuume that $E \in R^2(P^2, C^2, 2P)$ is defined by $(s_1, s_2) \in H^0(C^2, O_{C^2}(2P)) \times H^0(C^2, O_{C^2}(2P))$. Put $|s_i| = P_{i1} + P_{i2}$. Take the line ℓ_i going through P_{i1} and P_{i2} (if $P_{i1} = P_{i2}$, ℓ_i touches to C^2 at P_{i1}). Then $\ell_1 = C_1 = C_2$, $\ell_2 = C'_1 = C'_2$ satisfy the conditions for s_1, s_2 in Theorem 3.10. Let us show the latter part. Note that P_1, P_2, P_3 are collinear if and only if Q_1, Q_2, Q_3 are collinear for any element $Q_1 + Q_2 + Q_3$ of $|P_1 + P_2 + P_3|$. Assume that $E \in R^2(P^2, C^3, P_1 + P_2 + P_3)$ is defined by $(s_1, s_2) \in H^0(C^3, O_{C^3}(P_1 + P_2 + P_3)) \times H^0(C^3, O_{C^3}(P_1 + P_2 + P_3))$ with $|s_i| = \sum_{j=1}^{3} Q_{ij}$. If E is not simple, then one of C_i in Theorem 3.10 is a line, whence Q_{i1}, Q_{i2}, Q_{i3} are collinear. Assume P_1, P_2, P_3 collinear, then Q_{i1}, Q_{i2}, Q_{i3} are collinear. Take a line ℓ_i going through Q_{i1}, Q_{i2}, Q_{i3}. Then $\ell_1 = C_1$, $\ell_2 = C'_1$, $2\ell_2 = C_2$, $2\ell_2 = C'_2$ satisfy the conditions of Theorem 3.10, whence E is not simple.

q. e. d.

As a matter of fact if P_1, P_2, P_3 are not collinear, then every element of $R^2(P^2, C^3, P_1 + P_2 + P_3)$ is isomorphic to the tangent bundle of P^2 (see Example 4.8).

Example 3.12. Let T be a non-singular surfaces of degree 4 of P^3 which contains a line ℓ and let $\{H_\lambda\}_{\lambda \in P^1}$ be the linear pencil which consist of hyperplanes of P^3 containing ℓ. Then $H_\lambda \cdot T = \ell + C_\lambda$, $C_\lambda \cdot C_\mu = 0$ because $p_a(C_\lambda) = 1$, $K_T \sim 0$ with a canonical divisor K_T of T. Thus $R^2(P^3, T, 2C_\lambda) \neq \phi$ by virtue of Remark 2.16. Let us show that every element of $R^2(P^3, T, 2C_\lambda)$ is indecomposable and not simple.

Proof. Since $C_\lambda^2 = 0$, $|2C_\lambda| = \{D_{\lambda\mu} = C_\lambda + C_\mu\}_{\lambda, \mu \in P^1}$. Assume that $E \in R^2(P^3, T, 2C_\lambda)$ is defined by $(s_1, s_2) \in H^0(T, O_T(2C_\lambda)) \times H^0(T, O_T(2C_\lambda))$ with $|s_i| = D_{\lambda_i \mu_i}$. Then $C_j = H_{\lambda_1} + H_{\mu_1}$, $C_j' = H_{\lambda_2} + H_{\mu_2}$ $(j = 1, 2)$ satisfy the conditions in Theorem 3.10. Thus E is not simple. Since $c_1(E) = T$, $c_2(E) = 2C_\lambda$, $\deg c_1(E) = 4$ and $\deg c_2(E) = 6$, whence E is indecomposable.

q. e. d.

As an application of the above theorem we shall give another proof of a theorem of Schwarzenberger ([19] Theorem 8).

Theorem 3.13. Let S be a non-singular projective surface over k, c_1 a divisor on S and let c_2 be an integer. For $r > 1$, there exists a non-simple vector bundle of rank r on S with Chern classes c_1, c_2.

In order to prove the theorem we need a lemma.

Lemma 3.14. Let H be a very ample divisor on a non-singular projective surface S and let $x_1, \ldots, x_n$ be mutually distinct points on S. There exists a positive integer a_0 such that for any $a \geq a_0$ there is a non-singular irreducible member of $|aH|$ going through all of $x_1, \ldots, x_n$.

Proof. Let $f : S' \longrightarrow S$ be the blowing up with centers $x_1, \ldots, x_n$ and let E_i be the exceptional curve $f^{-1}(x_i)$. Then $O_{S'}(-(E_1 + \ldots + E_n))$ is f-very ample (E, G. A. Chap. II, 8.1.7). Since H is very ample, there exists a positive integer a_0 such that $O_{S'}(-(E_1 + \dots + E_n)) \otimes f^*(O_S(H)^{\otimes a}) = O_{S'}(af^{-1}(H) - (E_1 + \dots + E_n))$ is very ample for any $a \geq a_0$ (E. G. A. Chap. II, 4.4.10). Then a general member H' of $|af^{-1}(H) - (E_1 + \dots + E_n)|$ is non-singular irreducible. Since $(H', E_i) = (af^{-1}(H) - (E_1 + \dots + E_n), E_i) = 1$, $f(H')$ goes through all x_i with multiplicity 1. Thus $f(H')$ is non-singular irreducible because $S' - \bigcup_{i=1}^{n} E_i$ is isomorphic to $S - \{x_1, \dots, x_n\}$. Clearly $f(H') \sim aH$, $f(H')$ is therefore a desired member of $|aH|$.

q. e. d.

Proof of Theorem 3.13. For $r > 2$ and vector bundle E of rank 2, $c_1(E) = c_1(O_S^{\oplus(r-2)} \oplus E)$, $c_2(E) = c_2(O_S^{\oplus(r-2)} \oplus E)$. Thus we have only to prove the theorem in the case of $r = 2$. Let H be a very ample divisor on S with $(H, H) = h$ and let n be a non-negative integer. Take integers α, β such that $\alpha \geq 0$, $0 \leq \beta < h$, $n = \alpha h + \beta$. Let $H_i (1 \leq i \leq 4)$ be general members in $|H|$ such that $H_1 \cdot H_2 = \sum_{i=1}^{h} x_i$, $H_3 \cdot H_4 = \sum_{j=1}^{h} y_i$ with mutually distinct points $x_1, \ldots, x_h, y_1, \ldots, y_h$. Then by virtue of Lemma 3.14 there exists a positive integer such that for any $a \geq a_0$ there is a non-singular irreducible member of $|aH|$ going through all of $x_1, \ldots, x_h, y_1, \ldots, y_{h-\beta}$. Take such a member H' for $a = 2r-1 \geq \max(a_0, 4\alpha + 3)$ with an even integer r. We may assume that H' goes through none of $y_{h-\beta+1}, \ldots, y_h$. If $H_i \cdot H' = \sum_{i=1}^{h} x_i + \sum_{k=1}^{(2r-2)h} z_{ik}$ $(i=1,2)$, $H_i \cdot H' = \sum_{j=1}^{h-\beta} y_j + \sum_{k=1}^{(2r-2)h+\beta} w_{ik}$ $(i = 3, 4)$, then $A_1 = \sum_{k=1}^{(2r-2)h} z_{1k} \sim A_2 = \sum_{k=1}^{(2r-2)h} z_{2k}$, $B_1 = \sum_{\ell=1}^{(2r-2)h+\beta} w_{1\ell} \sim B_2 = \sum_{\ell=1}^{(2r-2)h+\beta} w_{2\ell}$ and $z_{1k_1} \neq z_{2k_2}$ $(1 \leq k_1, k_2 \leq (2r-2)h)$, $w_{1\ell_1} \neq w_{2\ell_2}$ $(1 \leq \ell_1, \ell_2 \leq (2r-2)h+\beta)$. Take another general members H_5, H_6 in $|H|$ such that $A_1' = H_5 \cdot H'$ and $A_2' = H_6 \cdot H'$ contain no common point, and put $D_i = \alpha A_i' + (\frac{r}{2} - \alpha - 1) A_i + B_i$ $(i = 1, 2)$. Then D_1 and D_2 contain no common point and $D_1 \sim D_2$, whence $R^2(S, H', D_1) \neq \phi$. The element E' of $R^2(S, H', D_1)$ defined by $(s_1, s_2) \in H^0(H', O_{H'}(D_1)) \times H^0(H', O_{H'}(D_2))$ with $|s_i| = D_i$ $(i = 1, 2)$ is not simple because $C_1 = \alpha H_5 + (\frac{r}{2} - \alpha - 1)H_1 + H_3$, $C_1' = \alpha H_6 + (\frac{r}{2} - \alpha - 1)H_2 + H_4$, $C_2 = (\alpha + r - 1)H_5 + (\frac{r}{2} - \alpha - 1)H_1 + H_3$,

$C'_2 = \alpha H_6 + (r - 1)H_5 + (\frac{r}{2} - \alpha - 1)H_2 + H_4$ obviously satisfy the conditions of Theorem 3.10. On the other hand, $c_1(E') = (2r - 1)H$, $c_2(E') = \alpha(2r - 1)h + (\frac{r}{2} - \alpha - 1)(2r - 2)h + (2r - 2)h + \beta = r^2h - rh + \alpha h + \beta$ by virtue of Corollary 2.18.1 and Corollary 2.19.1. Thus we obtain $c_1(E' \otimes O_S(-(r - 1)H)) = H$, $c_2(E' \otimes O_S(-(r - 1)H)) = \alpha h + \beta = n$. Therefore if H is a very ample divisor on S and if n is a non-negative integer, then there is a non-simple vector bundle E of rank 2 on S with $c_1(E) = H$, $c_2(E) = n$. For a given c_1, c_2, take a very ample divisor H'' and a positive integer t such that $H = c_1 + 2rH''$ is very ample and $c_2 + r^2(H'', H'') + r(c_1, H'') = n > 0$ (these conditions are satisfied if one takes sufficiently large r for a very ample divisor H''). As for these H, n there is a non-simple vector bundle E'' of rank 2 with $c_1(E'') = H$, $c_2(E'') = n$ by virtue of the above argument. Then $c_1(E'' \otimes O_S(-rH'')) = c_1$, $c_2(E'' \otimes O_S(-rH'')) = c_2$. Thus $E = E'' \otimes O_S(-rH'')$ gives a desired vector bundle.

q. e. d.

Chapter IV. Some special cases.

In this chapter we shall study some special vector bundles on some special algebraic varieties, along the line developed in the preceding three chapters.

§ 1. Tangent bundle of P_k.

Let T_X be the tangent bundle of a non-singular variety X over k. Then we have

Theorem 4.1. Let H be an arbitrary hyperplane of P_k. Then $R^n(P_k^n, H, H^2)$ consists only of one element $T_{P_k^n}(-1)$.

Proof. Let $\mathbf{P}$ be the dual space of P_k^n, then $\pi : X = \mathbf{P} \times_k P_k^n \longrightarrow P_k^n$ is the trivial P^n-bundle on P_k^n. On the other hand, the P^{n-1}-bundle $\pi' : Y = P(T_{P_k^n}) \longrightarrow P_k^n$ may be regarded as the bundle whose fibre $\pi'^{-1}(x)$ at x is $(n-1)$-dimensional projective space consisting of all hyperplanes in P_k^n going through x. Thus Y is naturally a subbundle of X. Take linearly independent points $x_1, \ldots, x_n$ in H. The set consisting of all hyperplanes going through x_i (for each fixed i) forms a hyperplane Z_i in $\mathbf{P}$. Put $H_i = Z_i \times_k P_k^n$. Let us consider $H_1 \cdot \ldots \cdot H_n \cdot Y$ in Y. Since $\pi'^{-1}(y) \cap H_i = \{\text{hyperplanes of } P_k^n \text{ going through } x_i \text{ and } y\}$, $\pi'^{-1}(y) \cap (\bigcap_{i=1}^{n} H_i)$ is not empty if and only if y is contained in H; and if $y \in H$, then $\pi'^{-1}(y) \cap (\bigcap_{i=1}^{n} H_i)$ is the point corresponding to H. Now let $X_0, \ldots, X_n$ be a system of homogeneous coordinate of P_k^n such that H is defined by $X_0 = 0$ and let $\eta_0, \ldots, \eta_n$ be the system of homogeneous coordinate of $\mathbf{P}$ induced from $X_0, \ldots, X_n$. Let U_i be the affine open set of P_k^n defined by $X_i \neq 0$ and put $\xi_j^i = X_i/X_j$. We may assume that Z_i is defined by $\eta_i = 0$. On the other hand, Y_{U_i} is defined by $\sum_{j=0}^{n} \eta_j \xi_i^j = 0$ in X_{U_i}. Thus $H_1, \ldots, H_n$, Y are transversal to each other at any point of $(\bigcap_{i=0}^{n} H_i)$ Y and $H' = H_1 \cdot \ldots \cdot H_n \cdot Y$ is defined by the ideal generated by $\xi_i^0, \eta_1, \ldots, \eta_n$. By virtue of Proposition 1.9 we know therefore $\mathrm{elm}_{H'}^0(Y) \cong P_k^{n-1} \times_k P_k^n$. Let H'' be the center of $(\mathrm{elm}_{H'}^0)^{-1}$, then $H'' \subset P_k^{n-1} \times_k H$. Since the regular vector bundle E defined by H'' is isomorphic to $T_{P_k^n}(r)$ and since $c_1(E) = H$, we have $E \cong T_{P_k^n}(-1)$.

Moreover, since $c_2(E) = c_2(T_{P_k^n}(-1)) = H^2$, $T_{P_k^n}(-1)$ is contained in $R^n(P_k^n, H, H^2)$. Conversely, if $E \in R^n(P_k^n, H, H^2)$ is defined by $(s_1, \ldots, s_n) \in H^0(H, O_H(H^2)) \times \cdots \times H^0(H, O_H(H^2))$, then $s_1, \ldots, s_n$ are linearly independent because if $s_1, \ldots, s_n$ are linearly dependent, $s_1(x) = \cdots = s_n(x) = 0$ for some $x \in H$. Thus $SR^n(P_k^n, H, H^2) = R^n(P_k^n, H, H^2)$ when one defines $SR^n(P_k^n, H, H^2)$ as in Remark 3.5. On the other hand, there is a surjective map $Grass_{n-1}^{n-1}(k) \longrightarrow SR^n(P_k^n, H, H^2)$ (see the proof of Theorem 2.14). Since $Grass_{n-1}^{n-1}(k)$ has only one point, $SR^n(P_k^n, H, H^2) = R^n(P_k^n, H, H^2)$ consists of one element $T_{P_k^n}(-1)$ only.

q. e. d.

It goes without saying that the above result has something to do with the fact that $T_{P_k^n}(-1)$ is a homogeneous vector bundle on the homogeneous space P_k^n. Furthermore, this theorem shows that the sufficient condition for a regular vector bundle to be simple stated in Theorem 3.4 is not best possible (Note $H^2 \notin A_H$).

As a corollary to the above proof we have the following, which is well known.

Corollary 4.1.1. There is an exact sequence

$$0 \longrightarrow O_{P_k^n} \longrightarrow O_{P_k^n}(1)^{\oplus(n+1)} \longrightarrow T_{P_k^n} \longrightarrow 0.$$

Proof. Since $O_X(H_1)$ is the tautological linebundle of $O_{P_k^n}{}^{\oplus(n+1)}$ on $X \cong P(O_{P_k^n}{}^{\oplus(n+1)})$, $O_X(H_1) \otimes O_Y$ is the tautological line bundle of $T_{P_k^n}(-1)$ on $Y = P(T_{P_k^n})$ by virtue of Lemma 2.2 and the above proof and since Y is a subbundle of X, we have a surjective homomorphism $\varphi : \pi_*(O_X(H_1)) = O_{P_k^n}{}^{\oplus(n+1)} \longrightarrow \pi'_*(O_X(H_1) \otimes O_Y) = T_{P_k^n}(-1)$. On the other hand, $\operatorname{Ker} \varphi \cong (\overset{n+1}{\wedge} O_{P_k^n}{}^{\oplus(n+1)}) \otimes (\overset{n}{\wedge} T_{P_k^n}(-1))^{-1} \cong O_{P_k^n}(-1)$.

q. e. d.

§ 2. Vector bundles on P_k^2.

We shall begin with an easy lemma.

Lemma 4.2. If E is a simple vector bundle of rank 2 on P_k^2 and if $\deg E \geq -3$, then $H^2(P_k^2, E) = 0$.

Proof. Assume that $H^2(P_k^2, E) \neq 0$. By the Serre duality, $\dim_k H^2(P_k^2, E) = \dim_k H^0(P_k^2, \check{E} \otimes O_{P_k^2}(-3)) > 0$. Since $\dim_k H^0(P_k^2, \check{E}) \geq \dim_k H^0(P_k^2, \check{E} \otimes O_{P_k^2}(-3))$, we have $H^0(P_k^2, \check{E}) \neq 0$. On the other hand, since $H^0(P_k^2, \check{E} \otimes O_{P_k^2}(-3)) = H^0(P_k^2, E \otimes (\det E)^{\vee} \otimes O_{P_k^2}(-3)) \neq 0$ and since $\deg((\det E)^{\vee} \otimes O_{P_k^2}(-3)) \geq 0$ by our assumption $\deg E \geq -3$, we have $H^0(P_k^2, E) \neq 0$. Thus E is not simple by virtue of Lemma 3.8. This is contrary to the assumption that E is simple.

q. e. d.

Let E be a vector bundle of rank 2 with Chern classes $c_1(E)$, $c_2(E)$ on a non-singular projective surface S. Define an integer $\Delta(E)$ to be $c_1(E)^2 - 4c_2(E)$. $-\Delta(E)$ is the second Chern class of End(E), hence it plays an important role in the theory of simple vector bundles. The following lemma is essentially due to Schwarzenberger ([18] Theorem 10).

Lemma 4.3. Let E, S be as above and let K be a canonical divisor of S. If $|-K| \neq \phi$ and $\Delta(E) \geq -(4p_a(S) + 1)$, then E

is not simple.

Proof. Since End(E) is self-dual, $\dim_k H^2(S, \underline{End}(E)) = \dim_k H^0(S, \underline{End}(E) \otimes O(K))$ by the Serre duality. On the other hand, the assumption $|-K| \neq \phi$ implies $\dim_k H^0(S, \underline{End}(E) \otimes O_S(K)) \leq \dim_k H^0(S, \underline{End}(E))$. Thus $\chi(\underline{End}(E)) = \Sigma(-1)^i \dim_k H^i(S, \underline{End}(E)) \leq 2 \dim_k H^0(S, \underline{End}(E))$. Besides the Riemann-Roch theorem provides equalities $\chi(\underline{End}(E)) = \Delta(E) + \frac{1}{3}(K^2 + c_2(S))$ and $p_a(S) + 1 = \frac{1}{12}(K^2 + c_2(S))$ ($c_2(S)$ is the second Chern class of S). We obtain therefore $2 \dim_k H^0(S, \underline{End}(E)) \geq \Delta(E) + 4(1 + p_a(S)) > 2$, which is our assertion.

q. e. d.

Let us consider some special cases.

Corollary 4.3.1. Let E, S be as above.

(i) If S is a rational ruled surface or P_k^2 and if $\Delta(E) \geq -1$ then E is not simple.

(ii) If S is an abelian surface and if $\Delta(E) \geq 3$, then E is not simple.

(ii)' If S is an abelian surface, the characteristic of $k \neq 2$ and if $\Delta(E) \geq 1$, then E is not simple.

(iii) If $K \sim 0$, $\dim_k H^1(S, O_S) = 0$ (for example K3 surfaces over $\mathbb{C}$, a non-singular surface of degree 4 in P_k^3) and if $\Delta(E) \geq -5$, then E is not simple.

Proof. (i) Let S be a rational ruled surface with minimal section D. Assume $(D, D) = -n$, then $-2D - (n+2)\ell$ is a canonical divisor on S, where ℓ is a fibre (= a generator of S). Thus we have $|-K| \neq \phi$. Let $S = P_k^2$ and let C be a cubic curve, then $-C$ is a canonical divisor, whence $|-K| \neq \phi$. In any case $p_a(S) = 0$. Then (i) follows from the above lemma.

(ii) If S is an abelian surface, then $K \sim 0$, $p_a(S) = -1$. Thus we obtain (ii). As for (ii)' see [Ü] Corollary to Theorem 2.

(iii) In this case $K \sim 0$, $p_a(S) = 1$, whence our assertion is obvious by vertue of Lemma 4.3.

q. e. d.

Example 4.4. (i) If S is a general non-singular surface of degree 4 in P_k^3 which contains a line ℓ in P_k^3, then there is a simple vector bundle E of rank 2 on S with $\Delta(E) = -2r$ for any $r \geq 3$.

(ii) If S is a general surface of degree 4 in $P_{\mathbb{C}}^3$, then $\Delta(E) \equiv 0$ (4) for any vector bundle E of rank 2 on S and there is a simple vector bundle E of rank 2 on S with $\Delta(E) = -4r$ for any r

Proof. (i) Take H_λ, C_λ, $\lambda \in P^1$ as in Example 3.12. Then C_λ is an elliptic curve for a general $\lambda \in P^1$ and $(C_\lambda, C_\lambda) = 0$, $(\ell, \ell) = -2$. If one takes points $P_1, \ldots, P_r$ on C_λ and $Q_1, \ldots, Q_s$ on ℓ for arbitrary r (≥ 2), and s (≥ 1), then $\sum_{i=1}^{r} P_i$ and $\sum_{j=1}^{s} Q_j$ are divisors free from base points. Thus $R^2(S, C_\lambda, \sum_{i=1}^{r} P_i) \neq \phi$, $R^2(S, \ell, \sum_{j=1}^{s} Q_i) \neq \phi$ and every element in $R^2(S, C_\lambda, \sum_{i=1}^{r} P_i)$, $R^2(S, \ell, \sum_{j=1}^{s} Q_j)$ is simple by virtue of Corollary 3.10.1. Since $\Delta(E_r) = -4r$, $\Delta(E_s) = -2 - 4s$ for $E_r \in R^2(S, C_\lambda, \sum_{i=1}^{r} P_i)$, $E_s \in R^2(S, \ell, \sum_{j=1}^{s} Q_j)$, our assertion is proved.

(ii) Note that if S is a general surface of degree 4 in $P^3_{\mathbf{C}}$, then $\mathrm{Pic}(S) \cong Z$ whose generator is the class of hyperplane section ((iii) Lecture 13). Thus $D^2 \equiv 0$ (4) for any divisor D on S, which shows the former assertion. In order to prove the latter, take a general hyperplane section C. Then C is a non-singular plane curve of degree 4. Hence there is a positive divisor A_r of degree r free from base point on C for any $r \geq 3$. Thus $R^2(S, C, A_r) \neq \phi$. Every element E_r of $R^2(S, C, A_r)$ is simple by virtue of Corollary 3.10.1 and $\Delta(E_r) = -4(r - 1)$ because of $(C, C) = 4$.

q. e. d.

Now let us come back to vector bundles on P^2_k. The following lemma is very interest when one takes Corollary 4.3.1, (i) into account.

Lemma 4.5. If E is a simple vector bundle of rank 2 on P^2_k, then $\Delta(E) \neq -4$.

Proof. We assume that E is a simple vector bundle of rank 2 on P^2_k with $\Delta(E) = -4$ and shall show a contradiction. By the assumption $\Delta(E) = -4$ where is a linebundle L such that $c_1(E \otimes L) = 0$, $c_2(E \otimes L) = -1$, whence we may assume $c_1(E) = 0$, $c_2(E) = 1$. The Riemann-Roch theorem asserts for a vector bundle E' of rank 2 on P^2_k :

$$\chi(E') = \sum_{i=0}^{2} (-1)^i \dim_k H^i(P^2_k, E') = 2 + \frac{3c_1(E')}{2} + \frac{c_1(E')^2 - 2c_2(E')}{2} .$$

Applying this to E we have $\chi(E) = 1$. On the other hand, Lemma 4.2 implies $H^2(P^2_k, E) = 0$. Thus we obtain $H^0(P^2_k, E) \neq 0$. Moreover, since $E^\vee \cong E \otimes (\det E)^\vee \cong E$, we know $H^0(P^2_k, E^\vee) \neq 0$. By virtue of Lemma 3.8 this is a contradiction.

q. e. d.

We have an interesting corollary.

Corollary 4.5.1. Let C be a non-singular curve of degree n in P^2_k and $D = \sum_{i=1}^{r} P_i$ be a positive divisor on C such that $|D|$ is free from base point.

i) If n $(= 2m)$ is even and $r \leq m^2 + 1$, then there is a positive divisor C' of degree m in P^2_k such that $C \cdot C' - D > 0$.

ii) If n $(=2m + 1)$ is odd and $r \leq m^2 + m$, then there is a positive divisor C' of degree m in P_k^2 such that $C \cdot C' - D > 0$.

Proof i) Since $|D|$ is free from base point, there is $D' \in |D|$ which contains none of P_i. Let $E \in R^2(P_k^2, C, D)$ be defined by $(s, s') \in H^0(C, O_C(D)) \times H^0(C, O_C(D))$ with $|s| = D$, $|s'| = D'$. On the other hand, since $\Delta(E) = C^2 - 4r \geq 4m^2 - 4(m^2 + 1) = -4$, $4 | \Delta(E)$, we know that E is not simple by virtue of Corollary 4.3.1 and Lemma 4.5. Appling Theorem 3.10 to E, we obtain positive divisors C_1, C_2 such that $C_1 + C_2 \sim C$, $C_1 \cdot C - D > 0$, $C_2 \cdot C - D > 0$. Since either C_1 or C_2 has a degree not greater than n, (i) is proved.

ii) Similar argument as above is available in this case too.

q. e. d.

Now we come to a theorem of Schwarzenberger ([19] Theorem 8)

Theorem 4.6.[15] Let n, m be integers. There is a vector bundle E of rank 2 on P_k^2 with $c_1(E) = n$, $c_2(E) = m$ if and only if $n^2 - 4m < 0, \neq -4$.

Proof. We have proved that if $\Delta(E) \geq 0$ or $\Delta(E) = -4$, then E is not simple (Corollary 4.3.1, Lemma 4.5). Let us show the "if" part of the theorem. Take a point P on a line C^1 and a point P' on an irreducible conic C^2. Since C^1 and C^2 are rational curves, $R^2(P^2_k, C^1, rP) \neq \phi$, $R^2(P^2_k, C^2, sP') \neq \phi$ for any $r > 0$, $s > 0$. If $E_r \in R^2(P^2_k, C^1, rP)$, $E'_s \in R^2(P^2_k, C^2, sP')$, then E_r, E'_s are simple for any $r \geq 1$, $s \geq 3$ by virtue of Corollary 3.10.1. Put $r = m + (\frac{1 - n^2}{4})$ if n is odd and put $s = m + 1 - \frac{n^2}{4}$ if n is even. The condition $n^2 - 4m < 0, \neq -4$ implies $r \geq 1$, $s \geq 3$. Take E_r or E'_s and put $E = E_r \otimes O_{P^2_k}(\frac{n-1}{2})$ or $E'_s \otimes O_{P^2_k}(\frac{n}{2} - 1)$ according as n is odd or even. Then $c_1(E) = n$, $c_2(E) = m$.

q. e. d.

The following theorem is due to F. Takemoto [20], which can be proved along our line.

Theorem 4.7. If E is a simple vector bundle of rank 2 on P^2_k with $\Delta(E) = -3$, then $E \cong T_{P^2_k}(n)$.

Proof. By the assumption $\Delta(E) = -3$ there is a linebundle L such that $c_1(E \otimes L) = 1$, $c_2(E \otimes L) = 1$. Hence we may assume that $c_1(E) = 1$ and $c_2(E) = 1$. We know by virtue of the Riemann-Roch theorem and Lemma 4.2 that $\chi(E) = 3$, $\chi(E(-1)) = 0$, $H^2(P^2_k, E) = H^2(P^2_k, E(-1)) = 0$. Thus $\dim_k H^0(P^2_k, E) \geq 3$ and therefore $H^0(P^2_k, E(-1)) = 0$ because $E^{\vee} = E(-1)$. Consequently we have $H^1(P^2_k, E(-1)) = 0$. Let $O_X(1)$ be the tautological linebundle of E on the P^1-bundle $\pi : X = P(E) \longrightarrow P^2_k$. Leray's spectral sequence $E_2^{p,q} = H^p(P^2_k, R^q\pi_*(O_X(1)) \otimes O_{P^2_k}(-1)) \Longrightarrow E^n = H^n(X, O_X(1) \otimes \pi^*(O_{P^2_k}(-1)))$ provides $H^1(X, O_X(1) \otimes \pi^*(O_{P^2_k}(-1))) = 0$ because $E_2^{1,0} = E_2^{1,0} = 0$. Let ℓ be a line in P^2_k.

$$0 \longrightarrow O_X(1) \otimes \pi^*(O_{P^2_k}(-1)) \longrightarrow O_X(1) \longrightarrow O_X(1) \otimes O_{\pi^{-1}(\ell)} \longrightarrow 0$$

is an exact sequence, which yields another exact sequence

$$0 \longrightarrow H^0(X, O_X(1) \otimes \pi^*(O_{P_k^2}(-1))) \longrightarrow H^0(X, O_X(1)) \longrightarrow$$

$$\begin{array}{cc} \| & \| \\ 0 = H^0(P_k^2, E(-1)) & H^0(P_k^2, E) \end{array}$$

$$H^0(\pi^{-1}(\ell), O_X(1) \otimes O_{\pi^{-1}(\ell)}) \longrightarrow H^1(X, O_X(1) \otimes \pi^*(O_{P_k^2}(-1))) = 0.$$

Therefore $H^0(X, O_X(1)) \cong H^0(\pi^{-1}(\ell), O_X(1) \otimes O_{\pi^{-1}(\ell)})$. Since $\pi^{-1}(\ell)$ is a rational ruled surface, this isomorphism implies that $|O_X(1)|$ has no fixed fibre (a fixed fibre of $|O_X(1)|$ is that of $|O_X(1) \otimes O_{\pi^{-1}(\ell)}|$, which can not occur). If D is a fixed component of $|O_X(1)|$, then $O_X(D)$ is a tautological linebundle and D contains no fibre, whence D is a section of $\pi : X \longrightarrow P_k^2$. Then E is decomposable, which is impossible because E is simple. Thus $|O_X(1)|$ has no fixed component. Hence if D_1, D_2 are general members of $|O_X(1)|$, they are irreducible and have no common fibre (note that $O_X(D_i)$ is a tautological linebundle on X). Let I_{D_i} be the defining ideal of D_i in X and let J be the ideal generated by I_{D_1} and I_{D_2}. Then the exact sequence

$$0 \longrightarrow I_{D_i} \longrightarrow J \longrightarrow J/I_{D_i} \longrightarrow 0$$

yields an isomorphism $\pi_*(J) \cong \pi_*(J/I_{D_i})$ of ideals of $O_{P_k^2}$ because $\pi_*(I_{D_i}) \cong \pi_*(O_X(-1)) = 0$, $R^1\pi_*(I_{D_i}) \cong R^1\pi_*(O_X(-1)) = 0$. Since J/I_{D_i} are locally principal O_{D_i}-ideal and $D_{i,U_i} \cong U_i$ for some open covering $U_1 \cup U_2$ of P_k^2, $\pi_*(J)$ is locally principal. This and $c_1(E) = 1$ imply that $\pi_*(J)$ defines a line . Thus $D_1 \cdot D_2 = C$ is irreducible and $\pi(C)$ is non-singular because $D_{i,U_i} \cong U_i$, $U_1 \cup U_2 = P_k^2$. C therefore satisfies the condition (E_0) and $\mathrm{elm}_C^0(X) \cong P_k^1 \times P_k^2$ by virtue of Proposition 1.9. Thus E is regular and $E \in R^2(P_k^2, \ell, P)$ for a point $P \in \ell$ because $c_2(E) = 1$. Then $E \cong T_{P_k^2}(-1)$ by virtue of Theorem 4.1.

q. e. d.

Example 4.8. As was shown in Example 3.11, $E \in R^2(P_k^2, C^3, Q_1+Q_2+Q_3)$ is simple if and only if Q_1, Q_2, Q_3 are not collinear. On the other hand, if $E \in R^2(P_k^2, C^3, Q_1+Q_2+Q_3)$, then $c_1(E) = 3$, $c_2(E) = 3$ and therefore $\Delta(E) = -3$. Thus $R^2(P_k^2, C^3, Q_1+Q_2+Q_3)$

consists only of one element $T_{P_k^2}$ if Q_1, Q_2, Q_3 are not collinear.

Let E be a vector bundle of rank r on P_k^n and let $j : P_k^1 \longrightarrow P_k^n$ be an embedding such that $j(P_k^1)$ is a line of P_k^n. By a famous theorem of Grothendieck we get $j^*(E) \cong O_{P_k^1}(a_1) \oplus \cdots \oplus O_{P_k^1}(a_r)$ $(a_1 \geq a_2 \geq \cdots \geq a_r)$. Let us consider the map $\alpha_E : j(P_k^1) \longrightarrow (a_1, \ldots, a_r)$ of $\mathrm{Grass}_n^1(k)$ to $\mathbf{Z}^{\oplus r}$.

Lemma 4.9. There is a non-empty open set $U(E)$ of Grass_1^n such that $\alpha_E(x)$ is a constant for every $x \in U(E)(k)$ and if $\alpha_E(y) = \alpha_E(x_0)$ for an $x_0 \in U(E)(k)$, then $y \in U(E)(k)$.

Proof. Let G be the universal quotient bundle on $X = \mathrm{Grass}_1^n$ and let $p : O_X^{\oplus(n+1)} \longrightarrow G$ be the canonical surjective homomorphism. Then we have the following diagram :

$$\begin{array}{ccc} P_k^n \times X = P(O_X^{\oplus(n+1)}) & \xhookleftarrow{\ i\ } & P(G) \\ \downarrow f' & & \downarrow f \\ P_k^n & & X \end{array}$$

Then $P(G)$, f, $g = f' \cdot i$ are nothing but the graph of the incidence correspondence between P_k^n and Grass_1^n and the natural projections respectively. Put $E'(m) = g^*(E(m))$. Since f is flat and $E'(m)$ is locally free $O_{P(G)}$-module, $x \longrightarrow \dim_{k(x)} H^0(f^{-1}(x), E'(m)_x)$ is upper semi-continuous on X. Since X is a noetherian space, $\dim_{k(x)} H^0(f^{-1}(x), E'(m)_x)$ is bounded. Thus $b_1 = \inf_{x \in X}$(the first term of $\alpha_E(x)) > -\infty$. Put $U_1 = \{x \in X \mid \dim_{k(x)} H^0(f^{-1}(x), E'(-b_1-1)_x) = 0\}$. Then U_1 is a non-empty open set of X by virtue of above argument. Similary $b_2 = \inf_{x \in U_1}$(the second term of $\alpha_E(x)) > -\infty$ and $U_2 = \{x \in U_1 \mid \dim_{k(x)} H^0(f^{-1}(x), E'(-b_2-1)_x) = b_1 - b_2\}$ is a non-empty open set of U_2. Inductively we get U_r and U_r is the desired open set of Grass_1^n.

q. e. d.

Definition (Schwarzenberger). A line contained in Grass_1^n - $U(E)$ is called an exceptional line of E.

Van de Ven showed that if $U(E) = \mathrm{Grass}_1^n$, E is rank 2 and if the characteristic of k is 0, then $E \cong O_{P_k^n}(a_1) \oplus O_{P_k^n}(a_2)$ or $T_{P_k^2}(a)$ (see [21]).

Theorem 4.10 (Schwarzengerger [18]). Let E be a non-simple vector bundle of rank 2 on P_k^2. The exceptional lines of E form a finite number of linear penciles. If E has no exceptional line, then E is decomposable.

Proof. Since the set of exceptional lines of $E \otimes L$ is nothing but that of E, we may assume that E is regular (Proposition 2.3). If E is defined by (s, s') of $H^0(C^n, O_{C^n}(D)) \times H^0(C^n, O_{C^n}(D))$ (C^n : non-singular curve of degree n in P_k^2), there are $u, v \in H^0(P_k^2, O_{P_k^2}(m))$ $(m \leq [\frac{n}{2}])$ such that u, v induce $s\bar{a}$, $s'\bar{a}$ on C^n $(a \in H^0(C^n, O_{P_k^2}(m) \otimes O_{C^n}(-D)))$ because E is not simple (Theorem 3.10). Let $\overline{\eta}_0$, $\overline{\eta}_1$ be the system of homogeneous coordinate of $P^1_{C^n}$ induced from a system homogeneous coordinate η_0, η_1 of $P^1_{P_k^2}$. Then E is defined by $Y : s\overline{\eta}_0 + s'\overline{\eta}_1 = 0$. Let A be the positive divisor on $P_k^1 \times P_k^2$ defined by $u\eta_0 + v\eta_1 = 0$, then $A \supset Y$. If A is reducible, then there are an irreducible component A_1 and a positive divisor C with $\deg C \leq m < n$ such that $A = A_1 + P_C^1$. Since $\deg C < n$ and Y is irreducible, $Y \subset A_1$. Thus we may

{ Since the self-intersection number of $B'_\ell = A' \cdot \pi^{-1}(\ell)$ is independent }

assume that A is irreducible. Hence $\mathrm{elm}^0_Y[A] = A'$ contains only a finite number of fibres $\pi^{-1}(x_1), \ldots, \pi^{-1}(x_r)$ of $\pi : X = P(E) = \mathrm{elm}^0_Y(P^1_k \times P^2_k) \longrightarrow P^2_k$. Take a general line ℓ_0 in P^2_k, then $P^1_{\ell_0} \cdot A = B_{\ell_0}$ is a section with $(B_{\ell_0}, B_{\ell_0}) = 2m \leq n$ and $P^1_{\ell_0} \cdot Y = P_1 + \ldots + P_n$. By virtue of Proposition 1.8 we have $\pi^{-1}(\ell_0) \cong \mathrm{elm}_{\{P_1, \ldots, P_n\}}(P^1_{\ell_0})$. Put $\mathrm{elm}_{\{P_1, \ldots, P_n\}}[B_{\ell_0}] = B'_{\ell_0}$, then $(B'_{\ell_0}, B'_{\ell_0}) = 2m - n = -b \leq 0$. Thus B'_{ℓ_0} is a minimal section of $\pi^{-1}(\ell_0)$. of the choice of a line ℓ in P^2_k, $A' \cdot \pi^{-1}(\ell_0) = B'_{\ell_0}$ and since $B'_\ell = B''_\ell$ + fibres (B''_ℓ : section), we get $(B''_\ell, B''_\ell) = -b_\ell \leq -b$, whence B''_ℓ is a minimal section of $\pi^{-1}(\ell)$ and $\pi^{-1}(\ell) \cong F_{b_\ell} = \mathrm{Proj}(O_{P^1_k} \oplus O_{P^1_k}(b_\ell))$. Since if $\alpha_E(\ell) = (a_1, a_2)$, then $a_1 + a_2 = n$, $a_1 - a_2 = b_\ell$ and since $B'_\ell \neq B''_\ell$ if and only if ℓ contains one of $x_1, \ldots, x_r$, we have the set of exceptional lines of $E = \bigcup_{i=1}^{r}$ {lines containing x_i}. Therefore the first assertion is proved.

If E has no exceptional line, then $r = 0$ and therefore A' is a section of $P(E)$. Thus E is an extension of line bundles. Since $H^1(P^2_k, L) = 0$ for any line bundle L, E is decomposable.

q. e. d.

We shall finish this section with some examples of exceptional lines.

Example 4.11. Schwarzenberger conjectured in [18] that if a vector bundle E of rank 2 on P_k^2 is simple, then the set of exceptional lines of E does not form a finite number of linear pencils. But his conjecture is disproved. In fact let C^3 be a non-singular cubic in P_k^2, let ℓ_0, ℓ_1 be lines in P_k^2 whose intersection is not on C^3 and let $\ell_i \cdot C_3 = P_{i1} + P_{i2} + P_{i3}$. Let s_i be an element of $H^0(P_k^2, O_{P_k^2}(1))$ with $|s_i| = \ell_i$ and let $\bar{s}_i$ be the element of $H^0(C^3, O_{P_k^2}(1) \otimes O_{C^3})$ induced from s_i. Then the regular vecter bundle E defined by $(\bar{s}_1{}^2, \bar{s}_2{}^2)$ is simple and the set of exceptional lines forms a linear pencil. •

Proof. Take $\bar{\eta}_0$, $\bar{\eta}_1$, η_0, η_1 as in the proof of Theorem 4.10. Then the positive divisor A defined by $s_0^2 \eta_0 + s_1^2 \eta_1 = 0$ contains $Y : \bar{s}_0{}^2 \bar{\eta}_0 + \bar{s}_1{}^2 \bar{\eta}_1 = 0$, and $A' = elm_Y^0[A]$ contains only one fibre $\pi^{-1}(x)$ with $x = \ell_0 \cdot \ell_1$ and $\pi : P(E) \longrightarrow P_k^2$. If ℓ is a line not containing x, it is easy to see that $A' \cdot (\pi^{-1}(\ell))$

$= B_\ell$ is a section of $\pi^{-1}(\ell)$ and $(B_\ell, B_\ell) = 1$. Since n is the minimum of self-intersection numbers of sections of $F_n = \mathrm{Proj}\,(O_{P^1_k} \oplus O_{P^1_k}(n))$ with non-negative self-intersection number, we know $\pi^{-1}(\ell) \cong F_1$. On the ther hand, if ℓ' is a line containing x, then $A' \cdot \pi^{-1}(\ell') = B'_{\ell'} + 2\,\pi^{-1}(x)$, B'_ℓ is a section of $\pi^{-1}(\ell')$ and $(B'_\ell, B'_\ell) + 4 = (B_\ell, B_\ell) = 1$. Thus $\pi^{-1}(\ell') \cong F_3$. We see therefore that the set of exceptional lines of E is the linear pencil formed by lines containing x. Since $E \in R^2(P^2_k, C^3, 2(P_{01} + P_{02} + P_{03}))$, E is simple by virtue of Corollary 3.10.1.

Proofs of the following examples are similar as above.

Example 4.12. i) Let ℓ be a line in P^2_k, let P be a point on ℓ and let $E \in R^2(P^2_k, \ell, nP)$. If $n = 1$, then $E \cong T_{P^2_k}(-1)$ and therefore E has no exceptional line. If $n > 1$, then E has only one exceptional line ℓ.

ii) Let C^n be a non-singular curve in P^2_k of degree n. Let D_0, D_1 be general conics in P^2_k and let $D_i \cdot C^2 = \sum_{j=1}^{4} P_{ij}$.

If s_0, s_1 are element of $H^0(C^2, O_{P_k^2}(2) \otimes O_{C^2})$ with $|s_i| = \sum_{j=1}^{4} P_{ij}$, then the set of exceptional lines of the regular vector bundle of rank 2 defined by (s_0, s_1) can not form a finite number of linear pencils.

iii) Let $E \in R^2(P_k^2, C^3, P_1+P_2+P_3)$. If P_1, P_2, P_3 is not collinear, then $E \cong T_{P_k^2}$ (Example 4.8) and therefore E has no exceptional line. If P_1, P_2, P_3 is collinear and if E is defined by $(s_0, s_1) \in H^0(C^3, O_{P_k^2}(1) \otimes O_{C^3}) \times H^0(C^3, O_{P^2}(1) \otimes O_{C^3})$ with $|s_i| = \sum_{j=1}^{3} Q_{ij}$, then the set of exceptional lines of E is the linear pencil formed by lines containing the point P_E which is the common point of lines ℓ_i $(i = 0, 1)$ going through Q_{i1}, Q_{i2}, Q_{i3}. Thus if $P_E \neq P_{E'}$, then $E \ncong E'$. Conversely it is easy to see that if $P_E = P_{E'}$ then $E \cong E'$. Thus $R^2(P_k^2, C^3, P_1+P_2+P_3)$ is in bijective correspondence with $P_k^2 - C^3$ if P_1, P_2, P_3 are collinear.

§ 3. Vector bundles on rational ruled surfaces.

A rational ruled surface over k is isomorphic to $\pi_n : F_n = \mathrm{Proj}(O_{P_k^1} \oplus O_{P_k^1}(n)) \longrightarrow P_k^1$ for some non-negative integer n. There is a section M on F_n with $(M, M) = -n$. If $n > 0$, then M is the unique irreducible curve with negative self-intersection number (see [13]). M is called a minimal section of F_n. Let N be a fibre (= a generator) of F_n. By virtue of the seesaw theorem, every divisor D on F_n is linearly equivalent to $aM + bN$, where $a = (D, N)$, $b = (D, M) + an$. On the other hand, $-2M - (n+2)N$ is a canonical divisor of F_n.

Lemma 4.13. Let E be a vector bundle of rank 2 on F_n. If $c_1(E) = aM + bN$ for $a \geq -2$, $b \geq -(n+2)$ and if E is simple, then $H^2(F_n, E) = 0$.

Proof. If one notes that $-2M - (n+2)N$ is a canonical divisor on F_n, the proof is similar to that of Lemma 4.2.

Lemma 4.14. If E is a simple vector bundle of rank 2 on F_n with $c_1(E) = aM + bN$, $c_2(E) = c$, then one of the following conditions is satisfied :

(1) Both a and b are even and $2ab - a^2n - 4c = -4r$ $(r \geq 2)$.

(2) Both a and b are odd and $2ab - a^2n - 4c = -n + 2 - 4r$

(4) a is odd, b is even and $2ab - a^2n - 4c = -n - 4r$ $(r \geq 1)$

($r \geq 1$ if $n = 0$; $r \geq 2$ if $n = 0$)

(3) a is even, b is odd and $2ab - a^2n - 4c = -4r$ $(r \geq 1)$

Proof. In the first place, note that $c_1(E) = aM + bN$, $c_2(E) = c$ imply $\Delta(E) = 2ab - a^2n - 4c$. The Riemann-Roch theorem asserts the following equality for a vector bundle of E' of rank 2 on F_n;

$$\chi(E') = 2 + \left((2M + (n+2)N)c_1(E') + \left(c_1(E')^2 - 2c_2(E')\right)\right)/2$$

1) Assume that both a and b are even, then $c_1(E \otimes \mathcal{O}_{F_n}(-(a/2)M - (b/2)N)) = 0$ and $\Delta(E) = \Delta(E \otimes \mathcal{O}_{F_n}((a/2)M - (b/2)N))$. Thus we may assume that $c_1(E) = 0$ and $c_2(E) = c$. For such an E we have $\chi(E) = 2 - c$. On the other hand, $H^2(F_n, E) = 0$ by virtue of Lemma 4.13. Thus $\dim_k H^0(F_n, E) \geq 2 - c$. Since $E^\vee \cong E$ and E is simple, we have $H^0(F_n, E) = 0$. Hence $c \geq 2$, which implies $\Delta(E) = -4c = -4r$ $(r \geq 2)$.

2) Assume that both a and b are odd. By a similar reason as above we may assume that $c_1(E) = M + N$, $c_2(E) = c$. Since $c_1(E^\vee) = -(M + N)$, $c_2(E^\vee) = c$, we have $\chi(E^\vee) = 1 - c$, $H^2(F_n,$

$E^\vee) = 0$, whence $\dim_k H^0(F_n, E^\vee) \geq 1 - c$. On the other hand, since $H^0(F_n, E^\vee) = H^0(F_n, E \otimes O_{F_n}(-M-N))$ is a linear subspace of $H^0(F_n, E)$ and since E is simple, we have $H^0(F_n, E^\vee) = 0$. Thus we get $c \geq 1$, which implies $\Delta(E) = (M + N)^2 - 4c = -n + 2 - 4c = -n + 2 - 4r$ $(r \geq 1)$. Moreover, in the case where $n = 0$ and $\Delta(E) = -2$, consider $E_1 = E \otimes O_{F_n}(-M)$ and $E_2 = E \otimes O_{F_n}(-N)$. Then $E_1^\vee = E_2$, $c_1(E_1) = N - M$, $c_1(E_2) = M - N$ and $c_2(E_1) = c_2(E_1) = 0$. Thus $\chi(E_1) = \chi(E_2) = 1$ and $H^2(F_n, E_1) = H^2(F_n, E_2) = 0$. Therefore $\dim_k H^0(F_n, E_1) > 0$, $\dim_k H^0(F_n, E_1^\vee) = \dim_k H^0(F_n, E_2) > 0$, which is impossible. Hence if $n = 0$, then $\Delta(E) = -4r + 2$ $(r \geq 2)$.

3) Assume that a is even and b is odd. Then we may assume that $c_1(E) = N$, $c_2(E) = c$. Since $\Delta(E) = -4c$ and since $\Delta(E) \leq -1$ by virtue of Corollary 4.3.1, (i), we have $\Delta(E) = -4r$ $(r \geq 1)$.

4) Finally assume that a is odd and b is even. Then we may assume that $c_1(E) = M$, $c_2(E) = c$. Since $c_1(E^\vee) = -M$, $c_2(E^\vee) = c$, we have $\chi(E^\vee) = 1 - c$, $H^2(F_n, E^\vee) = 0$, whence

$\dim_k H^0(F_n, E^\vee) \geq 1 - c$. On the other hand, since $\dim_k H^0(F_n, E^\vee) \leq \dim_k H^0(F_n, E)$ and E is simple, we have $H^0(F_n, E^\vee) = 0$. Thus $c \geq 1$, which implies that $\triangle(E) = M^2 - 4c = -n - 4r$ $(r \geq 1)$.

q. e. d.

Each of the conditions of the above lemma is sufficient for the existence of a vector bundle E of rank 2 on F_n with $c_1(E) = aM + bN$, $c_2(E) = c$. In fact,

Theorem 4.15.[16] There is a vector bundle E of rank 2 on F_n with $c_1(E) = aM + bN$, $c_2(E) = c$ if and only if one of the conditions (1), (2), (3), (4) of Lemma 4.14 is satisfied.

Proof. By virtue of Lemma 4.14 we have only to prove the "if" part.

1) Assume that the condition (1) is satisfied. Take a general member C of $|2M + (2n + 2)N|$, then C is a non-singular curve because $2M + (2n + 2)N$ is very ample. Let $P_1, \ldots, P_{n+2+r}$ $(r \geq 2)$ be sufficiently general points on C and put $D_r = \sum_{i=1}^{n+2+r} P_i$. Since

the genus of C is $n+1$ and D_r is general, $\dim|D_r| = r-1 \geq 1$ and $|D_r|$ is free from base point. Let $E_r \in R^2(F_n, C, D_r)$ be defined by $(s_0, s_1) \in H^0(C, O_C(D_r)) \times H^0(C, O_C(D_r))$ with $|s_0| = D_r$. Assume that E_r is not simple, then there are positive divisors C_1, C_2 on F_n such that $C_1 + C_2 \sim C \sim 2M + 2(n+1)N$ and $C_i \cdot C - D_r > 0$ (see Theorem 3.10). C_i is linearly equivalent to $a_iM + b_iN$ with $a_i \geq 0$, $b_i \geq 0$, $a_1 + a_2 = 2$, $b_1 + b_2 = 2n + 2$. If one of a_i, for instance a_1, is 0, then $b_1 \geq n+2+r$ because $C_1 = N_1 + \ldots + N_{b_1}$ for some fibres $N_1, \ldots, N_{b_1}$ of F_n and C_1 goes through $P_1, \ldots, P_{n+2+r}$. Thus $C_2 = 2M + N'_1 + \ldots + N'_{b_2}$ for some fibres $N'_1, \ldots, N'_{b_2}$ because $2M + b_2N = 2M +$ fibres if $b_2 < n$. Then C_2 cannot go through $P_1, \ldots, P_{n+2+r}$. We may assume therefore that $C_1 \sim M + b_1N$, $b_1 \leq n+1$. Since $\deg(C_1 \cdot C) = 2b_1 + 2$, we have $\dim|C_1 \cdot C| \leq n+3$ by virtue of the Riemann-Roch theorem on C and Clifford's theorem (which asserts that if D is a special divisor of degree n on a curve, then $2\dim|D| \leq n$). Thus C_1 cannot go through $P_1, \ldots, P_{n+2+r}$ if $P_1, \ldots, P_{n+2+r}$ are sufficiently general with $r \geq 2$. This is a contradiction.

Therefore E_r is simple. On the other hand, $\triangle(E_r) = -4r$. Thus $E = E_r \otimes O_{F_n}((a-2/2)M + (b/2 - n-1)N)$ is the desired vector bundle.

2) Assume that the condition (2) is satisfied. Take a general member C of $|M + mN|$, where $m = n$ or $n+1$ according as n is odd or not. Then C is a non-singular curve because C is a section of F_n. For a general positive divisor D_r of degree $r + (m-1)/2$ $(r \geq 1)$, construct a vector bundle $E_r \in R^2(F_n, C, D_r)$ as in the proof (1) above. Assume that E_r is not simple, then there are positive divisors C_1, C_2 on F_n such that $C_1 + C_2 \sim C \sim M + mN$ and $C_1 \cdot C - D_r > 0$. We may assume that $C_1 \sim M + b_1N$, $C_2 \sim b_2N$ with $b_1 + b_2 = m$, $b_1, b_2 > 0$. We have $b_2 \geq r + (m-1)/2$ because C_2 goes through every point of $\mathrm{Supp}(D_r)$. Thus $b_1 \leq (m+1)/2 - r$, whence $M + b_1N = M +$ fibres if $n \neq 0$. Thus if $n \neq 0$, C_1 cannot go through every points of $\mathrm{Supp}(D_r)$ because $b_1 < r + (m-1)/2 = \deg D_r$. This is a contradiction. Therefore E_r is simple if $n \neq 0$, $r \geq 1$. On the other hand, if $n = 0$, then

$(C, C) = 2$ and therefore E_r is simple for any $r \geq 2$ by virtue of Corollary 3.10.1. Since $\Delta(E_r) = -n + 2 - 4r$, $E_r \otimes O_{F_n}((a-1/2)M + (b-m/2)N) = E$ is the desired vector bundle.

3) Assume that the condition (3) is satisfied. Let P be a point on N, then $R^2(F_n, N, rP) \neq \phi$ for any $r \geq 1$ because N is a non-singular rational curve. Since $(N, N) = 0$, an element E_r of $R^2(F_n, N, rP)$ is simple by virtue of Corollary 3.10.1. Thus $E = E_r \otimes O_{F_n}((a/2)M + (b-1/2)N)$ is the desired vector bundle.

4) Assume that the condition (4) is satisfied. Let P be a point of M, then $R^2(F_n, M, rP) \neq \phi$ for any $r \geq 1$ because M is a non-singular rational curve. Since $(M, M) = -n \leq 0$, and element E_r of $R^2(F_n, M, rP)$ is simple by virtue of Corollary 3.10.1. Thus $E = E_r \otimes O_{F_n}((a-1/2)M + (b/2)N)$ is the desired vector bundle.

q. e. d.

As an example let us consider the family of simple vector bundles of rank 2 with $\Delta(E) = -4$ on F_n.

Theorem 4.16. Let $S(n, a, b)$ be the set of isomorphism classes of simple vector bundles E of rank 2 on F_n with $c_1(E) = aM + bN$, $\Delta(E) = -4$. If (1) a is even, b is odd and $n \neq 0$ or (2) one of a and b is odd, the other is even and $n = 0$, then there is a bijective map $\varphi_{n,a,b} : P^1(k) \longrightarrow S(n, a, b)$. Moreover, there is a vector bundle $\widetilde{S}(a, b)$ on $F_0 \times P^1$ such that $\widetilde{S}(a, b)_x = \varphi_{0,a,b}(x)$ for any $x \in P^1(k)$.

Proof. First of all, note that if n, a, b satisfy the above conditions then $S(n, a, b) \neq \phi$ by virtue of Theorem 4.15. Since $F_0 = P^1 \times P^1$ and since $M = P^1 \times Q$, $N = R \times P^1$ for some $Q, R \in P^1$, we may assume that a is even and b is odd even if $n = 0$. Take an $E' \in S(n, a, b)$ and let us consider $E = E' \otimes O_{F_n}(-(a/2)M - (b-1/2)N)$. Then $c_1(E) = N$ and $c_2(E) = 1$. Since $\chi(E) = 2$, $\chi(E^\vee) = 0$, $H^2(F_n, E) = H^2(F_n, E^\vee) = 0$, we have $\dim_k H^0(F_n, E) \geq 2$ and therefore $H^0(F_n, E \otimes O_{F_n}(-N)) = H^0(F_n, E^\vee) = 0$, $H^1(F_n, E \otimes O_{F_n}(-N)) = H^1(F_n, E^\vee) = 0$. By a similar argument as in the proof of Theorem 4.7 we have $H^0(X, O_X(1)) \cong H^0(\pi^{-1}(N), O_X(1) \otimes$

$O_{\pi^{-1}(N)}$) with the tautological linebundle $O_X(1)$ of E on $\pi: X = P(E) \longrightarrow F_n$. Since $\pi^{-1}(N)$ is a rational ruled surface and since the fibre N is chosen arbitrarily in the above argument, the above isomorphism implies that $|O_X(1)|$ has no fixed fibre. Thus if D is the fixed component of $|O_X(1)|$, then D is a section of $\pi : X \longrightarrow F_n$, whence E is an extension of linebundles;

$$0 \longrightarrow O_{F_n}(aM + bN) \longrightarrow E \longrightarrow O_{F_n}(a'M + b'N) \longrightarrow 0.$$

Since $c_1(E) = (a + a')M + (b + b')N = N$, $c_2(E) = -aa'n + ab' + b'b = 1$, we obtain $a = 1$, $a' = -1$, $b = n/2$, $b' = 1 - n/2$ or $a = -1$ $a' = 1$, $b = 1 - n/2$, $b' = n/2$. Thus n is even, If $a = 1$, then $n = 0$ because $H^0(F_n, O_{F_n}(aM + (b - 1)N)) \subseteq H^0(F_n, E \otimes O_{F_n}(-N)) = 0$. Since $H^1(F_0, O_{F_0}(2M - N)) = 0$ by virtue of the Riemann-Roch theorem, the above extension splits in this case, and we obtained a contradiction. If $a = -1$, then $n = 0$ also because $H^0(F_n, O_{F_n}(a'M + (b'-1)N)) \subseteq H^1(F_n, O_{F_n}(aM + (b-1)N))$ and because $\dim_k H^0(F_n, O_{F_n}(a'M + (b'-1)N)) = n/2$, $\dim_k H^1(F_n, O_{F_n}(aM + (b-1)N)) = 0$. On the other hand, since the exact sequence

$$0 \longrightarrow O_{F_0}(N - M) \longrightarrow E \longrightarrow O_{F_0}(M) \longrightarrow 0$$

provides $\dim_k H^0(X, O_X(1)) = \dim_k H^0(F_0, E) = 2$ and since $O_X(1) = O_X(D) \otimes \pi^*(L)$ for some linebundle L on F_0, we have $O_X(1) \cong O_X(D + \pi^{-1}(M))$ or $O_X(D + \pi^{-1}(N))$. But, in any case, $\pi(D_1 \cdot D_2) \not\sim N$ for D_1, D_2 with $O_X(1) \cong O_X(D_1)$ because $\pi(D_1 \cdot D) \sim M$. This contradicts the fact that $c_1(E) = N$. We see therefore that $|O_X(1)|$ has no fixed component. Then by a similar argument as in the proof of Theorem 4.7, for any gaeneral members D_1, D_2 in $|O_X(1)|$, we see that $D_1 \cdot D_2 = Y$ is a non-singular curve satisfying the condition (E_0) such that $\pi(Y) =$ (a fibre N_1 of F_n). Thus $\mathrm{elm}_Y^0(P(E)) \cong P_k^1 \times F_n$ and $E \in R^2(F_n, N_1, p)$ for a point $p \in N_1$. Conversely every element of $R^2(F_n, N_1, p)$ is simple because $(N_1, N_1) = 0$. Since $\dim_k H^0(N_1, O_{N_1}(p)) = 2$, $R^2(F_n, N_1, p)$ consists only of one element by virtue of Theorem 2.14. Moreover, if N_1, N_2 are mutually distinct fibres of F_n, then $E_1 \not\cong E_2$ for $E_i \in R^2(F_n, N_i, \mathfrak{p}_i)$ $(i = 1, 2)$ by virtue of Theorem 2.13. Thus there is a bijective map Ψ : {fibres of F_n} $\longrightarrow S(n, 0, 1)$. Since F_n is a rational ruled surface, there is a canonical bijective map $\psi : P^1(k) \longrightarrow$ {fibres of F_n}. Therefore we obtain a bijective map $\varphi_{n,0,1} = \Psi \cdot \psi : P^1(k) \longrightarrow S(n, 0, 1)$. Since $S(n, a, b) = \{E \otimes O_{F_n}((a/2)M + (b-1/2)N) \mid E \in S(n, 0, 1)\}$, we obtain a bijective map $\varphi_{n,a,b} : P^1(k) \longrightarrow S(n, a, b)$.

In order to prove the last assertion, consider $Z = P^1_k \times P^1_k \times P^1_k$ whose system of coordinates is $(z_0^{(1)}, z_1^{(1)} ; z_0^{(2)}, z_1^{(2)} ; z_0^{(3)}, z_1^{(3)})$. Let Y be the subvariety defined by $z_0^{(1)} z_0^{(2)} + z_1^{(1)} z_1^{(2)} = 0$ and let $j : Z \longrightarrow P^1_k \times F_0 \times P^1_k = P^1_k \times P^1_k \times P^1_k \times P^1_k$ be the closed immersion defined by $j((x, y, z)) = (x, y, z, z)$. Then $j(Y) = Y'$ is a subvariety of P^1-bundle $P^1_k \times F_0 \times P^1_k \longrightarrow F_0 \times P^1_k$ satisfying the condition (E_0). Let $\widetilde{S}(0, 1)$ be the regular vector bundle on $F_0 \times P^1$ defined by Y', then it is clear that for any even integer a and odd integer b, $\widetilde{S}(a, b) = \widetilde{S}(0, 1) \otimes p^*O_{F_0}((a/2)M + (b-1/2)N)$ is the desired vector bundle with the natural projection

$$p : F_0 \times P_k^1 \longrightarrow F_0.$$

q. e. d.

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Footnotes.

1) In fact $g_*(O_{\tilde{X}}) = O_X$, and so $g_*(I_{\overline{X}_T})$ is an ideal of O_X, (see Lemma 1.5.).

2) Let $\pi : X \longrightarrow S$ be the projective bundle $P(E)$ associated with a vector bundle E of rank $N + 1$ $(N \geq 1)$. A linebundle L on X is, by abuse of language, called a tautological linebundle when L is the tautological linebundle of a vector bundle E' with $P(E') = X$. In the case where S is reduced, L is a tautological linebundle if and only if $L_s = L \otimes_{O_S} k(s)$ is the linebundle associated with the hyperplane of $\pi^{-1}(s) = P^N_{k(s)}$ for any $s \in S$. If L_1, L_2 are tautological linebundles on X, then there is a linebundle M on S such that $L_1 = L_2 \otimes \pi^{-1}(M)$.

3) The direct proof of this fact is easy. But geometric interpretation of this (i.e. the relation between Theorem 1.1 and Theorem 1.3) is very important.

4) For an affine scheme $Z = \mathrm{Spec}(B)$ and $b \in B$, $Z(b)$

denotes $\mathrm{Spec}(B_b)$.

5) Note that locally this complex $K_\bullet$ is isomorphic to the usual Koszul comples defined by $h_0, \ldots, h_N$ with a local equation h_i of H_i. Note also that if $K_\bullet(f_1, \ldots, f_r)$ is the Koszul complex defined by elements $f_1, \ldots, f_r$ of A and if $\{f_1, \ldots, f_r\}$ contains a unit element, then $H_i(K_\bullet(f_1, \ldots, f_r) \otimes M) = 0$ ($\forall i > 0$) for every A-module M.

6) As a matter of fact α is an immersion.

7) If $\dim S = 1$, then the theory in the sequal is trivial because we assume that Y is irreducible (cf. Remark 2.16)

8) Of course, $C_1 \supset Y_1$ means that the support of C_1 contains Y_1.

9) Very ample in the sense of Sumihiro : A vector bundle E on S is called very ample if the tautological linebundle of E on $P(E)$ is very ample in the sense of Grothendieck. H. Sumihiro proved the following; (i) For any vector bundle E there is a linebundle L such that $E \otimes L$ is very ample if S is projective

(see Lemma 1.11). (ii) $E = E_1 \oplus E_2$ is very ample if and only if both E_1 and E_2 are very ample. (iii) If $f : E \longrightarrow E'$ is a surjective homomorphism of vector bundles and if E is very ample, then E' is very ample. (iv) If E is ample in the sense of Hartshorne, then there is an integer n_0 such that $S^n(E)$ is very ample for any $n \geq n_0$ ($S^n(E)$ is the symetric tensor product of grade n). (v) If E is very ample, then E is generated by its global sections and the morphism $g : S \longrightarrow \text{Grass}$ defined by E is a closed immersion.

10) In the next section we shall show that $c_1(E) = T$, $c_2(E) = D$ for $E \in R^r(S, T, D)$.

11) This means that $f_0, \ldots, f_N$ form a basis of $H^0(X(Y), O_{X(Y)}(H_0'))$ if $(f_i) = H_i' - H_0'$.

12) In the next chapter we shall show that $SR^r(S, T, D)$ consists of all simple vector bundles in $R^r(S. T. D)$.

13) Note $\sum_{\substack{a+b=c \\ a \geq 0, b \geq 0}} (-1)^{b-1}/a!b! = (-1/c!) \sum_{b=0}^{c} (-1)^b \binom{c}{b} = (-1/c!)(1-1)^c = 0.$

14) For divisors D_1, D_2 on a non-singular surface, (D_1, D_2) denotes the intersection number of D_1, D_2.

15) In [19] Schwarzenberger says that there is a simple vector bundle E of rank 2 with $c_1(E) = n$, $c_2(E) = m$ if $n^2 - 4m < 0$. But this is not true as we have shown. His error comes from an incorrect statement (b) in the proof of his Theorem 7.

16) In [19] Schwarzenberger says without proof that for any a, b, c with $ab - 2c < 0$ there is a simple vector bundle E of rank 2 on F_0 with $c_1(E) = aM + bN$, $c_2(E) = c$. But this is not true (see the above conditions (1), (2)).

17) A simple vector bundle E with $\Delta(E) = -4$ which does not satisfy these conditions exists only on F_2 (see Theorem 4.15).