學位申請論文

小倉幸雄
Asymptotic Behavior of Multitype

Galton-Watson Processes

Yukio OGURA

0. Introduction

The asymptotic behavior of the distributions of multitype Galton-Watson processes has been studied by many mathematicians. According to the author's knowledge, Jirina [8] for subcritical processes is the first paper on this subject, and Chistyakov [4] and Mullikin [10] for critical processes followed. But they assumed that (i) the second moments (in the subcritical case) or the third moments (in the critical case) are finite and (ii) the mean matrix is positively regular. Joffe and Spitzer [9] obtained the results for discrete time processes without the hypothesis (i), and Sevastyanov [14] extended them for continuous time processes. Their results are final for the processes satisfying the condition (ii'). However, when the condition (ii) fails, somewhat different phenomena occur. Chistyakov [3] illustrated it for the continuous time subcritical processes with the hypothesis (i). For the continu-
ous time critical processes, the results of Savin and Chistyakov [12] for the processes with three particle types and the hypothesis (i) are very suggestive.

In this paper, we shall give the whole asymptotic behavior of discrete and continuous time multitype Galton-Watson processes without the hypotheses (i) and (ii) (but with some weaker hypotheses). The processes are decomposed into elementary subprocesses. When the elementary subprocesses have positively regular mean matrices, the results naturally coincide with those of [8], [9], [10] and [14]. But when they are reducible, the rate that the generating functions tend to the extinction probabilities are different from those of the positively regular cases. Furthermore for the processes with discrete time we must take care of the periodicity.

We shall give the definitions and notations in section 1. In section 2 we shall deal with the discrete time noncritical processes having aperiodic mean matrices, while we shall deal with those having periodic mean matrices in section 3. Sections
4 and 5 are devoted to the study of the discrete time critical processes. The results for the continuous time processes are summarized in section 6, and some examples are given in section 7.
Definitions and notations

We designate the set of all integers between m and n by \(<m,n>\) and put \( Z_+ = <0, \infty> \), \( S = Z_+^N \) (\( N < 1, \infty > \)). If two vectors \( s_1 = (s_1^1, \ldots, s_1^N) \) and \( s_2 = (s_2^1, \ldots, s_2^N) \) satisfy \( s_1^i > s_2^i \) for all \( i \in <1, N> \), we say that \( s_1 \) is larger [resp. not less] than \( s_2 \) and write as \( s_1 > s_2 \) [resp. \( s_1 \geq s_2 \)]. Thus we can naturally define the maximum, minimum, monotony, etc., of a sequence of vectors.

Further, these notions and notations are extended for matrices in the natural way. For example a matrix \( A \) is called nonnegative if all its components are nonnegative, and in this case we write as \( A \geq 0 \). Let \( A \) be a nonnegative square matrix of order \( k \). We call \( A \) positively regular if \( A^n > 0 \) for some \( n < 1, \infty > \), where \( A^n \) means the n-fold product of the matrix \( A \). Also the matrix \( A \) is called irreducible if for each \( i, j < 1, k \), \( i \neq j \), there is an \( n < 1, \infty > \) such that \( A_j^i(n) > 0 \), where \( A_j^i(n) \) is the \((i,j)\)-component of the matrix \( A^n \). Hence each nonnegative matrix of order 1 is always irreducible. We also call a square matrix \( A \) with nonnegative off-diagonal elements to be irreducible if the matrix \( a + \lambda I \) (> 0) is irreducible for some \( \lambda > 0 \) in the above sense, where
I is the identity matrix. For two vectors $s_1$ and $s_2$, we define new vectors $s_1 s_2$ and $s_1 / s_2$ (for $s_2 > 0$) by

$$s_1 s_2 = (s_1^1 s_2^1, \ldots, s_1^N s_2^N), \quad \frac{s_1}{s_2} = (s_1^1 / s_2^1, \ldots, s_1^N / s_2^N).$$

For each $s \in \mathbb{R}^N$ and $x \in S$ we set

$$s^x = (s^1)^x^1 \cdots (s^N)^x^N, \quad s = (s^1, \ldots, s^N), \quad x = (x^1, \ldots, x^N).$$

Finally we denote the $i$-th canonical unit basis by $e_i$, i.e.

$$e_i^j = \delta_i^j$$

where $\delta_i^j$ is the Kronecker's delta.

Now we shall call a Markov chain $X = (Z(n), P_x)$ on $S$ a discrete time $N$-type Galton-Watson process (DGWP for brevity), if its probability generating functions

$$F^X(n; s) = \sum_{y \in S} P_x \{Z(n) = y\} s^y, \quad x \in S, \quad n \in \mathbb{N}, \quad 0 < s \leq 1,$$

are given by

$$F^X(n; s) = F(n; s)^x,$$

for some vector functions $F(n; s) = (F^1(n; s), \ldots, F^N(n; s))$. Then it is clear that $F(n; s)$ is given by the $n$-fold iteration of the vector probability generating function $F(s) = F(1; s)$:

$$F(n+1; s) = F(F(n; s)), \quad n \in \mathbb{N}, \quad 0 < s \leq 1,$$

$$F(0; s) = s,$$

where
(1.3) \( F^i(s) = \sum_{y \in S} p^i(y) s^y, \quad i \in \langle 1, N \rangle, \)

with \( p^i(y) \geq 0 \) and \( \sum_{y \in S} p^i(y) \leq 1 \). Since the family of generating functions \( \{ F(n; s) \} \) uniquely determines a DGWP, we sometimes call \( \{ F(n; s) \} \) itself a DGWP.

Similarly a Markov process \( X = (Z(t), P_X) \) on \( S \) is called a continuous time \( N \)-type Galton-Watson process (CGWP), if its probability generating functions \( F^X(t; s) \) are given by

\[
(1.4) \quad F^X(t; s) = F(t; s)^x, \quad x \in S, \quad t \in [0, \infty), \quad 0 \leq s \leq 1,
\]

where \( F(t; s) = (F^1(t; s), \ldots, F^N(t; s)) \) is the unique solution of

\[
(1.5) \quad \frac{dF(t; s)}{dt} = f(F(t; s)), \quad t > 0,
\]

\[
(1.6) \quad f^i(s) = \sum_{y \in S} p^i(y) s^y, \quad i \in \langle 1, N \rangle,
\]

with \( p^i(y) \geq 0 \), \( y \neq \epsilon_1 \), and \( \sum_{y \in S} p^i(y) \leq 0 \). Also, we sometimes call the family of generating functions \( \{ F(t; s) \} \) itself a CGWP.

It is shown by Sevastyanov ([13],[14]) that for a DGWP [CGWP] there exists at least nonnegative fixed point \( q \) of \( F(s) \) [resp. zero point \( q \) of \( f(s) \)] in the cube \( 0 \leq s \leq 1 \), and it is stable in the sense of
\[
\lim_{n \to \infty} F(n; s) = q \quad \text{[resp. } \lim_{t \to \infty} F(t; s) = q], \quad 0 \leq s \leq q.
\]

Especially it holds
\[
P_{e_1} \{ T < \infty \} = \lim_{n \to \infty} F^1(n; 0) = q^1
\quad \text{[resp. } P_{e_1} \{ T < \infty \} = \lim_{t \to \infty} F^1(t; 0) = q^1],
\]

where \( T \) is the first hitting time for the trap state \( 0 \in S \), namely the extinction time. Hence we shall call \( q \) the extinction probability of the DGWP [resp. CGWP]. Let \( R(s) = f^1 - F^1(s) \) and \( R(\infty; s) = e - F(\infty; s) \).

An object of the present paper is to obtain an exact estimate of \( R(s) \) for critical and subcritical\( q \)

For a DGWP, we shall assume
\[
(D) \quad q > 0 \quad \text{and} \quad F^1_j(q) < \infty, \quad i, j \in \{1, N\},
\]

where \( F^1_j(s) = \partial F^1(s) / \partial s^j \) if it exists and \( F^1_j(s) = \lim_{\xi \to \infty} F^1_j(\xi) \) otherwise.

Note that when the DGWP is critical with no final classes or subcritical, \( q = 1 > 0 \) holds. We call the matrix
\[
A = [A^{1}_{j,1}, j = 1]^{N}_{1,1} = [F^1_j(q)]^{N}_{1,1}
\]

the \( q \)-mean matrix of the DGWP. Since \( A \geq 0 \), there exists a non-negative characteristic root \( \rho(A) \) of \( A \) which is not smaller in absolute value than any other characteristic roots (cf. Gantmacher [6]). We call it the Perron-Frobenius root (P-F root for brevity) of the matrix \( A \). From the definition of \( q \), the inequality \( \rho(A) \leq 1 \) easily follows. It is known that by a change
of suffixes the nonnegative matrix $A$ is represented as

$$
A = \begin{bmatrix}
A_1 & 0 & \cdots & 0 \\
A_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\# & \# & \cdots & A_g
\end{bmatrix},
$$

where each $A_\alpha$ is an irreducible square matrix of order $m_\alpha \in \{1, N\}$

$$
(\sum_{\alpha=1}^{\Sigma} m_\alpha = N).
$$

We set

$$
\Gamma^i = \{ j \in \{1, N\} ; A^i_j(n) > 0 \text{ for some } n \in \{1, \infty\} \cup \{1\},
$$

$$
\Delta_\alpha = \left< \sum_{\beta=1}^{\alpha-1} m_\beta + 1, \sum_{\beta=1}^{\alpha} m_\beta \right> \quad (\Delta_1 = \left< 1, m_1 \right>).
$$

Since every $A_\alpha$ is irreducible, $\Delta_\beta \subseteq \Gamma^i$ if $\Gamma^i \cap \Delta_\beta \neq \emptyset$, and $\Gamma^i = \Gamma^i'$ if $i, i' \in \Delta_\alpha$. Hence $\Gamma^i$ is a disjoint union of some $\Delta_\beta$'s and it is same for all $i \in \Delta_\alpha$, which we denote by $\Gamma_\alpha$. We also set $\overline{\Gamma}_\alpha = \Gamma_\alpha - \Delta_\alpha$.

The $\Gamma_\alpha$-part $(s^1)_i \in \Gamma_\alpha$ $[\overline{\Gamma}_\alpha$-part $(s^1)_i \in \overline{\Gamma}_\alpha$, $\Delta_\alpha$-part $(s^1)_i \in \Delta_\alpha]$ of a vector $s = (s^1, \ldots, s^N)$ is denoted by $s_\alpha$ [resp. $\overline{s}_\alpha, \overline{s}_\alpha$]. From (1.3) and (1.8) it follows that the generating function $F^i(s)$ for $i \in \Gamma_\alpha$ $[i \in \overline{\Gamma}_\alpha]$ only depends on $s_\alpha$ [resp. $\overline{s}_\alpha$]. Hence we can write as

$$
F(s)_\alpha = F(s_\alpha)_\alpha \quad [\text{resp. } \overline{F}(s)_\alpha = \overline{F}(\overline{s})_\alpha].
$$

Similarly, since $F^i(n; s)$ for $i \in \Gamma_\alpha$ $[i \in \overline{\Gamma}_\alpha]$ only depends on $s_\alpha$ $[\overline{s}_\alpha, \overline{s}_\alpha]$ by (1.2), we can write as

$$
(1.9) \quad F(n; s)_\alpha = F(n; s_\alpha)_\alpha, \quad 0 \leq s_\alpha \leq 1
$$

in $[\text{resp. } \overline{F}(n; s)_\alpha = \overline{F}(n; \overline{s}_\alpha)_\alpha, \quad 0 \leq \overline{s}_\alpha \leq 1].$
We set \( S_\alpha = \{ x_\alpha = (x_i)_{i \in \Gamma_\alpha} : x_i^i < 0, \infty \} \). The family of generating functions \( \{ F(n; s_\alpha) : n \leq 0, \infty \} \) forms a DGWP on \( S_\alpha \), which we denote by \( X_\alpha = (Z_\alpha(n), F_\alpha^X) \). Note that the extinction probability of the DGWP \( X_\alpha \) is equal to the \( \Gamma_\alpha \)-part \( q_\alpha \) of the extinction probability \( q \) of the original DGWP \( X \) by (1.7), and hence the submatrix\( A_\alpha = [A_{i,j}]_{i,j \in \Gamma_\alpha} \) coincides with the \( q \)-mean-matrix of \( X_\alpha \).

Further it follows

\[
F_{i,j}^1(n; q) = F_{i,j}^1(n; q_\alpha) = (A_\alpha(n))^{i,j} = A_{i,j}^1(n), \quad i,j \in \Gamma_\alpha.
\]

Since \( \rho(A) \leq 1 \), \( \rho_\alpha \equiv \rho(A_\alpha) \leq 1 \) holds. We call the DGWP \( X_\alpha \) critical if \( \rho_\alpha = 1 \) and noncritical if \( \rho_\alpha < 1 \).

For a CGWP, we assume

\[(C) \quad q > 0 \text{ and } f_j^1(q) < \infty, \quad i,j \in \{1, N\}.
\]

We call the matrix

\[
a = [a_{i,j}^1]_{i,j=1}^N = [f_j^1(q)]_{i,j=1}^N
\]

the infinitesimal \( q \)-mean matrix of the CGWP \( X \). Since (1.6) implies \( a + \lambda I \geq 0 \) for some \( \lambda > 0 \), there is a real characteristic root \( \rho(a) \) of \( a \) which is not smaller in real part than any other characteristic roots of \( a \). In this case \( \rho(a) \leq 0 \) holds (cf. Ogura [11]). By a change of suffixes the matrix \( a \) is represented as
where each $\tilde{a}_\alpha$ is an irreducible square matrix of order $m_\alpha$ ($\sum_{\alpha=1}^{m_\alpha} m_\alpha = N$). We define the sets $\Lambda_\alpha$, $\Gamma_\alpha$ and $\bar{\Gamma}_\alpha$ as in the discrete time case but from the matrix $a+\lambda I$ ($\lambda \geq 0$) instead of $A$. By (1.6) and (1.10) the function $f^I(s)$ for $i \in \Gamma_\alpha$ [$i \in \bar{\Gamma}_\alpha$] only depends on $s_\alpha$ [resp. $\bar{s}_\alpha$], and we write as

$$f(s)_\alpha = f(s_\alpha)_\alpha, \quad 0 \leq s_\alpha < 1$$

[resp. $f(s)_\alpha = f(\bar{s}_\alpha)_\alpha, \quad 0 \leq \bar{s}_\alpha < 1$.]

Hence $F^I(t;s)$ for $i \in \Gamma_\alpha$ [$i \in \bar{\Gamma}_\alpha$] only depends on $s_\alpha$ [resp. $\bar{s}_\alpha$] by (1.5), so that we can write as

$$F(t;s)_\alpha = F(t;s_\alpha)_\alpha, \quad 0 \leq s_\alpha < 1$$

[resp. $F(t;s)_\alpha = F(t;\bar{s}_\alpha)_\alpha, \quad 0 \leq \bar{s}_\alpha < 1$.]

We designate the CGWP $\{F(t;s_\alpha)_\alpha; t \in [0,\infty)\}$ by $X_\alpha = (Z_\alpha(t), P^\alpha_\alpha)$. The extinction probability of the CGWP $X_\alpha$ is equal to the $\Gamma_\alpha$-partition $q_\alpha$ of that of the CGWP $X$, and the submatrix $a_\alpha = [a^I_{ij}], i \in \Gamma_\alpha$ coincides with the infinitesimal $q$-mean matrix of $X_\alpha$. Moreover, setting
\[ A(t) \equiv [A_{ij}^a(t)]_{1, j=1}^{N} = \exp'(ta), \]
\[ A_\alpha(t) \equiv [A_{ij}^\alpha(t)]_{1, j=1}^{\Gamma_\alpha} = \exp'(ta_\alpha), \]

we have

\[ (1.13) \quad P^i_j(t; q) = P^i_j(t; q_\alpha) = A^i_j^\alpha(t) = A^i_j(t), \quad i, j \in \Gamma_\alpha. \]

Since \( \rho(a) \leq 0 \), \( \sigma = \rho(a) \leq 0 \) holds. We call the CGWP \( X_\alpha \) critical if \( \sigma = 0 \), and noncritical if \( \sigma < 0 \).

\[ \Box \text{Noncritical aperiodic DGWP} \]

In this section we shall deal with noncritical DGWP's with the assumption

\[ (DN) \quad P^i_j(y; q) = \log y^j < \infty, \quad 1, j \in \{1, N\}. \]

We shall also assume that all the matrices in this section are aperiodic, i.e.

\[ \{n \in \mathbb{N} ; A^i_j(n) > 0\} = 1, \quad 1, j \in \Gamma_\alpha. \]

Since \( \lambda_\alpha \) is irreducible, it is positively regular if it is not equal to the zero matrix of order 1. Hence there correspond positive right and left eigenvectors \( \tilde{u}_\alpha = (\tilde{u}_\alpha^i)_{i \in \Gamma_\alpha} \) and \( \tilde{v}_\alpha = (\tilde{v}_\alpha^i)_{i \in \Gamma_\alpha} \) to the P-F root \( \tilde{\rho}_\alpha \equiv \rho(\lambda_\alpha) \);

\[ \lambda_\alpha \tilde{u}_\alpha = \tilde{\rho}_\alpha \tilde{u}_\alpha, \quad \tilde{v}_\alpha \lambda_\alpha = \tilde{\rho}_\alpha \tilde{v}_\alpha, \]

with the normalizations
\[ i \in \sum_{\alpha} \bar{v}_\alpha \bar{u}_\alpha = 1, \quad i \in \sum_{\alpha} \bar{u}_\alpha = 1 \]

(Gantmacher [6]). It is also known that as \( n \to \infty \)

\[ \lambda^n = \beta^n (\lambda^* + o(1)), \]

where \( \lambda^* = [\lambda^*_{ij}] = [\lambda^*_{ij}]_{i, j} \Delta^* \). Of course it holds

\[ \lambda^* \lambda^* = \lambda^* \lambda^* = \beta^* \lambda^* \lambda^*, \quad \lambda^* \lambda^* = \lambda^*. \]

In order to define the 'rank \( v_\alpha \)' of \( \alpha \), we shall introduce the semiorder '\( \prec \)' in the space of indices \( \langle 1, g \rangle \) by

\[ \beta \prec \alpha \quad \text{if} \quad \Delta^*_\beta \Gamma^*_\alpha. \]

Next we define the rank \( v_\beta(r) \) of \( \beta \) w.r.t. \( r \) by

\[ v_\beta(r) = \begin{cases} \max \{ v_\gamma(r); \gamma \not\succ \beta \}, & \text{if } \tilde{\rho}_\beta \neq r, \\ \max \{ v_\gamma(r); \gamma \not\succ \beta \} + 1, & \text{if } \tilde{\rho}_\beta = r, \end{cases} \]

inductively, where we agree on \( \max add \phi = 1 \). Then the rank \( v_\alpha \) of \( \alpha \) is given by

\[ v_\alpha = v_\alpha(\rho_\alpha). \]

Note that \( v_\alpha \in \langle 0, g-1 \rangle \) since \( \tilde{\rho}_\beta = \rho_\alpha \) for some \( \beta \prec \alpha \).

To state the theorem we shall define one more set:

\[ I_+(x) = \{ \alpha \in \langle 1, g \rangle; x_\alpha \neq 0 \}, x.S. \]

**Theorem 2.1.** Let a DGWP \( X=(\mathbb{Z}(n), P_X) \) satisfy Conditions (D) and (DN) for each \( \alpha \in \langle 1, g \rangle \) with \( \rho_\alpha \leq 1 \), and the matrices \( \lambda^*_\alpha \) be all
aperiodic. Then, 1) for each \( \alpha < \lambda, g \) with \( \rho_\alpha < 1 \) there correspond monotone nonincreasing functions \( R^{1}(s_\alpha) \) in \( 0 \leq s_\alpha \leq q_\alpha \), \( \iota \in \Lambda_\alpha \)

such that as \( n \to \infty \)

\[
R^{1}(n; s) = n^{\alpha_\rho} n^{R^{1}(s_\alpha) + o(1)), \quad \iota \in \Lambda_\alpha,
\]

where \( o(1) \) is uniform in \( s \) on \( 0 \leq s_\alpha \leq q_\alpha \). The \( R^{1}(s_\alpha) \) are determined inductively w.r.t. the semiorder '\( \prec \)' from Lemmas 2.1 and 2.4 below. Further, if \( \rho_\alpha > 0 \), every \( R^{1}(s_\alpha), \quad \iota \in \Lambda_\alpha \), is not identically zero.

2) For each \( x \in S \) such that \( \rho_\alpha < 1 \) holds for all \( \alpha \in I_+(x) \), and \( \rho_\alpha > 0 \) for some \( \alpha \in I_+(x) \), there corresponds a probability distribution \( \{ P^\#(y) \} \) on \( S-\{0\} \) satisfying

\[
\lim_{n \to \infty} P^\#(Z(n) = y | n < T < \infty) = P^\#(y).
\]

We shall prove this theorem by the induction w.r.t. the semiorder '\( \prec \)'. When \( \alpha \) is minimal, \( \Gamma_\alpha = \Lambda_\alpha \) and \( A_\alpha = X_\alpha \).

Hence,

the q-mean matrix \( A_\alpha \) is positively regular, if \( \rho_\alpha > 0 \), i.e. \( A_\alpha \neq [0] \).

In this case there are the following excellent results given by Joffe and Spitzer [9].

Lemma 2.1 (Joffe and Spitzer). Let the q-mean matrix \( A_\alpha \) of the DGWP \( X_\alpha \) is positively regular and \( \rho_\alpha < 1 \). Then there exist a
monotone nonincreasing function $K^*_\alpha(s_\alpha)$ in $0 \leq s_\alpha \leq q_\alpha$ and a distribution $\{P^\alpha(y_\alpha)\}$ on $S_\alpha$, such that

\[
\lim_{n \to \infty} \frac{q_\alpha - F(n; s_\alpha)}{\rho^n_\alpha} = K^*_\alpha(s_\alpha) \tilde{u}_\alpha, \quad 0 \leq s_\alpha \leq q_\alpha.
\]

Further $K^*_\alpha(s_\alpha) \neq 0$ if and only if (DN) holds.

When $\alpha$ is not minimal, $\Gamma_\alpha \neq \emptyset$ and the $q$-mean matrix $A_\alpha$ is represented as

\[
A_\alpha = \begin{bmatrix}
\tilde{A}_\alpha & 0 \\
A_\alpha^t & \bar{A}_\alpha
\end{bmatrix},
\]

where

\[
\tilde{A}_\alpha = [A^1_{ij}]_{i,j \in \Gamma_\alpha}, \quad A_\alpha^t = [A^1_{ij}]_{i \in \Delta_\alpha, j \in \Gamma_\alpha} + 0.
\]

We put $\rho_\alpha = \rho(\bar{A}_\alpha)$. Then $\rho_\alpha$ is equal to the maximum $\bar{\rho}_\alpha \vee \bar{\rho}_\alpha$ of $\check{\rho}_\alpha$ and $\bar{\rho}_\alpha$.

Let $R(s) = q - E(s)$ and $R(n; s) = q - P(n; s)$. Then it is given by 4.6. Joffe and Spitzer [8] (4.6) that

\[
R(s) = (A - E(s)) (q - s), \quad 0 \leq s \leq q,
\]

\[
E^1_j(s) = \sum_{y \in S} P^1(y) y^j \left\{ q - E^1_j \int_{0}^{1} (q - (q-s) \xi)^{y-e_j} d\xi \right\},
\]

where we agree on $s^y = 0$ for $y \in S$. (2.11) implies
\[0 \leq E(s) \leq E(s) \leq A, \quad 0 \leq s \leq q,\]
\[(2.12)\]
\[E(s) \rightarrow 0, \quad \text{as } s \rightarrow q \text{ in } 0 \leq s \leq q.\]

We set \(E(n; s) = E(F(n; s))\) and \(C(n; s) = A - E(n; s)\). We define the matrices \(E(n; s)\), \(C(n; s)\), \(E(n; s)\), etc. in the natural way.

From (1.3), (1.8), (1.9) and (2.11) it follows
\[(2.13) \quad E(n; s) = E(n; s)\), \quad C(n; s) = C(n; s), \quad 0 \leq s \leq q.\]

Hence (2.10) implies
\[R(n+1; s) = R(n+1; s) = C(n; s) R(n; s), \quad 0 \leq s \leq q,\]
and with the aid of (2.9) and (2.13)
\[(2.14) \quad \tilde{R}(n+1; s) = \tilde{C}(n; s) \tilde{R}(n; s) + C(n; s) \tilde{R}(n; s).\]

Using (2.14) inductively, we obtain
\[(2.15) \quad \tilde{R}(n+1; s) = \tilde{D}(n, l)(\tilde{q} - \tilde{s}) + \sum_{l=0}^{n} \tilde{D}(n, l) C(l; s) \tilde{R}(l; s),\]
where
\[(2.16) \quad \tilde{D}(n, l) = \tilde{D}(n, l; s)\]
\[= \begin{cases} \tilde{C}(n; s) \tilde{C}(n-1; s) \cdots \tilde{C}(l+1; s), & l \leq -1, n-1, \\ I, & l = n. \end{cases}\]

Lemma 2.2. If Condition (DN) and the inequality \(p_{\alpha} < 1\) are satisfied, then it holds
\[(2.17) \quad \sum_{n=0}^{\infty} E(n; 0) < \infty.\]
Proof. From the convexity of the function $F^*(n; s+(q-s)\xi)$ in $0 \leq \xi \leq 1$, it follows $q_\alpha - F(n; 0)_\alpha \leq A_{\alpha}^n q_\alpha$. Applying the same arguments as in the proof of Lemma 2.5 below to the matrices $A^n$, we obtain

$$A_{\alpha}^n q_\alpha \leq n^{\alpha} \rho_\alpha^n K q_\alpha \leq r^n q_\alpha,$$

where $K$ is a positive square matrix with the indices in $\Gamma_\alpha$, and $r$ and $\theta$ are constants with $\rho_\alpha < r < 1$ and $\theta > 0$. Hence it follows $F(n; 0)_\alpha \geq (1 - \theta r^n) q_\alpha$, and we obtain the conclusion by the same arguments as in Joffe and Spitzer [9](pp.424-425) with the aid of (2.11).

Lemma 2.3. The relations $p_\alpha > 0$ and (2.17) imply the existence of the limit

$$\lim_{n \to \infty} D_\alpha(n, \ell; s_\alpha) \rho_\alpha^{-n+\ell} = D^*(\ell; s_\alpha)$$

uniformly in $0 \leq s_\alpha \leq q_\alpha$. Further it holds

$$0 < \frac{D^*(\ell; s_\alpha)}{A_{\alpha}^*} \leq \frac{A_{\alpha}^*}{q_\alpha}, \quad 0 \leq s_\alpha \leq q_\alpha, \quad \ell \in (-1, \infty).$$
Proof. Let

\[ \varepsilon_n = \max\{ E_j(n;0)/\Lambda^j; j, \Lambda \in \Delta \} \]

Then it is clear that

\[ 0 \leq \tilde{E}(n;s) \leq \tilde{E}(n;0) \leq \varepsilon_n \Lambda^k \]

\[ \sum_{n=0}^{\infty} \varepsilon_n \leq \sum_{n=0}^{\infty} \sum_{j, \Lambda \in \Delta} \tilde{E}_j(n;0)/\Lambda^j < \infty, \]

by (2.17). On the other hand, there is a sequence \( \alpha_n \to 0 \),

\[ \alpha_n \geq 0, \text{ by (2.1) satisfying} \]

\[ (1 - \alpha_n) \Lambda^k \leq \Lambda^n - \alpha_n \leq (1 + \alpha_n) \Lambda^k. \]

Hence it follows

\[ \bar{\rho}^{-n+\ell} \bar{D}(n,\ell) \leq \bar{\rho}^{-n+\ell} \Lambda^{n-\ell} \leq (1 + \alpha_{n-\ell}) \Lambda^k, \]

and with the aid of (2.2) and (2.21)

\[ \bar{\rho}^{-n+\ell} \bar{D}(n,\ell) = \frac{\bar{\Lambda}^k}{\bar{\rho}^{n+\ell}} \prod_{k=\ell+1}^{n} \frac{(1 - \varepsilon_k/\bar{\rho}) \Lambda^k}{\Lambda^k} \]

\[ = \bar{\Lambda}^{n-\ell} \bar{\rho}^{-n+\ell} \prod_{k=\ell+1}^{n} \left(1 - \frac{\varepsilon_k/\bar{\rho}}{n+\ell} \Lambda^k \right) \]

\[ \leq (1 - \alpha_{n-\ell} - \sum_{k=\ell+1}^{n} \frac{\varepsilon_k/\bar{\rho}}{n+\ell}) \Lambda^k, \]
for all large $l$ with $\frac{\varepsilon_k}{k^{1/2}} \leq 1$, $k \in \langle l, \infty \rangle$. Therefore we obtain

$$\text{(2.24)} \quad -(\alpha_{n-l} + \sum_{k=\frac{2}{\varepsilon}}^{n} \varepsilon_k / k) A \leq \tilde{D}(n, l) - A \leq \alpha_{n-l} A.$$  

Now take any $\varepsilon > 0$. Then by (2.22) we can choose an $n_0$ such that

$$\sum_{k=n_0+1}^{\infty} \varepsilon_k / k \leq \varepsilon.$$  

Further, it holds

$$\tilde{D}(n_1, l) - \tilde{D}(n_2, l) = (\tilde{D}(n_1, n_0) - \tilde{D}(n_2, n_0)) \tilde{D}(n_0, l),$$  

and $\tilde{D}(n_0, l)$ is bounded in $n_0$, because of (2.23).

Hence it follows that the sequence

$$\tilde{D}(n, l), \quad n \in \langle l, \infty \rangle,$$

is a Cauchy sequence uniformly in $0 \leq s_a \leq q_a$. So we obtain (2.18). Now we shall show (2.19). Letting $n \rightarrow \infty$ in (2.24), we have $\tilde{D}(n_0) > 0$ for all sufficiently large $n_0$. Since

$$\tilde{D}(n, l) = \tilde{D}(n_0) \tilde{D}(n_0, l),$$

it holds

$$\text{(2.25)} \quad \tilde{D}(l) = \tilde{D}(n_0) \tilde{D}(n_0, l).$$
On the other hand it follows from (2.11) that \( A^i_j > 0 \) implies

\[ A^i_j - E^i_j > 0, \]

so that

\[ C^i_j(k) > 0, \quad \text{if} \quad A^i_j > 0. \]

Since the matrix \( A \) is positively regular \( A^{n_0 - \ell} > 0 \) for a large \( n_0 \). Combining these facts with (2.25) we have \( D^*(\ell) > 0 \).

The relation \( D^*(\ell) \leq A^* \) is clear, if we let \( n \to \infty \) in (2.23).

**Corollary 2.1.** Suppose that Condition (DN) holds and \( \alpha \) is minimal w.r.t. the semiorder \( \prec \) with \( \rho_\alpha < 1 \). Then the limit of (2.7) is uniform in \( 0 < s_\alpha \leq q_\alpha \) and \( K^*_\alpha(0) > 0 \).

The proof is clear from Lemmas 2.2 and 2.3, since

\[ \tilde{R}(n;s_\alpha) = \tilde{D}_\alpha(n,-1;s_\alpha)(\tilde{g}_\alpha - \tilde{s}_\alpha) \]

in this case.

Now we assume that for all \( \beta \not< \alpha \)

\[ (2.26) \quad \tilde{R}(n;s_\beta) = n^\beta \rho^\beta_\beta(n(\tilde{R}^*_\beta(s_\beta) + o(1)), 0 < s_\beta \leq q_\beta, \]

as \( n \to \infty \), where \( o(1) \) is uniform in \( 0 < s_\alpha \leq q_\alpha \). Then it follows as \( n \to \infty \).
(2.27) \( \tilde{\mathbf{R}}(n; s_\alpha) = n^{v_\alpha} \rho_\alpha n(\tilde{R}_\alpha(s_\alpha) + o(1)), \ 0 \leq s_\alpha \leq q_\alpha, \)

for some vector valued function \( \tilde{R}_\alpha(s_\alpha) \), where \( o(1) \) is uniform in \( 0 \leq s_\alpha \leq q_\alpha \) and

\[
\overline{v}_\alpha = \max \{ v_\beta (\overline{R}_\alpha) ; \beta \leq \alpha \}.
\]

Hence, it is enough for (2.5) to prove the following

**Lemma 2.4.** Let (2.17), (2.27) and \( \rho_\alpha < \frac{1}{\beta} \) hold. Then it follows

(2.28) \( \tilde{\mathbf{R}}(n; s_\alpha) = n^{v_\alpha} \rho_\alpha n(\tilde{R}_\alpha(s_\alpha) + o(1)), \ 0 \leq s_\alpha \leq q_\alpha, \)

where \( o(1) \) is uniform in \( 0 \leq s_\alpha \leq q_\alpha \), \( v_\alpha \) and \( \tilde{R}_\alpha(s_\alpha) \) are given separately in the following three cases: (i) if \( \rho_\alpha = \tilde{\rho}_\alpha \), then \( v_\alpha = 0 \) and

(2.29) \( \tilde{R}_\alpha(s_\alpha) = \tilde{D}(-1; s_\alpha)(\tilde{\alpha}_\alpha - \tilde{s}_\alpha) + \sum_{l=0}^{\infty} \tilde{D}_\alpha(l; s_\alpha) C(k; s_\alpha) \tilde{R}(l; s_\alpha) \rho_\alpha^{-l-1}, \)

(ii) if \( \rho_\alpha = \tilde{\rho}_\alpha > \tilde{\rho}_\alpha \), then \( v_\alpha = \overline{v}_\alpha \) and

(2.30) \( \tilde{R}_\alpha(s_\alpha) = (\rho_\alpha I - \tilde{A}_\alpha)^{-1} \tilde{R}_\alpha(s_\alpha), \)

and (iii) if \( \rho_\alpha = \tilde{\rho}_\alpha = \overline{\rho}_\alpha > 0 \), then \( v_\alpha = \overline{v}_\alpha + 1 \) and
(2.31) \[ R^*(s, \alpha) = \frac{A^A A^R(s, \alpha)}{\alpha^A \alpha^R(s, \alpha)} \]

**Proof:** (i) When \( \rho = \rho > \rho_0 \), we divide the sum in (2.15) into \( \sum_{n=0}^{n_0} \) and \( \sum_{n=n_0+1}^{n} \). For each \( \rho > r > \rho_0 \) we have from (2.23) and (2.27) that

\[
\rho^{-n-1} D(n, \ell) C(\ell) R(\ell) \leq (r - 1)^{n_0} c,
\]

where \( c \) is a positive vector with the indices in \( \Delta \).

Hence it follows

\[
\rho^{-n-1} \sum_{\ell=n_0+1}^{n} \tilde{D}(n, \ell) C(\ell) R(\ell) \leq \left( \frac{(r - 1)^{n_0}}{1 - r \rho^{-1}} \right) c < \varepsilon, \quad n \in < n_0 + 1, \infty >
\]

for all sufficiently large \( n_0 \). Similarly, for all large \( n_0 \), it holds

\[
\sum_{\ell=n_0+1}^{n} \tilde{D}^*(\ell) C(\ell) R(\ell) \rho^{-\ell} \leq \varepsilon, \quad n \in < n_0 + 1, \infty >
\]

uniformly in \( 0 \leq s \leq q \). But for a fixed \( n_0 \), (2.18) implies

\[
\rho^{-n-1} \{ \tilde{D}(n, -1)(\tilde{q} - \tilde{s}) + \sum_{\ell=0}^{n_0} \tilde{D}(n, \ell) C(\ell) R(\ell) \}
\]

\[
\longrightarrow \tilde{D}^*(-1)(\tilde{q} - \tilde{s}) + \sum_{\ell=0}^{n_0} \tilde{D}^*(\ell) C(\ell) R(\ell) \rho^{-\ell-1}
\]

as \( n \to \infty \), uniformly in \( 0 \leq s \leq q \). Hence we have (2.28) with
\( v = 0 \) and \( R^* \) given by (2.29).

(ii). When \( \rho = \rho > \rho_0 \), we shall exploit (2.15) in the form of

\[
\tilde{R}(n+1) = \tilde{D}(n,-1)(q-\lambda) + \sum_{\ell=0}^{n} \tilde{D}(n,n-\ell)C(n-\ell)\tilde{R}(n-\ell),
\]

dividing the sum into \( \sum_{0}^{n_0} \) and \( \sum_{n_0+1}^{n} \). From (2.23) and (2.27) it follows

\[
(n+1)^{V-\rho} \sum_{\ell=0}^{n_0} \tilde{D}(n,n-\ell)C(n-\ell)\tilde{R}(n-\ell) \leq (\rho_0^{-1})^{q+1} c,
\]

so that

\[
(n+1)^{V-\rho} \sum_{\ell=n_0+1}^{n} \tilde{D}(n,n-\ell)C(n-\ell)\tilde{R}(n-\ell) \leq \frac{(\rho_0^{-1})^{n_0}}{1-\rho_0^{-1}} c < \epsilon, \quad n < n_0+1,00>.
\]

for all sufficiently large \( n_0 \). Similarly, it holds for all large \( n_0 \) that

\[
\sum_{\ell=n_0+1}^{n} \rho^{-n-1} \tilde{A}^\ell \tilde{R}^* \leq \epsilon, \quad \text{uniformly in} \quad 0 \leq s \leq q,
\]

by means of \( \tilde{R}^*(s) < \tilde{R}^*(0) < \infty \). Since

(2.32) \quad \tilde{A} \geq C(n) > A-E(n;0) \quad A

as \( n \to \infty \), we have for a fixed \( \ell \in (0,n_0) \) that

\[
\lim_{n \to \infty} \tilde{D}(n,n-\ell) = \tilde{A}^\ell, \quad \text{uniformly in} \quad 0 \leq s \leq q.
\]
Hence it follows from (2.27) that

\[
\lim_{n \to \infty} (n+1)^{-\nu} \rho^{-n-1} \sum_{\lambda=0}^{n_0} \tilde{R}(n, \lambda) \rho^{-1} \rho^{-1} = \sum_{\lambda=0}^{n_0} \rho^{-1} \lambda^\nu \Lambda^* \Lambda^*,
\]

uniformly in \(0 \leq s \leq q\). Finally (2.23) and the inequality \(\rho > \tilde{\rho}\) imply

\[
\lim_{n \to \infty} (n+1)^{-\nu} \rho^{-n-1} \tilde{D}(n, \nu) \bar{R}(\nu) = 0, \quad \text{uniformly in } 0 \leq s \leq q.
\]

Combining the above facts we obtain the conclusion.

(iii) Suppose that \(\rho = \tilde{\rho} = \rho_{\rho} > 0\). From (2.24), (2.22), (2.32) and (2.27) we can find \(n_0\) and \(n_1 \in \langle 1, \infty \rangle\) satisfying

\[
(2.33) -c \leq \nu \leq \rho^{-n_0} \tilde{D}(n, \nu) \bar{R}(\nu) - \Lambda^* \Lambda^* \nu \leq \tilde{c} \nu, \nu \in [n_0, n_1],
\]

for some vector \(\tilde{c} > 0\). Now we divide the sum in (2.15) like as

\[
\sum_{\nu} = \sum_{\nu=0}^{n_0} + \sum_{\nu=n_0+1}^{n-n_1} + \sum_{\nu=n-n_1+1} = I + II + III.
\]

Since the functions

\[
\rho^{-n-1}(n+1)^{-\nu} \tilde{D}(n, \nu) \bar{R}(\nu), \quad \nu \in \langle 0, n \rangle, \quad n \in \langle 0, \infty \rangle,
\]

are bounded in \(\nu\), \(n\) and \(s\) on \(0 \leq s \leq q\), it holds

\[
\lim_{n \to \infty} \rho^{-n-1}(n+1)^{-\nu-1}(I + III) = 0, \quad \text{uniformly in } s.
\]
Further it follows from (2.33) that

\[-\epsilon \rho^{-1} \leq \rho^{-n-1}(n+1)^{-1} \sum_{l=n_0+1}^{n-1} l^{-1} A A' R^{-1} (n+1)^{-1} l^{-1} \leq \epsilon \rho^{-1}.\]

Hence by the fact that

\[\lim_{n \to \infty} (n+1)^{-1} l^{-1} \sum_{l=n_0+1}^{n-1} l^{-1} = 1/(\bar{\nu}+1),\]

and the boundedness of \(\bar{R}^*\) in \(s\), we have

\[\lim_{n \to \infty} \rho^{-n-1}(n+1)^{-1} \sum_{l=0}^{n} \tilde{D}(n, l) C(l) \bar{R}(l) = \tilde{A} A' \bar{R}^*/\rho(\bar{\nu}+1),\]

uniformly in \(0 \leq s \leq q\). But since (2.23) implies

\[\lim_{n \to \infty} (n+1)^{-1} l^{-1} \sum_{l=0}^{n} \tilde{D}(n, l) (\bar{q}-\bar{s}) = 0, \quad \text{uniformly in} \quad 0 \leq s \leq q,\]

we obtain the conclusion.

Note that the routine to determine \(\nu_a\) from \(\bar{\nu}_a\) by Lemma 2.4 is the same as that of (2.3) - (2.4). Further, we have

\begin{lemma}
Under Condition (DN), the function \(R_a^*(s_a)\)
determined by Lemmas 2.1 and 2.4 for each \(i \in \Delta_a, a \in \langle 1, g \rangle\)
with \(0 < \rho_a < 1\), is not identically zero.
\end{lemma}

\textbf{Proof.} If \(a\) is minimal w.r.t. the semiorder \(\prec\), the
assertion is clear by Lemma 2.1. If \(\rho_a = \bar{\rho}_a > \bar{\rho}_a\), it is
also clear from (2.19) and Lemmas 2.4 and 2.2. To deal with
other cases, we assume that \( R^*_{\beta}(s_\beta) \neq 0 \) for all \( i \in \Delta_\beta \)
with \( \beta \prec \alpha \) satisfying \( \rho_\beta > 0 \). We choose a maximal element
\( \beta_0 \) in the set \( \{ \beta \prec \alpha ; \nu_\beta(\rho_\alpha) = \nu_\alpha \} \). This \( \beta_0 \) is also maxi-
mal in the set \( \{ \beta \prec \alpha \} \), since in general \( \beta \prec \alpha \) implies
\( \rho_\beta \leq \rho_\alpha \), and \( \beta \prec \alpha \), \( \rho_\beta = \rho_\alpha \) imply \( \nu_\beta \leq \nu_\alpha \). Indeed, if
it is not maximal in \( \{ \beta \prec \alpha \} \), there is a \( \beta \) such that
\( \beta_0 \prec \beta \prec \alpha \). Then it follows \( \rho_\beta = \rho_\beta = \rho_\alpha \), and so \( \nu_\beta \beta_0 = \nu_\beta = \nu_\alpha \),
which implies \( \nu_\alpha = \nu_\beta(\rho_\alpha) \) and leads a contradiction.

Now, since \( \nu_\alpha = \nu_\beta(\rho_\alpha) \), it follows
\[ R^*_{\beta}(s_\beta) = R^*_{\beta_0}(s_\beta) \neq 0, \quad i \in \Delta_\beta, \]
by (2.26) and (2.27), and since \( \beta_0 \) is maximal in the set \( \{ \beta \prec \alpha \} \)
it holds
\[ A_{ij} > 0, \quad \text{for some} \quad i \in \Delta_\alpha \quad \text{and} \quad j \in \Delta_\beta. \]

Hence the conclusion is clear from (2.30) - (2.31) since
\[ A_{ij} > 0 \quad \text{and, when} \quad \rho_\alpha > \beta_\alpha, \quad (\rho_\alpha I - A_\alpha)^{-1} > 0. \]

Proof of Theorem 2.1. Since 1) is clear from the previous
arguments, we have only to show 2). Combining the equality
\[ P_x\{T < \infty\} = \lim_{n \to \infty} F(n;0)^x = q^x \]

with the Markov property, we obtain

\[ \sum_{y \in S} P_x\{Z(n) = y, T < \infty\} s^y = \sum_{y \in S} P_x\{Z(n) = y\} q^y s^y \]

\[ = P(n;qs)^x. \]

Hence it follows

\[(2.34) \sum_{y \in S} P_x\{Z(n) = y|n < T < \infty\} s^y = 1 - \frac{q^x - F(n;qs)^x}{q^x - F(n;0)^x}. \]

Further by mean of (2.5) and (1.7) it holds as \( n \to \infty \)

\[(2.35) q^x - F(n;qs)^x = \sum_{\alpha \in I_+(x)} \sum_{i \in \Delta_{\alpha}} x_i^{i-x} q^{-e_i} n^\alpha \rho_\alpha^{-n(R^i(q_\alpha s_\alpha) + o(1))}, \]

where \( o(1) \) is uniform in \( 0 \leq s \leq 1 \). Hence there exists the limit

\[ F^*(s) = \lim_{n \to \infty} \sum_{y \in S} P_x\{Z(n) = y|n < T < \infty\} s^y, \]

uniformly in \( 0 \leq s \leq 1 \). Since \( R^i(q_\alpha) = 0 \), \( i \in \Delta_{\alpha} \), it is easily seen that \( F^*(1) = 1 \). Thus \( F^*(s) \) is a generating function of a probability distribution and we obtain the conclusions.

**Remark 2.1.** We can calculate the support of the limit distribution \( \{P_x^*(y)\} \) more precisely. Let \( \rho_x = \max\{\rho_\alpha ; \alpha \in I_+(x)\} \),
\[ \nu = \max \{ \nu \alpha ; \alpha \in I^*(x) \}, \quad \rho = \rho_x \] and \[ I^*(x) = \{ \alpha \in I^+_+(x) ; \rho = \rho_x, \nu = \nu_x \}. \]

Then it is clear from (2.5), (2.6), (2.34) and (2.35) that the support of the limit distribution \( \{ P^*_x(y) \} \) is contained in the set

\[ \{ x = (x^1, \ldots, x^N) \in S ; x^1 = 0, \quad 1 \notin \bigcup_{\alpha \in I^*(x)} \Gamma_\alpha \} - \{ 0 \}. \]

**Remark 2.2.** It can also be calculated how the limit distributions \( \{ P^*_x(y) \} \) depend on \( x \in S - \{ 0 \} \). Indeed, it follows from (2.34) and (2.35) that

\[
\sum_{y \in S} P^*_x(y) s^y = F_x(s) = 1 - \sum_{\alpha \in I^*(x)} \sum_{i \in \Delta_\alpha} x^i q^i R^*(q, s^i) \cdot \frac{\sum_{\alpha \in I^*(x)} \sum_{i \in \Delta_\alpha} x^i q^i R^*(q, s^i)(0)}{\sum_{\alpha \in I^*(x)} \sum_{i \in \Delta_\alpha} x^i q^i R^*(0)}.
\]

Further, if \( \bar{\rho}_\alpha \geq \underline{\rho}_\alpha \) or \( \alpha \) is minimal w.r.t. the semiorder \( \prec \), it holds

\[ \bar{R}^*_\alpha(s^\alpha) = K^*_\alpha(s^\alpha) \bar{u}_\alpha, \]

for some monotone nonincreasing function \( K^*_\alpha(s^\alpha) \), since (2.7) holds, and (2.29) and (2.31) imply \( \bar{R}^*_\alpha(s^\alpha) = \bar{\rho}_\alpha \bar{R}^*(s^\alpha) \). In the case of \( \bar{\rho}_\alpha \prec \bar{\rho}_\alpha \), (2.30) will give us the sufficient informations for the purpose.
Remark 2.3. From (2.5) it easily follows that

\[(2.36) \quad R^*(F(n; s)_{\alpha}) = \rho_{\alpha}^n R^*(s_{\alpha}), \quad i \in \Delta_{\alpha},\]

if $0 < \rho_{\alpha} < 1$. Hence the coefficients of the power series

\[(\log R^*(s)/R^*(0))/\log \rho_{\alpha} \]

give a stationary measure of the DGWP $X_{\alpha}$
on $S_{\alpha} - \{0\}$.

3. Noncritical periodic DGWP

In this section we shall deal with the noncritical DGWP's with the periodic matrices $A_{\alpha}$. It is known that, by a change of suffixes, an irreducible nonnegative matrix $M_{\alpha}$ is represented as

\[(3.1) \quad M = \begin{bmatrix}
0 & M_1 & 0 & \cdots & 0 \\
0 & 0 & M_2 & 0 & \cdots
\end{bmatrix}
\]

where every $0$ matrix on the diagonal is a square matrix and each $Q_{\alpha} = M_c \cdots M_1 M_{d-1}$ is positively regular (Doob[5])
We shall call the positive integer $d$ the period of the matrix $M$. Of course the $d$-fold product $M^d$ of $M$ is given by

$$M^d = \begin{bmatrix} Q_0 & \cdots & 0 \\ 0 & Q_0 & \cdots \\ \vdots & \vdots & \ddots \\ 0 & \cdots & Q_0 \end{bmatrix}.$$

Lemma 3.1. The P-F root of the matrix $Q_0$ is equal to $\rho(M)^d$.

Proof. The set of all characteristic roots of $M^d$ is the union of the sets of characteristic roots of $Q_0$, $\alpha \in \langle 1, d \rangle$, by means of (3.2). On the other hand it holds $\rho(M^d) = \rho(M)^d$ by the Frobenius' theorem on the characteristic roots of a polynomial in a matrix. Hence we have

$$\rho(M)^d = \max \{ \rho(Q_\alpha) \; ; \; \alpha \in \langle 1, d \rangle \}.$$ 

Suppose that $\rho(M)^d = \rho(Q_{\alpha_0})$. Then, because of the positive regularity of $Q_{\alpha_0}$, there corresponds a positive eigenvector $u_{\alpha_0}$ of $Q_{\alpha_0}$ to $\rho(M)^d$.
\[ Q_{\alpha_0} u_{\alpha_0} = M_{\alpha_0} \cdots M_d M_{\alpha_0-1} u_{\alpha_0} = \rho(M)^d u_{\alpha_0} \]

Operating the matrix \( M_{\alpha} \cdots M_{\alpha_0-1} \) if \( \alpha \in <1, \alpha_0 - 1> \) (and the matrix \( M_{\alpha} \cdots M_d M_{\alpha_0-1} \) if \( \alpha \in <\alpha_0 + 1, d> \) ) from the left, we have \( Q_{\alpha} u_{\alpha} = \rho(M)^d u_{\alpha} \), where \( u_{\alpha} = M_{\alpha} \cdots M_{\alpha_0-1} u_{\alpha_0} \) if \( \alpha \in <1, \alpha_0 - 1> \) (and \( u_{\alpha} = M_{\alpha} \cdots M_d M_{\alpha_0-1} u_{\alpha_0} \) if \( \alpha \in <\alpha_0 + 1, d> \) ). But \( u_{\alpha} \neq 0 \) since every \( Q_{\alpha} \) is positively regular, and hence \( \rho(M)^d \) is a characteristic root of \( Q_{\alpha} \).

Therefore \( \rho(Q_{\alpha}) \geq \rho(M)^d \) and so \( \rho(Q_{\alpha}) = \rho(M)^d \) by means of (3.3).

For each \( d \in <1, \infty> \), the family of generating functions \( \{ F(nd ; s_{\alpha})_{\alpha}; n \in <0, \infty> \} \) forms a DGWP on \( S_{\alpha} \), which we denote by \( X_{\alpha}^{(d)} \).

**Lemma 3.2.** The least nonnegative fixed point of \( F(d ; s_{\alpha})_{\alpha} \) is equal to the \( q_{\alpha} \)-part of the extinction probability \( q \) of the DGWP \( X_{\alpha} \). Hence the \( q \)-mean matrix of the DGWP \( X_{\alpha}^{(d)} \) coincides with the \( d \)-fold product \( A_{\alpha}^d \) of \( A_{\alpha} \), and if Conditions (D) and (DN) are satisfied for the DGWP \( X_{\alpha} \) then they are also satisfied for the DGWP \( X_{\alpha}^{(d)} \).
Proof. Let \( r_\alpha \) be the least nonnegative fixed point of \( F(d; s_\alpha) \). Then it holds \( r_\alpha \leq q_\alpha \) since \( q_\alpha \) is a nonnegative fixed point of \( F(d; s_\alpha) \). Hence it follows

\[
\lim_{n \to \infty} F(nd; r_\alpha) = q_\alpha
\]

from (1.7). The remaining assertions except for that on (DN) are clear. But the assertion on (DN) can be easily seen if we make use of the same arguments as in Athreya [1] or Sevastyanov [14] Chapter III, §3.

Now let \( \tilde{d}_\alpha \in \langle 1, m_\alpha \rangle \) be the period of the irreducible matrix \( \tilde{A}_\alpha \) in (1.8), and

\[
d_\alpha = \text{L.C.M.}\{\tilde{d}_\beta; \Delta_\beta \subseteq \Gamma_\alpha\}
\]

(we set \( \tilde{d}_\alpha = 1 \) if \( \tilde{\rho}_\alpha = 0 \)). Then by a change of the suffixes, we have

\[
(3.4) \quad \tilde{A}_\beta^d\alpha = \begin{bmatrix}
A^{(\alpha)}_{\beta 1} & 0 & \cdots & 0 \\
0 & A^{(\alpha)}_{\beta 2} & 0 & \cdots \\
& \cdots & \cdots & \cdots \\
0 & \cdots & 0 & A^{(\alpha)}_{\beta d_\beta} \\
\end{bmatrix},
\]

\( \Delta_\beta \subseteq \Gamma_\alpha \).
where each $A_{a}^{(a)}_{\beta}\gamma$ is an irreducible aperiodic nonnegative square matrix of order $m_{\beta}\gamma \in \langle 1, m_{\beta} \rangle$ ($\sum_{\beta=1}^{\gamma} m_{\beta}\gamma = m_{\beta}$). We define from (3.4) the sets $\Delta_{\beta}\gamma$, $\Gamma_{\beta}\gamma^{(a)}$, and $S_{\beta}\gamma^{(a)}$, the vectors $s_{\beta}\gamma^{(a)}$ and $g_{\beta}\gamma$, and the matrices $A_{a}^{(a)}_{\beta}\gamma$ as we defined $\Delta_{\beta}$, $\Gamma_{\alpha}$, etc., in section 1; for example

\[ \Delta_{\beta}\gamma = \left\{ \sum_{p=1}^{\beta-1} m_{p} + \sum_{q=1}^{\gamma-1} m_{q} + 1, \sum_{p=1}^{\beta-1} m_{p} + \sum_{q=1}^{\gamma-1} m_{q} \gamma \right\}. \]

Note that $m_{\beta}\gamma$ (and hence $\Delta_{\beta}\gamma$) is independent of $d_{\alpha}$ which satisfies $d_{\alpha}' d_{\alpha}$. We also define the DGWP $X_{\beta}\gamma^{(a)}$ by the family of vector generating functions $\{ F(n_{d}\alpha; s_{\beta}\gamma^{(a)})(a)_{\beta}\gamma; n \in \langle 0, \infty \rangle \}$.

By Lemma 3.2 and the representation (3.4), our DGWP $X_{\beta}\gamma^{(a)}$ satisfies the assumptions of Theorem 2.1. As in section 2, we shall introduce the semiorder '$\prec_{\alpha}$' in the space of the suffixes $\{(\beta,p)\}$ by

\[ (\delta,q) \prec_{\alpha} (\beta,p) \quad \text{if} \quad \Delta_{\delta\gamma} \subseteq \Gamma_{\beta\gamma}(a). \]

Then the rank $v_{\alpha\gamma}$ of $(a,\gamma)$ is defined by

\[ (3.5) \quad v_{\beta\gamma}(a)(r) = \begin{cases} \max\{v_{\delta\gamma}(r); (\delta,q) \prec_{\alpha}(\beta,p)\}, & \text{if} \quad \bar{\beta} \neq r, \\ \max\{v_{\delta\gamma}(r); (\delta,q) \prec_{\alpha}(\beta,p)\} + 1, & \text{if} \quad \bar{\beta} = r. \end{cases} \]
(\max_f \gamma = -1), and \nu_{\alpha \gamma} = \gamma^{(\alpha)}(\rho_{\alpha}).

Lemma 3.3. Let Conditions (D) and (DN) be satisfied for all \alpha \in <1, \gamma> with \rho_{\alpha} < 1. Then for each \alpha \in <1, \gamma> and \gamma \in <1, \delta_{\alpha}> with \rho_{\alpha} < 1, there correspond monotone non-increasing functions \mathcal{R}_{i}^{*}(s^{(\alpha)}_{\alpha \gamma}) in 0 \leq s^{(\alpha)}_{\alpha \gamma} \leq q^{(\alpha)}_{\alpha \gamma}, i \in \Delta_{\alpha \gamma},

such that it holds as n \to \infty

\begin{equation}
\mathcal{R}_{i}^{*}(n d_{\alpha} + s_{\alpha \gamma}) = n^{\nu_{\alpha \gamma}} \rho_{\alpha}^{n d_{\alpha}}(\mathcal{R}_{i}^{*}(s^{(\alpha)}_{\alpha \gamma}) + o(1)), i \in \Delta_{\alpha \gamma},
\end{equation}

where o(1) is uniform in s on 0 \leq s^{(\alpha)}_{\alpha \gamma} \leq q^{(\alpha)}_{\alpha \gamma}. Further, if \rho_{\alpha} > 0, every \mathcal{R}_{i}^{*}(s^{(\alpha)}_{\alpha \gamma}), i \in \Delta_{\alpha \gamma}, is not identically zero.

For each \chi \in S, we set

\[ d_{\chi} = \text{L.C.M.}(d_{\alpha}; \alpha \in I_{+}(\chi)). \]

Theorem 3.1. Let a DGWP \( X = (Z(n), P_{X}) \) satisfy Conditions (D) and (DN) for each \alpha \in <1, \gamma> with \rho_{\alpha} < 1. Then

1) for each \alpha \in <1, \gamma> with \rho_{\alpha} < 1 and \gamma \in <1, \delta_{\alpha}> , it holds as n \to \infty

\begin{equation}
\mathcal{R}_{i}^{*}(n d_{\alpha} + l + s_{\alpha \gamma}) = n^{\nu_{\alpha \gamma}} \rho_{\alpha}^{n d_{\alpha}}(\mathcal{R}_{i}^{*}(F(l; s^{(\alpha)}_{\alpha \gamma})) + o(1)), \quad l \in <0, d_{\alpha} - 1>, i \in \Delta_{\alpha \gamma}, 0 \leq s^{(\alpha)}_{\alpha \gamma} \leq q^{(\alpha)}_{\alpha \gamma},
\end{equation}

\]
where $o(1)$ is uniform in $s$ on $0 \leq s \leq q$. Further, if

If $\rho > 0$, then every $R^*(F(s; \alpha)(\alpha)), \alpha \in \Delta$, is not identically zero.

2) For each $x \in S$ such that $\rho > 1$ for all $\alpha \in I_+(x)$, and $\rho > 0$ for some $\alpha \in I_+(x)$, there correspond a probability distributions $\{P_{x\lambda}(y)\}$ on $S-\{0\}$ satisfying

$$\lim_{n \to \infty} P_x\{\lambda n_x + \alpha | n_x + \lambda < T < \infty\} = P_{x\lambda}(y), \lambda \in 0, d_x-1\}.$$

**Proof.** Repeating the arguments in the proof of Theorem 2.1, we have only to show the nontriviality of the functions

$$R^*(F(\lambda; s)(\alpha)), \alpha \in \Delta, \text{ for } \rho > 0.$$ It follows from (3.6)

that

$$R^*(F(m_d; s)(\alpha)) = \rho_{md}^* R^*(s(\alpha)), \alpha \in \Delta_\alpha.$$ Since $F(\alpha)(\lambda; 0) \leq F(\alpha)(m_d; 0), \lambda \leq m_d$, it is clear that $\rho > 0$ implies

$$R^*(F(\lambda; 0)(\alpha)) \geq R^*(F(m_d; 0)(\alpha)) = \rho_{md}^* R^*(0) > 0, \alpha \in \Delta_\alpha, \lambda \geq m_d,$$

and we obtain the conclusion.

**Remark 3.1.** With the aid of Lemmas 2.1 and 2.4, we can determine the functions $R^*(s(\alpha))$ inductively w.r.t. the semiorder $\leq_\alpha$ in the space of the suffixes $\{\beta; \Delta_{\beta} \subseteq (\alpha)\}$. 


4. Asymptotic behavior of critical DGWP

Since we have studied the noncritical DGWP's in the previous sections we shall study the critical ones in this and the next sections. We assume Condition (D) and

\[(DC) \quad F_{jk}^i(q) < \infty, \quad i, j, k \in \Gamma_\alpha,\]

where \(F_{jk}^i(s) = a^{2F_{jk}^i(s)}/a^j_ks^k\) if it exists and

\[F_{jk}^i(s) = \lim_{\xi \to s} F_{jk}^i(\xi) \quad \text{otherwise.}\]

We set

\[(4.1) \quad \nu_\alpha = 1/2\nu_\alpha(1), \quad \nu_\alpha^\gamma = 1/2\nu_\alpha^\gamma(1),\]

where \(\nu_\alpha(1)\) and \(\nu_\alpha^\gamma(1)\) are those defined by (2.3) and (3.5).

The object of this section is to prove the next two theorems:

Theorem 4.1. Let a DGWP \(X = (\mathbb{Z}(n), P_x)\) satisfy Conditions (D) and (DC) for each \(\alpha \in \langle 1, g \rangle\) with \(\rho_\alpha = 1\), and every matrix \(\Lambda_\alpha\) be aperiodic. Then, for each \(\alpha \in \langle 1, g \rangle\) with \(\rho_\alpha = 1\), there correspond constants \(R^{*1} > 0, \quad i \in \Delta_\alpha\), such that

\[(4.2) \quad \lim_{n \to \infty} n^{\mu_\alpha^i} R_{n; s}^i = R^{*1}, \quad i \in \Delta_\alpha,\]

for each \(s\) satisfying \(0 \leq s_\alpha \leq q_\alpha\) and
The constants $R^{*i}$ are determined inductively w.r.t. the semi-order '<' from Lemmas 4.2 and 4.7 below.

**Theorem 4.2.** Let a DGWP $X = (Z(n), P_x)$ satisfy Conditions (D) and (DC) for each $\alpha \in <\hat{\phi}, \gamma>$ with $\rho_\alpha = 1$. Then, for each $\alpha \in <\hat{\phi}, \gamma>$ with $\rho_\alpha = 1$, and $\gamma \in <\hat{\phi}, \delta_\alpha>$, there correspond constants $R^{*i} > 0$, $i \in \Delta_\alpha \gamma$, such that

\[
\lim_{n \to \infty} R^i(n; s) = R^{*i}, \quad i \in \Delta_\alpha \gamma,
\]

for each $s$ satisfying $0 < s_\alpha < q_\alpha$ and (4.3).

**Proof of Theorem 4.2 assuming Theorem 4.1.** By the same arguments as in the proof of Lemma 3.3, we have from Theorem 4.1 that

\[
\lim_{n \to \infty} (n_\alpha + \lambda)^{\alpha \gamma} R^i(n_\alpha + \lambda; s) = R^{*i}, \quad i \in \Delta_\alpha \gamma,
\]

for each $s$ satisfying $0 < s_\alpha < q_\alpha$ and (4.3). But since $F(\lambda; s)$ also satisfies $0 < F(\lambda; s)_\alpha < q_\alpha$ and (4.3) for such an $s$, it follows

\[
\lim_{n \to \infty} (n_\alpha + \lambda)^{\alpha \gamma} R^i(n_\alpha + \lambda; s) = \lim_{n \to \infty} (n_\alpha)^{\alpha \gamma} R^i(n_\alpha; F(\lambda; s)).
\]
Remark 4.1. Combining Theorems 3.1 and 4.2, we of course obtain the whole asymptotic behavior of a DGWP satisfying conditions (D) and (DC) for all \( a \in \langle 1, g \rangle \).

Now we shall prove Theorem 4.1 without haste. In the following in this section, we assume that the hypotheses of Theorem 4.1 are satisfied, unless otherwise is stated.

Lemma 4.1. If \( \tilde{p}_a = 1 \), then

\[
(4.5) \quad B_a \equiv \frac{1}{2} \sum_{1, j, k \in \Delta_a} v_{a_1} F^i_{j k}(q) \tilde{u}^j_a \tilde{u}^k_a > 0.
\]

Proof. Suppose first that \( \overline{F} = \phi \) and \( F(s) = F(0) + A s \).

Then it follows

\[ q = F(n; q) = F(n; 0) + A^n q. \]

Letting \( n \to \infty \) we have \( \lim_{n \to \infty} A^n q = 0 \) by (1.7), which implies \( \rho < 1 \). Next we shall assume that \( \overline{F} \neq \phi \) and \( \tilde{F}(s) = \tilde{F}_0(s) + H(s) \tilde{s} \) with \( \tilde{F}_0(s) \neq 0 \). Then it follows that \( H(q) = \tilde{A} \) and

\[
\tilde{F}(n; s) = \tilde{F}_0(F(n-1; s)) + \sum_{\lambda=1}^{n-1} H(F(n-1; s))...H(F(\lambda; s))\tilde{F}_0(F(\lambda-1; s))... + H(F(n-2; s))...H(F(0; s)) s.
\]
Hence it follows
\[
\hat{q} = F(n; q) = \frac{1}{\ln} \sum_{q=1}^{n} \hat{A}^{n-\hat{F}}(\hat{q}) + \hat{A}^{n-\hat{q}}.
\]
Since \( \frac{1}{\ln} \sum_{q=1}^{n} \hat{A}^{n-\hat{F}}(\hat{q}) > 0 \) for a large \( n \), it holds \( \hat{q} > \hat{A}^{n}q \).
Hence we have \( \rho(\hat{A})^{n} \sim 1 \) by the mini-max principle (cf. Gantmacher [6] II, p.65).

For an \( \alpha \in \langle 1, g \rangle \) which is minimal w.r.t. the semiorder '\(<\)', we exploit the following

**Lemma 4.2 (Joffe and Spitzer [9]):** If the q-mean matrix \( A_{\alpha} \) is positively regular with \( \rho_{\alpha} = 1 \), it holds
\[
(4.6) \quad R^{i}(n; s) = \frac{1}{l+nB} \frac{1}{\hat{\alpha}_{i}} \frac{v_{i} \cdot (l_{i} - s_{i})}{(1 + o(1))}, \quad i \in A_{\alpha},
\]
as \( n \to \infty \), where \( o(1) \) is uniform in \( 0 \leq s_{i} \leq l_{i}, \ s_{\alpha} \neq l_{\alpha} \).

Note that \( q_{\alpha} \) is equal to the \( \Gamma_{\alpha} \)-part \( l_{\alpha} \) of the vector \( l = (1, \ldots, 1) \) in this case.

To study the case when \( \alpha \) is not minimal, we prepare some lemmas.

**Lemma 4.3.** Let \( \rho_{\alpha} = 1 \) and \( 0 \leq s_{\alpha} \leq q_{\alpha}, \ s_{\alpha} \neq q_{\alpha} \). Then the relation
\begin{equation}
\lim_{n \to \infty} \frac{\mathcal{R}(n+k;\alpha)}{n^k} = 0, \quad k \in \{0, \infty\},
\end{equation}

implies

\begin{equation}
\lim_{n \to \infty} \frac{\mathcal{R}(n;\alpha)}{n^s} = \tilde{u}_\alpha.
\end{equation}

Proof: First of all we note that

\begin{equation}
\tilde{v} \cdot \mathcal{R}(n;s) > 0, \quad n \in \langle n_0, \infty \rangle, \quad 0 \leq s \leq q, \quad s \neq q,
\end{equation}

for some \( n_0 \in \langle 1, \infty \rangle \). Indeed, for each \( i \in A \) and \( j \in \Gamma \) there corresponds an \( n^i_j \in \langle 1, \infty \rangle \) such that \( A^i_j(n^i_j) > 0 \). Hence the positive regularity of \( \bar{A} \) implies

\[ A^i_j(n) > A^i_j(n-n^i_j)A^i_j(n^i_j) > 0 \]

for all sufficiently large \( n \). So such \( F^i(n;s) \) depends on every variable \( s^i \) with \( j \in \Gamma \), and we obtain (4.9). Now using (2.14) inductively, we obtain

\begin{equation}
\mathcal{R}(n+1) = \mathcal{D}(n,n-m-1) \mathcal{R}(n-m) + \sum_{l=n-m}^{n} \mathcal{D}(n,l) \mathcal{C}(l) \mathcal{R}(l).
\end{equation}

We take the sequences \( \epsilon_n \) and \( \alpha_n \) in the proof of Lemma 2.3.

In our case the sequence \( \epsilon_n \) may not satisfy (2.22), but it
tends to zero as \( n \to \infty \) and satisfies (2.24) with \( \rho = 1 \).

Combining (2.24) and (4.10) we have

\[
(1-\alpha_{m+1}^{-} - \sum_{k=n-m}^{n} \varepsilon_k) \overline{A} \overline{R}(n-m) + \sum_{\ell=n-m}^{n} (1-\alpha_{n-\ell}^{-} - \sum_{k=\ell+1}^{n} \varepsilon_k) \overline{A} \overline{C}(\ell) \overline{R}(\ell) \\
\leq \overline{R}(n+1) \leq (1+\alpha_{m+1}^{+}) \overline{A} \overline{R}(n-m) + \sum_{\ell=n-m}^{n} (1+\alpha_{n-\ell}^{+}) \overline{A} \overline{C}(\ell) \overline{R}(\ell). 
\]

Hence it follows, for each \( m \) and \( n \) with \( n-m \in \langle n_0, \infty \rangle \),

\[
(4.11) \quad \frac{(1-\alpha_{m+1}^{-} \sum_{k=n-m}^{n} \varepsilon_k) P(n,m) + \sum_{\ell=n-m}^{n} (1-\alpha_{n-\ell}^{-} \sum_{k=\ell+1}^{n} \varepsilon_k) Q(n,\ell)}{(1+\alpha_{m+1}^{+}) + \sum_{\ell=n-m}^{n} (1+\alpha_{n-\ell}^{+}) V \cdot Q(n,\ell)} \\
\leq \frac{\overline{R}(n+1)}{V \cdot \overline{R}(n+1)} \leq \frac{(1+\alpha_{m+1}^{+}) P(n,m) + \sum_{\ell=n-m}^{n} (1+\alpha_{n-\ell}^{+}) Q(n,\ell)}{(1-\alpha_{m+1}^{-} \sum_{k=n-m}^{n} \varepsilon_k) + \sum_{\ell=n-m}^{n} (1-\alpha_{n-\ell}^{-} \sum_{k=\ell+1}^{n} \varepsilon_k) V \cdot Q(n,\ell)}
\]

where

\[ P(n,m) = \frac{\overline{A} \overline{R}(n-m)}{V \cdot \overline{R}(n-m)}, \quad Q(n,\ell) = \frac{\overline{A} \overline{C}(\ell) \overline{R}(\ell)}{V \cdot \overline{R}(n-m)}. \]

But \( P(n,m) = \overline{u} \) by the definition of \( \overline{A} \), and \( Q(n,\ell) \to 0 \) as \( n \to \infty \) by (4.7) and (2.32). Hence, letting \( n \to \infty \) in (4.11), we have

\[
\frac{(1-\alpha_{m+1}^{-}) \overline{u}^i}{1+\alpha_{m+1}^{+}} \leq \lim_{n \to \infty} \frac{\overline{R}(n+1)}{V \cdot \overline{R}(n+1)} \leq \lim_{n \to \infty} \frac{\overline{u}^i(n+1)}{V \cdot \overline{R}(n+1)} \leq \frac{(1+\alpha_{m+1}^{+}) \overline{u}^i}{1-\alpha_{m+1}^{-}}, \quad i \in \Delta, \quad m \in \langle 1, \infty \rangle
\]
Now we obtain (4.8) by letting $m \to \infty$.

**Lemma 4.4.** There are functions $B_{jk}(s_\alpha)$ and $G_{j}(s_\alpha)$ in $0 < s_\alpha < q_\alpha$ such that

$$
(4.12) \quad R^i_{j}(s_\alpha) = \sum_{j \in \Delta_\alpha} A^i_j(q^j-s_\alpha^j) - \sum_{j,k \in \Delta_\alpha} B_{jk}(s_\alpha)(q^j-s_\alpha^j)(q^k-s_\alpha^k)
$$

$$
+ \sum_{j \in \tilde{T}_\alpha} (A^i_j-G_j^i(s_\alpha))(q^j-s_\alpha^j), \quad i \in \Delta_\alpha, \quad 0 \leq s_\alpha \leq q_\alpha,
$$

where

$$
0 \leq B_{jk}(s_\alpha^{(1)}) \leq B_{jk}(s_\alpha^{(2)}) \leq \frac{1}{2} F_{jk}(q), \quad 0 < s_\alpha^{(1)} \leq s_\alpha^{(2)} < q_\alpha,
$$

$$
(4.13) \quad B_{jk}(s_\alpha) \to \frac{1}{2} F_{jk}(q), \quad \text{as} \quad s_\alpha \to q_\alpha \quad \text{in} \quad 0 < s_\alpha < q_\alpha, \quad i,j,k \in \Delta_\alpha,
$$

$$
(4.14) \quad 0 \leq G_{j}(s_\alpha) \leq 2 E_{j}(s_\alpha), \quad i \in \Delta_\alpha, \quad j \in \tilde{T}_\alpha.
$$

**Proof.** Integrating by parts the integral in (2.11), we have

$$
E_{j}^{i}(s) = \sum_{k \in \tilde{T}} B_{jk}(s)(q^k-s_\alpha^k), \quad i \in \Delta, \quad 0 \leq s \leq q,
$$

$$
(4.15) \quad B_{jk}(s) = \sum_{y \in S} P_{y}^{i}(y) y^{j} y^{k} y^{l} \int_{0}^{1} (q-(q-s)\xi)^{y-e_{j}^{l}} e^{-\xi}(1-\xi) d\xi.
$$

Combining this with (2.10) we have
\[ R^i(s) = \sum_{j \in \Delta} A^i_j(q^j-s^j) - \sum_{j,k \in \Delta} B^i_{jk}(s)(q^j-s^j)(q^k-s^k). \]

\[ + \sum_{j \in \Gamma} (A^i_j - E^i_j(s))(q^j-s^j) - \sum_{j \in \Delta} \sum_{k \in \Delta} B^i_{jk}(s)(q^j-s^j)(q^k-s^k), \]

\[ 0 \leq s \leq q. \]

Since \( B^i_{jk}(s) = B^i_{kj}(s) \) by (4.15), the last term is equal to

\[ - \sum_{j \in \Gamma} \sum_{k \in \Delta} B^i_{jk}(s)(q^k-s^k)(q^j-s^j), \]

and we obtain (4.12) with

(4.16) \[ G^i_j(s) = E^i_j(s) + \sum_{k \in \Delta} B^i_{jk}(s)(q^k-s^k). \]

Further (4.13) follows from (4.15), and (4.14) follows from (4.15) - (4.16).

Note that, if we replace \( s_\alpha \) in (4.12) by \( F(n ; s_\alpha) \alpha \), we obtain

(4.17) \[ R^i(n+1;s_\alpha) = \sum_{j \in \Delta_\alpha} A^i_j R^j(n;s_\alpha) - \sum_{j,k \in \Delta_\alpha} B^i_{jk}(n;s_\alpha) R^j(n;s_\alpha) R^k(n;s_\alpha) \]

\[ + \sum_{j \in \Gamma_\alpha} (A^i_j - G^i_j(n;s_\alpha)) R^j(n;s_\alpha), \] \( i \in \Delta_\alpha, \ 0 \leq s_\alpha \leq q_\alpha. \)
where

\begin{equation}
B_{jk}^i(n; s\alpha_a) = B_{jk}^i(F(n; s\alpha_a)), \quad G_j^i(n; s\alpha) = G_j^i(F(n; s\alpha_a)).
\end{equation}

Hence it follows, when \( \tilde{\rho}_a = 1 \),

\begin{equation}
a_{n+1} - a_n = -b_n a_n^2 + c_n,
\end{equation}

where

\begin{equation}
\begin{cases}
    a_n = a_n(s\alpha) = \tilde{\varphi}_a \tilde{\tilde{R}}(n; s\alpha_a) \\
    b_n = b_n(s\alpha) = \sum_{i,j,k} \tilde{\varphi}_{ai} B_{jk}^i(n; s\alpha_a) R_j^i(n; s\alpha) R^k(n; s\alpha) a_n(s\alpha)^2 \\
    c_n = c_n(s\alpha) = \sum_{i,j,k} \tilde{\varphi}_{ai} (A_j^i - G_j^i(n; s\alpha_a)) R_j^i(n; s\alpha).
\end{cases}
\end{equation}

Note that (4.9) is rewritten as

\begin{equation}
a_n > 0, \quad n \in (n_0, \infty), \quad 0 \leq s\alpha \leq q\alpha, \quad s\alpha \neq q\alpha,
\end{equation}

for some \( n_0 \in (1, \infty) \). Further

\begin{equation}
\lim_{n \to \infty} a_n = 0
\end{equation}

by (1.7), and

\begin{equation}
0 \leq b^* = \lim_{n \to \infty} b_n \leq \lim_{n \to \infty} b_n = \overline{b}^* < \infty
\end{equation}

by (4.13) and the inequality \( \tilde{\varphi}_\alpha > 0 \). Finally it holds
for some $n_1 \in (0, \infty)$ by means of (4.14), (1.7) and the fact that $A_j^1 = 0$ implies $E_j^1(s_\alpha) = 0$.

Now we assume that

\[(4.25) \lim_{n \to \infty} n^\mu R^1(n; s) = R^1, \quad i \in \Delta_B,\]

for each $\beta < \alpha$ with $\rho_\beta = 1$ and $s$ satisfying $0 \leq s_\alpha \leq q_\alpha$ and (4.3), where $R^1$ are constants with $R^1 > 0$. Then we have

**Lemma 4.5.** 1) If $\overline{\rho}_\alpha < 1$, it holds

\[(4.26) c_n = o(1/n^2), \text{ as } n \to \infty.\]

2) If $\overline{\rho}_\alpha = 1$,

\[(4.27) \lim_{n \to \infty} n^\mu \overline{R}(n; s) = \overline{R}_\alpha,\]

for each $s$ with $0 \leq s_\alpha \leq q_\alpha$ and (4.3), where

\[(4.28) \overline{\mu}_\alpha = \min\{\mu_\beta ; \beta < \alpha, \rho_\beta = 1\}.\]

Further, it holds
(4.29) \[ \lim_{n \to \infty} n^{\alpha} c_n = \sqrt[n]{a \cdot b} \cdot e \cdot c^* > 0. \]

Proof. (4.26) is clear from (4.20) and Theorem 2.1. (4.27) is also clear by (4.25). Hence (4.29) except for the relation \( c^* > 0 \) follows with the aid of (4.14) and (1.7). But \( c^* > 0 \) is easily seen if we repeat the same arguments as in the proof of Lemma 2.5.

The next lemma plays an important role in the following.

Lemma 4.6. Let sequences \( \{a_n\}, \{b_n\} \) and \( \{c_n\} \) satisfy (4.19) and (4.21)-(4.24). Then, \( 1) \) (4.26) implies

\[
1/b^* \leq \lim_{n \to \infty} n^{1/2} a_n \leq \lim_{n \to \infty} n a_n \leq 1/b^*.
\]

2) Let

\[
0 < b^* < b^* < \infty,
\]

(4.32) \[ \lim_{n \to \infty} n^{\mu} c_n = c^*, \]

for some \( 0 < \mu < 1 \), then it holds

\[
\sqrt{c^*/b^*} \leq \lim_{n \to \infty} n^{\mu/2} a_n \leq \lim_{n \to \infty} n^{\mu/2} a_n \leq \sqrt{c^*/b^*}.
\]
Proof. 1) By (4.22) and (4.23), it holds

\[
\frac{b_n}{1-a_n b_n} \leq M, \quad n \in \langle n_2, \infty \rangle
\]

for some \( M > 0 \) and \( n_2 \in \langle 1, \infty \rangle \). Hence it follows from (4.19), (4.21) and (4.24) that

\[
\frac{1}{a_{n+1}} - \frac{1}{a_n} \leq \frac{b_n a_n}{a_{n+1}} \leq \frac{b_n}{(1-a_n b_n) + c_n/a_n} \leq M, \quad n \in \langle n_3, \infty \rangle,
\]

where \( n_3 = n_0 \lor n_1 \lor n_2 \). Summing up these inequalities from \( n_3 \) to \( n \) we have

\[
\frac{1}{a_n} \leq (n-n_3)M + \frac{1}{a_{n_3}}, \quad n \in \langle n_3, \infty \rangle,
\]

so that, by means of (4.26),

\[
\lim_{n \to \infty} c_n/a_n = \lim_{n \to \infty} c_n/a_n^2 = 0.
\]

Hence we obtain (4.30) since (4.19) implies

\[
\frac{1}{n} \left( \frac{1}{a_n} - \frac{1}{a_{n_3}} \right) = \frac{1}{n} \sum_{k=n_3}^{n-1} \frac{b_k - c_k/a_k^2}{1-b_k a_k + c_k/a_k}.
\]

2) Setting \( \xi_n = n^{\mu/2} a_n \), we have from (4.19) that
\[ b_n \varepsilon_n^2 - n^\mu c_n + n^{\mu/2} (\varepsilon_{n+1} - \varepsilon_n) = a_{n+1} O(n^{\mu-1}), \]

as \( n \to \infty \). Since \( 0 < \mu \leq 1 \), this with (4.22) implies the basic equality

\[ \lim_{n \to \infty} \{ b_n \varepsilon_n^2 - n^\mu c_n + n^{\mu/2} (\varepsilon_{n+1} - \varepsilon_n) \} = 0. \]

Now we shall show that the sequence \( \{ \varepsilon_n \} \) is bounded. Suppose that \( \{ \varepsilon_n \} \) is unbounded, and let

\[ n_1 = 1, \quad n_k = \min \{ n; \varepsilon_n > \varepsilon_{n_{k-1}} \} \forall k, \quad k \in \langle 2, \infty \rangle. \]

Then it follows

\[ \varepsilon_{n_k} > \varepsilon_{n_{k-1}} \forall k \geq n_{k-1}, \quad k \in \langle 2, \infty \rangle, \]

\[ \lim_{k \to \infty} \varepsilon_{n_k} = \infty. \]

By (4.35) we have \( \varepsilon_{n_k} > \varepsilon_{n_{k-1}} \), and hence by (4.34)

\[ \lim_{k \to \infty} \{ b_{n_k-1} \varepsilon_{n_k}^2 - (n_k-1)^\mu c_{n_k-1} \} \leq 0. \]

Hence, with the aid of (4.32) and (4.31) we have

\[ \lim_{k \to \infty} \varepsilon_{n_k}^2 \leq c^*/b^* < \infty, \]
and from \((4.34)\)

\[
\lim_{k \to \infty} (\xi_{n_k} - \xi_{n_k - 1}) = \lim_{k \to \infty} \frac{b_{n_k - 1}\xi_{n_k - 1}^2 - (n_k - 1)^{\mu/2} c_{n_k - 1}}{(n_k - 1)^{\mu/2}} = 0.
\]

\((4.37)\) and \((4.38)\) imply the boundedness of the sequence \(\{\xi_{n_k}\}\), which is a contradiction. We note that, by means of the boundedness of the sequence \(\{\xi_n\}\), \((4.38)\) is valid for any subsequence \(\{n_k\}\). To prove \((4.33)\), we set

\[
\xi^* = \lim_{n \to \infty} \xi_n, \quad \xi^*_* = \lim_{n \to \infty} \xi_n.
\]

First we shall show that \(\xi^* = \xi^*_* = \xi^*\) implies

\[
\sqrt{c^*/b^*} < \xi^* < \sqrt{c^*/b^*}
\]

Indeed if \(\xi^* < \sqrt{c^*/b^*}\) for example, it holds by \((4.34)\) and \((4.32)\) that

\[
n^{\mu/2}(\xi_{n+1} - \xi_n) \geq n^{\mu} c_n - b_n \xi^2_n - \varepsilon
\]

\[
\geq c^* - b^*(\xi^*)^2 - 2\varepsilon > 0, \quad n \in \langle N_0, \infty \rangle
\]

for some \(N_0 \in \langle 0, \infty \rangle\). Hence it follows

\[
\xi_n - \xi_{N_0} \geq (c^* - b^*(\xi^*)^2 - 2\varepsilon) \sum_{k=N_0}^{n-1} \frac{1}{k^{\mu/2}},
\]
which contracts the boundedness of \( \{ \xi_n \} \). Next we shall show that (4.33) holds even when \( \xi^* < \bar{\xi}^* \). Since the situations do not differ, we suppose \( \bar{\xi}^* > \sqrt{c^*/b^*} \) and lead a contradiction. Take a constant \( \xi \) in \( \bar{\xi}^* > \xi > \xi^* \vee \sqrt{c^*/b^*} \), and let

\[
n_0 = \min\{ n; \xi_n > \xi \},
\]

\[
m_k = \min\{ n; \xi < n_k - 1 + 1, \infty; \xi_n < \xi \},
\]

\[
n_k = \min\{ n; \xi < n_k + 1, \infty; \xi_n > \xi \}, \quad k \in <1, \infty>.
\]

Then it holds

(4.39) \( \xi_{n_k} > \xi_{n_k - 1} \vee \xi, \quad k \in <1, \infty> \).

Indeed, the inequality \( \xi_{n_k} > \xi \) is clear from the definitions, and \( \xi_{n_k} > \xi_{n_k - 1} \) is also clear since \( \xi_{n_k - 1} \leq \xi \) if \( n_{k - 1} \in <m_k + 1, \infty> \), and \( \xi_{n_k - 1} < \xi \) if \( n_{k - 1} = m_k \). Now it follows from (4.34) and (4.39) that for any \( \epsilon > 0 \) there is a \( k_1 \) satisfying

\[
\xi_{n_k - 1}^2 < \frac{(n_{k - 1})^{\mu_c} n_{k - 1}}{b_{n_k - 1}} + \frac{\epsilon}{b_{n_k - 1}}, \quad k \in <k_1, \infty>.
\]

Combining this inequality with (4.39), (4.38) and (4.32), we obtain
\[ \xi^2 \leq \lim_{k \to \infty} \xi_n^2 = \lim_{k \to \infty} \xi_{k-1}^2 \leq \frac{c^*}{b^*}, \]

which contradicts the inequality \( \xi > \sqrt{c^*/b^*} \).

Corollary 4.1. (4.33) is still valid even if we replace the assumption (4.19) by (4.34) where \( \xi_n = n^{\mu/2}a_n \).

Now we are ready to prove the next lemma which completes the proof of Theorem 4.1:

Lemma 4.7. Let \( \rho_\alpha = 1 \), and (4.25) hold. Then it follows

\[ \lim_{n \to \infty} n^{1/2} R(n;s)_\alpha = \bar{R}^*_\alpha, \]

for all \( s \) satisfying \( 0 < s < q_\alpha \) and (4.3), where \( \mu_\alpha \) and \( \bar{R}^*_\alpha \) are given separately in the following three cases; (i) if \( 1 = \bar{\rho}_\alpha > \bar{\rho}_\alpha \), then \( \mu_\alpha = 1 \) and

\[ \bar{R}^*_\alpha = \bar{\alpha}/B_\alpha, \]

(ii) if \( 1 = \bar{\rho}_\alpha > \bar{\rho}_\alpha \), then \( \mu_\alpha = \bar{\mu}_\alpha \) and

\[ \bar{R}^*_\alpha = (1 - \bar{A}_\alpha)^{-1} A_\alpha^* \bar{R}^*_\alpha, \]

and (iii) if \( 1 = \bar{\rho}_\alpha = \bar{\rho}_\alpha \), then \( \mu_\alpha = \bar{\mu}_\alpha /2 \) and

\[ \bar{R}^*_\alpha = \left( \frac{\bar{A}_\alpha^* \bar{u}_\alpha}{B_\alpha} \right)^{1/2} \bar{u}_\alpha. \]
Proof. (i) When $1 = \bar{\rho} > \bar{\rho}$, it holds (4.26) by Lemma 4.5. Hence it follows (4.30) by Lemma 4.6, and we have (4.7) by Theorem 2.1. Therefore (4.8) holds by Lemma 4.3, and so

\begin{equation}
\lim_{n \to \infty} b_n = B
\end{equation}

by (4.20), (4.18), (1.7) and (4.13). Now appealing to Lemma 4.6 l') again, we have \( \lim_{n \to \infty} n a_n = 1/B \) to obtain (4.40) with $\mu = 1$ and $\tilde{R}^*$ given by (4.41) from (4.8).

(ii) When $1 = \bar{\rho} > \bar{\rho}$, it holds

\begin{equation}
n^\mu \tilde{R}(n;s) \leq \tilde{c}, \quad n \in \mathbb{N},
\end{equation}

for $\mu = \bar{\mu}$. Indeed, combining (2.15) with (2.23) and (4.27) we have

\begin{equation}
(n+1)^\mu \tilde{R}(n+1) \leq (n+1)^\mu A^{n+1} \tilde{q} + (n+1)^\mu \sum_{\ell=0}^{n} A^{n-\ell} A^* \tilde{R}(\ell)
\end{equation}

\begin{equation}
= (n+1)^\mu \Theta_1 \tilde{\rho}^{n+1} A^* \tilde{q} + (n+1)^\mu \Theta_2 \tilde{\rho}^{n} \sum_{\ell=1}^{n} \tilde{\rho}^{\ell} \tilde{\rho}^{n} A^* A^* \tilde{R}^* + K,
\end{equation}

where $\Theta_1, \Theta_2$ and $K$ are positive constants. But since

\begin{equation}
\sum_{\ell=1}^{n} \tilde{\rho}^{\ell} \tilde{\rho}^{n} \sim n^{-\tilde{\rho} - n} / (-\log \tilde{\rho}), \quad \text{as} \quad n \to \infty,
\end{equation}

(4.45) follows.
Now by means of (4.10) it holds

$$\sum_{\ell=0}^{m} \bar{D}(n,n-\ell)C(n-\ell)'(n+1)^{\mu}R(n-\ell) \leq (n+1)^{\mu}R(n+1).$$

$$\leq \frac{(n+1)^{\mu}}{n-m}A^{m+1}(n-m)^{\mu}R(n-m) + \sum_{\ell=0}^{m} A^{\ell}A'(n+1)^{\mu}R(n-\ell).$$

Hence letting \( n \to \infty \) we have from (4.45) that

$$\sum_{\ell=0}^{m} A^{\ell}A'\bar{R}^* \leq \lim_{n\to\infty} n^{\mu}R(n) \leq \lim_{n\to\infty} n^{\mu}R(n) \leq A^{m+1} + \sum_{\ell=0}^{m} A^{\ell}A'\bar{R}^*.$$  

But \( A^{m+1} \to 0 \) as \( m \to \infty \) since \( \beta < 1 \), and we obtain the conclusion.

(iii) In the case of \( l = \beta = \bar{\rho} \), we shall first prove (4.44).

Since the sequence \( a_n(0) \) is monotone nonincreasing in \( n \), it follows from (4.19) and (4.20) that

$$0 \leq \frac{c_n(0)}{a_n(0)} \leq b_n(0)a_n(0) \to 0, \quad \text{as} \quad n \to \infty.$$  

Hence it holds from (4.29) that

$$\lim_{n\to\infty} 1/n^\mu a_n(0) = 0.$$  

Further, for each \( 0 \leq s \leq q \) satisfying (4.3) we can find an \( \ell \in (0,\infty) \) by (1.7) such that \( s \leq F(\ell;0) \leq q \), whence it
follows \( R(n; s) \geq R(n+\ell; 0) \) and

\[ \lim_{n \to \infty} 1/n^\mu a_n(s) = 0. \]

Hence we have (4.7) by (4.27), so that (4.8) and (4.44) by Lemma 4.3 and (4.20). Now since \( B > 0 \) by Lemma 4.1, it follows from (4.44) and Lemma 4.6 2) that

\[ \lim_{n \to \infty} n^{\mu/2} a_n = \sqrt{c*/B}. \]

Hence we have the conclusion with the aid of (4.8).

Remark 4.2. The vectors \( R_\alpha^{\#} \) given above are positive.

The proof is similar to that of Lemma 2.5.

Remark 4.3. It is clear from the proof that (4.40) holds for \( s \) with \( 0 \leq s_\alpha \leq q_\alpha, \ s_\alpha \neq q_\alpha \) in case of (i), and for all \( s \) satisfying \( 0 \leq s_\alpha \leq q_\alpha, \ s_\alpha \neq q_\alpha \) and (4.27). Further, it can be seen that if we assume Condition (DE) in the next section (4.40) and hence (4.2)) holds for all \( s \) with \( 0 \leq s_\alpha \leq q_\alpha, \ s_\alpha \neq q_\alpha \) in all cases.
In this section we shall give the asymptotic behavior of the distributions

\[ Q_x(n;u) = P_x\left\{ \frac{Z(n)}{n} \leq u \mid n < T < \infty \right\}, \quad u \in \mathbb{R}^N, \]

of critical DGWP's. We shall assume for each \( \alpha \in <1, g> \) with \( \tilde{p}_\alpha = 1 \) that

\[ (DE) \quad \sum_{i,j,k \in \mathcal{A}_\alpha} \hat{\psi}_{\alpha \gamma} F_{\alpha}^{i,j,k}(\theta) \xi^j \xi^k \geq c_{\alpha \gamma} (\sum_{i \in \mathcal{A}_\alpha} \hat{\psi}_{\alpha \gamma} \xi_i^1)^2, \]

where \( c_{\alpha \gamma} \) is a positive constant and \( \hat{\psi}_{\alpha \gamma} \) is the positive left eigenvector of \( \bar{A}_\alpha^{(\alpha)} \) corresponding to the P-F root 1.

When the matrix \( \bar{A}_\alpha \) is aperiodic, it is clear that \( \tilde{q}_\alpha = 1 \), and Condition (DE) is reduced to

\[ (5.1) \quad \sum_{i,j,k \in \mathcal{A}_\alpha} \hat{\psi}_{\alpha \gamma} F_{\alpha}^{i,j,k}(\theta) \xi^j \xi^k \geq c_{\alpha} (\sum_{i \in \mathcal{A}_\alpha} \hat{\psi}_{\alpha \gamma} \xi_i^1)^2, \quad \tilde{\xi}_\alpha = (\xi_i^1)_{i \in \mathcal{A}_\alpha} > 0, \quad \gamma \in <1, \tilde{d}_\alpha>, \]

for some \( c_{\alpha} > 0 \). We set

\[ s(n) = s(n, \lambda) = (q^1 \exp(-\lambda^1/n), \ldots, q^N \exp(-\lambda^N/n)), \]
for each $\lambda = (\lambda^1, \ldots, \lambda^N) \geq 0$. Our object in this section is
to prove the following

**Theorem 5.1.** Let a DGWP $X = (Z(n), P_x)$ satisfy Conditions
(D), (DC) for each $\alpha < 1, g >$ with $\rho_\alpha = 1$ and (DE) for each
$\alpha \in <1, g >$ with $\tilde{\rho}_\alpha = 1$, and the matrices $\tilde{A}_\alpha$ be aperiodic.

Then, 1) for each $\alpha < 1, g >$ with $\rho_\alpha = 1$, there correspond nontrivial
nonnegative functions $\psi^i(\lambda_\alpha), i \in \Delta_\alpha$, such that

$$\lim_{n \to \infty} n^{\alpha} R^i(n; s(n, \lambda)) = \psi^i(\lambda_\alpha), \quad i \in \Delta_\alpha,$$

for each $\lambda \geq 0$ satisfying

$$\bar{\alpha} > 0, \quad \text{if } \beta \prec \alpha, \quad \tilde{\rho}_\beta > 0.$$

The functions $\psi^i(\lambda_\alpha), i \in \Delta_\alpha$, are determined inductively
w.r.t. the semiorder ' $\prec$ ' from Lemmas 5.1 and 5.3 below. 2)

For each $x \in S$ with $\rho_\alpha = 1$ for some $\alpha \in I_+(x)$, the distribu-
tions $Q_x(n; u), n \in <1, \infty >$, converge as $n \to \infty$ to a proba-

bility distribution $Q_x(u)$ on $\mathbb{R}^N$ given by

$$\int_{\mathbb{R}^N} e^{-\lambda \cdot u} dQ_x(u) = 1 - \frac{\sum_{\alpha \in I_+(x)} \sum_{i \in \Delta_\alpha} x^{\alpha} q^{-i} \psi^i(\lambda_\alpha)}{\sum_{\alpha \in I_+(x)} \sum_{i \in \Delta_\alpha} x^{\alpha} q^{-i} R^i}.$$
where \( \mu_x = \min \{ \mu_a; \alpha \in I_+(x) \} \).

**Theorem 5.2.** Let a DGWP \( X = (Z(n), P_X) \) satisfy Conditions (B), (DC) for each \( \alpha < 1, g \) with \( \rho_\alpha = 1 \) and (DE) for each \( \alpha < 1, g \) with \( \tilde{\rho}_\alpha = 1 \). Then, 1) for each \( \alpha < 1, g \) with \( \rho_\alpha = 1 \) and \( \gamma < 1, \tilde{\alpha} \) , there correspond nonnegative functions \( \psi^1(\lambda_\alpha \gamma) \), \( i \in \Delta_{\alpha \gamma} \), such that

\[
\lim_{n \to \infty} (n_{\alpha \lambda}, \mu) \rho^{1}(n_{\alpha \lambda} + s_{\alpha \lambda}, \mu) = \psi^1(\omega_\lambda(\lambda_\alpha \gamma)),
\]

for each \( \lambda \geq 0 \) with (5.3), where \( \omega_\lambda(\lambda) = A^{\lambda}(q\lambda)/q \). 2)

For each \( x \in S \) with \( \rho_\alpha = 1 \) for some \( \alpha \in I_+(x) \), the distributions \( Q_x(n_{\alpha \lambda} + s_{\alpha \lambda}, \mu) \), \( u \in R^N_+ \), converge as \( n \to \infty \) to a probability distribution \( Q^* (u) \) on \( R^N_+ \).

Throughout in the following in this section we always assume the hypotheses of Theorem 5.2. Further, we shall assume for the moment that every \( \tilde{A}_\alpha \) is aperiodic. Then, for an \( \alpha < 1, g \) which is minimal w.r.t. the semiorder \( \prec \), there is the following excellent
Lemma 5.1 (Joffe and Spitzer [9]). If the q-mean matrix $A_\alpha$ is positively regular with $\rho_\alpha = 1$, it holds (5.2) with $\mu_\alpha = 1$ and

\[
\psi^i(\lambda_\alpha) = \frac{\bar{u}_i \cdot (q_{\lambda_\alpha})}{1 + B_{\alpha} \bar{v}_\alpha \cdot (q_{\lambda_\alpha})}.
\]

To deal with the case when $\alpha$ is not minimal, we prepare a lemma.

Lemma 5.2. Suppose that $\tilde{\rho}_\alpha = 1$ and $\lambda \geq 0$ satisfies (5.3).

Then the relation

\[
\lim_{n \to \infty} \frac{\bar{R}(n-m+l; s_{(n, \lambda)})_\alpha}{\bar{v}_\alpha \cdot \bar{R}(n-m; s_{(n, \lambda)})_\alpha} = 0, \; l \in <0, m>, \; m \in <0, \infty>,
\]

implies

\[
\lim_{n \to \infty} \frac{\bar{R}(n; s_{(n, \lambda)})_\alpha}{\bar{v}_\alpha \cdot \bar{R}(n; s_{(n, \lambda)})_\alpha} = \bar{u}_\alpha.
\]

Further the relation

\[
\limsup_{k \to \infty} \frac{\bar{R}(k-m+l; s_{(n, \lambda)})_\alpha}{\bar{v}_\alpha \cdot \bar{R}(k-m; s_{(n, \lambda)})_\alpha} = 0, \; l \in <0, m>, \; m \in <0, \infty>,
\]

implies

\[
\limsup_{k \to \infty} \max_{n \geq k} \frac{R^i(k; s_{(n, \lambda)}_\alpha)}{\bar{v}_\alpha \cdot \bar{R}(k; s_{(n, \lambda)}_\alpha)} - \bar{u}_\alpha^i = 0.
\]
The proof is similar to that of Lemma 4.3 and will be omitted.

Here we assume

\[ \lim_{n \to \infty} \mu B R^i(n; s(n, \lambda)) = \psi^i(\lambda_\beta), \quad i \in \Delta_\beta, \]

for all \( \beta \lesssim \alpha \) with \( \rho_\beta = 1 \). Then it follows, if \( \overline{\rho}_\alpha = 1 \), that

\[ \lim_{n \to \infty} \overline{\mu}_\alpha R(n; s(n, \lambda)) = \overline{\psi}_\alpha(\overline{\lambda}_\alpha), \]

for some \( \overline{\psi}_\alpha(\overline{\lambda}_\alpha) = (\overline{\psi}_\alpha(\overline{\lambda}_\alpha))_{i \in \Gamma_\alpha} \).

**Lemma 5.3.** Let \( \rho_\alpha = 1 \), and (5.11) hold if \( \overline{\rho}_\alpha = 1 \). Then it follows

\[ \lim_{n \to \infty} \mu_\alpha R(n; s(n, \lambda)) = \tilde{\psi}_\alpha(\lambda_\alpha), \]

for all \( \lambda \geq 0 \) with (5.3), where \( \tilde{\psi}_\alpha(\lambda_\alpha) \) are given separately are those in section 4 and in the following three cases: (1) if \( 1 = \overline{\rho}_\alpha > \rho_\alpha \), then

\[ \tilde{\psi}_\alpha(\lambda_\alpha) = \frac{\psi_\alpha(\frac{\overline{\alpha}}{\alpha}(\overline{\lambda}_\alpha)\overline{\lambda}_\alpha)}{1 + \psi_\alpha(\frac{\overline{\alpha}}{\alpha}(\overline{\lambda}_\alpha)\overline{\lambda}_\alpha)(B_\alpha - \chi_\alpha(\lambda_\alpha))}, \]

where

\[ \chi_\alpha(\lambda_\alpha) = \sum_{k=0}^{\infty} \frac{\psi_\alpha A_\alpha \overline{\lambda}_\alpha \overline{\alpha}}{\psi_\alpha A_\alpha \overline{\alpha}(\overline{\lambda}_\alpha)\overline{\alpha}} \overline{\alpha}^{-k} \overline{\alpha}(A_\alpha^{k+1} \{q_\alpha \lambda_\alpha\}^{-1}) \{\psi_\alpha(A_\alpha^{k+1} \{q_\alpha \lambda_\alpha\})^{-1}\}. \]
(ii) if $1 = \bar{p}_\alpha > \tilde{p}_\alpha$, then

$$\tilde{\psi}_\alpha(\lambda_\alpha) = (I - \tilde{A}_\alpha)^{-1} \tilde{A}_\alpha \bar{\psi}_\alpha(\bar{\lambda}_\alpha),$$

and (iii) if $1 = \tilde{p}_\alpha = \bar{p}_\alpha$, then

$$\tilde{\psi}_\alpha(\lambda_\alpha) = (\tilde{\psi}_\alpha \tilde{A}_\alpha \bar{\psi}_\alpha(\bar{\lambda}_\alpha))^{1/2} \bar{\psi}_\alpha.$$

**Proof.** (i) With the notations in (4.20), $a_n > 0$ holds

for all $n \in <0, \infty>$ since $\lambda \geq 0$ satisfies (5.3) and $\tilde{p}_\alpha > 0$.

Hence it follows from (4.19)

$$\frac{1}{n} \left\{ \frac{1}{a_n(s(n))} - \frac{1}{a_0(s(n))} \right\} = \frac{1}{n} \sum_{k=0}^{n-1} \frac{b_k(s(n))}{1 - b_k(s(n))a_k(s(n)) + c_k(s(n))/a_k(s(n))} - \frac{1}{n} \sum_{k=0}^{n-1} \frac{c_k(s(n))}{a_k(s(n))a_{k+1}(s(n))}.$$

By the same arguments as in the proof of Lemma 2.2, it holds

$$\bar{R}(k - m + \ell; \bar{s}(n)) \leq \bar{R}^{k-m+\ell}(\bar{q} - \bar{s}(n)) \leq \frac{\theta_{1r}^{k-m+\ell}}{n} \bar{q}, \quad k \in <m-\ell, \infty>,$$
for some $\eta_1 = \eta_1(\lambda) > 0$ and $\bar{p} < r < 1$. Similarly, by the convexity of the function $F^i(n; s + (q-s)\xi)$ in $0 \leq \xi \leq 1$, we have

\[(5.19) \quad \tilde{R}(k-m; s(n)) \geq \tilde{A}(k-m; s(n))(\bar{q}-s(n)), \quad k \in <m, \infty>,\]

where $\tilde{A}(k; s) = [F^j_1(k; s)]_{1, j \in \Delta}$. Further it can be seen that for each $r < \tilde{r} < 1$ there is a vector $0 < \eta \leq q$ satisfying (4.3) such that

\[(5.20) \quad F(\eta) > \eta \quad \text{and} \quad \rho(\tilde{A}(1; \eta)) > \tilde{r}.\]

Indeed, since $F^i(n; 0) \to q^i$ as $n \to \infty$, it is enough to take an $F(n; 0)$ with a sufficiently large $n$ as the vector $\eta$. Since the matrix $\tilde{A}(1; \eta)$ is also positively regular, it follows from (5.20) that

\[(5.21) \quad \tilde{A}(k; \eta) \geq \tilde{A}(1; \eta)^k \geq \tilde{r}^k(1-\delta^k)\tilde{A}^*(\eta),\]

where $\tilde{A}^*(\eta)$ is a positive matrix and $\{\delta^k\}$ is a sequence with $\delta^k \to 0$ as $k \to \infty$ and $0 \leq \delta^k \leq 1$. But since there is a $k_0 \in <1, \infty>$ with

\[\eta < s^2(k) \leq s(n) \leq q, \quad n \in <k, \infty>, \quad k \in <k_0, \infty>,\]

we have from (5.19) that
for some $\theta_2 = \theta_2(\lambda) > 0$. Combining (5.18) and (5.22) we obtain (5.9), and hence (5.10) by Lemma 5.2. Since $B_{jk}(k; s) \rightarrow \bar{F}_{jk}(q)/2$ as $k \rightarrow \infty$ uniformly in $0 \leq s \leq q$, it follows from (5.10) and (4.20) that

\begin{equation}
\lim_{k \rightarrow \infty} \sup_{n > k} |b_k(s(n)) - B| = 0.
\end{equation}

Hence it also follows from (4.22) that

\begin{equation}
\lim_{k \rightarrow \infty} \sup_{n > k} b_k(s(n))a_k(s(n)) = 0.
\end{equation}

Letting $m = \ell = 0$ in (5.18) and (5.22), we have

\[ \frac{c_k(s(n))}{a_k(s(n))} \leq \frac{\theta_2^k \bar{A}'(q)}{\theta_2^k (1-\delta_k) \bar{A}^*(n)q}, \quad n \geq k \geq k_0, \]

so that

\begin{equation}
\lim_{k \rightarrow \infty} \sup_{n > k} c_k(s(n))/a_k(s(n)) = 0.
\end{equation}

To estimate the sequence $c_k(s(n))/a_k(s(n))a_{k+1}(s(n))$, we shall exploit (5.22) for an $\bar{r}$ with

\[ \sqrt{r} < \bar{r} < 1. \]
Then it is clear from (5.18) and (5.22) that

\[ \frac{1}{n} \frac{c_k(s(n))}{a_k(s(n)) a_{k+1}(s(n))} \leq \theta_3 \frac{r^k}{r^{2k}}, \quad n \geq k \geq k_0, \]

for some \( \theta_3 > 0 \). As for \( k \notin <0, k_0> \), it is not difficult to see that

\[ \frac{1}{n} \frac{c_k(s(n))}{a_k(s(n)) a_{k+1}(s(n))} \leq M_k, \quad n \in <k, \infty> . \]

Since

\[ \sum_{k=0}^{k_0} M_k + \sum_{k=k_0+1}^{\infty} \theta_3 \frac{r^k}{3^r 2^k} < \infty , \]

we can apply the Lebesgue's convergence theorem, obtaining

\[ \lim_{n \to \infty} \sum_{k=0}^{n-1} \frac{1}{n} \frac{c_k(s(n))}{a_k(s(n)) a_{k+1}(s(n))} = \chi(\lambda), \]

with the help of

\[ \lim_{n \to \infty} n R(k; s(n, \lambda)) = A^k \{ q \lambda \}. \]

Combining (5.23) - (5.26) with (5.17), we have

\[ \lim_{n \to \infty} n a_n(s(n)) = \frac{\bar{v} \cdot (\bar{\lambda} \lambda)}{1 + \bar{v} \cdot (\bar{\lambda} \lambda) \{ B - \chi(\lambda) \}} . \]

Hence we have (5.12) with \( \psi^i(\lambda) \) given by (5.13) because of (5.8).
(11) By the convexity of the function $F_i(k; s(k)+(s(n+1)-s(k)))$, we have

$$R(k; s(k)) - R_i(k; s(n+1)) = F_i(k; s(n+1)) - F_i(k; s(k))$$

for each $i \in \Gamma$. Similarly it holds

$$R_i(k; s(n+1)) = F_i(k; q) - F_i(k; s(n+1))$$

Since

$$\theta \frac{1}{n+1} \frac{n+1-l}{l} \leq \theta (q - s(n+1)) \leq \frac{n+1-l}{l},$$

for some $\theta > 0$ and $n_0 \in <l, \infty>$, it follows from (5.28), (5.29) and (4.27) that

$$0 \leq R(k; s(l)) - R_i(k; s(n+1)) \leq \frac{(n+1-l)}{l} \bar{R}(l; 0)$$

for some vector $\bar{c}$. Hence, substituting $l = n-l$, we have for any fixed $m$
\begin{align*}
\lim_{n \to \infty} \sum_{\ell=0}^{\infty} D(n,n-\ell;s(n+1)) C(n-\ell;s(n+1)) R(n-\ell;s(n+1))
\end{align*}

\begin{align*}
= \lim_{n \to \infty} \sum_{\ell=0}^{\infty} D(n,n-\ell;s(n+1)) C(n-\ell;s(n+1)) R(n-\ell;s(n+1))
\end{align*}

\begin{align*}
= \sum_{\ell=0}^{\infty} \frac{A^\ell}{\ell!} \overline{\psi}(\lambda).
\end{align*}

Now we can obtain (5.12) with (5.15) by the same arguments as in the proof of Lemma 4.7 (ii).

(iii) By Lemma 4.7 (iii), the sequence \( n^{1/2} R(n;s(n+1)) \)

is bounded in \( n \in \langle 1, \infty \rangle \) so that we have by the same way as for (5.31) that

\begin{equation}
0 \leq R(n+1;s(n)) - R(n+1;s(n+1)) \leq \frac{\delta}{n^{1+\mu/2}}, \quad n \geq n_0,
\end{equation}

for some vector \( \delta \) and \( n_0 : \langle 1, \infty \rangle \). Let

\begin{align*}
\alpha_n = \alpha_n(\lambda) &= n^{1/2} a_n(s(n)), \\
\beta_n = \beta_n(\lambda) &= b_n(s(n)), \\
\gamma_n = \gamma_n(\lambda) &= n^{1/2} c_n(s(n)).
\end{align*}

Then (4.19) and (5.32) imply

\begin{align*}
\alpha_{n+1} - \alpha_n &= n^{1/2} (-\beta_n a_n^2 + \gamma_n) + O(\frac{1}{n}),
\end{align*}
as \( n \to \infty \), so that

\[
(5.33) \quad \lim_{n \to \infty} \{ \frac{n}{2} (\alpha_{n+1} - \alpha_n) + (\beta_n \alpha_n^2 - \gamma_n) \} = 0.
\]

Further, by means of (4.20) and assumptions (DC) and (DE), it holds

\[
\lim_{n \to \infty} \beta_n (\lambda) = \lim_{n \to \infty} \beta_n (\lambda) = \beta > 0,
\]

for some \( \beta = \beta(\lambda) \) and \( \beta = \beta(\lambda) \). Hence, appealing to Corollary 4.1, we obtain from (5.33) that

\[
(5.34) \quad \sqrt{\gamma^* / \beta} \leq \lim_{n \to \infty} \alpha_n \leq \lim_{n \to \infty} \alpha_n \leq \sqrt{\gamma^* / \beta},
\]

where

\[
\gamma^* = \lim_{n \to \infty} \gamma_n = \varphi A' \psi(\lambda).
\]

Combining (5.34), (5.32) and (4.29), we obtain (5.7). Hence

(5.8) follows by Lemma 5.2, and also

\[
\lim_{n \to \infty} \beta_n (\lambda) = \beta.
\]

Hence, again using Corollary 4.1, we obtain from (5.33)

\[
\lim_{n \to \infty} \alpha_n (\lambda) = \sqrt{\gamma^* / \beta}.
\]
Now (5.12) with (5.16) is proved, since (5.8) is valid.

Proof of Theorem 5.1. Since 1) is clear from Lemmas 5.1 and 5.3, we shall show 2). By the similar arguments as for (2.34), it is easily seen that

\[ \int_{\mathbb{R}^+_N} e^{-\lambda u} dQ_x(n; u) = 1 - \frac{q_x^{-F(n; s(n, \lambda))}}{q_x^{-F(n; 0)}}. \]

Further, it follows from (5.2), (4.2) and (1.7) that

\[ q_x^{-F(n; s(n))} = n^{-\mu x} \sum_{\alpha \in I^+(x)} \sum_{i \in \Delta_\alpha} x^i q_x^{-e_1 \psi_i^+(\lambda_\alpha) + o(n^{-\mu x})}, \]

\[ q_x^{-F(n; 0)} = n^{-\mu x} \sum_{\alpha \in I^+(x)} \sum_{i \in \Delta_\alpha} x^i q_x^{-e_1 R^+ + o(n^{-\mu x})}, \]

as \( n \to \infty \). Hence it follows

\[ \lim_{n \to \infty} \int_{\mathbb{R}^+_N} e^{-\lambda u} dQ_x(n; u) = \psi_x(\lambda), \]

where \( \psi_x(\lambda) \) is given by the right side of (5.4). Further \( \psi_x(\lambda) \) is a Laplace transform of a nonnegative measure \( dQ_x^*(u) \) on \( \mathbb{R}^+_N \). Since \( \lim_{\lambda \to 0} \psi_x^+(\lambda_\alpha) = 0 \) by (5.6) and (5.13) - (5.16)', it holds \( \lim_{\lambda \to 0} \psi_x(\lambda) = 1 \). Hence the nonnegative measure \( dQ_x^*(u) \) is a probability measure and we obtain the conclusion.
We note that the parallel assertions to those of Remarks 2.1 and 2.2 are also valid in this case. Further, we have

**Remark 5.1.** It holds

\[(5.35) \quad \psi^1(\omega_\lambda(\lambda),_a^a) = \psi^1(\lambda),_a^a, \]

where \( \omega_\lambda(\lambda) = A^q \{q\} /q. \)

**Proof.** From (5.6) and (5.13) - (5.16), it is enough to show (5.35) in the case of (5.13). But this is not difficult since

\[
\tilde{\psi}(\omega_\lambda(\lambda)) = \frac{\tilde{v} \cdot (A\{q\})^{-1}}{1 + \tilde{v} \cdot (A\{q\})^{-1} B - \tilde{v} \cdot (A\{q\})^{-1} \chi(\omega_\lambda(\lambda))}
\]

\[
= \frac{\tilde{v} \cdot (A\{q\})^{-1} \tilde{A} }{1 + \tilde{v} \cdot (A\{q\})^{-1} (B - \chi(\lambda)) + \tilde{v} A\{q\}/\tilde{v} \cdot (q\lambda) - \chi(\lambda))}
\]

\[
= \tilde{\psi}(\lambda).
\]

As to Theorem 5.2, we have the next lemma from Theorem 5.1 by the same arguments as those to lead Lemma 3.3 from Theorem 2.1.

**Lemma 5.4.** There exist nontrivial limits
(5.36) \( \lim_{n \to \infty} (n d_{\alpha})^\mu \gamma R^i(nd_{\alpha}; nd_{\alpha}; \lambda) = \psi^i(\lambda_{\alpha}), \lambda \in \Delta_{\alpha}, \)

for each \( \lambda \geq 0 \) with (5.3), \( \alpha \in \langle 1, g \rangle \) with \( p_{\alpha} = 1 \) and \( \gamma \in \langle 1, \tilde{p}_{\alpha} \rangle \).

**Proof of Theorem 5.2.** First we set

\[ F(\lambda) = F(n; s(nd + \ell, \lambda)), \quad s(\omega) = s(nd, \omega_{\lambda}(\lambda)), \]

\[ \mathcal{F} = (F^1(\lambda) \forall \lambda, \ldots, F^N(\lambda) \forall \lambda). \]

Then it is clear that

(5.37) \( R^i(nd + \ell; nd_{\alpha}; \lambda) = R^i(nd; F(\lambda)). \)

Further by the differentiability of the function \( F^i(nd; F(\lambda)) \)

+ \( (s(\omega) - F(\lambda)) \xi \) it holds

(5.38) \[ |R^i(nd; F(\lambda)) - R^i(nd; s(\omega))| \leq \sum_{j \in \Gamma} F^i_j(nd; c) |F^j(\lambda) - s^j(\omega)|, \]

where \( c \) is a vector with \( c \leq \mathcal{F}. \) Similarly

(5.39) \[ R^i(nd; \mathcal{F}) \geq \sum_{j \in \Gamma} F^i_j(nd; \mathcal{F}) (q^j - F^j(\lambda) \forall s^j(\omega)). \]
On the other hand, since

\[ F_j^*(\lambda) = q^j - \sum A_k^j(\lambda)q^k\lambda^k/nd + O\left(\frac{1}{n^2}\right) \]

\[ = q^j(1 - \omega_j^*(\lambda)/nd) + O\left(\frac{1}{n^2}\right), \]

\[ s_j^*(\omega) = q^j(1 - \omega_j^*(\lambda)/nd) + O\left(\frac{1}{n^2}\right), \]

as \( n \to \infty \), it follows

\[ |F_j^*(\lambda) - s_j^*(\omega)| \leq k_1/n^2, \]

(5.40)

\[ q^j - F_j^*(\lambda) \leq s_j^*(\omega) \geq k_2/n, \quad n \in \langle n_0, \infty \rangle, \quad j \in \Gamma, \]

for some \( k_1, k_2 > 0 \) and \( n_0 \in \langle 1, \infty \rangle \). Combining (5.37)-(5.40),

we have

(5.41) \[ |R^1(nd + l; s(nd+\lambda)) - R^1(nd; s(\omega))| \leq \frac{k_1}{nk_2} R^1(nd; FVs) \]

\[ \leq \frac{k_1}{nk_2} R^1(nd; 0), \quad n \in \langle n_0, \infty \rangle . \]

Hence it follows from (5.36) and (5.37) that
\[ \lim_{n \to \infty} \left( \psi_{d+2}(\lambda), \quad 1 \in \Delta, \quad \lambda \in \mathbb{C}, \quad d \in \mathbb{Z}, \quad d \geq 0 \right). \]

The assertion of 2) is easily seen from (4.4) and (5.5) by the same arguments as in the proof of Theorem 5.1.

6. Asymptotic behavior of CGWP

In this section we shall deal with CGWP's \( X = (Z(t), P_x) \) satisfying Condition (C). Since the matrix

\[ A(t) = [A_j^i(t)]_{i,j}, \quad \Lambda = \exp(t\Lambda), \quad t > 0, \]

is always positive by the irreducibility of \( \Lambda \), the periodicity does not appear. There also correspond positive right and left eigenvectors \( \tilde{\mu}_\alpha = (\tilde{\mu}_\alpha^i)_{i \in \Lambda} \) and \( \tilde{\nu}_\alpha = (\tilde{\nu}_\alpha^i)_{i \in \Lambda} \) of the matrix \( \tilde{\alpha}_\alpha \) to the P-F root \( \tilde{\sigma}_\alpha \equiv \rho(\tilde{\alpha}_\alpha); \)

\[ \tilde{\alpha}_\alpha \tilde{\mu}_\alpha = \tilde{\sigma}_\alpha \tilde{\mu}_\alpha, \quad \tilde{\nu}_\alpha \tilde{\alpha}_\alpha = \tilde{\sigma}_\alpha \tilde{\nu}_\alpha, \]

with the normalizations

\[ \sum_{i \in \Lambda} \tilde{\nu}_\alpha^i \tilde{\mu}_\alpha^i = 1, \quad \sum_{i \in \Lambda} \tilde{\mu}_\alpha^i = 1. \]
We set $\delta_p = 1/2^p$, $p < 0, \infty$. Then the family of the generating functions $\{F(n\delta_p; s) ; n \in <0, \infty>\}$ forms a DGWP on $S$, which we shall denote by $X(\delta_p)$. The extinction probability of $X(\delta_p)$ is equal to $\exp(\delta_p)$. Similarly, the family of the generating functions $\{F(n\delta_p; s_\alpha) ; n \in <0, \infty>\}$ forms a DGWP $X(\delta_p)^\alpha$ with the q-mean matrix $A(\delta_p) = \exp(\delta_p a)$. Here we set the condition

$$\sum_{y \in S} p^i(y) y^i q^j \log y^j < \infty, \quad 1, j \in \Gamma_\alpha,$$

where $p^i(y)$ are those in (1.6).

**Lemma 6.1.** It is necessary and sufficient for Condition (CN) to hold that

$$(6.1) \quad \mathbb{E}_1 \{Z_i(t) q Z(t) \log Z_j(t)\} < \infty, \quad 1, j \in \Gamma_\alpha, \quad t > 0.$$ 

**Proof.** For a $j \in \{1, N\}$ with $q_j < 1$, both (CN) and (6.1) are automatically satisfied since the function

$$y^i q^j \log y^j = \{y^j(q^j)^y \log y^j\} \prod_{i \neq j} (q^j)^y_i$$
is bounded in $\gamma \in S$. But, for a $j \in \langle 1, N \rangle$ with $q^j = 1$, it is not difficult to show the necessity by the similar arguments as in the proof of Sevastyanov [13] Theorem 2.4.7, and the sufficiency from the arguments as in Athreya [1] (pp. 49-50).

Now as in (2.3) - (2.4), we shall define $v_\beta(r)$ by

$$v_\beta(r) = \begin{cases} 
\max\{v_y(r) ; y \leq \beta\}, & \text{if } \bar{\sigma}_\beta \neq r, \\
\max\{v_y(r) ; y > \beta\} + (1), & \text{if } \bar{\sigma}_\beta = r,
\end{cases}$$

inductively ($\max \phi = -1$), and $v_\alpha$ by $v_\alpha = v_\alpha(\sigma_\alpha)$. Then setting $R(t;s) = q - F(t;s)$, we have the following Theorem 6.1.

Let a CGWP $X = (Z(t), P_x)$ satisfy Conditions (C) and (CN) for each $\alpha \in \langle 1, \gamma \rangle$ with $\sigma_\alpha < 0$. Then, (1) for each $\alpha \in \langle 1, \gamma \rangle$ with $\sigma_\alpha < 0$, there correspond monotone nonincreasing functions $R^\#(s_\alpha)$ in $0 \leq s_\alpha \leq q_\alpha$, $1 \in \Lambda_\alpha$, such that as $t \to \infty$

$$R^\#(t;s) = t^{\alpha} \sigma_\alpha (R^\#(s_\alpha) + o(1)), \quad 1 \in \Lambda_\alpha,$$

where $o(1)$ is uniform in $s$ on $0 \leq s_\alpha \leq q_\alpha$. Further every $R^\#(s_\alpha)$ is not identically zero. 2) For each $x \in S$ such that $\sigma_\alpha < 0$ for all $\alpha > I^+(x)$, there corresponds a probability distribution $\{P^\#_x(y)\}$ on $S \setminus \{0\}$ satisfying
Proof. By means of Theorem 2.1 and (6.1), there are monotone nonincreasing functions $R^i(s)$, $i \in \Delta$, which are independent of the choice of $p \in (0, \infty)$, such that

$$R^i(n\delta_p; s) = (n\delta_p)^\gamma e^{n\delta_p \sigma} (R^i(s) + o(1)), \quad i \in \Delta,$$

as $n \to \infty$, where $o(1)$ is uniform in $0 \leq s \leq q$. Hence it holds by (2.36) that

$$R^i(F(t; s)) = e^{\sigma R^i(s)}, $$

for each $t \geq 0$ with the form of $n/2^p$ first, and then for all $t \geq 0$ by means of the continuity of $R^i(s)$ in $0 \leq s \leq q$ and of $F(t; s)$ in $t$. Now (6.4) and (6.5) imply

$$\lim_{n \to \infty} \frac{R^i(n; F(t; s))}{(n+\tau)^\gamma e^{(n+\tau)\sigma}} - R^i(s) = 0$$

uniformly in $0 \leq s \leq q$ and $0 \leq \tau < 1$. Since each $t \geq 0$ is represented as $t = n + \tau$, $0 \leq \tau < 1$, where $n \to \infty$ as $t \to \infty$, we obtain (6.2) from (6.6). The assertion 2) is clear from (6.2) if we repeat the arguments in the proof of Theorem 2.1.
Remark 6.1. The routine to determine the $v_\alpha$ and $R^i(s_\alpha)$, $i \in \Delta_\alpha$, is not complicated. Indeed we have only to repeat the analogous way along Lemmas 2.1 and 2.5 in the case of DGWP. Of course the parallel assertions to those of Remarks 2.1 - 2.3 are also valid in this case.

To deal with the critical CGWP, we shall assume

(CC) $f_{jk}(q) < \infty$, $i,j,k \in <1, N>$,

(CE) $\sum_{i,j,k \in \Delta_\alpha} \tilde{v}_a f_{jk}^i (q) \xi^j \xi^k \geq c_\alpha (\sum_{i \in \Delta_\alpha} \tilde{v}_a \xi^i)^2$, $\xi_\alpha = (\xi^i)_{i \in \Delta_\alpha} > 0$, for some $c_\alpha > 0$.

Lemma 6.2. Condition (CC) implies

(6.7) $F_{jk}^i (t;q) < \infty$, $i,j,k \in <1, N>$, $t > 0$.

Further, (CE) and $\delta_\alpha = 0$ imply

(6.8) $\sum_{i,j,k \in \Delta_\alpha} \tilde{v}_a F_{jk}^i (t;q) \xi^j \xi^k \geq c_\alpha(t) (\sum_{i \in \Delta_\alpha} \tilde{v}_a 1)^2$, $\xi_\alpha = (\xi^i)_{i \in \Delta_\alpha} > 0$, for some $c_\alpha(t) > 0$.

Proof. The first assertion is well known (eg. Sevastyanov [12] Theorem 4.7.3). To show the second assertion, we shall use the relations
\begin{equation*}
F_{jk}(t;q) = \sum_{l,m,n} \int_0^t A_l(t-\tau)f_{mn}(q)A_j^m(\tau)A_k^n(\tau)\,d\tau
\end{equation*}

(\textit{ibid.} (4.7.16)). Then it follows

\begin{equation*}
\sum_{i,j,k} i^i F_{jk}(t;q) \xi^i \xi^j \xi^k \geq \sum_{i,j,k} \int_0^t i^i F_{jk}(q)A_j^j(\tau)A_k^k(\tau)\xi^j \xi^k \,d\tau,
\end{equation*}

which implies (6.8), since \(A_j^j(\tau) \to 1\) as \(\tau \to 0\).

Setting \(\mu = 1/2\nu(0)\), we have the following

\textbf{Theorem 6.2.} Let a CGWP \(X = (Z(t), P_X)\) satisfy Conditions (C) and (CC). Then for each \(\alpha < 1, \gamma >\) with \(\sigma_\alpha = 0\), there correspond constants \(R^*_{\alpha}, i \in \Delta_\alpha\), such that

\begin{equation}
\lim_{t \to \infty} t^{\mu} R^*_{\alpha}(t;s) = R^*_{\alpha}, \quad i \in \Delta_\alpha, \quad 0 \leq s < q.
\end{equation}

The proof is clear from Theorem 4.1 and (6.7), and will be omitted.

\textbf{Theorem 6.3.} Let a DGWP \(X = (Z(t), P_X)\) satisfy Conditions (C), (CC) and (CE) for each \(\alpha < 1, \gamma >\) with \(\sigma_\alpha = 0\). Then, l) for each \(\alpha < 1, \gamma >\) with \(\sigma_\alpha = 0\), there correspond nonnegative functions \(\psi^i(\lambda_\alpha), i \in \Delta_\alpha\), such that
(6.9) \( \lim_{t \to \infty} t^{\mu} \alpha R^{i}(t; s(t, \lambda)) = \psi^{i}(\lambda), \ i \in \Delta, \ \lambda > 0. \)

2) For each \( x \in S \) with \( \sigma_{a} = 0 \) for some \( \alpha \in I_{+}(x) \), the distributions

\[ Q_{x}(t, u) = P_{x}\left\{ \frac{Z(t)}{t} \leq u \mid t < T < \infty \right\}, \ u \in \mathbb{R}_{+}, \]

converge as \( t \to \infty \) to a probability distribution \( Q^{x}(u) \) on \( \mathbb{R}_{+}^{N} \).

**Proof.** By means of Theorem 5.1 and (6.8), there are non-negative functions \( \psi^{i}(\lambda), \ i \in \Delta, \) which are independent of the choice of \( p \in <0, \infty> \), such that

\[
(6.10) \lim_{n \to \infty} n^{\delta}_{p} \mu R^{i}(n^{\delta}_{p}; s(n^{\delta}_{p}, \lambda)) = \psi^{i}(\lambda), \ i \in \Delta, \ \lambda > 0. 
\]

Further, (5.35) implies

\[
(6.11) \psi^{i}(\omega_{t}(\lambda)) = \psi^{i}(\lambda),
\]

for each \( t \geq 0 \) with the form of \( n/2p \), where \( \omega_{t}(\lambda) = A(t)(q\lambda)/q. \)

Since the function \( 1 - \psi^{i}(\lambda)/R^{i} \) is a Laplace transform of a probability distribution, it is continuous in \( \lambda > 0. \) Hence the function \( \psi^{i}(\omega_{t}(\lambda)) \) is continuous in \( t \), and so (6.11) holds for all \( t \geq 0. \) Now representing each \( t \geq 0 \) as \( t = n + \tau, \)

\( 0 \leq \tau < 1, \) we have
But by the same reason as of (5.41) it holds

\[ |R^1(n;F(t;s(t',X)) \rightarrow R^1(n;s(n,\omega_T(\lambda)))| = \frac{K^1 R^1(n;0), n \in \mathbb{N}, \infty} \]

Hence it follows from (6.8) and (6.10) - (6.12) that

\[ \lim_{t \to \infty} R^1(t;s(t,\lambda)) = \lim_{n \to \infty} n^\mu R^1(n;s(n,\omega_T(\lambda))) \]

\[ = \psi^1(\omega_T(\lambda)) = \psi^1(\lambda). \]

The assertion of 2) is clear from (6.9) and (6.8).

7. Examples

In this section we shall give four examples. The first two are those proposed by Jirina [8] and Sevastyanov [14] as examples which, because of the failure of the positive regularity, do not satisfy their theorems. But these are contained in our scheme, and the direct calculations show that the asymptotic forms coincide with those given by our theorems: Example 3 is of reducible cases, where the asymptotic behaviors are also calculated directly and coincide with those given by
our theorems. However, all the marginal distributions of $Q^*(u)$ in Examples 1 - 3 are of exponential type. In Example 4 we shall show with aid of our theorems that there really exists a case when a certain marginal distribution of $Q^*(u)$ is not of exponential type. Naturally the distribution is the same type of that in Savin and Chistyakov [12].

Example 1. Let $\phi(\xi) = \sum_{j=0}^{\infty} p_j \xi^j$ be an one-dimensional probability generating function with $p_0 > 0$, $\phi''(1)<\infty$ if $\phi'(1) = 1$, and consider the two-type DGWP $X$ with the generating functions

\begin{equation}
F^1(s^1, s^2) = \phi(s^2), \quad F^2(s^1, s^2) = \phi(s^1).
\end{equation}

Let $q_0$ be the least nonnegative fixed point of $\phi(\xi)$ and set $\rho = \phi'(q_0)$. Then it is well known that $\phi'(1) \neq 1$ implies $\rho \neq 1$, and $\phi'(1) = 1$ implies $\rho = 1$. The extinction probability $q$ of $X$ is equal to $(q_0, q_0)$, and the q-mean matrix $A$ is given by $[Q^\rho_0]$. Hence it follows that $A_{\downarrow} = \Gamma_{\downarrow} = \{1, 2\}$ and $\rho_{\downarrow} = \hat{\rho}_{\downarrow} = \rho$. We can calculate the n-step generating functions $F(n; s)$ precisely:
(7.2) \( F^1(n; s) = \begin{cases} 
\phi(n; s^1), & \text{if } n \text{ is even, } i = 1, 2, \\
\phi(n; s^{i+1}), & \text{if } n \text{ is odd, } i = 1, 2, 
\end{cases} \)

where \( \phi(n; \xi) \) is the n-step iteration of \( \phi(\xi) \) and \( i+1 \) is identified with 1 if \( i = 2 \). Here we shall divide it into three cases.

(i) When \( \rho = 0 \), it follows \( F(n; s) = 1 \), \( n \in <1, \infty> \), and all the situations are trivial.

(ii) When \( 0 < \rho < 1 \), the one-dimensional (or positively regular case) arguments assure the existence of a nonincreasing function \( K^*(\xi) \) and a distribution \( \{ P^*(j) \} \) on \( <1, \infty> \) such that

\[
\lim_{n \to \infty} \left\{ q_0 - \phi(n; \xi) \right\} / \rho^n = K^*(\xi), \quad 0 \leq \xi \leq q_0,
\]

(7.3)

\[
1 - \lim_{n \to \infty} \frac{q_0 - \phi(n; q_0 \xi)}{q_0 - \phi(n; 0)} = \sum_{j=1}^{\infty} P^*(j) \xi^j, \quad 0 \leq \xi \leq 1.
\]

Combining (7.2) and (7.3) we obtain

\[
\lim_{n \to \infty} \frac{R^1(2n; s)}{\rho^{2n}} = K^*(s^1), \quad 0 \leq s \leq q, \quad i = 1, 2,
\]

(7.4)

\[
\lim_{n \to \infty} \frac{R^1(2n+1; s)}{\rho^{2n}} = \rho K^*(s^{i+1}), \quad 0 \leq s \leq q, \quad i = 1, 2,
\]
\[ \lim_{n \to \infty} P \{ Z(2n) = y \mid 2n < T < \infty \} = \frac{x^{-1}P*(y) + x^2P*(y^2)}{x^{-1} + x^2}, \quad x = (x^1, x^2) \neq 0. \] (7.5)

\[ \lim_{n \to \infty} P \{ Z(2n+1) = y \mid 2n+1 < T < \infty \} = \frac{x^{-1}P*(y^2) + x^2P*(y^4)}{x^{-1} + x^2}, \quad x = (x^1, x^2) \neq 0. \]

(iii) Let \( \rho = 1 \). Also in this case the one-dimensional arguments tell us

\[ \lim_{n \to \infty} n \{ 1 - \phi(n; \xi) \} = 2/\phi''(1), \quad 0 \leq \xi < 1, \] (7.6)

\[ \lim_{n \to \infty} \frac{n}{1 + \phi''(1) n/2} = \frac{\eta}{(1 + \phi''(1))}, \quad n \geq 0. \]

Hence by means of (7.2) it follows

\[ \lim_{n \to \infty} nR_1(n; s) = 2/\phi''(1), \quad 0 \leq s < 1, \] (7.7)

\[ \lim_{n \to \infty} E_x \{ \exp(-\lambda \cdot Z(2n)/2n \mid 2n < T \} = \frac{1}{x^{-1} + x^2} \frac{1}{1 + \phi''(1) \lambda^{1/2}} + \frac{1}{1 + \phi''(1) \lambda^{2/2}}. \] (7.8)

\[ \lim_{n \to \infty} E_x \{ \exp(-\lambda \cdot Z(2n+1)/(2n+1)) \mid 2n+1 < T \} = \frac{1}{x^{-1} + x^2} \frac{x^1}{1 + \phi''(1) \lambda^{1/2}} + \frac{x^2}{(1 + \phi''(1) \lambda^{2/2})}. \]
for each \( x = (x^1, x^2) \neq 0 \) and \( \lambda = (\lambda^1, \lambda^2) > 0 \). From (7.8) it follows

\[
Q^*_{x^0}(u) = \frac{1}{x^1 + x^2} \{ x^1(1 - e^{-2u^1/\phi''(1)}) + x^2(1 - e^{-2u^2/\phi''(1)}) \}, \tag{7.9}
\]

for each \( x = (x^1, x^2) \neq 0 \) and \( u = (u^1, u^2) \in \mathbb{R}^2_+ \).

**Example 2.** Let \( \phi(x), q_0 \) and \( p \) be those given in Example 1. We consider the two-type DGWP \( X \) with the generating functions

\[
(7.10) \quad F^1(s^1, s^2) = \phi(s^2), \quad F^2(s^1, s^2) = s^1.
\]

The extinction probability is equal to \( (q_0, q_0) \) and the q-mean matrix is \( A = [0 \rho; \rho 0] \). Hence \( \Delta_1 = \Gamma_1 = \{1, 2\} \) and \( \rho_1 = \rho_2 = \sqrt{\rho} \).

The n-step generating functions \( F(n; s) \) is given by

\[
(7.11) \quad F^i(n; s) = \begin{cases} 
\phi(n/2; s^i), & \text{if } n \text{ is even, } i = 1, 2, \\
\phi\left(\frac{n-(-1)^i}{2}; s^{1+i}\right), & \text{if } n \text{ is odd, } i = 1, 2.
\end{cases}
\]

(i) When \( \rho = 0 \), \( F(n;s) = 1 \) for all \( n \in \mathbb{N} \).

(ii) When \( 0 < \rho < 1 \), it holds
\[ \lim_{n \to \infty} R_i^{r}(2n;s)/\rho^n = K^*(s^i), \quad 0 \leq s \leq q, \quad i = 1, 2, \]

(7.12)

\[ \lim_{n \to \infty} R_i^{r}(2n+1;s)/\rho^n = \rho(1-(-1)^i)/2K^*(s^i+1), \quad 0 \leq s \leq q, \quad i = 1, 2, \]

where \( K^*(\xi) \) is that of (7.3). Here we assume

(7.13) \[ \sum_{j=0}^{\infty} \frac{1}{j!} \log j < \infty, \quad \text{if} \quad \phi'(1) < 1. \]

Then \( K^*(\xi) \neq 0 \) and we have

\[ \lim_{n \to \infty} P_x \{ Z(2n) = y \mid 2n < T < \infty \} = \frac{x^1 P^*(y^1) + x^2 P^*(y^2)}{x^1 + x^2}, \]

(7.14)

\[ \lim_{n \to \infty} P_x \{ Z(2n+1) = y \mid 2n+1 < T < \infty \} = \frac{x^1 P^*(y^2) + x^2 P^*(y^1)}{x^1 + x^2}, \quad x = (x^1, x^2) \neq 0. \]

(iii) When \( \rho = 1 \), we also have (7.7) - (7.9) but with

\( \phi''(1) \) replaced by \( \phi''(1)/2 \).

Example 3. Let \( \phi(\xi) \) be an one-dimensional infinitesimal generating function with \( \phi''(1) < \infty \) and \( \phi(0) > 0 \). We consider the two-type CGWP with the infinitesimal generating functions

(7.15) \[ f^1(s^1, s^2) = \phi(s^1), \quad f^2(s^1, s^2) = b(s^1-1)+c(1-s^2), \]

where \( b \) and \( c \) are constants with \( 0 < b \leq c \). Let \( q_1 \) be the
least nonnegative zero point of $\phi(\xi)$ and put $\sigma = \phi'(q_1)$. Then $\phi'(1) \neq 0$ implies $\sigma < 0$, and $\phi'(1) = 0$ implies $\sigma = 0$. The extinction probability is given by $q = (q_1^1, q_2)$ where $q_1^1 = q_1$ and $q_2^2 = (1 - b(1 - q_1))/c$, and the infinitesimal $q$-mean matrix is $a = \begin{bmatrix} \sigma & 0 \\ b & c \end{bmatrix}$. Hence it follows $\Delta_1 = \{1\}$, $\Delta_2 = \{2\}$, $\Gamma_1 = \{1\}$ and $\Gamma_2 = \{1, 2\}$. Now we can define the one-type CGWP $\{\phi(t;\xi)\}$ with the infinitesimal generating function $\phi(\xi) :$

$$\frac{d\phi}{dt}(t;\xi) = \phi(\phi(\xi)), \quad \phi(0;\xi) = \xi, \quad 0 \leq \xi \leq 1.$$ 

Then our CGWP $\{F(t;s)\}$ is given by

$$F^1(t;s) = \phi(t;\xi^1),$$

$$F^2(t;s) = e^{-ct} \int_0^t e^{c\tau}(b\phi(s^1 \tau) + c - b) d\tau + s^2 - q_2 + e^{-ct} \int_0^t e^{c\tau}(\phi(s^1 \tau) - q_1^1) d\tau + s^2 - q_2.$$ 

The CGWP $X_1 = \{F^1(t;s)\}$ is divided into two cases.

(i) When $\sigma < 0$, the one-dimensional arguments assure the existence of a monotone nonincreasing function $K^*(\xi)$ and a distribution $\{P^*(j)\}$ on $[1, \infty)$ satisfying
\[ \lim_{t \to \infty} \frac{q_1 - \phi(t; \xi)}{\exp(-n/t)} = \frac{n}{1 + \phi''(1)n/2}, \quad n \geq 0. \]

Hence it follows that

\[ \lim_{t \to \infty} tR^1_1(t; s) = 2/\phi''(1), \quad 0 \leq s^1 \leq 1, \]

\[ \lim_{t \to \infty} \mathbb{P} \{ \frac{Z(t)}{t} \leq \{u^1, u^2\} \mid t \leq T \} = \lim_{t \to \infty} \mathbb{P} \{ Z(t) \mid t \leq T \} = 1 - e^{-2u^1/\phi''(1)}, \]

for each \( x^1 \in [1, \infty) \) and \( u \in \mathbb{R}_+^2 \).
The CGWP \( X_2 = X = \{P(t;s)\} \) is divided into four cases.

(i) When \(-c < \sigma < 0\), the P-F root \( \sigma_2 = \rho(a) \) is equal to \( \sigma \).

It follows from (7.16) and (7.17) that

\[
\lim_{t \to \infty} R^2(t; s)e^{\sigma t} = \frac{b}{c+\sigma} K^*(s), \quad 0 \leq s \leq q,
\]

(7.21)

\[
\lim_{t \to \infty} P \{Z(t) = (y_1, y_2) \mid t < T < \infty\} = \begin{cases} P^*(y), & y_2 = 0, \\ 0, & \text{otherwise}, \ x \neq 0. \end{cases}
\]

(ii) When \( \sigma = -c < 0 \), it holds \( \sigma_2 = -c \), and

\[
\lim_{t \to \infty} R^2(t; s)e^{-\sigma t} = b\int_0^\infty e^{\sigma t}(q_1-\phi(t; s^1))dt + q^2 - s^2,
\]

(7.22)

\[
\lim_{t \to \infty} P \{Z(t) = (y_1, y_2) \mid t < T < \infty\} = P^*(y), \quad x^2 \neq 0,
\]

where the distribution \( \{P^*(y)\} \) is given by

\[
\sum_{y \neq 0} P^*(y)s^y = 1 - \frac{b\int_0^\infty e^{\sigma t}(q_1-\phi(t; s^1))dt + q^2 - s^2}{b\int_0^\infty e^{\sigma t}(q_1-\phi(t; 0))dt + q^2}, \quad 0 \leq s \leq 1.
\]

(iii) In case of \( \sigma = -c < 0 \), it holds \( \sigma_2 = \sigma = -c \), and

\[
\lim_{t \to \infty} R^2(t; s)e^{\sigma t} = bK^*(s), \quad 0 \leq s \leq q,
\]

(7.23)

\[
\lim_{t \to \infty} P \{Z(t) = (y_1, y_2) \mid t < T < \infty\} = \begin{cases} P^*(y), & y^2 = 0, \\ 0, & \text{otherwise}, \ x \neq 0. \end{cases}
\]
(iv) When $-c < 0$, it follows $\sigma_2 = 0$ and the CGWP $X$ is critical with $q = 1$. By means of (7.16) and (7.19) it holds

$$
\lim_{t \to \infty} tR^2(t;s) = \frac{2b}{c\phi''(1)}, \quad 0 < s < 1,
$$

(7.24)

$$
\lim_{t \to \infty} tR^2(t;e^{-\lambda_1/t}, e^{-\lambda_2/t}) = \frac{b}{c} \frac{\lambda^2}{(1+\phi''(1)\lambda_1^{1/2})}, \quad (\lambda_1, \lambda_2) > 0.
$$

Hence with the aid of (7.19) and (7.20) it follows

$$
Q_x(u_1, u_2) = 1 - e^{-2u_1/\phi''(1)} \quad x \neq 0, \quad u \in \mathbb{R}^2_+.
$$

Example 4. Let $\phi(\xi)$ be a one-dimensional probability generating function with $\phi'(1) = 1$ and $0 < \phi''(1) \equiv 2B_1 < \infty$.

We consider two-type DGWP $X$ given by the generating functions

$F^1(s_1, s_2) = \phi(s_1)$ and $F^2(s_1, s_2)$ with $F^2_1(1) = A' > 0$, $F^2_2(1) = 1$, and $0 < F^2_{22}(1) \equiv 2B_2 < \infty$. Then the extinction probability is equal to $1 = (1,1)$ and the $q$-mean matrix is

$A = [0, 1 \ 1 0]$. Hence $A_1 = \{1\}$, $A_2 = \{2\}$, $\Gamma_1 = \{1\}$, $\Gamma_2 = \{1,2\}$ and $\rho_1 = \rho_2 = \rho_1 = \rho_2 = 1$. From (7.6), we have
\[
\lim_{n \to \infty} nR^1(n; s) = \frac{1}{B_1}, \quad 0 < s^1 < 1,
\]
(7.26)

\[
\lim_{n \to \infty} nR^1(n; s, \lambda) = \frac{\lambda^1}{1+B_1\lambda^1}, \quad \lambda^1 > 0.
\]

Now by Lemmas 4.7 and (5.3)

\[
\lim_{n \to \infty} n^{1/2}R^2(n; s) = \sqrt{\frac{A'}{B_1B_2}}, \quad 0 < s < \frac{1}{1+B_1},
\]
(7.27)

\[
\lim_{n \to \infty} n^{1/2}R^2(n; s, \lambda) = \sqrt{\frac{A'\lambda^1}{B_2(1+B_1\lambda^1)}}, \quad \lambda > 0.
\]

Hence by Theorem 5.2 2), it follows

\[
\lim_{n \to \infty} E_{x_1, x_2} \{\exp(-\lambda \cdot Z(n)/n) \mid n < T\} =
\]

\[
= \begin{cases} 
\frac{1}{1+B_1\lambda^1}, & x_2 = 0, \\
1 - (1 - \frac{1}{1+B_1\lambda^1})^{1/2}, & x_2 \neq 0,
\end{cases}
\]

that is

\[
Q^*(x_1, x_2)(u) =
\]

\[
= \begin{cases} 
1 - e^{-u_1/B_1}, & x_2 = 0, \\
\frac{1}{2B} \int_0^u e^{-\xi} \frac{\xi}{B} F_1\left(-\frac{1}{2}; -2; \frac{\xi^2}{B}\right) d\xi, & x_2 \neq 0,
\end{cases}
\]
(7.29)
where \( \genfrac{[}{]}{0pt}{}{1}{-1/2;-2;\xi} \) is the Barnes' generalized hypergeometric function:
\[
\genfrac{[}{]}{0pt}{}{1}{-1/2;-2;\xi} = \sum_{k=0}^{\infty} \frac{(-1/2)_k \xi^k}{(-2)_k k!}
\]
\[
= \sum_{k=0}^{\infty} \frac{(k-1/2)(k-1-1/2)\ldots(1-1/2)}{(k+1)!} \xi^k.
\]

DEPARTMENT OF MATHEMATICS

FACULTY OF SCIENCES AND ENGINEERING

SAGA UNIVERSITY
References


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Footnotes

1. If $A_a = [0]$, (2.5) is always satisfied.

2. Or equivalently, we may use the Jordan's normal form of $A$ reminding the asymptotic forms of its products.

3. In the proofs of the following theorems and lemmas we shall often abbreviate the suffix $a$ and the variable $s$ where there are no confusions.

4. More precisely, one may take $\lambda_a$ with the form of $\bar{\lambda}_a = \theta \bar{q}$, $\bar{\lambda}_a = \theta^2 \bar{q}$ where $\theta > 0$, $\theta > 0$, in the case of $q = \bar{\beta}_a > \bar{\rho}_a$. 

5. This means, in terms of measures,

$$Q^*(E_1 \times E_2) = \frac{1}{x_1 + x_2} \left\{ \int_{E_1} e^{-2u_1^2/\phi''(l)} du \right\}^{\frac{1}{1}}_{E_2(0)} + \frac{2x_2}{\phi''(l)} \int_{E_2} e^{-2u_2^2/\phi''(l)} du \right\}^{\frac{1}{1}}_{E_1(0)} I_E(\cdot),$$

where $I_E(\cdot)$ is the indicator function.