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主論文

Spontaneous Emission of Plasma Waves in the Presence of a Finite Amplitude Wave

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~~PART I~~

Introduction and Summary

In 1929, Langmuir and Tonks found a systematic oscillation in a plasma and showed that its angular frequency is given by $\omega_p=((4\pi n_0 e^2)/m)^{1/2}$, where m, e and n_0 are the mass, charge and average density of the particle. Since then, this oscillation has been studied in many aspects both experimentally and theorectically. The first theoretical work is the one by Vlasov[1], who proposed to describe wave processes in a rarefied plasma by means of a kinetic equation with a self-consistent field, starting from the collisionless Boltzmann equation and the Poisson equation. After his theory, Landau pointed out the mathematical error of his treatment and then theoretically predicted the phenomenon of Landau damping[2]. Both treatments are based on the theory linear with respect to the fluctuation field. If we go beyond the linear theory, however, those particles whose velocities are close to the phase velocity of the wave continue to be accelerated (or decelerated) by the wave field and show a secular behavior in time. As a result, there is a time above which the linear theory becomes invalid. Let us consider two time scales. One is the Landau damping time of the wave, (to be referred to as τ_D) and the other is the period of bouncing motion of a particle in the bottom of potential trough of the wave (referred as τ_β).

The latter time scale is inversely proportional to the square root of the field strength of the wave. When the amplitude of the wave field is sufficiently small, so that $\tau_D \ll \tau_\beta$, we can use the linear theory in respect of the interaction between the wave and particles. As the amplitude of the wave becomes larger and τ_β gets nearly equal or larger than τ_D, the nonlinear coupling of the wave and particles begins to play important roles. Such a situation was first discussed by Bohm and Gross[3]. They divided the particles into two groups: one is composed of the particles which are trapped in the potential-energy troughs of the wave and which are necessary to be treated nonlinearly, and the other is composed of those which are untrapped. These authors have studied the relation between the distribution of the two groups of particles and the stationary character of the wave. As an extension of their work, a theory is developed by Bernstein, Greene and Kruskal[4] who discussed a stationary exact solution of the nonlinear Vlasov equation. The stability of the solution was not discussed, however.

There are other nonlinear theories of plasmas, for instance, the weak turbulence theory, the orbit-modification theory, and soon. We shall not refer to these theories here.

Experimentally, a considerable progress has been made in plasma physics since 1955 and we are now able to

produce very quiescent plasmas in laboratories and also to excite waves of arbitrary frequencies by applying high frequency oscillations to a Langmuir probe. As a result the dispersion relation for plasma waves have been studied in detail and many of the results of the linear theory have been experimentally checked. In particular, the phenomenon of Landau damping has been observed by Malmberg, Wharton and Drummond[5]. Further progress in experimental techniques has made it possible to excite a large-amplitude coherent wave for which the relation $\tau_B > \tau_D$ holds. The phenomena due to the presence of trapped particles have been widely studied both experimentally and theoretically. Among them, the phenomena of the amplitude oscillation and the trapped-particle instability are noticeable. The former is a phenomenon that the amplitude of a large-amplitude wave (hereafter referred to as the carrier wave) excited in a plasma oscillates in space with a period approximately equal to $v_p \tau_B$ (v_p is the carrier phase velocity) and was observed by Malmberg and Wharton[6] for the first time. The particles trapped in the potential troughs of the wave make the bouncing motion and, therefore, they periodically exchange their energies with the carrier wave. It is the physical reason for this phenomenon. Theoretically, it has been investigated by O'Neil[7] and by Al'thul and Karpman[8] using different methods. The trapped particle

instability is a phenomenon in which waves of frequencies different from the carrier-wave frequency by approximately integral multiples of the bounce frequency $2\pi/\tau_B$ (i.e. sideband waves to the carrier) become unstables, and has been observed by Wharton[7] et al. This phenomenon is due to the energy transfer from the trapped particles bouncing in the periodic potential troughs of the carrier wave to sideband waves via resonance coupling and it has been studied theoretically by Krüer-Dawson-Sudan[18] and Mima-Nishikawa[19]. This problem has raised a question regarding the stability of the B-G-K solution mentioned above.

The above problems are all concerned with the phase coherent response of a plasma to the electric field. In plasmas, however, there also exists spontaneous emission of plasma waves. It comes from the discreteness of the plasma, namely, charged particles moving in the dielectric medium emit collective oscillations by Cerenkov emission. The fluctuation level in a thermal-equilibuium plasma is derivable from a balance between the rate of spontaneous emission and that of Landau damping. This corresponds to the fluctuation dissipation theorem. In a homogeneous plasma, the emission rate can be calculated by making use of a particle orbit of free motion and a dielectric function derived by the linear theory. In an inhomogeneous system where a large-amplitude

coherent wave is propagating, however, both of the particle orbit and the dielectric character of the plasma are modified by the existence of the carrier wave. The principal subject of this work is to develop a theory of the spontaneous Cerenkov emission in which the effects of the carrier wave mentioned above are taken into account.

The main motivations of this work are the following. First, the theory of a moderately strong ion-wave turbulence given by Nishikawa and Wu[12] indicates that in collisionless plasmas, particle-trapping (discussed in a statistical sense) due to low-level ion-wave fluctuations could be significant even for stable plasmas. According to their considerations, it may be necessary to take into account the effect of electron-trapping for the spontaneous emission even in the discussion of the thermal level of ion-wave fluctuations. The second motivation is connected with the work on the anomalous resistivity by Dupree[13], Kadomtsev and Pogutse[14] and others.[15] According to the theory of Balescu[16], Lenard[17], and Guernsey[18] on the collisions between the particles and waves in a plasma, there exists an effect of the slowing down of the particles due to the reaction of the medium against the spontaneous Cerenkov emission by the particles. Dupree and others proposed that this slowing-down effect would become much larger if we take into account the effect of coherent emission due to the localization of the

particles in phase-space (Dupree called it "macroparticle"), and consequently, its effect would give rise to an anomalous effective collision frequency on the particles. Following their proposal, we want to consider the influence of the carrier wave on the spontaneous Cerenkov emission and the coherence effect of this emission due to the existence of the carrier wave. Finally, although the experimentally observed sideband excitation in the presence of a large-amplitude wave is considered to be explained by the theory of trapped particle instabilities, there also exists a possibility of spontaneous emission of sideband waves as a result of a periodic modulation of the particle orbits. For a complete comparison of the theory with experiment, it is necessary to clarify a qualitative difference between the sideband excitation via a trapped particle instability and that via spontaneous emission.

Let us present a brief summary of the results obtained in this work. In the case of the emission of an untrapped particle, waves of many different frequencies are emitted. They include the one which satisfies the ordinary Cerenkov condition, but its emission rate is weakened as compared with the emission by a freely-moving particle. Also included is an emission corresponding to the scattering of the carrier wave by the particle (for example, the Thomson scattering). A trapped particle emits sideband waves in addition to the carrier wave itself. The frequencies of

the sideband waves are different from the carrier-wave frequency by approximately integral multiples of the bounce frequency and the spectrum of these emissions in the wave-number space becomes discrete and symmetric around the wave-number of the carrier wave. The effect of the presence of the carrier wave on the dielectric property of the system appears mainly in the emission by the trapped particle. Namely, the dispersion relation and emission rate of the sideband waves are modified in comparison with the result of the primary treatment where this effect is neglected. For the emission by an aggregate of trapped particles, there appears a coherence effect because of the localization of particles in the potential trough and also because of a periodic array of groups of localized particles which reflects the periodicity of the carrier-wave potential. These effects occur only for the emission of the carrier wave itself.

We discuss the problem of spontaneous emission of plasma waves in the presence of a finite-amplitude wave, dividing the consideration into two parts. In the first, the theory of the spontaneous emission in a homogeneous plasma is briefly reviewed. We also refer to a theory[19] which takes into account only the effect of the orbit modification, but neglects the effect of the carrier wave on the dielectric character of the plasma. Such a theory may be used in the case when the amplitude of the carrier

wave is sufficiently small. In the second part, we develop a theory which takes into account the effect of the carrier wave on the dielectric character of the plasma as well as on the particle orbit. The theory is based on the Klimontovich formalism[20] and is an extension of the theory of spontaneous emission in a uniform plasma to an inhomogeneous plasma.

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Part I

§1. Spontaneous Emissions in Plasmas

First of all, let us derive a general expression for the rate of spontaneous Cerenkov emission of longitudinal waves per unit time by a single particle. This rate is equal to the work done by the medium as a reaction to the wave emission by the particle because of the energy conservation in the process of the emission. Therefore, the rate of wave emission, denoted by I_σ, is given by

$$I_\sigma = -\int d\vec{r}\ \vec{j}_\sigma(r,t)\cdot\vec{E}_\sigma(r,t) \quad , \tag{1.1}$$

where σ represents the species of the particle, and $\vec{j}_\sigma$, $\vec{E}_\sigma$ are the current density corresponding to the particle motion and the electric field due to the particle, respectively. The integration is carried out over the whole plasma. Let us Fourier-analyse in space and time as

$$\vec{j}_\sigma(\vec{r},t) = \int d\vec{k}\ d\omega\ \vec{j}_\sigma(\vec{k},\omega)\ e^{i\vec{k}\cdot\vec{r}-i\omega t}$$

and (1.2)

$$\vec{E}_\sigma(\vec{r},t) = \int d\vec{k}\ d\omega\ \vec{E}_\sigma(\vec{k},\omega)\ e^{i\vec{k}\cdot\vec{r}-i\omega t} \quad .$$

Then, I_σ is written in the Fourier representation as

$$I_\sigma = -(2\pi)^3 \int d\vec{k}\, d\omega d\omega'\, \vec{E}_\sigma(\vec{k},\omega)\cdot\vec{j}_\sigma(-\vec{k},\omega')\, e^{-i(\omega+\omega')t} \quad . \qquad (1.3)$$

By means of the test-particle method, we can relate $\vec{j}_\sigma$ and $\vec{E}_\sigma$ with the charge density ρ_σ corresponding to the moving particle. The Poisson equation in the Fourier representation is given by

$$i\vec{k}\cdot\vec{D}_\sigma(\vec{k},\omega) = 4\pi\rho_\sigma(\vec{k},\omega) \quad , \qquad (1.4)$$

where $\vec{D}_\sigma$ is the corresponding electric displacement. The electric displacement is related to the electric field as

$$\vec{D}_\sigma(\vec{k},\omega) = \overleftrightarrow{\epsilon}^e(\vec{k},\omega)\, \vec{E}_\sigma(\vec{k},\omega) \qquad (1.5)$$

where $\overleftrightarrow{\epsilon}^e(\vec{k},\omega)$ is the dielectric tenser. In this treatment, however, we use the dielectric function for a homogeneous and isotropic plasma since we suppose that the amplitude of the carrier wave is small enough for its effect on the dielectric property of the system to be negligible. From the equation of continuity, $\vec{j}_\sigma(\vec{k},\omega)$ is given by

$$\vec{k}\cdot\vec{j}_\sigma(\vec{k},\omega) = \omega\rho_\sigma(\vec{k},\omega) \qquad (1.6)$$

By making use of eqs.(1.3)~(1.6), the expression for I_σ is rewritten as

$$I_\sigma = -2(2\pi)^4 i \int d\vec{k}d\omega d\omega' \frac{\omega' \rho_\sigma(\vec{k},\omega)\rho_\sigma(-\vec{k},\omega')}{k^2 \epsilon^\ell(\vec{k},\omega)} e^{-i(\omega+\omega')t} \quad . \tag{1.7}$$

We apply eq.(1.7) to the case of a homogeneous unmagnetized plasma. In this case, we may adopt the orbit of free motion for the test particle. The charge density corresponding to this motion is

$$\rho(r,t) = e\delta(\vec{r}-\vec{v}_0 t) \quad , \tag{1.8}$$

$\vec{v}_0$ being the unperturbed velocity of the particle. Substituting the Fourier component of this quantity into eq.(1.7), we obtain the emission rate of the particle with the velocity $\vec{v}_0$ as

$$I(\vec{v}_0) = -\frac{4\pi e^2}{(2\pi)^3} \int d\vec{k} \frac{\vec{k}\cdot\vec{v}_0}{k^2} \mathrm{Im} \frac{1}{\epsilon^\ell(\vec{k},\omega)} \quad . \tag{1.9}$$

This expression can approximately be written as

$$I(\vec{v}_0) = \frac{e^2}{(2\pi)} \int d\vec{k} \frac{\omega_k}{k^2 \left(\frac{\partial \epsilon^e}{\partial \omega}\Big|_{\omega=\omega_k}\right)} \delta(\vec{k}\cdot\vec{v}_0 - \omega_k) \quad , \tag{1.10}$$

where ω_k satisfies the equation $\mathrm{Re}\epsilon^e(\vec{k},\omega_k)=0$.

§2. Effects of a Finite Amplitude Wave

We consider the spontaneous Cerenkov emission in an unmagnetized plasma in the presence of a finite-amplitude wave. For simplicity, we confine ourselves to a one-dimensional treatment. The equation of motion for the test-particle is

$$m \frac{d^2x}{dt^2} = -eE_0 \sin(k_0 x - \Omega_0 t) \quad , \tag{2.1}$$

where $E_0 \sin(k_0 x - \Omega_0 t)$ represents the carrier wave with the amplitude E_0, the frequency Ω_0 and the wave number k_0. Considering this equation in the wave frame, we find that the total energy of the particle W is conserved in this frame and the whole particles can be divided into two groups according to their total energies; one is a group of untrapped particles whose total energies are larger than eE_0/k_0 and the other is a group of trapped ones whose total energies are smaller than eE_0/k_0. Therefore, we separate that problem into two cases as follows

(i) Untrapped-particle case:

For simplicity, we consider only those particles whose kinetic energies are much larger than their potential energies. We then adopt the following orbit for the test-particle in the laboratory frame:

$$x(t) = \alpha \sin(k_0 x - \Omega_0 t) + v_0 t \quad , \tag{2.2}$$

where x_0 and v_0 are the initial position and velocity of the test particle and

$$\alpha \equiv \frac{\frac{e}{m} E_0}{(k_0 v_0 - \Omega_0)^2}$$
$$\Omega_D \equiv \Omega_0 - k_0 v_0 \qquad (2.3)$$

This orbit is correct to the first order with respect to the expansion parameter $eE_0/k_0 W$. The charge density corresponding to this motion is given by

$$\rho(x,t) = e\delta(x - v_0 t - \alpha \sin(k_0 x_0 - \Omega_D t)) \quad . \qquad (2.4)$$

Substituting the Fourier component of this quentity into eq.(1.7) and averaging over the initial position, we obtain the emission rate by the untrapped particle after some calculations as

$$I^{WT}(v_0) = 2\pi e^2 \sum_n \int dk \frac{\omega_k}{k^2 (\frac{\partial \varepsilon^e}{\partial \omega}|_{\omega=\omega_k})} J_n^2(k\alpha)\delta(\omega_k - kv_0 - n\Omega_D) \quad , \qquad (2.5)$$

where J_n is the Bessel function. In the above expression, the term $n=0$ shows that the wave which satisfies the ordinary Cerenkov condition is emitted but its rate is reduced by a factor $J_0^2(k\alpha)$. This reduction is due to the fact that the particle velocity can be equal to the

phase velocity of the modulation of the particle orbit by the wave field. The term $n=1$ corresponds to the Thomson scattering of the wave (k_0,Ω_0) by the particle of velocity v_0 as seen by the resonance condition $\omega_k - \Omega_0 = (k - k_0)v_0$.

(ii) Trapped-particle case:

For simplicity we confine our treatment only to those particles which are trapped in the bottom of the potential trough. We obtain the orbit for such a particle perturbationally as

$$x(t) = V_0(t) + \frac{1}{\omega_B}\sqrt{\frac{2W}{m}}\sin(\omega_0 t + \beta) \quad , \tag{2.6}$$

where V_0 is the phase velocity of the carrier wave, ω_B is the bounce frequency in the bottom of the potential trough and β is the initial phase of the particle. The above expression is correct to the first order with respect to the expansion parameter k_0W/eE_0 . The corresponding charge density is given by

$$\rho(x,t)=e\delta\left(x-V_0t-\frac{1}{\omega_B}\sqrt{\frac{2W}{m}}\sin(\omega_B t+\beta)\right) \quad . \tag{2.7}$$

The same procedure as done for the case of the untrapped particle brings the final expression for the emission rate of the trapped particle with the total energy W to the form:

$$I^T(W)=2\pi e^2\sum_n\int dk\frac{\omega_k}{k^2(\frac{\partial\epsilon^e}{\partial\omega}|_{\omega=\omega_k})}J_n^2(\frac{k}{\omega_B}\sqrt{\frac{2W}{m}})\delta(\omega_k-kV_0-n\omega_B)\ . \quad (2.8)$$

In this case, the term $n=0$ corresponds to the emission under the ordinary Cerenkov condition. The emission rate is, however, reduced by a factor $J_0^2(\frac{k}{\omega_B}\sqrt{\frac{2W}{m}})$ compared with that of a freely moving particle. The resonance condition for the term $n\neq0$ shows that the trapped particles also emit side-band waves whose frequencies are different from the carrier-wave frequency by approximately integral multiples of ω_B and consequently, the spectrum of emission becomes discrete in k space. These situations reflects the bouncing motion of the particle in the potential well. If we treat the motion of trapped particles more exactly, however, the bounce frequency is no longer independent of the particle energy and the result mentioned above would be modified correspondingly. We neglect the particles which exist near the boundary between the trapped and untrapped regions. We find, however, from the equation of motion that these particles spend almost all of their time near the top of the potential with the velocity nearly equal to the phase velocity of the carrier wave. Therefore, they will emit the carrier wave on the average as a freely moving particle with the velocity V_0 does.

§3 Coherence effect

In the present system the trapped particles are localized to each potential trough because of the existence of the carrier wave. Consequently, the coherence effect can occur for the case of the emission by an aggregate of trapped particles. There exists another coherent effect due to a periodic array of groups of localized particles which reflects the periodicity of the carrier-wave potential. Let us consider these coherence effects in the following. We specify the trapped particle with a pair of suffixes (J,p), which denotes the p-th particle trapped in the J-th potential trough. The Fourier component of the charge density corresponding to the particle (J,p) is

$$\rho^{(J,p)}(k,\omega) = \frac{e}{2\pi} \sum_n J_n\left(\frac{k}{\omega_B}\sqrt{\frac{2W(J,p)}{m}}\right) e^{-in\beta(J,p)} \delta(\omega - kV_0 - n\omega_B)\, e^{-ikx_J} \tag{3.1}$$

where $W(J,p)$ and $\beta(J,p)$ are the total energy and initial phase of the particle (J,p) in the wave frame respectively and x_J is the initial position of the bottom of the J-th potential trough. The charge density corresponding to an aggregate of trapped particles is obtained by summing eq.(3.1) over (J,p). Making use of eq.(1.7), we get the expression for the emission rate by all of the trapped particles as

$$I^T=2\pi e^2\Big\{\sum_{(J,p)}\sum_n\int dk\frac{\omega_k}{k^2(\frac{\partial\epsilon^e}{\partial\omega}|_{\omega=\omega_k})}J_n^2(\frac{k}{\omega_B}\sqrt{\frac{2W(J,p)}{m}})\delta(\omega_k-kV_0-n\omega_B)$$

$$+\sum_{\substack{J\\p\neq s}}\int dk\frac{\omega_k}{k^2(\frac{\partial\epsilon^e}{\partial\omega}|_{\omega=\omega_k})}J_0(\frac{k}{\omega_B}\sqrt{\frac{2W(J,p)}{m}})\ J_0(\frac{k}{\omega_B}\sqrt{\frac{2W(J,s)}{m}})\delta(\omega_k-kV_0)$$

$$+\sum_{\substack{J\neq L\\p,s}}\int dk\frac{\omega_k}{k^2(\frac{\partial\epsilon^e}{\partial\omega}|_{\omega=\omega_k})}J_0(\frac{k}{\omega_B}\sqrt{\frac{2W(J,p)}{m}})\ J_0(\frac{k}{\omega_B}\sqrt{\frac{2W(L,s)}{m}})$$

$$\delta(\omega_k-kV_0)\ e^{-ik(x_J-x_L)}\Big\}$$

Let us discuss each term of the right hand side of the above expression separately.

(i) First term:

This term is the simple sum of the result for a single particle obtained in §2, and represents an ordinary incoherent contribution. Let us consider the emission by the aggregate of particles trapped in the same potential and denote the number of them as N_J. Then, the emission rate corresponding to this term (denoted by I_1^T) is N_J times as large as that of a single particle if we neglect the difference of the energy between different particles:

$$I_1^T \sim N_J\ I_\mu^T$$

where I_{μ}^{T} is the emission rate by a single particle.

(ii) second term:

This term represents the coherent contribution of the group of particles in the same potential trough, and arises only for the emission of the carrier wave itself. The order of magnitude of this contribution (denoted by I_2^T) is N_J^2 times as large as I_{μ}^{T} :

$$I_2^T \sim N_J^2 \, I_{\mu}^{T} \quad .$$

(iii) third term:

This term corresponds to the coherence effect between the emissions by the particles trapped in different potential troughs. As seen from the factor $\exp[-ik(x_J-x_L)]$, this effect appears only for the emission of the carrier wave itself.

If we treat the problem three-dimensionally, these coherence effects will considerably be reduced except for the emission of the wave whose direction of propagation is equal to that of the carrier wave, since the localization of trapped particles mentioned above is essentially one-demensional and so there is no localization over the directions perpendicular to the wave vector of the carrier wave.

Part II

Synopsis

The spontaneous Cerenkov emission of longitudinal waves in a plasma with small but finite amplitude wave (hereafter referred as the carrier wave) is investigated. Two effects of the carrier wave are taken into account; one is the modification of the test-particle orbit and the other that of the dielectric character of the plasma. It is shown that particles trapped in the potential-energy troughs of the carrier wave emit a discrete spectrum of sideband waves (in addition to the carrier wave itself) whose frequencies are different from the carrier-wave frequency by approximately integral multiples of the bounce frequency. An untrapped particle also emits waves at several different frequencies, including the one which satisfies the ordinary Cerenkov condition, but these frequencies depend on the particle velocity. The results are compared with those reported in our previous paper where only the effect of the orbit modification is taken into consideration.

§1. Introduction

In the usual treatment[1] of the spontaneous Cerenkov emission of longitudinal waves, the plasma is assumed to be spatially homogeneous and a particle orbit of free motion is used for the test particle. In this paper, we consider the situation in which a finite-amplitude wave (hereafter to be referred to as the carrier wave) is stationarily travelling in the plasma. Two effects of the carrier wave are to be taken into consideration; one is the modification of the test-particle orbit and the other is that of the dielectric character of the plasma.

We have recently presented a theory[2] in which only the modification of the test-particle orbit is taken into account, the dielectric character being assumed to be the same as that for a homogeneous system. We found that i) the particles trapped at the bottoms of the potential-energy troughs of the carrier wave emit the carrier wave itself, but at the same time its sideband waves whose frequencies are different from the carrier frequency by the integral multiples of the bounce frequency of the trapped particles, and ii) an untrapped particle also emits waves at several different frequencies, including the one which satisfies the ordinary Cerenkov condition, but these frequencies depend on the velocity of the test particle.

In the present paper, we take into account the effect on the dielectric character of the plasma as well as that on the test-particle orbit. Two important modifications to our previous result[2] are obtained regarding the emission of sideband waves by the trapped particles. First, the dispersion relation (the frequency versus wavenumber relation) of sideband waves is modified from the one in a homogeneous system. Secondly the emission rates of the sideband waves are modified. In addition to the sidebands the trapped particles also emit the carrier wave itself. For the emission by the untrapped particles, there is no substantial modification by the change in the dielectric character of the system.

We start from the assumption that there exists a so-called B.G.K. state[3) 4)]. Although the stability of a B.G.K. state is open to question[5], we do not concern this problem. In fact, our results can equally be used for both stable and weakly unstable systems, as in the case of a homogeneous system[6]. We then consider a microscopic charge-density fluctuation to this B.G.K. state and calculate the electric field induced by this fluctuation. This is done in §2 and §3. The relation between the electric field and the charge-density fluctuation is expressed in terms of the generalized susceptibilities of the plasma which are calculated in §4. The spontaneous Cerenkov emission rate is then calculated in §5 with the use of a kinetic wave equation for the spectral density of the electric field. The results are discussed in the last section.

§2. Basic Equations

For simplicity, we consider a one-dimensional and non-magnetized plasma. We confine our attention to high-frequency electrostatic waves and regard the ions as a uniform background of positive charge.

In order to discuss spontaneous Cerenkov emission, the contribution from microscopic fluctuations must be taken into account in the kinetic wave equation. Therefore, we start from the Klimontovich equations:

$$\left[\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} - \frac{e}{m}E(x,t)\frac{\partial}{\partial v}\right]f(x,v,t) = 0 \qquad (2\text{-}1)$$

$$\frac{\partial}{\partial x}E(x,t) = 4\pi e n_o\left\{1 - \int dv\, f(x,v,t)\right\} , \qquad (2\text{-}2)$$

where m , $-e$ and n_o are the mass, charge and average density of the electron and $f(x,v,t)$ is the microscopic electron distribution function which satisfies the initial condition:

$$f(x,v,o) = \frac{1}{n_o}\sum_j \delta[x - x_j(0)]\delta[v - v_j(0)] , \qquad (2\text{-}3)$$

$x_j(0)$, $v_j(0)$ being the initial position and velocity of the j-th particle respectively. Averaging eqs. (2-1) and (2-2) with respect to the initial preparation of the system, we obtain

$$\left[\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} - \frac{e}{m}\langle E(x,t)\rangle\frac{\partial}{\partial v}\right]\langle f(x,v,t)\rangle = \frac{e}{m}\frac{\partial}{\partial v}\langle E(x,t)f(x,v,t)\rangle_c \qquad (2\text{-}4)$$

$$\frac{\partial}{\partial x}\langle E(x,t)\rangle = 4\pi e n_o\left[1 - \int dv\langle f(x,v,t)\rangle\right] , \qquad (2\text{-}5)$$

where the angular bracket means the ensemble average and

$$\langle E(x,t)f(x,v,t)\rangle_c = \langle E(x,t)f(x,v,t)\rangle - \langle E(x,t)\rangle\langle f(x,v,t)\rangle .$$

If we adopt the Vlasov approximation, eq. (2-4) is reduced to

$$[\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} - \frac{e}{m}\langle E(x,t)\rangle\frac{\partial}{\partial v}]\langle f(x,v,t)\rangle = 0 . \qquad (2-4')$$

As the average state satisfying eqs. (2-4') and (2-5), we consider a situation in which there is a monochromatic stationarily traveling wave of the form,

$$E_0(x,t) = E_0\cos(k_0 x - \omega_0 t) , \qquad (2-6)$$

eqs. (2-4') and (2-5) are then written in the wave frame (moving with wave phase velocity $V_0=\frac{\omega_0}{k_0}$) as follows:

$$[v\frac{\partial}{\partial x} - \frac{e}{m}E_0(x)\frac{\partial}{\partial v}]F(x,v) = 0 \qquad (2-7)$$

$$\frac{\partial}{\partial x}E_0(x) = 4\pi e n_0[1-\int dv F(x,v)] , \qquad (2-8)$$

where $E_0(x)$ and $F(x,v)$ are the average electrostatic field and distribution function in the wave frame.

From eqs. (2-1), (2-2), (2-7) and (2-8) we can obtain the equations governing the fluctuating quantities

$$f^\mu(x,v,t) = f(x,v,t) - F(x,v) , \quad E^\mu(x,t) = E(x,t) - E_0(x)$$

in the wave frame as follows:

$$[\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} - \frac{e}{m}E_0(x)\frac{\partial}{\partial v}]f^\mu(x,v,t) = \frac{e}{m}E^\mu(x,t)\frac{\partial}{\partial v}F(x,v) \qquad (2-9)$$

$$\frac{\partial}{\partial x}E^\mu(x,t) = -4\pi e n_0\int dv\, f^\mu(x,v,t) , \qquad (2-10)$$

where the superscript μ denotes the fluctuation from the average. In the above equation we have neglected the term $\frac{e}{m}E^{\mu}(x,t)\frac{\partial}{\partial v}f^{\mu}(x,v,t)$.[7)†] The general solution of eq. (2-9) consists of two parts; the particular solution of this equation and the general solution of the corresponding homogeneous equation, i.e.,

$$[\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} - \frac{e}{m}E_o(x)\frac{\partial}{\partial v}]f^{\mu}(x,v,t) = 0 \ . \qquad (2\text{-}11)$$

Using the solution of eqs. (2-9) and (2-10) we can derive an equation for the spectral density of the electric field in the form

$$\frac{\partial}{\partial t}<|E_k^{\mu}(t)|^2> = 2\gamma_k<|E_k^{\mu}(t)|^2> + S_k \qquad (2\text{-}12)$$

where γ_k is the damping rate of the fluctuation of wave number k and S_k is a source term. It is this source term which gives the time rate of spontaneous emission of waves.

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† Neglect of this term is essentially equivalent to the random phase approximation in the equation for $\langle f^{\mu}(x,v,t)f^{\mu}(x',v',t')\rangle$.

§3. Solution of the Basic Equations

We first investigate the particular solution of eq. (2-9). It can be derived by the Green's function method as done by M. V. Goldman.[6] Integrating the solution over the velocity, the charge-density fluctuation $\rho^{\mu}(x,t)$ is derived as

$$4\pi\rho^{\mu}(x,t) = -\int_{-\infty}^{+\infty} dx' \int_{0}^{t} dt' \frac{\partial}{\partial x} \chi(x,x';t-t')E^{\mu}(x',t') \tag{3-1}$$

$$\frac{\partial}{\partial x}\chi(x,x';t-t') \equiv \omega_p^2 \theta(t-t') \int_{-\infty}^{+\infty} dv' \delta[x-x(x',v';t-t')] \times \frac{\partial}{\partial v'} F(x',v') \ , \tag{3-2}$$

where $\chi(x,x';t-t')$ represents the electrostatic susceptibility of the system, $\theta(t-t')$ is the usual step function and $\omega_p \equiv (\frac{4\pi n_0 e^2}{m})^{1/2}$ (electron plasma frequency). In the above expression, $x(x',v';t-t')$ and $v(x',v';t-t')$ represent the orbit satisfying the following equation with the condition $x=x'$, $v=v'$ at $t=t'$:

$$\frac{dx}{dt} = v \tag{3-3}$$

$$\frac{dv}{dt} = -\frac{e}{m} E_0(x) \ . \tag{3-4}$$

We introduce a Fourier-Laplace component $A(k,\omega)$ of a physical quantity $A(x,t)$ defined by

$$A(k,\omega) = \int_{-\infty}^{+\infty} dx \int_{0}^{\infty} dt \ e^{i\omega t - ikx} A(x,t) \tag{3-5}$$

$A(k,\omega)$ being assumed to be regular in the region $Im\omega > 0$.

In this representation, eqs. (3-1) and (3-2) are rewritten as

$$\rho^{\mu}(k,\omega) = -i\frac{k}{4\pi}\sum_{n=-\infty}^{+\infty}\chi_n(k,\omega)E^{\mu}(k+nk_0,\omega) \tag{3-6}$$

$$\chi_n(k,\omega) = -i\frac{\omega_p^2}{k}\int_{-\infty}^{+\infty}dv'\int_0^{\lambda_0}\frac{dx'}{\lambda_0}\int_0^{\infty}dt\, e^{i\omega t}\, e^{i(k+nk_0)x'} \times e^{-ikx(x',v';t)}\frac{\partial}{\partial v}F(x',v') \ , \tag{3-7}$$

where we have used the fact that our system is periodic in space with the period λ_0 $(\equiv \frac{2\pi}{k_0})$.

Next, the general solution of the homogeneous equation (2-11) can be written as

$$f_s^{\mu}(x,v.t) = \frac{1}{n_0}\sum_j \delta[x-x_j(t)]\ [v-v_j(t)] - F(x,v) \ , \tag{3-8}$$

where $x_j(t)=x(x_{0j},v_{0j};t)$, $v_j(t)=v(x_{0j},v_{0j};t)$ in the previous notation and suffix s is used to indicate the solution of the homogeneous equation. The corresponding charge-density fluctuation becomes

$$\rho_s^{\mu}(x,t) \equiv -n_0 e\int_{-\infty}^{+\infty}dv\ f_s^{\mu}(x,v,t) = -e\sum_j \delta[x-x_j(t)] - \rho_0(x) \ , \tag{3-9}$$

$\rho_0(x)$ being the charge density due to $F(x,v)$. The Fourier-Laplace component of eq. (3-9) is

$$\rho_s^{\mu}(k,\omega) = -e\sum_j\int_0^{\infty}dt\, e^{-ikx_j(t)+i\omega t} - \rho_0(k_0)\delta_{k,k_0} \ , \tag{3-10}$$

Adding this term to eq. (3-6) and using Poisson's equation, we obtain the equation for electric field $E^{\mu}(k,\omega)$ as follows:

$$E^{\mu}(k,\omega) + \sum_{n=-\infty}^{+\infty} \chi_n(k,\omega)E^{\mu}(k+nk_o,\omega) = \frac{4\pi i}{k} \rho_s^{\mu}(k,\omega) \quad . \qquad (3\text{-}11)$$

We rewrite eq. (3-11) in terms of Fourier-Laplace components in the laboratory frame. The Fourier-Laplace component in the laboratory frame $\tilde{E}^{\mu}(k,\omega)$ is related to the component in the wave frame $E^{\mu}(k,\omega)$ through

$$\tilde{E}^{\mu}(k,\omega+kV_o) = E^{\mu}(k,\omega) \qquad (3\text{-}12)$$

(this can be derived using the relation between the positions in both frames.) Therefore, eq. (3-11) is written in the laboratory frame as

$$\tilde{E}^{\mu}(k,\omega) + \sum_{n} \chi_n(k,\omega-kV_o)\tilde{E}^{\mu}(k+nk_o,\omega+n\omega_o) = \frac{4\pi i}{k} \tilde{\rho}_s^{\mu}(k,\omega) \quad . \qquad (3\text{-}13)$$

The above equation is a matrix equation of infinite-dimension.

Let us restrict ourselves to the case of small amplitude field E_o in the sense that

$$E_o \ll \sqrt{4\pi n_o T} \quad , \qquad (3\text{-}14)$$

T being the electron temperature. In this case the dispersion relation of this system is not much different from that of the homogeneous system. Therefore, we may truncate the above matrix equation, retaining only those components $\tilde{E}^{\mu}(k+nk_o,\omega+n\omega_o)$ in eq. (3-13) whose wavenumbers and frequencies nearly satisfy the dispersion relation of the homogeneous system. In the present case we have only to retain two component $\tilde{E}^{\mu}(k,\omega)$ and $\tilde{E}^{\mu}(k-2k_o,\omega-2\omega_o)$, because only these can simultaneously satisfy the dispersion relation

of the homogeneous system approximately. Then, eq. (3-11) is reduced to the coupled equations for these two components:

$$\begin{pmatrix} \{1+\chi_o(k,\omega-kV_o)\} & \chi_{-2}(k,\omega-kV_o) \\ \chi_2(k-2k_o,\omega-kV_o) & \{1+\chi_o(k-2k_o,\omega-kV_o)\} \end{pmatrix} \begin{pmatrix} \tilde{E}^{\mu}(k,\omega) \\ \tilde{E}^{\mu}(k-2k_o,\omega-2\omega_o) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4\pi i}{k}\tilde{\rho}_s^{\mu}(k,\omega) \\ \frac{4\pi i}{k-2k_o}\tilde{\rho}_s^{\mu}(k-2k_o,\omega-2\omega_o) \end{pmatrix} \qquad (3\text{-}15)$$

From the condition that the corresponding homogeneous equations have a non-trivial solution, we can obtain the dispersion equation in the present approximation as

$$D(k,\omega) = 0 , \qquad (3\text{-}16)$$

where

$$D(k,\omega) \equiv \{1+\chi_o(k,\omega-kV_o)\}\{1+\chi_o(k-2k_o,\omega-kV_o)\} - \chi_{-2}(k,\omega-kV_o)\chi_2(k-2k_o,\omega-kV_o) . \qquad (3\text{-}17)$$

The solution of eq. (3-16) specifies the collective oscillation of the system. Here we are not concerned with the details of the solution of (3-16).

Using (3-10) one can derive the charge-density correlation function $\langle\rho_s^{\mu}(k,\omega)\rho_s^{\mu *}(k',\omega')\rangle$ as

$$\langle\rho_s^{\mu}(k,\omega)\rho_s^{\mu *}(k',\omega')\rangle = \int_{-\infty}^{+\infty} dxdx' \int_0^{\infty} dtdt' \langle\rho_s^{\mu}(x,t)\rho_s^{\mu}(x',t')\rangle e^{-ikx+i\omega t+ik'x'-i\omega' t'}$$

$$= e^2 \int_0^{\infty} dtdt' \sum_{i,j} \langle e^{-ikx_i(t)+ik'x_j(t')}\rangle_c e^{i\omega t-i\omega' t'} , \qquad (3\text{-}18)$$

where the asterisk denotes the complex conjugate. If we neglect correlations between different particles, this reduces to

$$\langle \rho_s^{\mu}(k,\omega)\rho_s^{\mu *}(k',\omega')\rangle = e^2 \int_0^{\infty} dt dt' \sum_j \langle e^{-ikx_j(t)+ik'x_j(t')}\rangle e^{i\omega t - i\omega' t'} . \tag{3-19}$$

In the above derivation we used the relation

$$e\langle \sum_j \delta[x-x_j(t)]\rangle = -\rho_o(x) . \tag{3-20}$$

§4. Explicit Expression for Susceptibility

In this section we wish to derive an explicit expression for the susceptibility. For this purpose we must assume explicit forms for the orbit $x(x',v';t)$ and the average distribution function $F(x',v')$ in eq. (3-7).

We first derive the orbit satisfying eqs. (3-3) and (3-4). We can put $E_0(x)=-E_0\sin k_0x$ without loss of generality. Then, the first integration of eq. (3-4) gives the energy conservation law in the wave frame:

$$W \equiv \frac{1}{2}mv^2 + \frac{eE_0}{k_0}\cos k_0x = \text{constant}. \tag{4-1}$$

As we can see from this law, particles can be split into two groups according to their energies; one is a group of untrapped particles whose total energies are larger than eE_0/k_0 and the other is a group of trapped ones whose total energies are smaller than eE_0/k_0. Although we can get the exact orbit of a particle by using an elliptic integral, we shall here be content with an approximate orbit for simplicity.

First for trapped particles, we approximate their motion by a small oscillation near the bottom of the potential trough; that is, we derive the orbit by perturbational approach with small expansion parameter $W/\frac{eE_0}{k_0}$ $(<<1)$, retaining only the lowest order term. More explicitly, we use the following approximate equation for eq. (3-4)

$$\frac{dv}{dt} = -\frac{eE_0}{m}k_0\left(x-\frac{\lambda_0}{2}\right), \tag{4-2}$$

where we chose a particle trapped in the trough at $x=\frac{\lambda_0}{2}$ as a representative of the trapped particles.

The solution of the above equation is of the form

$$x(t) = \frac{1}{\omega_B}\sqrt{\frac{2W}{m}}\sin(\omega_B t+\alpha) + \frac{\lambda_0}{2} \qquad (4-3)$$

$$v(t) = \sqrt{\frac{2W}{m}}\cos(\omega_B t+\alpha) \ , \qquad (4-4)$$

where $\omega_B=\sqrt{eE_0k_0/m}$ is the bounce frequency and the energy W and the phase α are determined by the initial conditions. For the initial values of $x=x'$, $v=v'$, we have

$$x(t) \equiv x_0^T(x',v';t) = \frac{v'}{\omega_B}\sin\omega_B t + (x'-\frac{\lambda_0}{2})\cos\omega_B t + \frac{\lambda_0}{2} \qquad (4-5)$$

$$v(t) \equiv v_0^T(x',v';t) = v'\cos\omega_B t - \omega_B(x'-\frac{\lambda_0}{2})\sin\omega_B t \ . \qquad (4-6)$$

The quantities $x_0^T(x',v';t)$ and $v_0^T(x',v';t)$ are the expressions for $x(x',v';t)$ and $v(x',v';t)$ of a trapped particle in the present approximation.

For untrapped particles, we can also solve the equation perturbationally with a small expansion parameter $\frac{eE_0}{k_0}/W$ ($<<1$) and retain only the lowest order term. The approximate equation is

$$\frac{dv}{dt} = \frac{eE_0}{m}\sin k_0[v_0t+x(0)] \qquad (4-7)$$

and its solution is written as follows:

$$x(t) = v_* t + \frac{eE_0}{mk_0^2 v_0^2} \sin[k_0 v_0 t + \beta] \qquad (4\text{-}8)$$

$$v(t) = v_0 + \frac{eE_0}{mk_0 v_0} \cos[k_0 v_0 t + \beta] \ , \qquad (4\text{-}9)$$

where v_0 is the unperturbed velocity of a particle in the wave frame and β is determined by the initial position of the particle. In the calculation of χ_n, however, it is sufficient to use the orbit of a freely moving particle;† that is,

$$x(t) \equiv x_0^{UT}(x',v';t) = v't + x' \qquad (4\text{-}10)$$

$$v(t) \equiv v_0^{UT}(x',v';t) = v' \ , \qquad (4\text{-}11)$$

x_0^{UT} and v_0^{UT} being the expressions for $x(x',v';t)$ and $v(x',v';t)$ of an untrapped particle in the present approximation.

Next, we consider the average distribution function $F(x,v)$. Detailed discussions about this function are found in the references 3) and 4). Here we use their result in the limit of small amplitude harmonic waves, i.e.

$$F(x,v) = \begin{cases} f_{01}\left(\sqrt{v^2 - \frac{2}{m}\gamma(x)}\right)\theta\left(v^2 - \frac{2}{m}\gamma(x)\right) & \text{for } v>0 \\ f_{01}\left(-\sqrt{v^2 - \frac{2}{m}\gamma(x)}\right)\theta\left(v^2 - \frac{2}{m}\gamma(x)\right) & \text{for } v<0 \end{cases}$$

$$+ \left[f_{01}(0) - \frac{\mu}{\pi}\sqrt{\frac{2}{m}\gamma(x) - v^2}\right]\theta\left(\frac{2}{m}\gamma(x) - v^2\right) \ , \qquad (4\text{-}12)$$

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† Error incurred by this approximation is of orders of $[eE_0/(k_0 W)]^2$ and $[eE_0/(k_0 T)]^2$.

where $f_{01}(v)=f_{01}(0,v)$ is the distribution function of untrapped electrons at the point $x=0$ in the wave frame, $y(x)=\frac{eE_0}{k_0}\{1-\cos k_0 x\}$ and μ is a constant which is given by the dispersion relation:

$$1 = \frac{\omega_p^2}{k_0^2}\left[P\int_{-\infty}^{+\infty} dv\ \frac{f_{01}'(v)}{v} + \mu\right] . \qquad (4\text{-}13)$$

We can see from the relation between W and $y(x)$ that particles whose velocities at the point x exceed $\frac{2}{m}y(x)$ are untrapped and those whose velocities at the point x are smaller than $\frac{2}{m}y(x)$ are trapped.

From now on, we choose a Maxwellian distribution for $f_{01}(v)$ in the laboratory frame. That is,

$$f_{01}(v) = C_0 \exp\left[-\frac{m}{2T}(v+V_0)^2\right] , \qquad (4\text{-}14)$$

C_0 being a suitable normalization constant.

We are now ready to carry out the calculation of the susceptibility $\chi_n(k,\omega)$ given by eq. (3-7). We rewrite eq. (3-7) as follows:

$$\chi_n(k,\omega) = -\frac{\omega_p^2}{k}\int_{-\infty}^{+\infty} dv' \int_0^{\lambda_0} \frac{dx'}{\lambda_0}\, \mathcal{J}(x',v';\omega)\, e^{i(k+nk_0)x'}\, \frac{\partial F(x',v')}{\partial v'} , \qquad (4\text{-}15)$$

where we put

$$\mathcal{J}(x',v';\ ) \equiv i\int_0^{\infty} dt\ e^{i\omega t - ikx(x',v';t)} . \qquad (4\text{-}16)$$

(regular at $\mathrm{Im}\omega>0$)

In our approximation this becomes the following form; for an untrapped particle

$$\mathcal{Q}^{UT}(x',v';\omega) = -\frac{e^{-ikx'}}{\omega-kv'} \tag{4-17}$$

and for a trapped particle

$$\mathcal{Q}^{T}(x',v';\omega) = -e^{-i\frac{k\lambda_0}{2}} \sum_{n,\ell=-\infty}^{+\infty} (-1)^n \frac{J_n[k(x'-\frac{\lambda_0}{2})]J_\ell(\frac{kv'}{\omega_B})}{\omega+(n-\ell)\omega_B} . \tag{4-18}$$

We used the expansion formula by the Bessel function in calculation of $\mathcal{Q}^T$. Let us split $\chi_n(k,\omega)$ into two parts $\chi_n^{UT}(k,\omega)$ and $\chi_n^T(k,\omega)$ corresponding to contributions from two groups of electrons. That is, $\chi_n(k,\omega)=\chi_n^{UT}(k,\omega)+\chi_n^T(k,\omega)$, where

$$\chi_n^{UT}(k,\omega) = -\frac{\omega_p^2}{k}\int_0^{\lambda_0}\frac{dx}{\lambda_0}\left\{\int_{-\infty}^{-\sqrt{\frac{2y(x)}{m}}} dv + \int_{\sqrt{\frac{2y(x)}{m}}}^{+\infty} dv\right\}\mathcal{Q}^{UT}(x,v;\omega)e^{i(k+nk_0)x}\frac{\partial F(x,v)}{\partial v} \tag{4-19}$$

$$\chi_n^{T}(k,\omega) = -\frac{\omega_p^2}{k}\int_0^{\lambda_0}\frac{dx}{\lambda_0}\int_{-\sqrt{\frac{2y(x)}{m}}}^{+\sqrt{\frac{2y(x)}{m}}} dv\,\mathcal{Q}^{T}(x,v;\omega)e^{i(k+nk_0)x}\frac{\partial F(x,v)}{\partial v} . \tag{4-20}$$

Substituting the explicit form for $F(x,v)$ into eqs. (4-19) and (4-20) and carrying out the integration over the velocity (see Appendix A), we obtain the following expressions correct to the first order in $y(x)$ (or E_0):

$$\chi_n^{UT}(k,\omega) = -\frac{m}{T}\cdot\frac{\omega_p^2}{k^2}Z_0(\tfrac{\omega}{k})\delta_{n,0} - (\tfrac{m}{T})^2\cdot\frac{\omega_p^2}{k^2}\cdot\frac{\omega_B^2}{k_0^2}\{Z_1(\tfrac{\omega}{k})+Z_2(\tfrac{\omega}{k})\}\{\delta_{n,0}-\tfrac{1}{2}(\delta_{n,-1}+\delta_{n,1})\} , \tag{4-21}$$

where

$$Z_o\left(\frac{\omega}{k}\right) = \begin{cases} \int_{-\infty}^{+\infty} dv \, \frac{v+V_o}{\omega/k-v} f_{01}(v) & \text{for } \frac{\omega}{k} \neq 0 \\ -P \int_{-\infty}^{+\infty} dv \, \frac{v+V_o}{v} f_{01}(v) & \text{for } \frac{\omega}{k} = 0 \end{cases}$$

and P in front of the integral sign means to take the principal value of integration at $v=0$, and

$$\chi_n^T(k,\omega) = (-1)^n \left(\frac{\omega_p}{k_o}\right)^2 \frac{\mu}{2} \sum_{s=-\infty}^{+\infty} \frac{\omega_B}{\omega+s\omega_B} \left\{ H_{s+1,n}(k) - H_{s-1,n}(k) \right\} \qquad (4\text{-}22)$$

where

$$H_{s,n}(k) = (-i)^s [B_{s,n}(k) + \tfrac{1}{2}\{B_{s,n+1}(k) + B_{s,n-1}(k)\}]$$

$$B_{s,n}(k) = \int_{-\lambda_o/2}^{\lambda_o/2} \frac{dx}{\lambda_o} J_s(kx) e^{i(k+nk_o)x} .$$

From the property of the Bessel function, we can verify that $H_{s,n}(k)$ is purely real and

$$\begin{aligned} H_{-s,n}(k) &= H_{s,n}(k) \\ B_{s,n}(k) &= B_{s,-n}(-k) . \end{aligned} \qquad (4\text{-}23)$$

Note that the right-hand side of (4-22) does not contain the term corresponding to $s=0$ (see (4-23)). The dielectric constant of a homogeneous system in the laboratory frame $\epsilon_o(k,\omega)$ is written as follows in terms of the quantities mentioned above;

$$\epsilon_o(k,\omega) = 1 + \chi_\varphi^{UT}(k, \omega - kV_o) , \qquad (4\text{-}24)$$

but in this case the function $Z_0\left(\frac{\omega}{k}\right)$ is given by $\int_{-\infty}^{+\infty} dv \frac{v+V_0}{\omega/k-v} f_{01}(v)$ always.

Equations (4-21) and (4-22) are our final results obtained for the susceptibility.

§5. Time Rate of Spontaneous Emission

We start from the coupled equation (3-15). Solving this equation for $\tilde{E}^{\mu}(k,\omega)$, we get

$$\tilde{E}^{\mu}(k,\omega) = \frac{4\pi i}{D(k,\omega)}\Big[\frac{1}{k}\{1+\chi_0(k-2k_0,\omega-kV_0)\}\tilde{\rho}_s^{\mu}(k,\omega) - \frac{1}{k-2k_0}\chi_{-2}(k,\omega-kV_0)\tilde{\rho}_s^{\mu}(k-2k_0,\omega-2\omega_0)\Big] \quad (5\text{-}1)$$

Using this solution, we can obtain the expression for the spectral density of electric field due to fluctuations, $U_k(t) \equiv <|\tilde{E}_k^{\mu}(t)|>$, as follows:

$$\begin{aligned} U_k(t) &= <\Big|\int_+ \frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{D(k,\omega)}\tilde{E}^{\mu}(k,\omega)\Big|^2> \\ &= (4\pi)^2 <\Big|\int_+ \frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{D(k,\omega)}\Big[\frac{1}{k}\{1+\chi_0(k-2k_0,\omega-kV_0)\}\tilde{\rho}_s^{\mu}(k,\omega) \\ &\qquad - \frac{1}{k-2k_0}\chi_{-2}(k,\omega-kV_0)\tilde{\rho}_s^{\mu}(k-2k_0,\omega-2\omega_0)\Big]\Big|^2>, \end{aligned} \quad (5\text{-}2)$$

where the integral sign $\int_+ \frac{d\omega}{2\pi}$ is an abbreviation of $\int_{-\infty+i\sigma}^{+\infty+i\sigma}\frac{d\omega}{2\pi}$ $(\sigma>0)$.

Let us calculate the correlation function $<\tilde{\rho}_s^{\mu}(k,\omega)\tilde{\rho}_s^{\mu}{}^*(k',\omega')>$, making use of eqs. (3-19), (4-3) and (4-8). We neglect correlations between different particles. Therefore, we can split $<\rho_s^{\mu}\rho_s^{\mu}{}^*>$ into two parts as follows:

$$<\rho_s^{\mu}(k,\omega)\rho_s^{\mu}{}^*(k',\omega')> \equiv <\rho_s^{\mu}(k,\omega)\rho_s^{\mu}{}^*(k',\omega')>^{T} + <\rho_s^{\mu}(k,\omega)\rho_s^{\mu}{}^*(k',\omega')>^{UT},$$

where

$$\langle \tilde{\rho}_s^{\mu}(k,\omega)\tilde{\rho}_s^{\mu *}(k',\omega')\rangle^{T} = e^2 \int_0^{\infty} dt dt' \sum_{j=TP} e^{-ikx_j(t)+ik'x_j(t')} e^{i\omega t - i\omega' * t'}$$

$$\langle \tilde{\rho}_s^{\mu}(k,\omega)\tilde{\rho}_s^{\mu *}(k',\omega')\rangle^{UT} = e^2 \int_0^{\infty} dt dt' \sum_{j=UTP} e^{-ikx_j(t)+ik'x_j(t')} e^{i\omega t - i\omega' * t'} , \tag{5-3}$$

the symbols $\sum_{j=TP}$ and $\sum_{j=UTP}$ denoting the summations over the trapped and untrapped particles, respectively. After some calculations (see Appendix B), we obtain trapped-particle contribution in the wave frame

$$\langle \rho_s^{\mu}(k,\omega)\rho_s^{\mu *}(k',\omega')\rangle^{T} = \sum_{j=TP} e^2 e^{-i(k-k')\lambda_o/2} \sum_{\ell=-\infty}^{+\infty} J_\ell(ka_j) J_\ell(k'a_j) \times \frac{1}{(\omega-\ell\omega_B)(\omega'*-\ell\omega_B)} , \tag{5-4}$$

and for untrapped-particle contribution

$$\langle \rho_s^{\mu}(k,\omega)\rho_s^{\mu *}(k',\omega')\rangle^{UT} = \sum_{j=UTP} e^2 \sum_{\ell=-\infty}^{+\infty} J_\ell(kb_j) J_\ell(k'b_j) \times \frac{1}{\{\omega-(k+\ell k_o)v_{oj}\}\{\omega'*-(k'+\ell k_o)v_{oj}\}} , \tag{5-5}$$

where we put

$$a_j \equiv \frac{1}{\omega_B}\sqrt{\frac{2W_j}{m}} , \quad b_j \equiv \frac{1}{k_o}\left(\frac{\omega_B}{k_o v_{oj}}\right)^2 \tag{5-6}$$

and W_j and v_{oj} are the energy and unperturbed velocity of the j-th particle in the wave frame, respectively. In the above derivation we assumed random distribution for the initial phases of the particles.

Using the relations (5-4) and (5-5), we can rewrite (5-2) as

$$U_k(t) = \sum_{\ell=-\infty}^{+\infty} U_k^T(\ell;t) + U_k^{UT}(t) , \tag{5-7}$$

$$U_k^T(\ell;t) \equiv \sum_{j=TP} (4\pi e)^2 |A^T(j,\ell)|^2 , \tag{5-8}$$

$$U_k^{UT}(t) \equiv \sum_{j=UTP} (4\pi e)^2 \sum_{\ell=-\infty}^{+} |A^{UT}(j,\ell)|^2 , \tag{5-9}$$

$$A^T(j,\ell) \equiv \int_+ \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{D(k,\omega)(\omega-kV_o-\ell\omega_B)} \left[\{1+\chi_o(k-2k_o,\omega-kV_o)\}\cdot\frac{J_\ell(ka_j)}{k} - \chi_{-2}(k,\omega-kV_o)\cdot\frac{J_\ell((k-2k_o)a_j)}{k-2k_o}\right] , \tag{5-10}$$

$$A^{UT}(j,\ell) \equiv \int_+ \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{D(k,\omega)} \left[\frac{\{1+\chi_o(k-2k_o,\omega-kV_o)\}}{\omega-k(V_o+v_{oj})-\ell k_o v_{oj}}\cdot\frac{J_\ell(kb_j)}{k} - \frac{\chi_{-2}(k,\omega-kV_o)}{\omega-k(V_o+v_{oj})-(\ell-2)k_o v_{oj}}\cdot\frac{J_\ell((k-2k_o)b_j)}{k-2k_o}\right] \tag{5-11}$$

where we used the relation $\bar{\rho}_s^{\mu}(k,\omega)=\rho_s^{\mu}(k,\omega-kV_o)$.

In the calculation of the spontaneous emission rate, it is sufficient to know a behavior of the dielectric constant near the frequencies $\omega=kV_o+\ell\omega_B$ for emissions by trapped particles and $\omega=k(V_o+v_{oj})+\ell k_o v_{oj}$ for emissions by untrapped particles, because waves are spontaneously emitted only when a pole associated with the collective oscillation (i.e. $D(k,\omega)=0$) coincides with another pole associated with the individual particle motion (i.e. $\omega=kV_o+\ell\omega_B$ and $\omega=k(V_o+v_{oj})+\ell k_o v_{oj}$ for trapped and untrapped particles respectively). This situation is essentially the same as in the

usual treatment for a homogeneous case.[6] It should be noted that each term on the right-hand side of (5-7) has a definite physical meaning as the spectral density of each separate wave, since different terms in (5-7) correspond to fluctuations of different frequencies and wavenumbers, as we shall see below.[†] We shall calculate $U_k^T(\ell;t)$ and $U_k^{UT}(t)$ in the following.

(I) Trapped-particle contribution

As seen from (4-22) and (4-23), the susceptibility $\chi^T(k,\omega-kV_0)$ is a smooth funciton of ω near the frequency $\omega=kV_0$, while it becomes very large near $\omega=kV_0+\ell\omega_B$ $(\ell\neq 0)$. This is due to the fact that the bouncing motion of trapped particles can resonate with the oscillation of a perturbed electric field of frequency $\omega=kV_0+\ell\omega_B$ $(\ell\neq 0)$. Taking this resonance effect into account, we calculate $U_k^T(o;t)$ and $U_k^T(\ell;t)$ $(\ell\neq 0)$ separately in the following.

(i) Calculation of $U_k^T(0;t)$

We shall approximate the solution of $D(k,kV_0)=0$ by $k=k_0$. This approximation is based on the understanding that the dispersion relation (3-16) can be used to describe approximately the carrier wave itself by taking the limit $k\to k_0$. Comparison of the solution of $D(k,kV_0)=0$ with that of (4-13) shows that the error incurred by this approximation is of the order of $\mu(k_0/\omega_B)^2$ which is very small compared with unity, provided the trapped-particle density is sufficiently small compared with the total number density. Using the above approximation, we obtain after some approximate

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†) See the δ-functions contained in eqs. (5-13), (5-16) and (5-17).

calculations (see Appendix C)

$$A^T(j,0) = -i\frac{J_0(k_0a_j)}{k_0\alpha_0}e^{\delta t - i\omega_0 t}\left\{\frac{e^{-i(kV_0-\omega_0-i\delta)t}-1}{kV_0-\omega_0-i\delta}\right\}, \qquad (5\text{-}12)$$

where $\alpha_0=\frac{\partial\epsilon_0}{\partial\omega}\Big|_{\omega=\omega_0}$ and a small imaginary quantity $i\delta$ is introduced for convenience. Inserting (5-12) into (5-8) and applying an asymptotic formula for small δ ($\delta<<\omega_0$) ,[9] we obtain

$$U_k^T(0;t) = \sum_{j=TP}\frac{(4\pi e)^2J_0^2(k_0a_j)}{k_0^2}\frac{}{|\alpha_0|^2}\cdot 2\pi t\cdot\delta(kV_0-\omega_0) \ . \qquad (5\text{-}13)$$

(ii) Calculation of $U_k^T(\ell;t)$ ($\ell\neq 0$)

In the calculation, the following approximation will be used for $D(k,\omega)$:

$$D(k,\omega) \simeq \epsilon_0(k,\omega)\epsilon_0(k-2k_0,\omega-2\omega_0) + \epsilon_0(k,\omega)\chi_0^T(k-2k_0,\omega-kV_0) + \epsilon_0(k-2k_0,\omega-2\omega_0)\chi_0^T(k,\omega-kV_0) \ .$$

In this approximation the term $[\chi_0^T(k,\omega-kV_0)\chi_0^T(k-2k_0,\omega-kV_0) - \chi_{-2}^T(k,\omega-kV_0)\chi_2^T(k-2k_0,\omega-kV_0)]$ is neglected (see (3-17)), since this term is of order $(\bar{v}/v_t)^2\exp[-2V_0^2/\bar{v}^2]$, where $\bar{v}^2=T/2m$ and $v_t=\omega_B/k_0$. Near the frequency satisfying $\omega=kV_0+\ell\omega_B$ ($\ell\neq 0$) , the term corresponding to $s=-\ell$ in the expression for $\chi_n^T(k,\omega-kV_0)$ (see (4-22)) becomes very large, so that $D(k,\omega)$ can approximately be written as

$$D(k,\omega) \simeq \frac{\omega_B}{\omega-kV_0-\ell\omega_B}[\epsilon_0(k,\omega)\{H_{-\ell+1,0}(k-2k_0)-H_{-\ell-1,0}(k-2k_0)\} + \epsilon_0(k-2k_0,\omega-2\omega_0)\{H_{-\ell+1,0}(k)-H_{-\ell-1,0}(k)\}] \ .$$

Similar approximations can be used for the numerators of the integrand in (5-10). After some calculations (see Appendix C), we get

$$A^T(j,\ell) \simeq \frac{(-i)}{k\alpha_k\{1+h(\ell)\}}\{J_\ell(ka_j)+\frac{k}{2k_o-k}g(\ell)J_\ell((k-2k_o)a_j)\}$$
$$\times e^{\gamma_k(\ell)t-i\omega_k(\ell)t}\cdot\{\frac{e^{i(kV_o+\ell\omega_B-\omega_k(\ell)-i\gamma_k(\ell))t}-1}{kV_o+\ell\omega_B-\omega_k(\ell)-i\gamma_k(\ell)}\}, \tag{5-14}$$

where

$$\begin{aligned}
\omega_k(\ell) &\equiv \frac{\omega_k^o+h(\ell)\{2\omega_o+\omega_{k-2k_o}^o\}}{1+h(\ell)}\\
\gamma_k(\ell) &\equiv \frac{\gamma_k^o+h(\ell)\gamma_{k-2k_o}^o}{1+h(\ell)}\\
g(\ell) &\equiv \frac{H_{-\ell+1,-2}(k)-H_{-\ell-1,-2}(k)}{H_{-\ell+1,0}(k-2k_o)-H_{-\ell-1,0}(k-2k_o)}\\
h(\ell) &\equiv \frac{\alpha_{k-2k_o}}{\alpha_k}\cdot\frac{H_{-\ell+1,0}(k)-H_{-\ell-1,0}(k)}{H_{-\ell+1,0}(k-2k_o)-H_{-\ell-1,0}(k-2k_o)}\\
\alpha_k &\equiv \frac{\partial\varepsilon_o}{\partial\omega}\Big|_{\omega=\omega_k^o},
\end{aligned} \tag{5-15}$$

ω_k^o and γ_k^o being the real and imaginary parts of the zero of $\varepsilon_o(k,\omega)$, respectively. Inserting (5-14) into (5-8) and applying the same asymptotic formula for small $\gamma_k(\ell)$ ($\gamma_k(\ell)<<\omega_k(\ell)$) as in case (i), we obtain

$$U_k^T(\ell;t) = \sum_{j=TP}\frac{(4\pi e)^2}{k^2}\cdot\frac{1}{|\alpha_k\{1+h(\ell)\}|^2}[J_\ell(ka_j)+\frac{k}{2k_o-k}g(\ell)J_\ell((k-2k_o)a_j)]^2$$
$$\cdot\frac{\pi}{\gamma_k(\ell)}\{e^{2\gamma_k(\ell)t}-1\}\delta[kV_o+\ell\omega_B-\omega_k(\ell)]. \tag{5-16}$$

(II). Untrapped-particle contribution

We confine our attention to the emission by those particles whose velocities in the wave frame satisfy the condition:

$$v_{\circ j}^2 \gg \left(\frac{\omega_B}{k_\circ}\right)^2 .$$

We shall neglect the emission by other particles for simplicity. By the same reason as for the trapped-particle case, we replace the dielectric function by an approximate expression which is correct in the region of ω near $[k(V_\circ+v_{\circ j})+ k_\circ v_{\circ j}]$. We then find that the first term in the square bracket of the integrand in (5-11) gives the dominant contribution to $A^{UT}(j,\ell)$ (see Appendix C). The final result is given by

$$U_k^{UT}(t) = \sum_{j=UTP} \frac{(4\pi e)^2}{k^2|\alpha_k|^2}\cdot\frac{\pi}{\gamma_k^\circ}\{e^{2\gamma_k^\circ t}-1\}\sum_{\ell=-\infty}^{+\infty} J_\ell^2(kb_j) \times\delta[k(V_\circ+v_{\circ j})+\ell k_\circ v_{\circ j}-\omega_k^\circ] . \quad (5\text{-}17)$$

We now derive a kinetic wave equation for the spectral density by differentiating eqs. (5-13), (5-16) and (5-17) with time. That is,

$$\frac{\partial}{\partial t}U_k^T(0;t) = 2\pi \sum_{j=TP} \frac{(4\pi e)^2}{k^2}\cdot\frac{J_\circ^2(k_\circ a_j)}{|\alpha_\circ|^2}\,\delta(kV_\circ-\omega_\circ) \quad (5\text{-}18)$$

$$\left\{\frac{\partial}{\partial t}-2\gamma_k(\ell)\right\}U_k^T(\ell;t) = 2\pi \sum_{j=TP} \frac{(4\pi e)^2}{k^2}\cdot\frac{1}{|\alpha_k\{1+h(\ell)\}|^2} \times\left[J_\ell(ka_j)+\frac{k}{2k_\circ-k}g(\ell)J_\ell((k-2k_\circ)a_j)\right]^2\delta[kV_\circ+\ell\omega_B-\omega_k(\ell)] \quad (\ell\neq 0) \quad (5\text{-}19)$$

$$\left\{\frac{\partial}{\partial t}-2\gamma_k^\circ\right\}U_k^{UT}(t) = 2\pi \sum_{j=UTP} \frac{(4\pi e)^2}{k^2|\alpha_k|^2}\sum_{\ell=-\infty}^{+\infty} J_\ell^2(kb_j) \times\delta[k(V_\circ+v_{\circ j})+\ell k_\circ v_{\circ j}-\omega_k^\circ] . \quad (5\text{-}20).$$

These equations are of the form of eq. (2-12). As pointed out there, the right-hand sides give the time rates of spontaneous emission for the corresponding fluctuations.

Integrating the right-hand sides over k and summing up over ℓ for the trapped-particle case, we obtain the time rates of total emissions by the trapped and untrapped particles (to be denoted by I^T and I^{UT}, respectively). They are

$$I^T = \sum_{j=TP} \int dk \left[\frac{(4\pi e)^2}{k_o^2} \cdot \frac{J_o^2(k_o a_j)}{|\alpha_o|^2} \delta(kV_o - \omega_o) \right.$$
$$+ \sum_{\ell \neq 0} \frac{(4\pi e)^2}{k^2} \cdot \frac{1}{|\alpha_k\{1+h(\ell)\}|^2} \left\{ J_\ell(ka_j) + \frac{k}{2k_o - k} g(\ell) J_\ell((k-2k_o)a_j) \right\}^2$$
$$\left. \times \delta(kV_o + \ell\omega_B - \omega_k(\ell)) \right] \qquad (5\text{-}21)$$

and

$$I^{UT} = \sum_{j=UTP} \int dk \frac{(4\pi e)^2}{k^2 |\alpha_k|^2} \sum_{\ell=-\infty}^{+\infty} J_\ell^2(kb_j) \delta[k(V_o + v_{oj}) + \ell k_o v_{oj} - \omega_k^o] \ . \qquad (5\text{-}22)$$

Note that these results can be used not only for a stable case but also for a weakly unstable case, because we assumed only the smallness of the imaginary part of the frequency compared to the real part.[9)]

§6. Discussions

Let us discuss the results obtained in §5 by investigating the emission rate and the resonance condition (the condition which determines the frequency and wavenumber of the wave emitted).

We first consider the emission by trapped particles. In the integrand of (5-21), the first term gives the emission of the carrier wave itself. Its emission rate is the same as was obtained in our previous paper,[3] and can become quite large, since all the trapped particles take part in this emission. Strictly speaking, the presence of this emission produces a difficulty in the present formulation, since the carrier-wave amplitude continuously increases due to this emission. Note that we are considering a B.G.K. state as the unperturbed state, so that the carrier wave does not suffer any Landau damping. In reality, however, the spontaneous emission produces a modification of the average distribution and this modification in turn affects the stationary character of the B.G.K. state. Thus for a complete discussion of the stationary state, it appears necessary to consider a coupled equation for the spectral wave energy and the average distribution, as done in the quasilinear theory.

The resonance conditions in the terms with $\ell=0$ in (5-21) indicate emission of those sideband waves whose wavenumbers and frequencies are given by the relations $kV_0+\ell\omega_B=\omega_k(\ell)$ and $\omega=\omega_k(\ell)$. These sidebands constitute a discrete spectrum with the frequency separation of order ω_B around the carrier-wave frequency ω_0. A similar kind of sideband emission was also obtained in our previous work, but our new result includes two important

modifications. First, the frequency versus wavenumber relation is now given by $\omega=\omega_k(\ell)$, but not by the Bohm-Gross frequency; this modification arises from that of the effective dielectric function from $\epsilon_o(k,\omega)$ to $\{\epsilon_o(k,\omega)+h(\ell)\epsilon_o(k-2k_o,\omega-2\omega_o)$, which is a certain average of the dielectric functions for the two waves, (k,ω) and $(k-2k_o,\omega-2\omega_o)$, in the homogeneous system. Secondly, the emission rate is modified by two effects, one due to the above modification of the dielectric function and the other due to the presence of two charge fluctuations, $\tilde{\rho}_s^{\mu}(k,\omega)$ and $\tilde{\rho}_s^{\mu}(k-2k_o,\omega-2\omega_o)$, which contribute to the fluctuating field $\tilde{E}^{\mu}(k,\omega)$. Either of these two effects, when taken independently, is very large, but their effects tend to cancel each other, so that their resultant effect is not quite substantial. An estimate for the case of $\ell=\pm1$ and $k_o a_j \ll 1$ shows that the new emission rate is one order of magnitude less than the result obtained in our previous work.

Equation (5-22), obtained for the emission by untrapped particles, indicates that an untrapped particle also emits many waves including the one which satisfies the usual Cerenkov condition, i.e. the term with $\ell=0$ in the sum. The terms with $\ell=0$ correspond to the scattering of the carrier wave by the particle. These results for the emission by untrapped particles are the same as those obtained in our previous work.

To summarize, one can state that the effect of the modification of the dielectric property of the plasma appears mainly on the emission by the trapped particles. This is naturally untderstood since the effects mentioned above come mainly from the trapped-particle susceptibility χ^T.

In the present theory, we have ignored the correlation among different particles. In reality, the motions of those trapped particles which are localized in the same potential trough of the carrier wave can become strongly correlated to each other. This may cause a coherence effect on the spontaneous emission, namely the emission rate may become proportional to the square of the trapped particle density in a given potential trough. In addition to this, another type of coherence effect can be expected due to a periodic array of the trapped particles. Namely, waves emitted in different potential troughs can be superposed in phase. These coherence effects can be expected only for the emission of the carrier wave itself.

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Appendix A

— Integration over Velocity in χ_n —

Substituting the explicit form of eq. (4-12) for $F(x,v)$ into eqs. (4-19) and (4-20), we obtain

$$\chi_n^{UT}(k,\omega) = \frac{\omega_p^2}{k}\int_0^{\lambda_0}\frac{dx}{\lambda_0} e^{ink_0x}\left\{\int_{-\infty}^{-\sqrt{\frac{2y(x)}{m}}} dv \frac{1}{\omega-kv}(-\frac{m}{T})(-\sqrt{v^2-\frac{2y(x)}{m}}+V_0)\frac{(-v)f_{o1}(-\sqrt{v^2-\frac{2y(x)}{m}})}{\sqrt{v^2-\frac{2}{m}y(x)}}\right.$$
$$\left. + \int_{\sqrt{\frac{2y(x)}{m}}}^{+\infty} dv \frac{1}{\omega-kv}(-\frac{m}{T})(\sqrt{v^2-\frac{2y(x)}{m}}+V_0)\frac{vf_{o1}(\sqrt{v^2-\frac{2y(x)}{m}})}{\sqrt{v^2-\frac{2}{m}y(x)}}\right\} \tag{A1}$$

$$\chi_n^T(k,\omega) = \frac{\omega_p^2}{k} e^{-i\frac{k\lambda_0}{2}} \sum_{s,\ell=-\infty}^{+\infty}(-1)^s \frac{1}{\omega+(s-\ell)\omega_B}\int_0^{\lambda_0}\frac{dx}{\lambda_0} J_s[k(x-\frac{\lambda_0}{2})]e^{i(k+nk_0)x}$$
$$\times \frac{\mu}{\pi}\int_{-\sqrt{\frac{2y(x)}{m}}}^{+\sqrt{\frac{2y(x)}{m}}} dv J_\ell(\frac{kv}{\omega_B})\cdot\frac{v}{\sqrt{\frac{2}{m}y(x)-v^2}} . \tag{A2}$$

(i) Calculation in χ_n^T:

$$A_\ell(x) \equiv \frac{\mu}{\pi}\int_{-\sqrt{\frac{2y(x)}{m}}}^{\sqrt{\frac{2y(x)}{m}}} dv\ J_\ell(\frac{kv}{\omega_B})\frac{v}{\sqrt{\frac{2}{m}y(x)-v^2}}$$
$$= \frac{\mu}{\pi}(\frac{\omega_B}{k})^2\sqrt{\frac{m}{2y(x)}}\int_{\frac{k}{\omega_B}\sqrt{\frac{2y(x)}{m}}}^{\frac{k}{\omega_B}\sqrt{\frac{2y(x)}{m}}} dt\ \frac{t\ J_\ell(t)}{\sqrt{1-(\frac{\omega_B}{k})^2\frac{m}{2y(x)}t^2}} . \tag{A3}$$

Making use of properties of the Bessel function, we obtain

$$A_\ell(x) = \begin{cases} 2\cdot\frac{\mu}{\pi}\cdot\frac{\omega_B}{k}\cdot\frac{1}{\alpha}\int_0^\alpha dt\,\frac{t\,J_\ell(t)}{\sqrt{1-t^2/\alpha^2}} & \text{for } \ell=\text{odd} \\ 0 & \text{for } \ell=\text{even} \end{cases} \tag{A4}$$

and

$$A_{-\ell}(x) = (-1)^\ell A_\ell(x)\ , \tag{A5}$$

where we put $\alpha=\frac{k}{\omega_B}\sqrt{2y(x)/m}$

$$[\alpha^2 = \frac{k^2}{\omega_B^2}\cdot\frac{2}{m}\cdot\frac{eE_0}{k_0}(1-\cos k_0 x) = 4\cdot(\frac{k}{k_0})^2\sin^2(\frac{k_0 x}{2})]\ . \tag{A6}$$

Expanding $J_\ell(t)$ in power series of t and integrating over t, we obtain

$$\int_0^\alpha dt\frac{t\,J_{2\ell+1}(t)}{\sqrt{1-t^2/\alpha^2}} = \frac{\alpha^{2\ell+3}}{2^{2\ell+1}}\sum_{n=0}\frac{(-1)^n}{n!(2\ell+n+1)!}\cdot(\frac{\alpha}{2})^{2n}\frac{\{2(\ell+n+1)-1\}!!}{\{2(\ell+n+1)\}!!}\cdot\frac{\pi}{2}\ . \tag{A7}$$

Then, substituting this result into the expression of χ_n^T and carrying out the x-integration, we find that contributions from higher-order terms in α become smaller (see the factor $\sin^2(\frac{k_0 x}{2})$ in eq. (A6)). So we retain only the lowest-order terms in α and obtain eq. (4-25).

(ii) Calculation in χ_n^{UT}:

In case that $y(x)/T\ll 1$, we can expand the functions in the integrand of eq. (A1) as follows:

$$\pm\sqrt{v^2-\frac{2}{m}y(x)} = v\{1-\frac{y(x)}{mv^2}+\cdots\}$$

$$\pm 1/\sqrt{v^2-\frac{2}{m}y(x)} = \frac{1}{v}\{1+\frac{y(x)}{mv^2}+\cdots\} ,$$

and

$$f_{o1}(\pm\sqrt{v^2-\frac{2}{m}y(x)}) = f_{o1}(v)\{1+\frac{v+v_o}{v}\cdot\frac{y(x)}{T}+\cdots\} .$$

Substituting these expansions into the corresponding factors and considering the fact $y(x)/T<<1$, we obtain eq. (4-21).

Appendix B

— Calculation of Charge-Density Correlation Functions —

We can calculate these quantities by substituting the explicit orbits of eqs. (4-3) and (4-8) into eq. (3-19).

(i) Trapped-particle case

Using eq. (4-3), we get

$$e^{-ikx_j(t)} = e^{ik\frac{\lambda\alpha}{2}} e^{-ika_j\sin(\omega_B t+\alpha)}$$

$$= e^{-ik\frac{\lambda}{2}\alpha} \sum_{\ell} J_\ell(ka_j) e^{-i\ell\omega_B t - i\ell\alpha} . \qquad \text{(B1)}$$

Then,

$$\langle e^{-ikx_j(t)+ik'x_j(t')} \rangle = e^{-i(k-k')\frac{\lambda\alpha}{2}} \sum_{\ell,s} J_\ell(ka_j) J_s(k'a_j)$$

$$\times e^{-i(\ell\omega_B t - s\omega_B t')} \langle e^{-i(\ell-s)\alpha} \rangle .$$

Assuming no initial-phase correlation, we find that this reduces to

$$\langle e^{-ikx_j(t)+ik'x_j(t')} \rangle$$

$$= e^{-i(k-k')\frac{\lambda\alpha}{2}} \sum_{\ell} J_\ell(ka_j) J_\ell(k'a_j) e^{-i\ell\omega_B(t-t')} . \qquad \text{(B2)}$$

Then, we obtain

$$\int_0^\infty dtdt' \langle e^{-ikx_j(t)+ik'x_j(t')} \rangle e^{i\omega t - i\tilde{\omega}^* t'}$$

$$= e^{-i(k-k')\frac{\lambda\alpha}{2}} \sum_{\ell} J_\ell(ka_j) J_\ell(k'a_j) \frac{1}{(\omega-\ell\omega_B)(\tilde{\omega}^*-\ell\omega_B)} \qquad \text{(B3)}$$

$$(\mathrm{Im}\omega, \tilde{\omega} > 0)$$

(ii) Untrapped-particle case

Using the orbit of eq. (4-8), we get under the assumption of no initial-phase correlation,

$$\langle e^{-ikx_j(t)+ik'x_j(t')} \rangle = e^{-i(k+\ell k_0)v_{0j}t+i(k'+\ell k_0)v_{0j}t'} \sum_{\ell} J_\ell(kb_j)J_\ell(k'b_j) \quad . \tag{B4}$$

Then, we obtain

$$\int_0^\infty dtdt' \langle e^{-ikx_j(t)+ik'x_j(t')} \rangle e^{i\omega t-i\tilde{\omega}^* t'} = \sum_{\ell} J_\ell(kb_j)J_\ell(k'b_j) \frac{1}{\{\omega-(k+\ell k_0)v_{0j}\}\{\tilde{\omega}^*-(k'+\ell k_0)v_{0j}\}} \quad . \tag{B5}$$

$(\mathrm{Im}\omega, \tilde{\omega} > 0)$

Appendix C

— Calculation of A^T and A^{UT} —

(i) $A^T(j,0)$

When $k\sim k_o$ and $\omega\sim kV_o\sim \omega_o$, we can show the following relations making use of eqs. (4-21) and (4-22):

$$\chi_o(k,\omega-kV_o) \simeq \chi_o(k-2k_o,\omega-kV_o)$$

$$\chi^T_{-2}(k,\omega-kV_o)\simeq \chi^T_2(k-2k_o,\omega-kV_o)$$

and therefore

$$D(k,\omega) \simeq \{1+\chi_o(k,\omega-kV_o)+\chi^T_{-2}(k,\omega-kV_o)\}\{1+\chi_o(k-2k_o,\omega-kV_o) -\chi^T_2(k-2k_o,\omega-kV_o)\} .$$

Using the above relation, we obtain

$$A^T(j,0) \simeq \frac{J_o(k_o a_j)}{k_o}\int_+ \frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{(\omega-kV_o)\{1+\chi_o(k,\omega-kV_o)-\chi^T_{-2}(k,\omega-kV_o)\}}. \tag{C1}$$

Near $\omega=\omega_o$ the function $\{1+\chi_o(k,\omega-kV_o)-\chi^T_{-2}(k,\omega-kV_o\}$ can be approximated as follows;

$$[1+\chi_o(k,\omega-kV_o)-\chi^T_{-2}(k,\omega-kV_o)] \simeq \alpha_o\{\omega-\omega_o-i\delta\} , \tag{C2}$$

where the coefficient α_o can approximately be written as $\alpha_o\simeq\frac{\partial\varepsilon_o}{\partial\omega}\Big|_{\omega=\omega_o}$. Substituting (C2) into (C1) and closing the integration path in the lower-half plane, we obtain eq. (5-12).

(ii). $A^T(j,\ell)$

Using the approximate forms for the integrand of eq. (5-10), we obtain

$$A^T(j,\ell) = \int_+ \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega - kV_o - \ell\omega_B} \cdot \frac{\frac{1}{k}J_\ell(ka_j) - \frac{q(\ell)}{k-2k_o}J_\ell((k-2k_o)a_j)}{\epsilon_o(k,\omega) + \frac{\alpha_k}{\alpha_{k-2k_o}} \cdot h(\ell)\epsilon_o(k-2k_o, \omega-2\omega_o)} \quad . \tag{C3}$$

We make the resonance approximation for ϵ_o;

$$\begin{aligned} &\epsilon_o(k,\omega) + \frac{\alpha_k}{\alpha_{k-2k_o}} h(\ell)\epsilon_o(k-2k_o, \omega-2\omega_o) \\ &\quad = \alpha_k\{\omega - \omega_k^o - i\gamma_k^o\} + \alpha_k h(\ell)\{\omega - 2\omega_o - \omega_{k-2k_o}^o - i\gamma_{k-2k_o}^o\} \\ &\quad = \alpha_k\{1+h(\ell)\}\{\omega - \omega_k(\ell) - i\gamma_k(\ell)\} \quad . \end{aligned}$$

Substituting this relation into (C3) and closing the contour in the lower-half plane, we get eq. (5-14).

(iii) $A^{UT}(j,\ell)$

The susceptibility χ_n^T can be estimated as follows at the point $\omega = k(V_o + v_{oj}) + \ell k_o v_{oj}$ for those particles which satisfy the condition $v_{oj}^2 >> (\frac{\omega_B}{k_o})^2$: For $\ell \neq -1$

$$\begin{aligned} \chi_n^T[k, kv_{oj} + \ell k_o v_{oj}] &= (-1)^n \sum_s \frac{2\omega_B^2}{\{(k+\ell k_o)v_{oj} + s\omega_B\}^2 - \omega_B^2} H_{s,n}(k) \\ &\sim \frac{1}{(\ell+1)^2} \cdot \frac{\omega_p^2}{(k_o v_{oj})^2} \cdot \mu (\frac{\omega_B}{k_o})^2 \\ &\sim \frac{1}{(\ell+1)^2} \frac{|n_B|}{n_o} << 1 \end{aligned}$$

and for $\ell=-1$ $(k\neq k_o)$

$$\chi_n^T[k,(k-k_o)v_{oj}] \simeq (-1)^n \frac{1}{(k-k_o)v_{oj}}\{H_{1,n}(k)-H_{-1,n}(k)\}$$

$$= 0 \quad .$$

where

$$n_T \equiv n_o \int_{-\lambda_o/2}^{\lambda_o/2} \frac{dx}{\lambda_o} \int_{-\sqrt{2y(x)/m}}^{\sqrt{2y(x)/m}} dv\, F(x,v) \quad \text{(trapped-particle number)}$$

$$= n_o\left\{\frac{8f_{o1}(0)}{\pi} - \mu\cdot\frac{\omega_B}{k_o}\right\}\frac{\omega_B}{k_o}$$

$$n_B \equiv \mu\cdot\left(\frac{\omega_B}{k_o}\right)^2 \quad \text{(effective-oscillator number)}$$

$$= \frac{8}{\pi}f_{o1}(0)\cdot\frac{\omega_B}{k_o} - n_T/n_o$$

(n_B can be negative)

and we used the following evaluation;

$$(k_o v_{oj})^2 \simeq k_o^2(V_o-\bar{v})^2$$

$$= \omega_o^2(1-k_o\lambda_o)^2$$

$$\sim \omega_p^2 \quad .$$

We, therefore, neglect χ_n^T as being small compared to ϵ_o in this case.

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Figure Caption

Fig. 1 The curve of $\omega=\tilde{\omega}_k(\ell)$ and $\omega=kV_o+\ell\omega_B$. The intersecting point $(\omega_\ell(k),k_\ell)$ gives the frequency and wave number of the wave which satisfies the resonance condition $kV_o+\ell\omega_B=\tilde{\omega}_k(\ell)$.

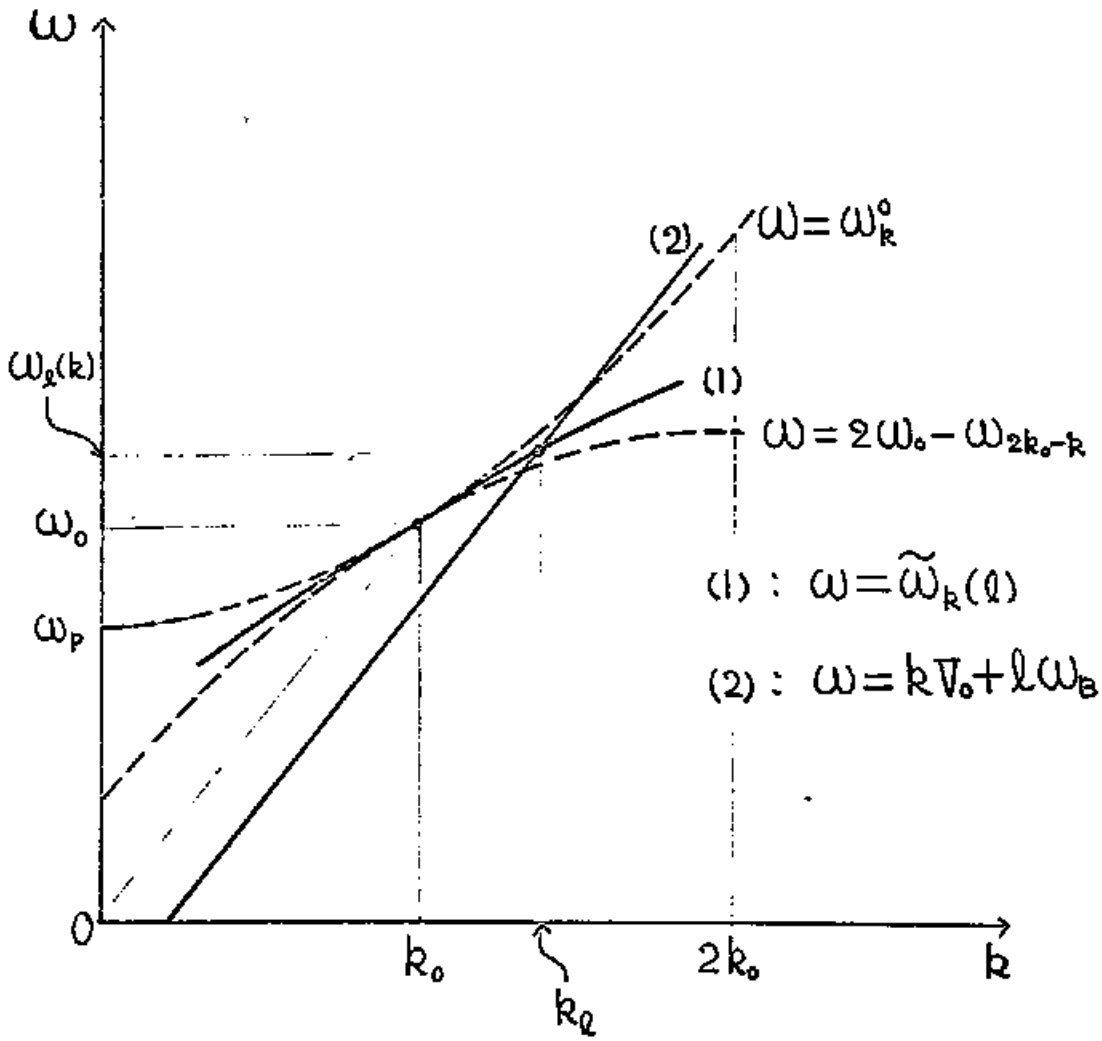

Fig. 1.