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主論文 1.

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A Study of Kripke-type Models for Some Modal Logics by Gentzen's Sequential Method

By

Masahiko SATO

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京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto Japan

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Research Institute for Mathematical Sciences

Kyoto University

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INTRODUCTION

The main objective of the present paper is to clarify a close relationship between Gentzen-type sequential formulation of formal systems (especially of modal calculi) and Kripke-type semantics. Though the investigations by Schütte [31], Maehara [20], Fitting [3], Prawitz [27], etc. have suggested this relationship either explicitly or implicitly, the usefulness of Gentzen systems for the semantical studies of modal calculi seems to be less recognized that it deserves. In this paper, we wish to establish its usefulness in a decisive way. We now proceed to explain the background motivation for our study.

When an interpretation, or semantics, of a formal system is given, we are always interested in the question: "Is it complete?" Indeed, the completeness of the semantics is essential so that it is really useful for the study of the formal system in question. The naturalness of the semantics is fundamental as well. For instance, in the case of modal calculi, we know such semantics as algebraic, topological and Kripke-type. (See Cresswell [2], Lemmon [18], Rasiowa [28], Rasiowa-Sikorski [29], Segerberg [34] etc.) Among these, Kripke-type semantics introduced by Kripke [15, 16] has proved to be most successful.

On the other hand, the method of formulating a formal system is not unique. Formulations such as Hilbert-type natural deduction, Gentzen's sequent system and Smullyan's analytic tableau are well-known. And each formulation has its own merits for both syntactical and semantical study of formal systems. (See, e.g., Kreisel [13, 14], Prawitz [25, 26], Zucker [39], Takeuti [38] and Smullyan [35].) In this paper, however, we take the standpoint of regarding that Gentzen-type sequential formulation is best fitted for the Kripke-type semantical study of formal systems. We have slightly modified the notion of a sequent in order to establish the natural correspondence between Gentzen systems and Kripke models. I.e., we define a sequent as a pair of two (possibly infinite) sets of well formed formulas.

Though our method is general enough to admit applications to, for example, intermediate logics and other modal calculi, we will, in this paper, only concentrate on three modal systems KT3, KT4 and KT5 of knowldge as introduced by McCarthy [21, 22]. However, since these systems are generalizations of bi-modal logics S4-T, S4-S4 and S5-S5, which in turn are generalizations of T, S4 and S5, our results applies directly to these modal calculi. In fact, we have so designed the languages that our argument will always be relative to a particular choice of the language, and that by a suitable choice of the language we will be able to obtain the specific result for any one of these logics. We leave applications of our method to other logics to the interested reader.

There are many known proof-techniques of completeness results. See, e.g., Gödel [6], Henkin [10], Takahashi [37], Fitting [3], Smullyan [35], Kripke [15, 16], Lemmon-Scott [18], Segerberg [34], Schütte [31] and Maehara [20]. In the present paper, we prove the completeness theorem in two different ways. The first one is the so-called Henkin-style proof. However, our proof is new in that it is relative to a set Ω of wffs which is closed under subformulas, so that we can at the same time prove compactness by letting Ω to be the whole set of wffs and decidability by letting Ω to be the set of subformulas of a certain formula. Our second proof is based on cut-free formulations of the systems. Especially, a cut-free system for S5 is obtained by a close inspection of the first proof. The cut-elimination theorem of these systems yields our second proof of the decidability of KT3, KT4 and S5. For KT3 and KT4, it also gives a proof of the disjunction property of these logics.

As we mentioned above, in our first proof of the completeness theorem, we construct a model $U(\Omega)$, called the universal model over Ω, for any Ω which is closed under subformulas. By means of this fundamental model, we will define a category $\mathbb{K}(\Omega)$ of Kripke-type models over Ω. In this category, $U(\Omega)$ will be characterized as "the" terminal object of the category. The classification problem of models will also be conveniently treated in this category. For the modal logic S5, we can obtain a complete classification of models. This result easily shows the normal form theorem for S5, and the structure of Lindenbaum algebra of S5 will also be determined.

We now briefly sketch the content of each chapter.

In Chapter 1, we first define the languages upon which our formal systems will be built. The main reason for introducing many languages rather than a single language is that we can explain the difference between certain logics (such as S4 and S4-T) as the mere difference of languages. We then define Hilbert-type axiomatizations of the three modal systems KT3, KT4 and KT5. Corresponding to these, three equivalent Gentzen-type sequential systems GT3, GT4 and GT5 will be defined. Though our notion of a sequent admitts an infinite set of wffs both in the antecedent and in the succedent, a theorem to the effect that this generalization is superficial will be proved. Nevertheless, the importance of the generalization will be fully exhibited in the subsequent chapters.

In Chapter 2, we introduce a topology, which is homeomorphic to Scott's $P\omega$ topology, on the whole set Wff of wffs. Several syntactic notions concerning deducibility will be expressed in topological terminology.

In Chapter 3, we define the Kripke-type semantics for KTi (i = 3, 4, 5). Two completeness proofs will be given there. Compactness, decidability and cut-elimination theorem will be proved as by-products. The first completeness proof furnishes us with a basis for subsequent studies, while the importance of the second proof lies in giving cut-free systems as by-products.

Chapter 4 is devoted to the category theory of Kripke models. In contrast to the notion of p-morphism due to Segerberg [34], which is defined by referring to the relational structure of models, our notion of homomorphism is defined without any explicit reference to the relational structure of models. Roughly speaking, we define an (Ω-) homomorphism as a mapping which preserves the semantics in $U(\Omega)$ of a model. Thus for each Ω, we obtain a distinct category $\mathbb{K}(\Omega)$. In case Ω is equal to Wff, our notion of homomorphism contains the notion of p-morphism.

In Chapter 5, we study the modal calculus S5 as an application of the results obtained in Chapter 4. A complete classification of S5 models under a certain equivalence relation on models will be given. Our method gives another proof of normal form theorems by Itoh [12] and the result of Bass [1] which determines the Lindenbaum algebra of S5 with finite generators.

The final chapter, Chapter 6, is devoted to the study of two well-known puzzles, the puzzle of wise men and the puzzle of unfaithful wives. It was McCarthy [21] who first attacked these puzzles in a formal manner. The second puzzle, however, remained almost untouched. The difficulties which arise in the formal presentation of the puzzle are twofold. Firstly, the puzzle involves the self-referential statements. Secondly, the totality of one's knowledge is difficult to characterize. We will present a solution which we think successfully gets over these difficulties. The notion of knowledge set and knowledge base to be defined in this chapter will play an important role in characterizing the totality of one's knowledge. A model-theoretic solution of the puzzle of wise men will also be given there.

CHAPTER 1

THE FORMAL SYSTEMS

1.1. *Basic language*

The *basic language* $\mathbf{L}$ is a triple $(\mathbb{Pr}, \mathbb{Sp}, \mathbb{N}^+)$, where

$$\mathbb{Pr} = p_1, p_2, \cdots \quad ;$$
$$\mathbb{Sp} = S_0, S_1, \cdots \quad ;$$
$$\mathbb{N}^+ = \bar{1}, \bar{2}, \cdots$$

are denumerable sequences of distinct symbols. $\mathbb{N}^+$ is the set of numerals denoting the corresponding positive integers. But, for simplicity, we will identify n with its denotation $\bar{n}$. $S_0 \in \mathbb{Sp}$ will also be denoted by 0 and will be called "FOOL."

1.2. *Languages*

A *language* L is a triple (Pr, Sp, T) where

$$\mathrm{Pr} \subseteq \mathbb{Pr} \quad ;$$
$$\mathrm{Sp} \subseteq \mathbb{Sp} \quad ;$$
$$\mathrm{T} \subseteq \mathbb{N}^+ .$$

Elements in Pr, Sp and T denote *propositional variables*, *persons* and *time*, respectively. Our arguments henceforth will, unless stated otherwise, always be relative to a language L. So the reader may choose any language he likes and read the following by fixing his favorite language. For example, if he is only interested in the classical propositional calculus, he should take $L = (\mathbf{Pr}, \emptyset, \emptyset)$. When an explicit mention of the language L to be considered is necessary, we will express it by explicitly writing L somewhere as a suffix etc.

1.3. *Well formed formulas*

The set of *well formed formulas* is defined to be the least set Wff such that:

(W1) $\perp \in \mathrm{Wff}$;

(W2) $\mathrm{Pr} \subseteq \mathrm{Wff}$;

(W3) $\alpha, \beta \in \mathrm{Wff}$ implies $\supset\alpha\beta \in \mathrm{Wff}$;

(W4) $S \in \mathrm{Sp}$, $t \in T$, $\alpha \in \mathrm{Wff}$ implies $St\alpha \in \mathrm{Wff}$.

The symbols $\perp$ and $\supset$ denote "false" and "implication", respectively.

We will make use of the following abbreviations:

$\alpha\supset\beta$	=	$\supset\alpha\beta$	read "α implies β"
$\neg\alpha$	=	$\alpha\supset\perp$	read "not α"
$\top$	=	$\neg\perp$	read "true"
$\alpha\vee\beta$	=	$\neg\alpha\supset\beta$	read "α or β"
$\alpha\wedge\beta$	=	$\neg(\alpha\supset\neg\beta)$	read "α and β"
$[St]\alpha$	=	$St\alpha$	read "S knows α at time t"
$\langle St\rangle\alpha$	=	$\neg[St]\neg\alpha$	read "α is possible for S at time t"

$\{St\}\alpha \quad = [St]\alpha \vee [St]\neg\alpha$ read "S knows whether α at time t"

Remark. If L is the simplest language ($\emptyset$, $\emptyset$, $\emptyset$), the conditions (W2) and (W4) in the definition of Wff become vacuous, so that we have Wff = $\{ \perp, \perp\supset\perp, \perp\supset(\perp\supset\perp), (\perp\supset\perp)\supset\perp, \cdots\}$. We will not repeat this sort of remarks in the sequel. However, the reader should always be alert and notice that the definitions or proofs may become simpler for a particular choice of L. We also remark that the cardinality of Wff is ω irrespective of L.

For any $\alpha \in$ Wff, we define Sub(α) $\subseteq$ Wff inductively as follows:

(S1) $\alpha \in Pr\cup\{\perp\} \quad \Rightarrow \quad Sub(\alpha) = \{\alpha\}$;

(S2) $\alpha = \beta\supset\gamma \quad \Rightarrow \quad Sub(\alpha) = \{\alpha\}\cup Sub(\beta)\cup Sub(\gamma)$;

(S3) $\alpha = [St]\beta \quad \Rightarrow \quad Sub(\alpha) = \{\alpha\}\cup Sub(\beta)$.

We say β is a *subformula* of α if $\beta \in Sub(\alpha)$.

1.4. *Hilbert-type systems*

We now define three modal systems KT3, KT4 and KT5 of knowledge due to McCarthy [22]. We begin with the definition of KT3.

The axiom schemata for KT3 are:

(A1) $\neg\neg\alpha\supset\alpha$

(A2) $\alpha \supset (\beta \supset \alpha)$

(A3) $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$

(A4) $[St]\alpha \supset \alpha$

(A5) $[Ot]\alpha \supset [Ot][St]\alpha$

(A6) $[St](\alpha \supset \beta) \supset ([Su]\alpha \supset [Su]\beta)$, where $t \leq u$ [1)]

In (A1)-(A6), α, β, γ denote arbitrary wffs, S denotes arbitrary element in Sp, and t, u denote arbitrary elements in T.

The notion of a *proof* in KT3 is defined by:

Definition 1.1. Let $\alpha \in$ Wff. A finite sequence of wffs $\alpha_1, \cdots, \alpha_n$ $(n \geq 1)$ is a proof of α in KT3 if $\alpha_n = \alpha$ and for each i one of the following three conditions holds:

(i) α_i is an instance of (A1)-(A6)

(ii) there exist $j, k < i$ such that $\alpha_k = \alpha_j \supset \alpha_i$ (In this case, we say α_i is obtained from α_j and $\alpha_j \supset \alpha_i$ by *modus ponens*.)

(iii) there exists $j < i$ such that $\alpha_i = [St]\alpha_j$ for some $S \in$ Sp and $t \in T$ (In this case, we say $[St]\alpha_j$ is obtained from α_j by ($[St]$-) *necessitation*.)

We write $\vdash \alpha$ if there exists a proof of α. When we wish to emphasize that it is a proof in KT3, we write

$\vdash \alpha$ (in KT3). Furthermore, for any $\Gamma \subseteq$ Wff we write $\Gamma \vdash \alpha$ if $\vdash \beta_1 \supset (\beta_2 \supset (\cdots \supset (\beta_m \supset \alpha) \cdots))$ for some $\beta_1, \cdots, \beta_m \in \Gamma$.

As an exercise we show the following

Lemma 1.2. *Let* KT3* *be the logical system obtained from* KT3 *by replacing* (A6) *by the following two axiom schemata:*

$$(*) \quad [St]\alpha \supset [Su]\alpha \quad , \text{ where } t \le u$$

$$(**) \quad [St]\alpha \wedge [St](\alpha \supset \beta) \supset [St]\beta$$

Then KT3 *and* KT3* *are equivalent. I.e., for any* $\alpha \in$ Wff,

$$\vdash \alpha \text{ (in KT3) } \textit{ iff } \vdash \alpha \text{ (in KT3}^*\text{)},$$

where the notion of a proof in KT3* *is defined similarly as in Definition 1.1.*

Proof. It is sufficient to show $\vdash (*)$, $\vdash (**)$ (in KT3) and $\vdash$ (A6) (in KT3*). Now, suppose $t \le u$. Then, putting $\tau = \neg\neg\bot\supset\bot$, we have

1 $[St](\tau\supset\alpha)\supset([Su]\tau\supset[Su]\alpha)$ (instance of (A6))

2 $([St](\tau\supset\alpha)\supset([Su]\tau\supset[Su]\alpha))\supset(([St](\tau\supset\alpha)\supset[Su]\tau)$
$\supset([St](\tau\supset\alpha)\supset[Su]\alpha))$ (instance of (A3))

3 $([St](\tau\supset\alpha)\supset[Su]\tau)\supset([St](\tau\supset\alpha)\supset[Su]\alpha)$

(modus ponens (MP) from 1, 2)

4	τ	(instance of (A1))
5	$[Su]\tau$	([Su]-necessitation from 4)
6	$[Su]\tau\supset([St](\tau\supset\alpha)\supset[Su]\tau)$	(instance of (A2))
7	$[St](\tau\supset\alpha)\supset[Su]\tau$	(MP from 5, 6)
8	$[St](\tau\supset\alpha)\supset[Su]\alpha$	(MP from 7, 3)
9	$\alpha\supset(\tau\supset\alpha)$	(instance of (A2))
10	$[St](\alpha\supset(\tau\supset\alpha))$	([St]-necessitation from 9)
11	$[St](\alpha\supset(\tau\supset\alpha))\supset([St]\alpha\supset[St](\tau\supset\alpha))$	(instance of (A6))
12	$[St]\alpha\supset[St](\tau\supset\alpha)$	(MP from 10, 11)
13	$([St](\tau\supset\alpha)\supset[Su]\alpha)\supset([St]\alpha\supset([St](\tau\supset\alpha)\supset[Su]\alpha))$	(instance of (A2))
14	$[St]\alpha\supset([St](\tau\supset\alpha)\supset[Su]\alpha)$	(MP from 8, 13)
15	$[St]\alpha\supset([St](\tau\supset\alpha)\supset[Su]\alpha)\supset(([St]\alpha\supset[St](\tau\supset\alpha))\supset([St]\alpha\supset[Su]\alpha))$	(instance of (A3))
16	$([St]\alpha\supset[St](\tau\supset\alpha))\supset([St]\alpha\supset[Su]\alpha)$	(MP from 14, 15)
17	$[St]\alpha\supset[Su]\alpha$	(MP from 12, 16)

This is a proof of (*) in KT3. We now give an outline of a proof of (**) in KT3. Let $\alpha' = [St]\alpha$, $\beta' = [St]\beta$ and $\gamma = [St](\alpha\supset\beta)$. We wish to prove

$$(\gamma\supset(\alpha'\supset\beta'))\supset((\alpha'\wedge\gamma)\supset\beta'), \quad \text{i.e.,}$$

$$(\gamma\supset(\alpha'\supset\beta'))\supset(((\alpha'\supset(\gamma\supset\bot))\supset\bot)\supset\beta')$$

in KT3. We will make use of the following rules which may be easily asecrtained.

$$\vdash \beta\supset\gamma \quad \Rightarrow \quad \vdash (\alpha\supset\beta)\supset(\alpha\supset\gamma)$$

$$\vdash \alpha\supset\beta,\ \vdash \beta\supset\gamma \quad \Rightarrow \quad \vdash \alpha\supset\gamma$$

We then proceed as follows.

1 $(\gamma\supset(\alpha'\supset\beta'))\supset(\alpha'\supset(\gamma\supset\beta'))$

2 $(\alpha'\supset(\gamma\supset\beta')\supset(\alpha'\supset(\neg\beta'\supset\neg\gamma))$

3 $(\alpha'\supset(\neg\beta'\supset\neg\gamma))\supset(\neg\beta'\supset(\alpha'\supset\neg\gamma)$

4 $(\neg\beta'\supset(\alpha'\supset\neg\gamma))\supset(\neg(\alpha'\supset\neg\gamma)\supset\neg\neg\beta')$

5 $(\neg(\alpha'\supset\neg\gamma)\supset\neg\neg\beta')\supset(\neg(\alpha'\supset\neg\gamma)\supset\beta')$

6 $(\gamma\supset(\alpha'\supset\beta')\supset(\neg(\alpha'\supset\neg\gamma)\supset\beta')$

In the above proof we have omitted several easy steps of the derivation. The proof of (A6) in KT3 is left to the reader.

Now, KT4 is defined to be the system obtained from KT3 by adding the following

(A7) $[St]\alpha\supset[St][St]\alpha$

This axiom will be referred to as the positive introspective axiom.

KT5 is obtained by adjoining the following

(A8) $\neg[St]\alpha\supset[St]\neg[St]\alpha$

This axiom will be called the negative introspective axiom.

Remarks.

(1) Axioms (A1)-(A3) give an axiomatization of classical propositional calculus. (See, e.g., Lyndon [19].) Axioms (A4)-(A6) may be intuitively understood as follows.

(A4): What is known is true.

(A5): What FOOL knows at time t, FOOL knows at time t that everyone knows it at time t.

(A6): The meaning of (A6) is better explained in terms of (*) and (**) in Lemma 1.2.

(*): What is known remains to be known.

(**): Everybody can do modus ponens.

(2) If Sp contains 0, the condition (iii) of Definition 1.1 may be restricted to: Infer $[0t]\alpha$ from α.

(3) The relation of the systems KTi to the other modal system may be illustrated as below. We do not include Hintikka's knowledge system [11] in the following figure. However, we note that it is a special case of K4 with the language so restricted as not to contain 0 in Sp. For any set S, $|S|$ will denote its cardinality.

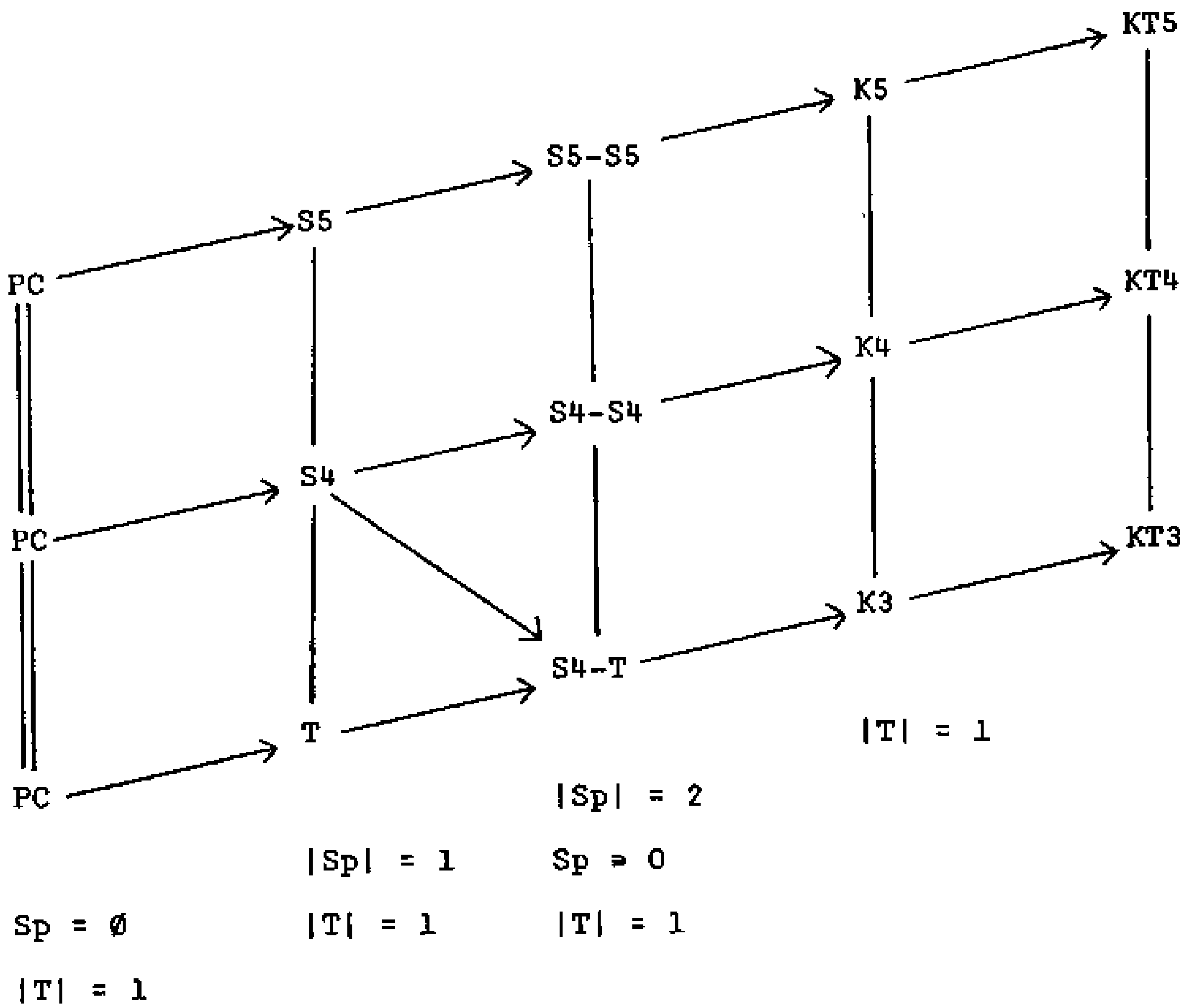

Fig. 1.1. Relation of KTi to other modal logics

In the above diagram, K3, K4 and K5 are the systems in McCarthy [21], Sato [30], and PC denotes the classical propositional calculus. The restrictions imposed on the language to obtain a desired logical system is shown below the name of the system. Furthermore, an arrow $A \longrightarrow B$ indicates that A is a subsystem of B. For example, the modal system S4 is obtained from KT4 by restricting Sp and T to be singleton sets. The systems on the same vertical line are arranged according to their deductive power. Thus, for exaple, anything provable in S4 is provable in S5.

(4) Hayashi's remark [8] is still valid. Namely, KT3+(A8) is already equivalent to KT5 (= KT3+(A7)+(A8)).

1.5. *Gentzen-type systems*

We now define Gentzen-type systems GTi (i = 3, 4, 5)[2] which are equivalent to KTi. By a *sequent* we will mean an element in the set $2^{Wff} \times 2^{Wff}$. Namely, it is a pair of (possibly infinite) sets of wffs. Note that our notion of a sequent differs from the original one due to Gentzen [4] at least in the following points. Gentzen defines a sequent as a finite figure of the form $\alpha_1, \cdots, \alpha_m \to \beta_1, \cdots, \beta_n$ while we define a sequent more abstractly and admits infinite sets of wffs.

In order to match with Gentzen's notation, we will denote a sequent by $\Gamma \to \Delta$ rather than by (Γ, Δ), where

Γ, $\Delta \subseteq$ Wff. Like this, subsets of Wff will be denoted by Greek capitals. Furthermore, we will employ the abbreviations such as:

$$\Gamma \to \Delta, \Pi = \Gamma \to \Delta \cup \Pi ,$$

$$\alpha, \Gamma, \beta \to \ = \{\alpha\} \cup \Gamma \cup \{\beta\} \to \emptyset .$$

Thus, for example, $\alpha, \beta \to \gamma, \delta, \gamma$, $\beta, \alpha \to \delta, \delta, \gamma$ and $\alpha, \alpha, \beta, \beta \to \gamma, \delta$ denote the same sequent $(\{\alpha, \beta\}, \{\gamma, \delta\})$.

We will also use the following notation:

(1) $\Gamma_0 \to \Delta_0 \subseteq \Gamma \to \Delta$ iff $\Gamma_0 \subseteq \Gamma$ and $\Delta_0 \subseteq \Delta$. (In this case, we say $\Gamma_0 \to \Delta_0$ is a *restriction* of $\Gamma \to \Delta$, or $\Gamma \to \Delta$ is an *extension* of $\Gamma_0 \to \Delta_0$.)

(2) $\Gamma_0 \Subset \Gamma$ iff $\Gamma_0 \subseteq \Gamma$ and Γ_0 is finite.

(3) $\Gamma_0 \to \Delta_0 \Subset \Gamma \to \Delta$ iff $\Gamma_0 \Subset \Gamma$ and $\Delta_0 \Subset \Delta$.

Now, we give the definition of GT3.

Axioms: $\alpha \to \alpha$

$\bot \to$

Rules:

$$\frac{\Gamma \to \Delta}{\Pi, \Gamma \to \Delta, \Sigma} \quad \text{(extension)}$$

$$\frac{\Gamma \to \Delta, \alpha \qquad \alpha, \Pi \to \Sigma}{\Gamma, \Pi \to \Delta, \Sigma} \quad \text{(cut)}$$

$$\frac{\Gamma \to \Delta, \alpha \qquad \beta, \Pi \to \Sigma}{\alpha \supset \beta, \Gamma, \Pi \to \Delta, \Sigma} \quad (\supset\to)$$

$$\frac{\alpha,\ \Gamma \to \Delta,\ \beta}{\Gamma \to \Delta,\ \alpha \supset \beta} \qquad (\to\supset)$$

$$\frac{\alpha,\ \Gamma \to \Delta}{[St]\alpha,\ \Gamma \to \Delta} \qquad ([St]\to)$$

$$\frac{\Gamma,\ [Ou]\Pi \to \alpha}{[Su]\Gamma,\ [Ou]\Pi \to [St]\alpha} \qquad (\to u,\ [St])_3, \text{where} \quad u \le t$$

In the above, the rules $([St]\to)$ and $(\to u,\ [St])_3$ are rule schemata, where S is an arbitrary element in Sp and t, u are arbitrary elements in T. One may apply the rule $(\to u,\ [St])_3$ only when $u \le t$. Also in the above for any $\Gamma \subseteq Wff$, $S \in Sp$ and $t \in T$, $[St]\Gamma$ denotes the set $\{[St]\alpha \mid \alpha \in \Gamma\}$. The notion of a *proof* in GT3 is defined similarly as in Gentzen's LK [4]. Note, however, that we allow the sequent $\bot \to$ as a beginning sequent. We write $\vdash \Gamma \to \Delta$ (in GT3) if it is provable in GT3.

The following inference rules are easily seen to be admissibe in GT3:

$$\frac{\Gamma \to \Delta}{\alpha,\ \Gamma \to \Delta} \qquad (\text{thinning}\to)$$

$$\frac{\Gamma \to \Delta}{\Gamma \to \Delta,\ \alpha} \qquad (\to\text{thinning})$$

$$\frac{\alpha,\ \alpha,\ \Gamma \to \Delta}{\alpha,\ \Gamma \to \Delta} \qquad (\text{contraction}\to)$$

$$\frac{\Gamma \to \Delta, \alpha, \alpha}{\Gamma \to \Delta, \alpha} \qquad (\to\text{contraction})$$

$$\frac{\Gamma, \alpha, \beta, \Pi \to \Delta}{\Gamma, \beta, \alpha, \Pi \to \Delta} \qquad (\text{interchange}\to)$$

$$\frac{\Gamma \to \Delta, \alpha, \beta, \Sigma}{\Gamma \to \Delta, \beta, \alpha, \Sigma} \qquad (\to\text{interchange})$$

$$\frac{\Gamma \to \Delta, \alpha}{\neg\alpha, \Gamma \to \Delta} \qquad (\neg\to)$$

$$\frac{\alpha, \Gamma \to \Delta}{\Gamma \to \Delta, \neg\alpha} \qquad (\to\neg)$$

$$\frac{\alpha, \Gamma \to \Delta \qquad \beta, \Gamma \to \Delta}{\alpha\vee\beta, \Gamma \to \Delta} \qquad (\vee\to)$$

$$\frac{\Gamma \to \Delta, \alpha}{\Gamma \to \Delta, \alpha\vee\beta} \qquad \frac{\Gamma \to \Delta, \beta}{\Gamma \to \Delta, \alpha\vee\beta} \qquad (\to\vee)$$

$$\frac{\alpha, \Gamma \to \Delta}{\alpha\wedge\beta, \Gamma \to \Delta} \qquad \frac{\beta, \Gamma \to \Delta}{\alpha\wedge\beta, \Gamma \to \Delta} \qquad (\wedge\to)$$

$$\frac{\Gamma \to \Delta, \alpha \qquad \Gamma \to \Delta, \beta}{\Gamma \to \Delta, \alpha\wedge\beta} \qquad (\to\wedge)$$

For example, the following proof figure shows that $(\vee\to)$ is admissible in GT3:

$$\cfrac{\cfrac{\cfrac{\alpha,\ \Gamma \to \Delta}{\alpha,\ \Gamma \to \Delta,\ \bot}\ (\text{extension})}{\Gamma \to \Delta,\ \alpha\supset\bot}\ (\to\supset) \qquad \beta,\ \Gamma \to \Delta}{(\alpha\supset\bot)\supset\beta,\ \Gamma \to \Delta}\ (\supset\to)$$

This means that, in spite of the difference in the definition of a sequent, every proof figure in (propositional) LK may itself be considered as one in GT3.

Now, GT4 is obtained from GT3 by replacing the rule $(\to u,\ [St])_3$ by the following:

$$\frac{[Su]\Gamma,\ [Ou]\Pi \to \alpha}{[Su]\Gamma,\ [Ou]\Pi \to [St]\alpha}\quad (\to u,\ [St])_4,\ \text{where}\quad u \le t$$

GT5 is obtained from GT4 by changing the rule $(\to u,\ [St])_4$ to:

$$\frac{[Su]\Gamma,\ [Ou]\Pi \to [Ou]\Sigma,\ [Su]\Delta,\ \alpha}{[Su]\Gamma,\ [Ou]\Pi \to [Ou]\Sigma,\ [Su]\Delta,\ [St]\alpha}\quad (\to u,\ [St])_5,$$

where, $u \le t$

1.6. *Some metatheorems*

Let us call a sequent $\Gamma \to \Delta$ *finite* if both Γ and Δ are finite. Then the following lemma is easily obtained.

Lemma 1.3. *If a finite sequent* $\Gamma \to \Delta$ *is provable (in* GTi*) then each sequent occurring in any proof of* $\Gamma \to \Delta$ *is finite.*

Theorem 1.4. *If* $\vdash \Gamma \to \Delta$ (in KTi) *then there exist some* $\Gamma_0 \Subset \Gamma$ *and* $\Delta_0 \Subset \Delta$ *such that* $\vdash \Gamma_0 \to \Delta_0$ (in KTi).

Proof. By induction on the number n of sequents occurring in the proof of $\Gamma \to \Delta$.

($n = 1$): Since $\Gamma \to \Delta$ is a beginning sequent, $\Gamma \to \Delta$ itself is finite.

($n > 1$): We consider the case that the last (i.e., downmost) inference is $(\supset\to)$. The proof then is of the form:

$$\dfrac{\begin{matrix}\vdots \\ \Pi \to \Sigma, \alpha\end{matrix} \qquad \begin{matrix}\vdots \\ \beta, \Phi \to \Psi\end{matrix}}{\alpha \supset \beta, \Pi, \Phi \to \Sigma, \Psi} \ (\supset\to)$$

By induction hypothesis, we have finite Π_0, Σ_0, Φ_0, Ψ_0 such that

$$\dfrac{\begin{matrix}\vdots \\ \Pi_0 \to \Sigma_0\end{matrix}}{\Pi \to \Sigma, \alpha} \ \text{(extension)} \quad \text{and}$$

$$\dfrac{\begin{matrix}\vdots \\ \Phi_0 \to \Psi_0\end{matrix}}{\beta, \Phi \to \Psi} \ \text{(extension)}$$

Then we construct the following proof figure.

$$
\dfrac{\dfrac{\dfrac{\vdots}{\Pi_0 \to \Sigma_0}}{\Pi_0 \to \Sigma_0 - \alpha,\ \alpha} \qquad \dfrac{\dfrac{\vdots}{\Phi_0 \to \Psi_0}}{\beta,\ \Phi_0 - \beta \to \Psi_0}}{\dfrac{\alpha \supset \beta,\ \Pi_0,\ \Phi_0 - \beta \to \Sigma_0 - \alpha,\ \Psi_0}{\alpha \supset \beta,\ \Pi,\ \Phi \to \Sigma,\ \Psi}}
$$

We see that $\alpha \supset \beta,\ \Pi_0,\ \Phi_0 - \beta \to \Sigma_0 - \alpha,\ \Psi_0$ serves as the desired sequent. Other cases may be dealt with similarly.

Theorem 1.5. *For any* $\alpha \in \mathrm{Wff}$, $\vdash \alpha$ (in KTi) *if and only if* $\vdash \to \alpha$ (in GTi).

Proof. We only prove the case $i = 5$.
Proof of only if part: We prove by induction on the construction of a proof of α in KT5. Namely, we assume that we are given a proof of α. Then each formula occurring in the proof is either an axiom or the result of an application of an inference rule to previously obtained formula(s). We first show that every axiom of KT5 is provable in GT5.

(A1):

$$\dfrac{\dfrac{\dfrac{\alpha \to \alpha}{\to \alpha,\ \neg\alpha}}{\neg\neg\alpha \to \alpha}}{\to\ \neg\neg\alpha \supset \alpha}$$

(A2):

$$\dfrac{\dfrac{\dfrac{\alpha \to \alpha}{\beta,\ \alpha \to \alpha}}{\alpha \to \beta \supset \alpha}}{\to \alpha \supset (\beta \supset \alpha)}$$

(A3):

$$\dfrac{\dfrac{\dfrac{\alpha \to \alpha \qquad \dfrac{\alpha \to \alpha \qquad \dfrac{\beta \to \beta \qquad \gamma \to \gamma}{\beta,\ \beta \supset \gamma \to \gamma}}{\alpha,\ \alpha \supset \beta,\ \beta \supset \gamma \to \gamma}}{\alpha,\ \alpha \supset \beta,\ \alpha \supset (\beta \supset \gamma) \to \gamma}}{\dfrac{\alpha \supset \beta,\ \alpha \supset (\beta \supset \gamma) \to \alpha \supset \gamma}{\alpha \supset (\beta \supset \gamma) \to (\alpha \supset \beta) \supset (\alpha \supset \gamma)}}}{\to (\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))}$$

(A4):

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \to \alpha}{\alpha \to \beta,\ \alpha}}{\to \alpha \supset \beta,\ \alpha} \qquad \alpha \to \alpha}{(\alpha \supset \beta) \supset \alpha \to \alpha}}{\to ((\alpha \supset \beta) \supset \alpha) \supset \alpha}$$

(A5):

$$\dfrac{\dfrac{\alpha \to \alpha}{[St]\alpha \to \alpha}}{\to [St]\alpha \supset \alpha}$$

(A6):

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \to \alpha}{[Ot]\alpha \to \alpha}}{[Ot]\alpha \to [St]\alpha}}{[Ot]\alpha \to [Ot][St]\alpha}}{\to [Ot]\alpha \supset [Ot][St]\alpha}$$

(A7):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha\supset\beta \to \alpha\supset\beta}{[St](\alpha\supset\beta) \to \alpha\supset\beta}}{[St](\alpha\supset\beta) \to [Su](\alpha\supset\beta)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\alpha \to \alpha \qquad \beta \to \beta}{\alpha,\ \alpha\supset\beta \to \beta}}{[Su]\alpha,\ \alpha\supset\beta \to \beta}}{[Su]\alpha,\ [Su](\alpha\supset\beta) \to \beta}}{[Su]\alpha,\ [Su](\alpha\supset\beta) \to [Su]\beta}}{[St](\alpha\supset\beta), [Su]\alpha \to [Su]\beta}}{[St](\alpha\supset\beta) \to [Su]\alpha\supset[Su]\beta}}{\to [St](\alpha\supset\beta)\supset([Su]\alpha\supset[Su]\beta)}$$

We now consider the inference rules. We can express modus ponens ($\vdash\alpha$, $\vdash \alpha\supset\beta$ => $\vdash\beta$) and necessitation ($\vdash\alpha$ => $\vdash[St]\alpha$) in GT5 as follows:

(modus ponens):

$$\dfrac{\begin{array}{c}\vdots\\ \to \alpha\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \to \alpha\supset\beta\end{array} \qquad \dfrac{\alpha\to\alpha \qquad \beta\to\beta}{\alpha\supset\beta,\ \alpha\to\beta}}{\alpha\to\beta}}{\to\beta}$$

(necessitation):

$$\dfrac{\begin{array}{c}\vdots\\ \to \alpha\end{array}}{\to [St]\alpha}$$

Proof of if part: We prove that if a finite sequent $\Gamma \to \Delta$ is provable in GT5 then $\top\wedge\alpha_1\wedge\cdots\wedge\alpha_m\supset\beta_1\vee\cdots\vee\beta_n\vee\bot$ is provable in KT5, where $\alpha_1, \cdots, \alpha_m$ $(\beta_1, \cdots, \beta_n)$ is any enumeration of $\Gamma(\Delta$, resp.) with possible repetitions. First note that $(\top\wedge\alpha_1\wedge\cdots\wedge\alpha_m\supset\beta_1\vee\cdots\vee\beta_n\vee\bot)\supset(\top\wedge\alpha_1'\wedge\cdots\wedge\alpha_p'\supset\beta_1'\vee\cdots\vee\beta_q'\vee\bot)$ is provable in KT5 if $\{\alpha_1, \cdots, \alpha_m\} = \{\alpha_1', \cdots, \alpha_p'\}$ and $\{\beta_1, \cdots, \beta_n\} = \{\beta_1', \cdots, \beta_q'\}$. The proof is carried out by induction on the construction of the proof. We only deal with the rules ($[St]\to$) and $(\to u, [St])_5$. Suppose $[St]\alpha, \alpha_1, \cdots, \alpha_m \to \beta_1, \cdots, \beta_n$ is obtained from $\alpha, \alpha_1, \cdots, \alpha_m \to \beta_1, \cdots, \beta_n$ by an application of ($[St]\to$). Then by induction hypothesis, $\vdash(\top\wedge\alpha\wedge\alpha_1\wedge\cdots\wedge\alpha_m)\supset(\beta_1\vee\cdots\vee\beta_n\vee\bot)$ (in KT5). Since $\vdash[St]\alpha\supset\alpha$, we have $\vdash(\top\wedge[St]\alpha\wedge\alpha_1\wedge\cdots\wedge\alpha_m)\supset(\top\wedge\alpha\wedge\alpha_1\wedge\cdots\wedge\alpha_m)$. Hence, $\vdash(\top\wedge[St]\alpha\wedge\alpha_1\wedge\cdots\wedge\alpha_m)\supset(\beta_1\vee\cdots\vee\beta_n\vee\bot)$. Next, suppose $[St]\alpha_1, \cdots, [St]\alpha_m, [Ot]\gamma_1, \cdots, [Ot]\gamma_p \to [Ot]\delta_1, \cdots, [Ot]\delta_q, [St]\beta_1, \cdots, [St]\beta_n, [Su]\alpha$ is obtained from $[St]\alpha_1, \cdots, [St]\alpha_m, [Ot]\gamma_1, \cdots, [Ot]\gamma_p \rightarrow [Ot]\delta_1, \cdots, [Ot]\delta_q, [St]\beta_1, \cdots, [St]\beta_n, \alpha$ by an application of $(\rightarrow u, [St])_5$. By induction hypothesis,

$$\vdash (\top \wedge [St]\alpha_1 \wedge \cdots \wedge [St]\alpha_m \wedge [Ot]\gamma_1 \wedge \cdots \wedge [Ot]\gamma_p) \supset ([Ot]\delta_1 \vee \cdots \vee [Ot]\delta_q \vee [St]\beta_1 \vee \cdots \vee [St]\beta_n \vee \bot) \qquad (1)$$

Noting that

$$\vdash [St](\alpha \supset \beta) \supset ([St]\alpha \supset [St]\beta)$$

and

$$\vdash [St]\sigma_1 \wedge \cdots \wedge [St]\sigma_k \supset [St](\sigma_1 \wedge \cdots \wedge \sigma_k)$$

we have from (1), by necessitation and above,

$$\vdash \top \wedge [St][St]\alpha_1 \wedge \cdots \wedge [St][St]\alpha_m \wedge [St][Ot]\gamma_1 \wedge \cdots \wedge [St][Ot]\gamma_p \wedge [St]\neg [Ot]\delta_1 \wedge \cdots \wedge [St]\neg [Ot]\delta_q \wedge [St]\neg [St]\beta_1 \wedge \cdots \wedge [St]\neg [St]\beta_n \supset [St]\alpha.$$

Since

$$\vdash [St]\alpha_i \supset [St][St]\alpha_i,$$

$$\vdash [Ot]\gamma_i \supset [St][Ot]\gamma_i,$$

$$\vdash \neg [Ot]\delta_i \supset [St]\neg [Ot]\delta_i$$

and

$$\vdash \neg [St]\beta_i \supset [St]\neg [St]\beta_i$$

we have

$$\vdash \top\wedge[St]\alpha_1\wedge \cdots \wedge[St]\alpha_m\wedge[Ot]\gamma_1\wedge \cdots \wedge[Ot]\gamma_p\supset$$
$$[Ot]\delta_1\vee \cdots \vee[Ot]\delta_q\vee[St]\beta_1\vee \cdots \vee[St]\beta_m\vee[St]\alpha\vee\bot,$$

which was to be proved.

Corollary 1.6. *Let* $\Gamma \subseteq \mathrm{Wff}$ *and* $\alpha \in \mathrm{Wff}$. *Then* $\Gamma \vdash \alpha$ (in KTi) *if and only if* $\vdash \Gamma \to \alpha$ (in GTi).

Proof. Only if part: By definition, $\Gamma \vdash \alpha$ implies the existence of some $\beta_1, \cdots, \beta_n \in \Gamma$ such that $\vdash \beta_1\supset(\beta_2\supset \cdots (\beta_n\supset\alpha) \cdots)$. Hence $\vdash \beta_1, \cdots, \beta_n \to \alpha$. By (extension) we have $\vdash \Gamma \to \alpha$.
If part: By Lemma 1.4, there exist some $\beta_1, \cdots, \beta_n$ such that $\vdash \beta_1, \cdots, \beta_n \to \alpha$. Hence $\vdash \to \beta_1\supset(\beta_2\supset \cdots (\beta_n\supset\alpha) \cdots)$. By Theorem 1.5, $\vdash \beta_1\supset(\beta_2 \cdots \supset(\beta_n\supset\alpha) \cdots)$. This means $\Gamma \vdash \alpha$.

For any $\Gamma \subseteq \mathrm{Wff}$, we let $\neg\Gamma = \{\neg\alpha \mid \alpha \in \Gamma\}$. The following lemma is easy to ascertain.

Lemma 1.7.

$$\vdash \Gamma \to \Delta \quad \text{(in GTi)}$$
$$\textit{iff} \;\; \vdash \to \Delta, \neg\Gamma \text{ (in GTi)}$$
$$\textit{iff} \;\; \vdash \neg\Delta, \Gamma\to \text{ (in GTi)}.$$

CHAPTER 2

TOPOLOGY ON 2^{Wff}

Scott [33] has introduced $P\omega$ as a model for type-free lamda calculus. It is also designed as a universal domain of computation. In this chapter we introduce a topology on 2^{Wff} which is homeomorphic to $P\omega$ topology. We then show that several syntactical properties of our logical systems may be conveniently expressed in terms of topological languages. The result in this chapter tells us the naturalness of considering infinite sequents. This chapter is independent of the remaining chapters.

2.1. *Definition of topology*

We now define a topology on 2^{Wff}. For any *finite* $\Gamma \subseteq \mathrm{Wff}$, we put $U_\Gamma = \{ \Delta \in 2^{\mathrm{Wff}} \mid \Gamma \subseteq \Delta\}$. $\{U_\Gamma \mid \Gamma\text{: finite}\}$ forms a basis of open sets. I.e., $X \subseteq 2^{\mathrm{Wff}}$ is, by definition, *open* if and only if it may be written as a union of some U_Γ's. Since Wff is a denumerable set it is clear that under this topology 2^{Wff} is homeomorphic to Scott's $P\omega$. Following Scott, we write $\top$ for Wff and $\perp$ for the empty set $\emptyset$, since these are top and bottom elements of the Boolean lattice 2^{Wff} (under the inclusionship ($\subseteq$) ordering). We define several functions on 2^{Wff} as follows.

$$\mathrm{not} : 2^{\mathrm{Wff}} \longrightarrow 2^{\mathrm{Wff}}$$

is defined by:

$\mathrm{not}(\Gamma) = \neg\Gamma$.

(2) $\mathrm{isinconsistent}_i$[3]$: 2^{Wff} \longrightarrow 2^{Wff}$

is defined by:

$$\mathrm{isinconsistent}_i(\Gamma) = \begin{cases} \top & (\text{if } \Gamma \vdash \bot \text{ (in KTi)}) \\ \bot & (\text{otherwise}), \end{cases}$$

where $i = 3, 4, 5$.

(3) $\mathrm{istheorem}_i : 2^{Wff} \longrightarrow 2^{Wff}$

is defined by:

$$\mathrm{istheorem}_i(\Gamma) = \begin{cases} \top & (\text{if } \vdash \alpha_1 \vee \cdots \vee \alpha_n \text{ (in KTi) for} \\ & \quad \text{some } \{\alpha_1, \cdots, \alpha_n\} \subseteq \Gamma) \\ \bot & (\text{otherwise}) \end{cases}$$

(4) $\mathrm{DC}_i : 2^{Wff} \longrightarrow 2^{Wff}$ (deductive closure)

is defined by:

$\mathrm{DC}_i(\Gamma) = \bar{\Gamma} = \{\alpha \mid \Gamma \vdash \alpha \text{ (in KTi)}\}$.

(5) $\mathrm{isprovable}_i : 2^{Wff} \times 2^{Wff} \longrightarrow 2^{Wff}$

is defined by:

$$\mathrm{isprovable}_i(\Gamma \to \Delta) = \begin{cases} \top & (\text{if GTi} \vdash \Gamma \to \Delta) \\ \bot & (\text{otherwise}) \end{cases}$$

(6) $\mathrm{left} : 2^{Wff} \times 2^{Wff} \longrightarrow 2^{Wff}$

is defined by:

$\mathrm{left}(\Gamma \to \Delta) = \neg\Delta \cup \Gamma$.

(7) $\mathrm{right} : 2^{Wff} \times 2^{Wff} \longrightarrow 2^{Wff}$

is defined by:

$$\mathrm{right}(\Gamma \to \Delta) = \Delta \cup \neg\Gamma.$$

2.2. *Topological characterization of syntactical properties*

2^{Wff}, with the above topology, is a *continuous lattice* in the sense of Scott [32], and so is $2^{\mathrm{Wff}} \times 2^{\mathrm{Wff}}$ with product topology. Then the functions defined in 2.1 are all continuous functions. More precisely, we have the following:

Theorem 2.1. *The following diagrams are commutative in the category of continuous lattices with continuous maps.*

$$\begin{tikzcd}
 & 2^{\mathrm{Wff}} \arrow[dr, "\mathrm{isincons}_i"] & \\
2^{\mathrm{Wff}} \times 2^{\mathrm{Wff}} \arrow[ur, "\mathrm{left}"] \arrow[rr, "\mathrm{isprovable}_i"] \arrow[dr, "\mathrm{right}"'] & & 2^{\mathrm{Wff}} \\
 & 2^{\mathrm{Wff}} \arrow[ur, "\mathrm{istheorem}_i"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
2^{\mathrm{Wff}} \arrow[dd, shift left, "\mathrm{not}"] \arrow[dr, "\mathrm{isincons}_i"] & \\
 & 2^{\mathrm{Wff}} \\
2^{\mathrm{Wff}} \arrow[uu, shift left, "\mathrm{not}"] \arrow[ur, "\mathrm{istheorem}_i"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
2^{Wff} \arrow[dr, "isincons_i"] & \\
 & 2^{Wff} \\
2^{Wff} \arrow[uu, "DC_i"] \arrow[ur, "isincons_i"'] &
\end{tikzcd}$$

Proof. Commutativity follows from results in 1.6. Continuity is also immediate. For example,

$$\begin{aligned}
\text{isprovable}_i(\Gamma \to \Delta) &= \bigsqcup\{\text{isprovable}_i(\Gamma_0 \to \Delta) \mid \Gamma_0 \Subset \Gamma\} \\
&= \bigsqcup\{\text{isprovable}_i(\Gamma \to \Delta_0) \mid \Delta_0 \Subset \Delta\}
\end{aligned}$$

by Lemma 1.4. Then by definition in Scott [33], we see isprovable_i is continuous.

The following result is also straightforward. For the definition of retracts and the least fixed point operator Y, we refer to Scott [33].

Theorem 2.2.

(1) istheorem_i, isinconsistent_i, *and* DC_i *are retracts.*

(2) $Y(DC_i)$ *is equal to the set of theorems in* KTi.

Remark. Theorem 1.4 is equivalent to the continuity of isprovable_i.

CHAPTER 3

KRIPKE-TYPE SEMANTICS

3.1. *Definition of Kripke-type models*

Let W be any nonvoid set (of *possible worlds*). A *model* M on W is a triple

$$<W; r, v>,$$

where

$$r : Sp\times T \longrightarrow 2^{W\times W}$$

and

$$v : Pr\cup\{\bot\} \longrightarrow 2^{W}.$$

Given any model M, we define a relation $\models \subseteq W\times Wff$ as follows:

(E1) If $\alpha \in Pr\cup\{\bot\}$ then $w \models \alpha$ iff $w \in v(\alpha)$

(E2) If $\alpha = \beta\supset\gamma$ then $w \models \alpha$ iff not $w \models \beta$ or $w \models \gamma$

(E3) If $\alpha = [St]\beta$ then $w \models \alpha$ iff for all $w' \in W$ such that $(w, w') \in r(S, t)$, $w' \models \alpha$

We will write "$w \models \alpha$ (in M)" if we wish to make M explicit. An informal meaning of (E3) is that $[St]\alpha$ is true in w if and only if α is true in any world accessible to S at time t from w. A formula α is said to be *valid* in M, denoted by $M \models \alpha$, if $w \models \alpha$ for all $w \in M$. (By $w \in M$, we of course mean $w \in W$.) We will write $w \xrightarrow{St} w'$ instead of $(w, w') \in r(S, t)$ when r is understood. Furthermore, we will employ the following notations:

$w \models \Gamma$ (read "w *realizes* Γ") iff $w \models \alpha$ for all $\alpha \in \Gamma$

$w \mathrel{=\!\!|} \alpha$ iff not $w \models \alpha$

$w \mathrel{=\!\!|} \Gamma$ iff $w \mathrel{=\!\!|} \alpha$ for all $\alpha \in \Gamma$

$w \mathrel{=\!\!|} \Gamma \to \Delta$ (read "w *realizes* $\Gamma \to \Delta$ ") iff $w \models \Gamma$ and $w \mathrel{=\!\!|} \Delta$

$w \models \Gamma \to \Delta$ iff not $w \mathrel{=\!\!|} \Gamma \to \Delta$

$M \models \Gamma \to \Delta$ iff $w \models \Gamma \to \Delta$ for all $w \in M$

A model M is a KT3-*model* if

(M1) $r(\imath) = \emptyset$

(M2) $r(O, t) \supseteq r(S, t)$ for any $S \in Sp$ and $t \in T$

(M3) $r(S, u) \supseteq r(S, t)$ for any $S \in Sp$ and $u, t \in T$ such that $u \leq t$

(M4) $r(S, t)$ is a reflexive relation for any $S \in Sp$ and $t \in T$

(M5) $r(O, t)$ is a transitive relation for any $t \in T$

A model M is a KT4-*model* if it satisfies (M1)-(M3) and

(M6) $r(S, t)$ is a reflexive and transitive relation for any $S \in Sp$ and $t \in T$

A model M is a KT5-*model* if it satisfies (M1)-(M3) and

(M7) $r(S, t)$ is an equivalence relation for any $S \in Sp$

and $t \in T$

3.2. *Soundness of* KTi-*models*

We now wish to show that each formula provable in KTi is valid in any Ki-model. First we prepare some terminology. We say $\Gamma \to \Delta$ is i-*provable* (i-*consistent*, resp.) if it is provable (unprovable, resp.) in GTi. We say $\Gamma \to \Delta$ is i-*realizable* if there exists some Ki-model M and $w \in M$ such that $w \models \Gamma \to \Delta$. $\Gamma \to \Delta$ is said to be i-*valid* if it is not i-realizable.

Theorem 3.1. *(Soundness Theorem) Any* i-*provable sequent is* i-*valid.*

Proof. The proof is by induction on the construction of a proof of the given sequent. That any beginning sequent is i-valid is immediate from the definition. As for the inference rules, we only treat $(\to u, [St])_5$ of GT5, since other cases are either similar or easier. So, consider:

$$\frac{[Su]\Gamma,\ [Ou]\Pi \to [Ou]\Sigma,\ [Su]\Delta,\ \alpha}{[Su]\Gamma,\ [Ou]\Pi \to [Ou]\Sigma,\ [Su]\Delta,\ [St]\alpha}\ ,$$

where $u \leq t$.

By induction hypothesis, the upper sequent is 5-valid. Suppose, for the sake of contradiction, that the lower sequent is not 5-valid. Then there exist some KT5-model M

and $w \in M$ such that

$$w \models\!\!\!| \; [Su]\Gamma, [Ou]\Pi \rightarrow [Ou]\Sigma, [Su]\Delta, [St]\alpha.$$

This implies $w =\!| \; [St]\alpha$. Hence, for some w' such that $w \xrightarrow{St} w'$,

(1) $\quad w' =\!| \; \alpha$

holds. Since $u \leq t$, we have

(2) $\quad w \xrightarrow{Su} w'$

by (M3). Then, we have

(3) $\quad w \xrightarrow{Ou} w'$

by (M2). Let $\beta \in \Gamma$ and take any w'' such that $w' \xrightarrow{Su} w''$. Since $r(S, u)$ is transitive by (M7), we have $w \xrightarrow{Su} w''$. Since $w \models [Su]\beta$, we have $w'' \models \beta$. This means $w' \models [Su]\beta$ by (E3). Hence

(4) $\quad w' \models [Su]\Gamma$.

Next, take any β in Δ. Then, since $w =\!| \; [Su]\beta$ there exists some w''' such that

(5) $\quad w \xrightarrow{Su} w'''$.

Since $r(S, u)$ is an equivalence relation we have $w' \xrightarrow{Su} w'''$ from (2) and (5). Hence, $w' =\!| \; [Su]\beta$ by (E3), so that

(6) $\quad w' \mathrel{=\!|} [Su]\Delta.$

From (3) we obtain, similarly as above,

(7) $\quad w' \models [Ou]\Pi,$

(8) $\quad w' \mathrel{=\!|} [Ou]\Sigma.$

(1), (4), (6), (7) and (8) means

$$w' \mathrel{=\!|} [Su]\Gamma,\ [Ou]\Pi \to [Ou]\Sigma,\ [Su]\Delta, \alpha.$$

This is a contradiction.

Corollary 3.2. *If* $\vdash \alpha$ (in KTi) *then* $M \models \alpha$ *for any* Ki *-model* M.

Corollary 3.3. *(Consistency of* KTi *and* GTi*) The empty sequent* $\to$ *is not provable in* GTi.

3.3. *Completeness of* Ki*-models*

We begin by a syntactical result, which is a kind of Lindenbaum's Lemma.

Lemma 3.4. *Let be that* $\nvdash \Gamma \to \Delta$ (in GTi) *and* $\Phi \supseteq \Gamma \cup \Delta$. *Then there exist* $\tilde{\Gamma}, \tilde{\Delta}$ *such that*

(i) $\nvdash \tilde{\Gamma} \to \tilde{\Delta}$ (in GTi)

(ii) $\tilde{\Gamma} \to \tilde{\Delta} \supseteq \Gamma \to \Delta$

(iii) $\tilde{\Gamma} \cup \tilde{\Delta} = \Phi$

Proof. Let $\alpha : \mathbb{N}^+ \longrightarrow \Phi$ be a surjection. We write α_i for $\alpha(i)$. We define $\Gamma_n \to \Delta_n$ $(n \geq 0)$ as follows:

$$\Gamma_0 \to \Delta_0 = \Gamma \to \Delta$$

$$\Gamma_{n+1} \to \Delta_{n+1} = \begin{cases} \Gamma_n \to \Delta_n, \alpha_{n+1} & (\text{if } \not\vdash \Gamma_n \to \Delta_n, \alpha_{n+1}) \\ \alpha_{n+1}, \Gamma_n \to \Delta_n & (\text{otherwise}) \end{cases}$$

We show by induction that $\not\vdash \Gamma_n \to \Delta_n$ $(n \geq 0)$. The case $n = 0$ is verified by the assumption of the lemma. Consider the case $n = m+1$, and suppose $\vdash \Gamma_{m+1} \to \Delta_{m+1}$. Then, by the definition of $\Gamma_{m+1} \to \Delta_{m+1}$, we have $\vdash \Gamma_m \to \Delta_m, \alpha_{m+1}$ and $\vdash \alpha_{m+1}, \Gamma_m \to \Delta_m$. From these we obtain $\vdash \Gamma_m \to \Delta_m$ by (cut), which contradicts the induction hypothesis.

Now we put $\bar{\Gamma} \to \bar{\Delta} = \bigcup_{n=0}^{\infty} \Gamma_n \to \bigcup_{n=0}^{\infty} \Delta_n$. Then we have $\bar{\Gamma} \to \bar{\Delta} \supseteq \Gamma \to \Delta$ and $\bar{\Gamma} \cup \bar{\Delta} = \Phi$. What remains to be shown is that $\bar{\Gamma} \to \bar{\Delta}$ is i-consistent. Suppose the contrary. Then by Lemma 1.4, we have $\Gamma' \to \Delta' \Subset \bar{\Gamma} \to \bar{\Delta}$ such that $\vdash \Gamma' \to \Delta'$. Now, let $N = \max\{ n(\beta) \mid \beta \in \Gamma' \cup \Delta' \}$, where $n(\beta) = \min\{ i \mid \beta = \alpha_i \}$. Then we have $\Gamma' \cup \Delta' \subseteq \Gamma_N \cup \Delta_N$. We prove $\Gamma' \subseteq \Gamma_N$. Suppose $\alpha_i \in \Gamma'$ and $\alpha_i \notin \Gamma_N$. Then we have $\alpha_i \in \bar{\Gamma}$ and $\alpha_i \in \Delta_N \subseteq \bar{\Delta}$. But $\bar{\Gamma} \cap \bar{\Delta} = \emptyset$. This proves $\Gamma' \subseteq \Gamma_N$. Similarly, $\Delta' \subseteq \Delta_N$. Since $\vdash \Gamma' \to \Delta'$. we have $\vdash \Gamma_N \to \Delta_N$, which is a contradiction.

A set Ω of wffs is said to be *closed under subformulas* if $\perp \in \Omega$ and $\mathrm{Sub}(\alpha) \subseteq \Omega$ for all $\alpha \in \Omega$. Now take any such Ω and fix it. We say a sequent $\Gamma \to \Delta$ is Ω, *i-complete* if $\Gamma \to \Delta$ is i-consistent and $\Gamma \cup \Delta = \Omega$. We denote by $C_i(\Omega)$ the set of all Ω, i-complete sequents. I.e.,

$$C_i(\Omega) = \{\Gamma \to \Delta \mid \Gamma \cup \Delta = \Omega,\ \Gamma \to \Delta \text{ is i-consistent}\}.$$

We observe that $\Gamma \cap \Delta = \emptyset$ since $\Gamma \to \Delta$ is i-consistent. For any $\Gamma \subseteq \mathrm{Wff}$, $S \in \mathrm{Sp}$ and $t \in T$, we put $\Gamma_{St} = \{\alpha \mid [St]\alpha \in \Gamma\}$. We now define the *universal model* $U(\Omega) = \langle U;\ R,\ V\rangle$ *over* Ω as follows. (Since our definition will depend on the logical system KTi, we will call $U(\Omega)$ the Ω, *i-universal model* when necessary, and will denote it as $U_i(\Omega)$.)

(1) $U = C_i(\Omega)$

(2) $V(\alpha) = \{\Gamma \to \Delta \in U \mid \alpha \in \Gamma\}$, where $\alpha \in \mathrm{Pr} \cup \{\perp\}$

(3) Let $w = \Gamma \to \Delta \in U$, $w' = \Gamma' \to \Delta' \in U$.

(i = 3): $(w, w') \in R(S, t)$ iff $\Gamma_{Su} \subseteq \Gamma'$ and $\Gamma_{0u} \subseteq \Gamma'_{0u}$ for any $u \leq t$.

(i = 4): $(w, w') \in R(S, t)$ iff $\Gamma_{Su} \subseteq \Gamma'_{Su}$ and $\Gamma_{0u} \subseteq \Gamma'_{0u}$ for any $u \leq t$.

(i = 5): $(w, w') \in R(S, t)$ iff $\Gamma_{Su} = \Gamma'_{Su}$ and $\Gamma_{0u} = \Gamma'_{0u}$ for any $u \leq t$.

Lemma 3.5. *$U_i(\Omega)$ is a KTi-model.*

Proof. First, since $\bot \in \Omega$ and $\nvdash \ \rightarrow \bot$ (Corollary 3.3), Lemma 3.4 assures us that $U = C_i(\Omega) \neq \emptyset$.

$(i = 3)$:

(M1) Suppose $w = \Gamma \rightarrow \Delta \in V(\bot)$. Then $\bot \in \Gamma$. Since $\vdash \bot \rightarrow$, we have $\vdash \Gamma \rightarrow \Delta$, which is a contradiction. Hence $V(\bot) = \emptyset$.

(M2), (M3) are immediate from the definition of R.

(M4) Let $w = \Gamma \rightarrow \Delta \in U$. Suppose $u \leq t$ and take any $\alpha \in \Gamma_{Su}$. Since $[Su]\alpha \in \Gamma$ and Ω is closed under sub-formulas, we have $\alpha \in \Gamma \cup \Delta$. Suppose $\alpha \in \Delta$. Then, since $\vdash [Su]\alpha \rightarrow \alpha$, we have $\vdash \Gamma \rightarrow \Delta$, which is a contradiction. Hence $\alpha \in \Gamma$. This proves $\Gamma_{Su} \subseteq \Gamma$. Since $\Gamma_{Ou} \subseteq \Gamma_{Ou}$, we see $R(S, t)$ is reflexive.

(M5) Let $(\Gamma \rightarrow \Delta, \Gamma' \rightarrow \Delta'), (\Gamma' \rightarrow \Delta', \Gamma'' \rightarrow \Delta'') \in R(O, t)$. Suppose $u \leq t$. Then since $\Gamma_{Ou} \subseteq \Gamma'_{Ou} \subseteq \Gamma''_{Ou}$, we have $\Gamma_{Ou} \subseteq \Gamma''_{Ou}$. We can prove $\Gamma''_{Ou} \subseteq \Gamma''$ as in the proof of (M4), whence $\Gamma_{Ou} \subseteq \Gamma''$. Thus we see $R(O, t)$ is transitive.

The cases $(i = 4)$ and $(i = 5)$ are now easily seen.

The following theorem will play a key role in the subsequent studies.

Theorem 3.6. *(Fundamental Theorem of Universal Model)* *For any* $\alpha \in \Omega$ *and* $w = \Gamma \rightarrow \Delta \in U(\Omega)$, $w \models \alpha$ (in $U(\Omega)$) *if* $\alpha \in \Gamma$ *and* $w \mathrel{=\!\!|} \alpha$ (in $U(\Omega)$) *if* $\alpha \in \Delta$.

Proof. By induction on the construction of formulas.

(1) $\alpha \in \mathrm{Pru}\{\iota\}$: Immediate from the definition of R.

(2) $\alpha = \beta \supset \gamma$: Suppose $\alpha \in \Gamma$. We must show that $w \mathrel{=\!|} \beta$ or $w \models \gamma$. Suppose, by way of contradition, that $w \models \beta$ and $w \mathrel{=\!|} \gamma$. Then, by induction hypothesis, we have $\beta \in \Gamma$ and $\gamma \in \Delta$. Since $\vdash \beta, \beta \supset \gamma \to \gamma$ (in GTi), we have $\vdash \Gamma \to \Delta$ (in GTi), a contradiction. Suppose now $w \mathrel{=\!|} \alpha$. We can prove $w \models \beta$ and $w \mathrel{=\!|} \gamma$, similarly.

(3) $\alpha = [St]\beta$: Suppse $\alpha \in \Gamma$ and take any $w' = \Gamma' \to \Delta'$ such that $w \xrightarrow{St} w'$. We show $\beta \in \Gamma'$. First, we consider the case $i = 3$. Since $\beta \in \Gamma_{St} \subseteq \Gamma'$ we have $\beta \in \Gamma'$. Next, we treat the case $i = 4, 5$. We have $\Gamma_{St} \subseteq \Gamma'_{St} \subseteq \Gamma'$ (see the proof of (M4) in Lemma 3.5). Hence $\beta \in \Gamma'$. Thus we see $w \models [St]\beta = \alpha$.

Now suppose $\alpha \in \Delta$.

(i = 3): The sequent $\{[Su]\gamma \in \Gamma \mid u \leq t\}, \{[Ou]\gamma \in \Gamma \mid u \leq t\} \to [St]\beta$ is 3-consistent, since it is a restriction of $\Gamma \to \Delta$. By $(\to u, [St])_3$, we see $\{\gamma \mid [Su]\gamma \in \Gamma, u \leq t\}, \{[Ou]\gamma \in \Gamma \mid u \leq t\} \to \beta$ is also 3-consistent. Since Ω is closed under subformulas, we can extend this sequent to an Ω, 3-complete sequent $w' = \Gamma' \to \Delta'$, by Lemma 3.4. Then for any $u \leq t$, we have $\Gamma_{Su} \subseteq \Gamma'$ and $\Gamma_{Ou} \subseteq \Gamma'_{Ou}$. Therefore, we have $w' \xrightarrow{St} w'$. Since $\beta \in \Delta'$, by induction hypothesis, we have $w' \mathrel{=\!|} \beta$. Hence $w \mathrel{=\!|} [Su]\beta = \alpha$.

(i = 4): Similar to the case (i = 3).

(i = 5): Since $\{[Su]\gamma \in \Gamma \mid u \leq t\}, \{[Ou]\gamma \in \Gamma \mid$

$u \leq t\} \to \{[Ou]\gamma \in \Delta \mid u \leq t\}, \{[Su]\gamma \in \Delta \mid u \leq t\}, [St]\beta$ is 5-consistent as a restriction of $\Gamma \to \Delta$, we see $\{[Su]\gamma \in \Gamma \mid u \leq t\}, \{[Ou]\gamma \in \Gamma \mid u \leq t\} \to \{[Ou]\gamma \in \Delta \mid u \leq t\}, \{[Su]\gamma \in \Delta \mid u \leq t\}, \beta$ is also 5-consistent. Take an Ω, 5-complete extension $w' = \Gamma' \to \Delta'$ of this sequent. Clearly, for any $u \leq t$, we have $\Gamma_{Su} \subseteq \Gamma'_{Su}$, $\Delta_{Su} \subseteq \Delta'_{Su}$, $\Gamma_{Ou} \subseteq \Gamma'_{Ou}$ and $\Delta_{Ou} \subseteq \Delta'_{Ou}$. We have $\Gamma_{Su} = \Gamma'_{Su}$ because $\Gamma_{Su} \subseteq \Gamma'_{Su} = \Omega_{Su}-\Delta'_{Su} \subseteq \Omega_{Su}-\Delta_{Su} = \Gamma_{Su}$. Similarly, we have $\Gamma_{Ou} = \Gamma'_{Ou}$. By virtue of the definition of R, we have $w \xrightarrow{St} w'$. Since $\beta \in \Delta'$, we have by induction hypothesis $w' \mathrel{=\!\!|} \beta$, which proves $w \mathrel{=\!\!|} [St]\beta = \alpha$.

From this theorem we at once have the following results.

Theorem 3.7. *(Generalized Completeness Theorem)*
Any i-*consistent sequent is* i-*realizable.*

Proof. Let an i-consistent sequent $\Gamma \to \Delta$ be given. We put $\Omega = \{\perp\} \cup \bigcup \{Sub(\alpha) \mid \alpha \in \Gamma \cup \Delta\}$. We construct the Ω, i-universal model $U_i(\Omega)$. Then by Lemma 3.4 and Theorem 3.6, there exists $w \in U$ such that $w \mathrel{=\!\!|} \Gamma \to \Delta$.

Corollary 3.8. *(Compactness Theorem)*
Let $\Gamma \subseteq Wff$. *Then,* Γ *is* i-*realizable if and only if any* $\Gamma_0 \Subset \Gamma$ *is* i-*realizable.*

Theorem 3.9. *(Completeness and Decidability Theorem)* *For any* $\alpha \in \mathrm{Wff}$, α *is a theorem of* KTi *if and only if* α *is valid in all* KTi*-models whose cardinality* $\leq 2^n$, *where* n *is the cardinality of the finite set* $\mathrm{Sub}(\alpha)\cup\{\perp\}$.

Proof. Let $\Omega = \mathrm{Sub}(\alpha)\cup\{\perp\}$. Then the result easily follows from Lemma 3.4 and Theorem 3.6.

Remark. Our definition of universal models differs from that of canonical models due to Lemmon-Scott [18], in the following points. Firstly, we define models relative to Ω, while canonical models are defined only for $\Omega = \mathrm{Wff}$. So that we need not use filtration method due to Segerberg [34] to secure decidability of the systems. Secondly, relational structures are defined differently. The naturalness of universal models will become clear in the next chapter.

3.4. *Cut-free system for* S5

In this and next section, we give our second proof of completeness. It is based on cut-free formulations of the systems, and in this section we first formulate a cut-free system GS5 which is equivalent to GT5 with the language restricted to $|Sp| = |T| = 1$. Hence GS5 is a cut-free system for the modal calculus S5. In GS5, a *sequent* is defined to be an element of the set $2^{\mathrm{Wff}}\times 2^{\mathrm{Wff}}\times 2^{\mathrm{Wff}}\times 2^{\mathrm{Wff}}$. Thus a sequent is of the form $(\Gamma, \Pi, \Sigma, \Delta)$. However we denote this as $\Gamma; \Pi \rightarrow \Sigma; \Delta$. Further we will denote $\Gamma; \rightarrow ; \Delta$ (= $(\Gamma, \emptyset, \emptyset, \Delta)$) simply as $\Gamma \rightarrow \Delta$. A sequent of this form will be called *proper*. Other sequents will be called *improper*. The idea of considering this kind of sequents is due to Sonobe [36]. Since our language is subject to the condition $|Sp| = |T| = 1$, we will denote $[St]\alpha$ as $\Box\alpha$. GS5 is defined as follows:

Axioms: $\alpha \rightarrow \alpha$

$\bot \rightarrow$

Rules:

$$\frac{\Gamma \rightarrow \Delta}{\Gamma', \Gamma \rightarrow \Delta, \Delta'} \quad \text{(extension: out)}$$

$$\frac{\Gamma; \quad \Pi \rightarrow \Sigma \quad ; \Delta}{\Gamma; \Pi', \Pi \rightarrow \Sigma, \Sigma'; \Delta} \quad \text{(extension: in)}$$

$$\frac{\Gamma \rightarrow \Delta, \alpha \qquad \alpha, \Pi \rightarrow \Sigma}{\Gamma, \Pi \rightarrow \Delta, \Sigma} \quad \text{(cut)}$$

$$\frac{\Gamma; \rightarrow \alpha; \Delta}{\Gamma; \rightarrow \; ; \Box\alpha, \Delta} \quad (\rightarrow\text{exit})$$

$$\frac{\Gamma, \Box\alpha; \Pi \rightarrow \Sigma; \Delta}{\Gamma; \Box\alpha, \Pi \rightarrow \Sigma; \Delta} \quad (\text{enter}\rightarrow)$$

$$\frac{\Gamma; \Pi \rightarrow \Sigma; \Box\alpha, \Delta}{\Gamma; \Pi \rightarrow \Sigma, \Box\alpha; \Delta} \quad (\rightarrow\text{enter})$$

$$\frac{\Gamma \to \Delta,\ \alpha,\ \beta \qquad \beta,\ \Phi \to \Psi,\ \alpha \qquad \alpha,\ \beta,\ \Xi \to \Lambda}{\alpha\supset\beta,\ \Gamma,\ \Phi,\ \Xi \to \Delta,\ \Psi,\ \Lambda} \quad (\supset\to\text{: out})$$

$$\frac{\alpha,\ \Gamma \to \Delta,\ \beta}{\Gamma \to \Delta,\ \alpha\supset\beta} \quad (\to\supset\text{: out})$$

$$\frac{\Gamma;\ \Pi \to \Sigma,\ \alpha,\ \beta;\ \Delta \qquad \Gamma;\ \beta,\ \Phi \to \Psi,\ \alpha;\ \Delta \qquad \Gamma;\ \alpha,\ \beta,\ \Xi \to \Lambda;\ \Delta}{\Gamma;\ \alpha\supset\beta,\ \Pi,\ \Phi,\ \Xi \to \Sigma,\ \Psi,\ \Lambda;\ \Delta}$$

$(\supset\to\text{: in})$

$$\frac{\Gamma;\ \alpha,\ \Pi \to \Sigma,\ \beta\ ;\Delta}{\Gamma;\ \Pi \to \Sigma,\ \alpha\supset\beta\ ;\Delta} \quad (\to\supset\text{: in})$$

$$\frac{\alpha,\ \Gamma \to \Delta}{\Box\alpha,\ \Gamma \to \Delta} \quad (\Box\to\text{: out})$$

$$\frac{\Box\Gamma \to \Box\Delta,\ \alpha}{\Box\Gamma \to \Box\Delta,\ \Box\alpha} \quad (\to\Box\text{: out})$$

The following lemma shows the equivalence of GS5 with GT5 (over the language restricted as above).

Lemma 3.10. *Let* $\Phi \to \Psi$ *be a proper sequent. Then* $\vdash \Phi \to \Psi$ (in GT5) *if and only if* $\vdash \Phi \to \Psi$ (in GS5).

Proof. Only if part: We have only to prove that the rule $(\supset\to)$ in GT5 is admissible in GS5. To see this we construct the following proof figure:

$$\dfrac{\dfrac{\Gamma \to \Delta,\ \alpha}{\Gamma \to \Delta,\ \alpha,\ \beta} \quad \dfrac{\beta,\ \Pi \to \Sigma}{\beta,\ \Pi \to \Sigma,\ \alpha} \quad \dfrac{\beta,\ \Pi \to \Sigma}{\alpha,\ \beta,\ \Pi \to \Sigma}}{\alpha\supset\beta,\ \Gamma,\ \Pi \to \Delta,\ \Sigma} \quad (\supset\to\text{: out})$$

If part: Suppose that $\vdash \Phi \to \Psi$ (in GS5). We note that Lemmas 1.3 and 1.4 hold also for GS5. Then, by Lemma 1.4, there exists $\Phi_0 \to \Psi_0 \subset \Phi \to \Psi$ such that $\vdash \Phi_0 \to \Psi_0$ (in GS5). Let $\mathbf{F}$ be a proof figure of $\Phi_0 \to \Psi_0$. Then by Lemma 1.3, any sequent occurring in $\mathbf{F}$ is finite, where $\Gamma;\ \Pi \to \Sigma;\ \Delta$ is *finite* if so are Γ, Π, Σ, Δ. We convert $\mathbf{F}$ to a proof figure in GT5 whose end-sequent is $\Phi_0 \to \Psi_0$. Let $\Gamma;\ \Pi \to \Sigma;\ \Delta$ be any improper sequent occurring in $\mathbf{F}$. We replace this sequent by the proper sequent $\Gamma \to \Delta,\ \Box\alpha$, where $\alpha = (\top\wedge\pi_1\wedge\cdots\wedge\pi_m)\supset(\sigma_1\vee\cdots\vee\sigma_n\vee\bot)$ ($\Pi = \{\pi_1, \cdots, \pi_m\}$, $\Sigma = \{\sigma_1, \cdots, \sigma_n\}$). We do this replacement for all improper sequents in $\mathbf{F}$. By this replacement, for example, an application of the rule

$$(\text{enter}\to) \qquad \frac{\Gamma,\ \Box\alpha;\ \Pi \to \Sigma;\ \Delta}{\Gamma;\ \Box\alpha,\ \Pi \to \Sigma;\ \Delta}$$

will become

$$(\#) \qquad \frac{\Gamma,\ \Box\alpha \to \Delta,\ \Box(\pi\supset\sigma)}{\Gamma \to \Delta,\ \Box(\Box\alpha\wedge\pi\supset\sigma)} \quad ,$$

where $\pi = \top\wedge\pi_1\wedge\cdots\wedge\pi_m$ ($\Pi = \{\pi_1, \cdots, \pi_m\}$) and $\sigma = \sigma_1\vee\cdots\vee\sigma_n\vee\bot$ ($\Sigma = \{\sigma_1, \cdots, \sigma_n\}$). We change (#) to the following:

$$
\dfrac{
\dfrac{\Gamma, \Box\alpha \to \Delta, \Box(\pi\supset\sigma)}{\Gamma \to \Delta, \Box\alpha\supset\Box(\pi\supset\sigma)}
\qquad
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\Box\alpha \to \Box\alpha}{\Box\alpha \to \Box\alpha, \sigma}}{\Box\alpha\wedge\pi \to \Box\alpha, \sigma}}{\to \Box\alpha, \Box\alpha\wedge\pi\supset\sigma}}{\to \Box\alpha, \Box(\Box\alpha\wedge\pi\supset\sigma)}
\quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\pi\to\pi \qquad \sigma\to\sigma}{\pi, \pi\supset\sigma \to \sigma}}{\Box\alpha\wedge\pi, \pi\supset\sigma \to \sigma}}{\pi\supset\sigma \to \Box\alpha\wedge\pi\supset\sigma}}{\Box(\pi\supset\sigma) \to \Box\alpha\wedge\pi\supset\sigma}}{\Box(\pi\supset\sigma) \to \Box(\Box\alpha\wedge\pi\supset\sigma)}
}{\Box\alpha\supset\Box(\pi\supset\sigma) \to \Box(\Box\alpha\wedge\pi\supset\sigma)}
}{\Gamma \to \Delta, \Box(\Box\alpha\wedge\pi\supset\sigma)}
$$

We must also consider the rules other than (enter→). But they can be treated similarly. Therefore we can obtain a proof of $\Phi_0 \to \Psi_0$ in GT5. From this we obtain a proof of $\Phi \to \Psi$ in GT5 by (extension).

We say a sequent is *strictly provable* (in GS5) if it is provable in GS5 without using (cut). A sequent is *weakly consistent* if it is not strictly provable. By Lemma 3.10 and Theorem 3.1, we have

Theorem 3.11. *If a proper sequent is provable (in* GS5*) then it is* 5-*valid.*

We now construct a KT5-model $M = \langle W; r, v \rangle$ which realizes any proper weakly consistent sequent. For any $\alpha \in \mathrm{Wff}$ we put $\mathrm{Sub}_{\Box}(\alpha) = \{\Box\beta \mid \Box\beta \in \mathrm{Sub}(\alpha)\}$. For any *finite* sequent $\Gamma \to \Delta$, we say $\Gamma \to \Delta$ is *saturated* if:

(i) $\Gamma \to \Delta$ is weakly consistent

(ii) $\beta \supset \gamma \in \Gamma \cup \Delta$ implies $\{\beta, \gamma\} \subseteq \Gamma \cup \Delta$

(iii) $\Box\beta \in \Gamma$ implies $\beta \in \Gamma$

(iv) $\Box\beta \in \Delta$ implies $\mathrm{Sub}_{\Box}(\beta) \subseteq \Gamma \cup \Delta$

Lemma 3.12. *Let a finite sequent* $\Gamma \to \Delta$ *be weakly consistent. Then there exists* $\bar{\Gamma} \to \bar{\Delta}$ *such that* $\Gamma \to \Delta \subseteq \bar{\Gamma} \to \bar{\Delta}$ *and* $\bar{\Gamma} \to \bar{\Delta}$ *is saturated.*

Proof. Let $\Omega = \bigcup\{\mathrm{Sub}(\alpha) \mid \alpha \in \Gamma \cup \Delta\}$. This is a finite set. Let $\mathbb{C} = \{\Pi \to \Sigma \mid \Pi \to \Sigma$ is weakly consistent and $\Pi \cup \Sigma \subseteq \Omega\}$. $\mathbb{C}$ is also finite. We construct a sequence $\{\Gamma_n \to \Delta_n\}_{n \geq 0}$ in $\mathbb{C}$ as follows. We put $\Gamma_0 \to \Delta_0 = \Gamma \to \Delta$. By assumption, we have $\Gamma_0 \to \Delta_0 \in \mathbb{C}$. Suppose that $\Gamma_n \to \Delta_n$ $\mathbb{C}$ has been defined. If $\Gamma_n \to \Delta_n$ is saturated, we put $\Gamma_{n+1} \to \Delta_{n+1} = \Gamma_n \to \Delta_n$. Suppose otherwise. Then one of (ii)-(iv) in the above definition of being saturated fails.

(1) Suppse there exists some $\beta \supset \gamma \in \Gamma_n \cup \Delta_n$ such that $\{\beta, \gamma\} \not\subseteq \Gamma_n \cup \Delta_n$. Suppose $\beta \supset \gamma \in \Gamma_n$. Then by $(\supset\to$: out) we have that one of $\Gamma_n \to \Delta_n, \beta, \gamma$, $\gamma, \Gamma_n \to \Delta_n, \beta$ or $\beta, \gamma, \Gamma_n \to \Delta_n$ is weakly consistent. We define $\Gamma_{n+1} \to \Delta_{n+1}$ as the first weakly consistent sequent among these three sequents.

(2) Suppose that there exists some $\Box\beta \in \Gamma_n$ such that

$\beta \notin \Gamma_n$. We put $\Gamma_{n+1} \to \Delta_{n+1} = \beta, \Gamma_n \to \Delta_n$. By ($\Box\!\to$: out), we have $\Gamma_{n+1} \to \Delta_{n+1} \in \mathbb{C}$.

(3) Suppose that there exists some $\Box\beta \in \Delta_n$ such that $\mathrm{Sub}_\Box(\beta) \not\subseteq \Gamma_n \cup \Delta_n$. Let $\Box\gamma$ be an element of the set $\mathrm{Sub}_\Box(\beta) - (\Gamma_n \cup \Delta_n)$ with maximal degree, where the *degree* of a formula is defined to be the number of logical connectives (i.e, $\supset$ and $\Box$) occurring in it. Let $\Box\delta$ be an element of $\Gamma_n \cup \Delta_n$ such that $\Box\gamma \in \mathrm{Sub}(\delta)$ and with minimal degree. The existence of such $\Box\delta$ is guaranteed by the fact that $\Box\delta \in \mathrm{Sub}(\beta)$ and $\Box\beta \in \Delta_n$. Then we have two cases.

$\Box\delta \in \Gamma_n$: Since $\Gamma_n \to \Delta_n = \Box\delta, \Gamma_n \to \Delta_n$ is weakly consistent, so is $\delta, \Gamma_n \to \Delta_n$ by ($\Box\!\to$: out). Then using ($\supset\to$: out), ($\to\supset$: out) and (extension: out), we see, by *reductio ad absurdum* , that either $\Box\gamma, \Gamma_n \to \Delta_n$ or $\Gamma_n \to \Delta_n, \Box\gamma$ is weakly consistent. So, we define $\Gamma_{n+1} \to \Delta_{n+1}$ as the first weakly consistent sequent of the two.

$\Box\delta \in \Delta_n$: Since $\Gamma_n \to \Delta_n = \Gamma_n \to \Delta_n, \Box\delta$ is weakly consistent, so is $\Gamma_n; \to \delta; \Delta_n$ by ($\to$exit). Then by ($\supset\to$: in), ($\to\supset$: in) and (extension: in), we see either $\Gamma_n; \Box\gamma \to ; \Delta_n$ or $\Gamma_n; \to \Box\gamma; \Delta_n$ is weakly consistent. Since the argument goes similarly, we suppose the first case. Then by (enter$\to$), $\Gamma_n, \Box\gamma \to \Delta_n$ is weakly consistent. In this case we put $\Gamma_{n+1} \to \Delta_{n+1} = \Gamma_n, \Box\gamma \to \Delta_n$.

In any of the above three cases, we have $\Gamma_{n+1} \to \Delta_{n+1} \in \mathbb{C}$ and $|\Gamma_n \cup \Delta_n| < |\Gamma_{n+1} \cup \Delta_{n+1}|$. Therefore, since $\mathbb{C}$ is finite, we obtain a saturated $\Gamma_n \to \Delta_n$ for some n. Putting

$\Gamma \to \bar{\Delta} = \Gamma_n \to \Delta_n$ we have the desired result.

We now define a model $M = \langle W; r, v\rangle$. Let $W = \{\Gamma \to \Delta \mid \Gamma \to \Delta$ is saturated$\}$. W is nonempty since $\to \bot \in W$. Let $w = \Gamma \to \Delta$, $w' = \Gamma' \to \Delta' \in W$. We define $(w, w') \in r$ iff $\Gamma_{\Box} = \Gamma'_{\Box}$. (Since $|Sp\times T| = 1$, we may consider $r : Sp\times T \longrightarrow 2^{W\times W}$ as an element of $2^{W\times W}$. $\Gamma_{\Box}$ denotes the set $\{\alpha \mid \Box\alpha \in \Gamma\}$.) $v : Pr\cup\{\bot\} \longrightarrow 2^{W}$ is defined by that $w = \Gamma \to \Delta \in v(\alpha)$ iff $\alpha \in \Gamma$. The following lemma is proved similarly as Lemma 3.5.

Lemma 3.13. M *is a* KT5-*model.*

Just like $U(\Omega)$, M has the following important property:

Theorem 3.14. *Let* $w = \Gamma \to \Delta \in M$ *and* $\alpha \in \Gamma\cup\Delta$. *Then* $w \models \alpha$ (in M) *if* $\alpha \in \Gamma$ *and* $w \Dashv \alpha$ *if* $\alpha \in \Delta$.

Proof. By induction on the construction of formulas. We only consider the case that $\alpha = \Box\beta \in \Delta$, since other cases may be handled similarly as in the proof of Thorem 3.6. Now, $\Gamma_0 \to \Delta_0 = \{\Box\gamma \mid \Box\gamma \in \Gamma\} \to \{\Box\delta \mid \Box\delta \in \Delta\}, \Box\beta$ is weakly consistent since it is a restriction of $\Gamma \to \Delta$. By $(\to\Box$: out), we see $\Gamma_1 \to \Delta_1 = \{\Box\gamma \mid \Box\gamma \in \Gamma\} \to \{\Box\delta \mid \Box\delta \in \Delta\}, \beta$ is also weakly consistent. By Lemma 3.12, we can extend this sequent to a saturated sequent $w' = \Gamma' \to \Delta' \in W$. By this construction, it is clear that $\Gamma_{\Box} \subseteq \Gamma'_{\Box}$. Suppose $\sigma \in \Gamma'_{\Box} - \Gamma_{\Box}$. Then by inspecting the construction method in Lemma 3.12, we see that $\Box\sigma \in \mathrm{Sub}_{\Box}(\gamma_1)$ for some $\gamma_1 \in \Gamma_1 \cup \Delta_1$. Hence, $\Box\sigma \in \mathrm{Sub}_{\Box}(\gamma_0)$ for some $\gamma_0 \in \Gamma_0 \cup \Delta_0 \subseteq \Gamma \cup \Delta$. (If $\gamma_1 = \beta$ then let $\gamma_0 = \Box\beta \in \Delta_0$, otherwise let $\gamma_0 = \gamma_1$.) Since $\Gamma \to \Delta$ is saturated, we have $\Box\sigma \in \Gamma \cup \Delta$. Since $\sigma \notin \Gamma_{\Box}$ we have $\Box\sigma \in \Delta$. Hence we have $\Box\sigma \in \Gamma' \cap \Delta'$. This contradicts the consistency of $\Gamma' \to \Delta'$. Thus we see $\Gamma_{\Box} = \Gamma'_{\Box}$, so that $(w, w') \in r$. Now since $\beta \in \Delta'$, we have $w' \dashv \beta$ by induction hypothesis. Hence we have $w \dashv \Box\beta$.

It is now easy to establish:

Theorem 3.15. *(Cut-elimination Theorem)*
If a proper sequent is provable in GS5 *then it is strictly provable in* GS5.

Proof. By Lemma 1.4 it suffices to consider only finite sequents. We prove the contraposition. Suppose that a finite sequent $\Gamma \to \Delta$ is not strictly provable. $\Gamma \to \Delta$ has a saturated extension $\bar{\Gamma} \to \bar{\Delta}$ by Lemma 3.12. Then $\bar{\Gamma} \to \bar{\Delta}$ is 5-realizable by Theorem 3.14. Then $\bar{\Gamma} \to \bar{\Delta}$ is not provable by Theorem 3.11. Hence $\Gamma \to \Delta$ is not provable.

3.5. *Cut-elimination theorem for* GT3 *and* GT4

In this section we consider only KT3 and KT4, so that when we refer to KTi or GTi, i is always 3 or 4. If a sequent $\Gamma \to \Delta$ is provable in GTi without cut, we say $\Gamma \to \Delta$ is *strictly provable*. We wish to establish this:

Theorem 3.16. *(Cut-elimination Theorem)*
If a sequent is provable (in GTi*) then it is strictly provable.*

We prove this by an argument similar to that in 3.3. Let $\Omega \subseteq$ Wff be closed under subformulas. Let us call a sequent $\Gamma \to \Delta$ Ω, i-*maximal* if it is maximal in the set $\{\Pi \to \Sigma \mid \Pi \to \Sigma$ is i-weakly consistent and $\Pi \cup \Sigma \subseteq \Omega\}$, where a sequent is i-*weakly consistent* if it is not strictly provable in GTi. We can show that if a sequent is i-weakly consistent and $\Gamma \cup \Delta \subseteq \Omega$ then it has a maximal extension $\bar{\Gamma} \to \bar{\Delta} \in W_i(\Omega) = \{\Pi \to \Delta \mid \Pi \to \Sigma$ is Ω, i-maximal$\}$, by means of Zorn's Lemma and Lemma 1.4. Now, we define a model $M_i(\Omega) = \langle W_i(\Omega); r, v\rangle$, where r and v are defined just as in the definition of $U_i(\Omega)$. That $M_i(\Omega)$ is a KTi-model is proved similarly as in Lemma 3.5. We now have the following lemma.

Lemma 3.17. *Let* $w = \Gamma \to \Delta \in M_i(\Omega)$ *and* $\alpha \in \Gamma \cup \Delta$. *Then* $w \models \alpha$ (in $M_i(\Omega)$) *if* $\alpha \in \Gamma$ *and* $w \dashv \alpha$ (in $M_i(\Omega)$) *if* $\alpha \in \Delta$.

Proof. By induction on the construction of formulas. The base step of $\alpha \in \mathrm{Pr} \cup \{\bot\}$ is trivial.

$\alpha = \beta \supset \gamma$: Suppose $\alpha \in \Gamma$. Then $\Gamma \to \Delta, \beta$ or $\gamma, \Gamma \to \Delta$ is i-weakly consistent. By the maximality of $\Gamma \to \Delta$, we have $\Gamma \to \Delta, \beta = \Gamma \to \Delta$ or $\gamma, \Gamma \to \Delta = \Gamma \to \Delta$. In any case, we have $w \models \alpha$ by induction hypothesis and definition of $\models$. The case $\alpha \in \Delta$ is similar.

$\alpha = [St]\beta$: If $\alpha \in \Gamma$, then the result follows similarly as in Theorem 3.6. Suppose $\alpha \in \Delta$.

(i = 3): $\{[Su]\gamma \in \Gamma \mid u \leq t\}, \{[Ou]\gamma \in \Gamma \mid u \leq t\} \to [St]\beta$ is i-weakly consistent as a restriction of $\Gamma \to \Delta$. Hence $\{\gamma \mid [Su]\gamma \in \Gamma, u \leq t\}, \{[Ou]\gamma \in \Gamma \mid u \leq t\} \to \beta$ is also i-weakly consistent. Extend this sequent to $w' = \Gamma' \to \Delta'$ in $M_i(\Omega)$. It is clear that $w \xrightarrow{St} w'$. Since $\beta \in \Delta'$ we have $w' \Dashv \beta$ by induction hypothesis. Hence $w \Dashv \alpha$.

(i = 4): Similar to the case (i = 3).

Now we can complete the proof of Theorem 3.16. Suppose $\Gamma \to \Delta$ is i-weakly consistent. Let $\Omega = \{\bot\} \cup \bigcup \{Sub(\alpha) \mid \alpha \in \Gamma \cup \Delta\}$. Let $\tilde{\Gamma} \to \tilde{\Delta} \in M_i(\Omega)$ be an extension of $\Gamma \to \Delta$. Then by Lemma 3.17, $M_i(\Omega) \Dashv \Gamma \to \Delta$. Hence by the Soundness Theorem 3.1, $\Gamma \to \Delta$ is not provable.

Remarks.

(1) Our method does not work for GT5, because, except for

the obvious fact that GT5 is not cut-free;[4] if we construct a model $M_5(\Omega)$ it does not always give w' such that $w \xrightarrow{St} w'$ and $w' =\!| \beta$ for w such that $w =\!| [St]\beta$. However, as a partial result, we gave a cut-free system for S5 in 3.4.

(2) By Theorem 3.16, we observe that $M_i(\Omega)$ is identical with $U_i(\Omega)$ (for $i = 3, 4$).

The following theorem will have some significance in Chapter 6.

Theorem 3.18. *(Disjunction property of* KT3 *and* KT4*)*[5] *Suppose* $\vdash [S^1t_1]\alpha_1 \vee \cdots \vee [S^nt_n]\alpha_n$ (in KTi) $(n \geq 1)$. *Then for some* j $(1 \leq j \leq n)$ *we have* $\vdash [S^jt_j]\alpha_j$ (in KTi), *where* $i = 3$ *or* 4.

Proof. Consider a cut-free proof of $\to [S^1t_1]\alpha_1, \cdots, [S^nt_n]\alpha_n$. Let $N = |\{[S^1t_1]\alpha_1, \cdots, [S^nt_n]\alpha_n\}|$. If $N = 1$ then we see that $\vdash \to [S^1t_1]\alpha_1$. Let $N > 1$. Then the last inference rule must be (extension). Furthermore we may assume without losing generality that the cardinality $|\Delta|$ of the upper sequent $\to \Delta$ of the last inference is less than N. Hence the result follows by induction hypothesis.

In this and last §, we have seen that GS5, GT3 and GT4 are cut-free. Using this fact, we obtain our second proof of the decidability of these systems as follows.

Theorem 3.19. *KT3, KT4 and S5 are decidable.*

Proof. Since the proof goes similarly, we only prove the theorem for S5. We first note that any proof figure may be represented as a pair $(\mathbb{P}, f)$, where $\mathbb{P} = (P, \leq_P)$ is a tree partially ordered by $\leq_P$ and f is a function $f : P \longrightarrow 2^{Wff} \times 2^{Wff} \times 2^{Wff} \times 2^{Wff}$. Suppose a formula $\alpha \in Wff$ is given. Let $\Omega = Sub(\alpha)$ and $|\Omega| = n$. Suppose α is provable. Then it has a cut-free proof $(\mathbb{P}, f)$. Then we have

$$(1) \qquad \mathrm{Image}(f) \subseteq 2^{\Omega} \times 2^{\Omega} \times 2^{\Omega} \times 2^{\Omega}.$$

(*Subformula property* of a cut-free proof!) Furthermore, we may assume without losing generality that $f(p) \neq f(q)$ if $p <_P q$. (For, othewise, we can obtain a *smaller* proof figure with the same end-sequent $\rightarrow \alpha$.) Thus we see that any linearly ordered subset Q of P has cardinality less than or equal to $2^{2^n \cdot 2^n \cdot 2^n \cdot 2^n} = m$. Since the number of the upper sequents of each inference rule is at most 3, it follows that

$$(2) \qquad |P| \leq 3^m.$$

By (1) and (2), we can construct an algorithm which determines the provability of α.

CHAPTER 4

CATEGORIES OF KRIPKE MODELS[6)]

4.1. *Definition of* $\mathbb{K}_i(\Omega)$

Let Ω be closed under subformulas. Let us take any i $(3 \le i \le 5)$ and fix it. We define the *category* $\mathbb{K}_i(\Omega)$ *of* KTi-*models over* Ω as follows:

(1) Objects ($\mathbb{M}$) are KTi-models.

(2) Let $M, N \in \mathbb{M}$, then $\mathrm{Hom}(M, N) = [M \to N]$ consists of homomorphisms (from M to N) as defined below.

(3) Composition of homomorphisms is defined by the usual function composition, i.e., $(f \circ g)(x)$ is defined by $f(g(x))$.

For any $M \in \mathbb{M}$, we define its *characteristic function*

$$\chi_M : M \longrightarrow U(\Omega)$$

by $\chi_M(w) = \Gamma \to \Delta$, where $\Gamma = \{\alpha \in \Omega \mid w \models \alpha\}$ and $\Delta = \{\alpha \in \Omega \mid w \dashv \alpha\}$. It is clear that $\Gamma \to \Delta$ is Ω-complete and hence χ_M is well-defined. ($U(\Omega)$ means $U_i(\Omega)$ and Ω-complete means Ω, i-complete.) A mapping

$$h : M \longrightarrow N$$

is a *homomorphism* (from M to N) if the diagram below commutes:

$$\begin{tikzcd} M \arrow[rr, "h"] \arrow[dr, "\chi_M"'] & & N \arrow[dl, "\chi_N"] \\ & U(\Omega) & \end{tikzcd}$$

Informally speaking, for $w \in M$, $\chi_M(w)$ denotes the scene (restricted to Ω) as seen from w. Thus a homomorphism is a mapping which preserves scenes. It is an easy task to verify that $\mathbb{K}_i(\Omega)$ defined above is indeed a category. As an example, consider the simplest case of $\Omega = \{\bot\}$. Then any mapping $f : M \longrightarrow N$ is a homomorphism.

4.2. *Properties of* $\mathbb{K}_i(\Omega)$

First of all, by the Fundamental Theorem of Universal Model, we see that $\chi_{U(\Omega)} : U(\Omega) \longrightarrow U(\Omega)$ is the identity mapping $1_{U(\Omega)}$. Hence, for any $M \in \mathbb{M}$, by the following commutative diagram we observe that χ_M itself is a homomorphism.

$$\begin{tikzcd} M \arrow[rr, "\chi_M"] \arrow[dr, "\chi_M"'] & & U(\Omega) \arrow[dl, equal, "\chi_{U(\Omega)}=1_{U(\Omega)}"] \\ & U(\Omega) & \end{tikzcd}$$

On the other hand, let $h \in [M \to U(\Omega)]$. Then since the diagram below commutes, we have $h = \chi_M$.

$$\begin{tikzcd}
M \arrow[rr, "h"] \arrow[dr, "\chi_M"'] & & U(\Omega) \arrow[dl, equal] \\
& U(\Omega) &
\end{tikzcd}$$

Thus we obtain:

Theorem 4.1. $U(\Omega)$ *is a terminal object*[7] *of* $\mathbb{K}(\Omega)$.

We now list up several basic properties of $\mathbb{K}(\Omega)$.

Lemma 4.2. *If* $f \in [M \to N]$ *is a monomorphism then* f *is an injection.*

Proof. We prove the contraposition. Let $x, y \in M$ be such that $x \neq y$ and $f(x) = f(y)$. Define $g : M \longrightarrow N$ by:

$$g(z) = \begin{cases} x & \text{if } z = y \\ y & \text{if } z = x \\ z & \text{otherwise} \end{cases}$$

Then we have:

$$\chi_M(g(z)) = \begin{cases} \chi_M(x) = \chi_N(f(x)) = \chi_N(f(y)) = \chi_M(y) & \text{if } z = y \\ \chi_M(y) = \chi_N(f(y)) = \chi_N(f(x)) = \chi_M(x) & \text{if } z = x \\ \chi_M(z) & \text{otherwise} \end{cases}$$

Hence, $g \in [M \to M]$. Now, clearly $f \circ g = f \circ 1_M$, but $g \neq 1_M$. This means f is not a monomorphism.

Lemma 4.3. *If* $f \in [M \to N]$ *is an epimorphism then* f *is a surjection.*

Proof. We prove the contraposition. Let $N = \langle W; r, v\rangle$. Let $x \in N$ be such that $x \notin \text{Image}(f)$. Take y such that $y \notin N$. We define a model $\tilde{N} = \langle\tilde{W}; \tilde{r}, \tilde{v}\rangle$ such that $\tilde{W} = W \cup \{y\}$ as follows: Let $g : \tilde{W} \longrightarrow W$ be defined by:

$$g(z) = \begin{cases} x & \text{if } z = y \\ z & \text{otherwise} \end{cases}$$

We define $\tilde{r}$ by $(w, w') \in \tilde{r}(S, t)$ iff $(g(w), g(w')) \in r(S, t)$. We define $\tilde{v}$ by $w \in \tilde{v}(p)$ iff $g(w) \in v(p)$. It is easy to verify that $\tilde{N}$ is a KTi-model. We can prove, by induction, that for any $w \in W$ and $\alpha \in$ Wff,

$$w \models \alpha \text{ (in } \tilde{N}) \text{ iff } g(w) \models \alpha \text{ (in } N).$$

I.e., $g \in [\tilde{N} \to N]$. Let $h : N \longrightarrow \tilde{N}$ be the inclusion map, and let $h' : N \longrightarrow \tilde{N}$ be defined by:

$$h'(z) = \begin{cases} y & \text{if } z = x \\ z & \text{otherwise} \end{cases}$$

We have $g \circ h = g \circ h' = 1_N$.

$$\begin{tikzcd}
M \arrow[r, "f"] & N \arrow[r, shift left=2, "h"] \arrow[r, shift right=2, "h'"'] \arrow[d, "\chi_N"'] & \tilde{N} \arrow[l, shift right=4, "g"'] \arrow[dl, "\chi_{\tilde{N}}"] \\
& U(\Omega) &
\end{tikzcd}$$

Then we have

$$\chi_{\tilde{N}}(h(z)) = \chi_N(g(h(z))) = \chi_N(z),$$

so that $h \in [N \to \tilde{N}]$. Similarly, $h' \in [N \to \tilde{N}]$. Now, clearly, $h \circ f = h' \circ f$ but $h \neq h'$. This means h is not an epimorphism.

Remark. The reader familiar with the notion of p-morphism might have noticed that g in the above proof is a p-morphism. By the p-morphism theorem [34], every p-morphism is a homomorphism (for any Ω), but the converse is not valid. In this sense our notion of homomorphism is more general than that of p-morphism. Note also that we defined homomorphisms without referring to the relational structure of models.

Lemma 4.4. *If* $f \in [M \to N]$ *is an epimorphism,* f *is a retraction.*

Proof. By Lemma 4.2, f is onto. Let $g : N \longrightarrow N$ be any mapping such that $f \circ g = 1_N$. Let $x \in N$. Then $\chi_M(g(x)) = (\chi_N \circ f)(g(x)) = \chi_N(f \circ g(x)) = \chi_N(x)$, i.e., $\chi_M \circ g = \chi_N$. Hence $g \in [N \to M]$. This means f is a retraction.

We cite the following easy lemma from Mitchell [23].

Lemma 4.5. *If* $f \in [M \to N]$ *is a retraction and also a monomorphism, then it is an isomorphism.*

By Lemmas 4.4 and 4.5, we have

Theorem 4.6. $\mathbb{K}(\Omega)$ *is balanced, i.e., every homomorphism which is both a monomorphism and an epimorphism is also an isomorphism.*

Lemma 4.7. *Let* $M \in \mathbb{M}$. *Then the following conditions are equivalent:*

(i) χ_M *is a monomorphism.*

(ii) *For any* $N \in \mathbb{M}$, $|[N \to M]| \leq 1$

(iii) $\mathrm{End}(M) = \{1_M\}$

(iv) $\mathrm{Aut}(M) = \{1_M\}$

where End(M) *denotes the endomorphism semigroup of* M *and* Aut(M) *denotes the automorphism group of* M.

Proof. The implications (i) => (ii) => (iii) => (iv) are trivial. To show (iv) => (i), we prove the contraposition. Suppose χ_M is not a monomorphism. Then there exist $N \in \mathbb{M}$ and $f, g \in [N \to M]$ such that $f \neq g$ and $\chi_M \circ f = \chi_M \circ g$. Take $x \in N$ such that $f(x) \neq g(x)$. We put $u = f(x)$, $v = g(x)$. We define $h : M \longrightarrow M$ by:

$$h(z) = \begin{cases} v & \text{if } z = u \\ u & \text{if } z = v \\ z & \text{otherwise} \end{cases}$$

It is easy to see that $h \in \mathrm{Aut}(M)$, so that $|\mathrm{Aut}(M)| > 1$.

4.3. *Structure of* $\mathbb{K}_i(\Omega)$

A model $M \in \mathbb{M}$ is said to be *reduced* if χ_M is a monomorphism.

Theorem 4.8. *Let* $M = \langle W; r, v\rangle$ *be any model in* $\mathbb{M}$, *and suppose* $(x, y) \in r(S, t)$. *Then* $(\chi_M(x), \chi_M(y)) \in R(S, t)$.

Proof. (i = 3): Let $\chi_M(x) = \Gamma \to \Delta$ and $\chi_M(y) =$

$\Gamma' \to \Delta'$. Suppose, by way of contradiction, that $(\chi_M(x), \chi_M(y)) \notin R(S, t)$. Then, by the definition of R, for some $u \leq t$, we have $\Gamma_{Su} \nsubseteq \Gamma'$ or $\Gamma_{Ou} \nsubseteq \Gamma'_{Ou}$. Suppose $\Gamma_{Su} \nsubseteq \Gamma'$. Then there exists an α such that $[Su]\alpha \in \Gamma$ and $\alpha \notin \Gamma'$. Then by the Fundamental Theorem of Universal Model, we have $\chi_M(x) \models [Su]\alpha$ and $\chi_M(y) \mathrel{=\!\!|} \alpha$. Hence, by the definition of χ_M, we have $x \models [Su]\alpha$ and $y \mathrel{=\!\!|} \alpha$. Since $(x,y) \in r(S, t) \subseteq r(S, u)$, this is a contradiction. Next, suppose $\Gamma_{Ou} \nsubseteq \Gamma'_{Ou}$. Then, similarly as above, for some α we have $x \models [Ou]\alpha$ and $y \mathrel{=\!\!|} [Ou]\alpha$. Since $(x, y) \in r(O, u)$ and $r(O, u)$ is transitive, we have a contradiction.

The cases $(i = 4)$ and $(i = 5)$ may be treated likewise.

Let $M, N \in \mathbf{M}$. We write $M \equiv N \pmod{\Omega}$ if $\mathrm{Image}(\chi_M) = \mathrm{Image}(\chi_N)$. (We should write χ_M^Ω (or χ_N^Ω) in place of χ_M (or χ_N) if we wish to emphasize the dependence of χ on Ω.) We say M is *equivalent (modulo Ω)* to N if $M \equiv N \pmod{\Omega}$. Among the models equivalent to M, we will be interested in finding the simplest one. Let $M = \langle W; r, v\rangle \in \mathbf{M}$. We define its *relational closure* $\bar{M} = \langle W; \bar{r}, v\rangle$ by letting $(w, w') \in \bar{r}(S, t)$ iff $(\chi_M(w), \chi_M(w')) \in R(S, t)$. By the above theorem we see $r \subseteq \bar{r}$ (, i.e., $r(S, t) \subseteq \bar{r}(S, t)$ for any S, t.) We can prove by induction that $1_W : M \longrightarrow \bar{M}$ is an isomorphism. Thus, $\bar{r}$ is the largest among the relations r' on W such that $\langle W; r', v\rangle$ is equivalent to M. We say $M \in \mathbf{M}$ is *relationally closed* if $M = \bar{M}$. Now, let $M = \langle W; r, v\rangle$ be relationally closed. An equivalence $\sim$ on W is called a *congruence* if $w \sim w'$ implies $\chi_M(w) = \chi_M(w')$. In this case, we can naturally define its *quotient model* $M/\sim\, = \langle \tilde{W}; \tilde{r}, \tilde{v}\rangle$ by:

(1) $\tilde{W} = W/\sim\, = \{[w] \mid w \in W\}$

(2) $([w], [w']) \in \tilde{r}(S, t)$ iff $(w, w') \in r(S, t)$

(3) Let $p \in Pr^{u}\{\bot\}$. If $p \in \Omega$ then $[w] \in \tilde{v}(p)$ iff $w \in v(p)$, otherwise $\tilde{v}(p)$ is arbitrary

where $[w]$ denotes the equivalence class containing w. It is easy to see that $M/\sim$ is well-defined (up to the arbitrariness of $\tilde{v}(p)$ for $p \notin \Omega$) and $M \equiv M/\sim$. (The canonical map $[\ ] : M \longrightarrow M/\sim$ is a p-morphism if $\Omega = \mathrm{Wff}$, and it is a homomorphism in any case.)

Suppose M, N are relationally closed, and let $f \in [M \to N]$ be an epimorphism. Then, $\sim\, \subseteq M \times M$ defined by $w \sim w'$ iff $f(w) = f(w')$ is a congruence, and we see $M/\sim$ is isomorphic to N. We write this as $M/f \cong N$.

Let $M \in \mathbf{M}$. By definition of χ_M, χ_M $(= \chi_{\bar{M}})$ induces the largest congruence among the congruences on M. Hence we have:

Theorem 4.9. *For any* $M \in \mathbf{M}$, *there uniquely (up to isomorphism) exists a reduced* $N \in \mathbf{M}$ *such that* $M \equiv N$. *Namely,* N *is given by* $N \cong \bar{M}/\chi_M$.

Schematically, we have the following diagram:

$$M \xrightarrow{1_W} \bar{M} \xrightarrow{\chi_M} \bar{M}/\chi_M \xrightarrow{\text{inclusion map}} U(\Omega)$$

Our argument in this chapter has been relative to Ω. We end this chapter by giving a definition which does not depend on Ω. Let $M = \langle W; r, v\rangle$ and $M' = \langle W'; r', v'\rangle$ be two KTi-models. We say M and M' are *strongly isomorphic* if there is a bijection $f : M \longrightarrow M'$ which preserves the model structure, i.e., f is a bijection such that

(1) For any $x, y \in W$, $(f(x), f(y)) \in f'(S, t)$ iff $(x, y) \in r(S, t)$.

(2) For any $p \in Prv\{\iota\}$ and $w \in W$, $w \in v(p)$ iff $f(w) \in v'(p)$.

CHAPTER 5

S5 MODEL THEORY

In this chapter we give a complete classification of S5 models under the equivalence $\equiv$ (mod Wff). First, we need some general discussions.

5.1. *Lindenbaum albebra of* KTi

Let us define a relation $\leq^* \subseteq \mathrm{Wff}\times\mathrm{Wff}$ by $\alpha \leq^* \beta$ iff $\vdash \alpha \to \beta$ (in GTi). (As usual, we discuss by fixing a logical system KTi.) Furthermore, define $\sim \subseteq \mathrm{Wff}\times\mathrm{Wff}$ by $\alpha \sim \beta$ iff $\alpha \leq^* \beta$ and $\beta \leq^* \alpha$. $\leq^*$ is reflexive since $\vdash \alpha \to \alpha$. $\leq^*$ is transitive since $\vdash \alpha \to \beta$ and $\vdash \beta \to \gamma$ implies $\vdash \alpha \to \beta$. Hence $\sim$ is an equivalence relation. We may regard Wff as an algebra $\langle \mathrm{Wff}; \wedge, \vee, \neg, \supset, \{[St] \mid S \in Sp, t \in T\}\rangle$. By the following lemma, we see that $\sim$ is a congruence on the algebra Wff. (For the definition of algebra and congruence, we refer to Grätzer [7].)

Lemma 5.1. *Suppose* $\alpha \sim \alpha'$ *and* $\beta \sim \beta'$. *Then,*

(i) $\alpha\wedge\beta \sim \alpha'\wedge\beta'$

(ii) $\alpha\vee\beta \sim \alpha'\vee\beta'$

(iii) $\neg\alpha \sim \neg\alpha'$

(iv) $\alpha\supset\beta \sim \alpha'\supset\beta'$

(v) $[St]\alpha \sim [St]\alpha'$ *(for any* $S \in Sp$, $t \in T$*)*

Proof. Left to the reader.

By this lemma, one can define the quotient algebra $\mathbb{B} = \langle B; \wedge, \vee, \neg, \supset, \{[St] \mid S \in Sp, t \in T\}\rangle$, where $B = Wff/\sim$. We will call this algebra the *Lindenbaum algebra* of KTi. Let $[\![\ ]\!] : Wff \longrightarrow \mathbb{B}$ denote the canonical homomorphism. We put $1 = [\![\top]\!]$ and $0 = [\![\bot]\!]$.

Theorem 5.2. $\langle B; \wedge, \vee, \neg, 0, 1\rangle$ *is a Boolean algebra.*

Proof. Left to the reader.

Let $\leq_B \subseteq B \times B$ denote the partial ordering induced by the Boolean structure of B, i.e., $a \leq_B b$ if and only if $a = a \wedge b$. Then we can easily verify that for any $\alpha, \beta \in Wff$, $\alpha \leq^* \beta$ if and only if $[\![\alpha]\!] \leq_B [\![\beta]\!]$.

We will use the term *theory* as a synonym for a subset of Wff. Let Γ be any theory. We say Γ is *consistent (or inconsistent)* if so is the sequent $\Gamma \rightarrow$. If $\Gamma = \bar{\Gamma} = DC(\Gamma)$, we say Γ is *(deductively) closed* Let $\mathbb{C}$ denote the set of all closed theories, i.e.,

$$\mathbb{C} = \{\Gamma \subseteq Wff \mid \Gamma = \bar{\Gamma}\}.$$

$\mathbb{C}$ is the set of fixed points of the retract $DC : 2^{Wff} \longrightarrow 2^{Wff}$. $\mathbb{C}$ is partially ordered by the set inclusionship relation $\subseteq$. We define a mapping $\phi : Wff \longrightarrow \mathbb{C}$ by $\phi(\alpha)$

$= \overline{\{\alpha\}}$. We say Γ is *finitely axiomatizable* if $\bar{\Gamma} = \phi(\alpha)$ for some $\alpha \in \mathrm{Wff}$.

Lemma 5.3. $[\alpha] \leq_B [\beta]$ *if and only if* $\phi(\alpha) \supseteq \phi(\beta)$.

Proof. Only if part: By the assumption we have $\alpha \leq^* \beta$. Hence $\vdash \alpha \to \beta$. Take any $\pi \in \phi(\beta) = \overline{\{\beta\}}$. Then $\vdash \beta \to \pi$. Hence $\vdash \alpha \to \pi$, so that $\alpha \vdash \pi$. This means $\pi \in \phi(\alpha)$.

If part: Suppose $\phi(\alpha) \supseteq \phi(\beta)$. Since $\beta \in \phi(\beta) \subseteq \phi(\alpha)$, we have $\alpha \vdash \beta$, i.e., $\vdash \alpha \to \beta$. Hence $[\alpha] \leq_B [\beta]$.

From this lemma we see that there uniquely exists an anti-order preserving injection $\iota : \mathbb{B} \longrightarrow \mathbb{C}$ such that the diagram below commutes:

$$\begin{tikzcd} & \mathrm{Wff} \arrow[dl, "{[\ ]}"'] \arrow[dr, "\phi"] & \\ \mathbb{B} \arrow[rr, "\iota"'] & & \mathbb{C} \end{tikzcd}$$

We note that ι is onto iff ϕ is onto. We give a sufficient condition for ι to be an anti-order isomorphism.

Lemma 5.4. *If* $\mathbb{B}$ *satisfies the descending chain* *condition, then* ι *is an anti-oder isomorphism.*

Proof. Let Γ be any element in $\mathbb{C}$. Let $\alpha_1, \alpha_2, \cdots$ be an enumeration of Γ. Let $\beta_n = \alpha_1 \wedge \cdots \wedge \alpha_n$. Let $\pi \in \phi(\beta_n)$. Then we have $\vdash \beta_n \to \pi$. Since $\vdash \Gamma \to \alpha_i$ $(i = 1, 2, \cdots, n)$, we have $\vdash \Gamma \to \beta_n$. Hence $\vdash \Gamma \to \pi$. This means $\pi \in \bar{\Gamma} = \Gamma$. Therefore,

$$(1) \qquad \phi(\beta_n) \subseteq \Gamma.$$

Let $\pi \in \Gamma$. Then $\pi = \alpha_n$ for some n. Since $\vdash \beta_n \to \alpha_n$, we have $\pi = \alpha_n \in \phi(\beta_n)$. Hence, together with (1), we have

$$(2) \qquad \Gamma = \bigcup_{n=1}^{\infty} \phi(\beta_n).$$

Since $\vdash \beta_{n+1} \to \beta_n$ for any n, we see $[\![\beta_1]\!] \geq_B [\![\beta_n]\!] \geq_B \cdots$. Since $\mathbb{B}$ satisfies descending chain condition, there exists an m such that $[\![\beta_m]\!] \leq_B [\![\beta_n]\!]$ for any n. Then, by Lemma 5.3, we have $\phi(\beta_m) \supseteq \phi(\beta_n)$ for any n. Thus, by (1) and (2),

$$(3) \qquad \Gamma \supseteq \phi(\beta_m) \supseteq \bigcup_{n=1}^{\infty} \phi(\beta_n) \supseteq \Gamma.$$

This establishes the sujectivity of ι. Thus we see that ι is an anti-order isomorphism.

5.2. S5 *model theory*

For any $n \geq 1$, we let the language $L_n = (Pr(n), Sp, T)$ be defined by:

(1) $Pr(n) = \{p_1, p_2, \cdots, p_n\}$,

(2) $Sp = \{0\}$,

(3) $T = \{1\}$.

Let us take any L_n and fix it. In this section, we study KT5 over the language L_n, which is none other than the modal calculus S5 as we have seen in Fig. 1.1. Hence a KT5-model over L_n will be called an S5-model. Our aim is to determine the structure of the Universal Model $U = U(n) = U_5(\mathrm{Wff})$. We employ the more conventional notation $\Box\alpha$ ($\Diamond\alpha$) in place of $[01]\alpha$ ($\langle 01\rangle\alpha$, resp.).

Let $\{\pm\}^n$ denote the n-fold cartesian product of the doubleton set $\{+, -\}$. For any $\alpha \in \mathrm{Wff}$ and $\delta \in \{\pm\} = \{+, -\}$, we put

$$\alpha^{\delta} = \begin{cases} \alpha & \text{if } \delta = + \\ \neg\alpha & \text{if } \delta = - \end{cases} .$$

We define a mapping

$$\pi : \{\pm\}^n \longrightarrow \mathrm{Wff}$$

by $\pi(\varepsilon) = p_1^{\varepsilon_1} \wedge \cdots \wedge p_n^{\varepsilon_n}$, where $\varepsilon = \varepsilon_1 \cdots \varepsilon_n$ ($\varepsilon_i \in \{\pm\}$). We put $\Pi = \mathrm{Image}(\pi)$. For any E ($\neq \emptyset$) $\subseteq \{\pm\}^n$, we define an S5-model $M(E) = \langle W_E; r_E, v_E\rangle$ as follows:

(1) $W_E = E \times \{E\}$,

(2) $r_E(0, 1) = 2^{W_E \times W_E}$,

(3) For any $(\varepsilon, E) \in W_E$, $(\varepsilon, E) \in v(p_i)$ iff $\varepsilon_i = +$, where $\varepsilon = \varepsilon_1 \cdots \varepsilon_n$, and $v(\bot) = \emptyset$.

Since $r_E(0, 1)$ is an equivalence relation, $M(E)$ is an S5-model. We call this model *the fragment model on* E. We define its *characteristic formula* $\chi(E)$ by:

$$\chi(E) = \bigwedge_{\varepsilon \in E} \Diamond \pi(\varepsilon) \wedge \bigwedge_{\varepsilon \in \{\pm\}^n - E} \neg\Diamond \pi(\varepsilon).^{8)}$$

For any $(\varepsilon, E) \in M(E)$, we define its *charcteristic formula* $\chi(\varepsilon, E)$ by:

$$\chi(\varepsilon, E) = \pi(\varepsilon) \wedge \chi(E).$$

Now, let $(M_\lambda)_{\lambda \in \Lambda}$ be an indexed family of S5-models, where $M_\lambda = \langle W_\lambda; r_\lambda, v_\lambda \rangle$. We define their *sum*

$$M = \langle W; r, v \rangle = \sum_{\lambda \in \Lambda} M_\lambda$$

by:

(1) $W = \sum_{\lambda \in \Lambda} W_\lambda$ (disjoint union),

(2) $(w, w') \in r(0, 1)$ iff both w and w' are in W_λ for some λ and $(w, w') \in r_\lambda(0, 1)$,

(3) $v(p) = \sum_{\lambda \in \Lambda} v_\lambda(p)$.

An S5-model $M = \langle W; r, v \rangle$ is said to be *connected* if $r(0, 1) = 2^{W \times W}$. It is easy to see that any S5-model M may be expressed as a sum $\sum_{\lambda \in \Lambda} M_\lambda$ of their connected components $(M_\lambda)_{\lambda \in \Lambda}$.

Let S be the sum of the family of all fragment models, i.e.,

$$S = \sum_{\emptyset \neq E \subseteq \{\pm\}^n} M(E).$$

We will show that S is strongly isomorphic to U.

Lemma 5.5. *Let an* S5-*model* $M = \langle W; r, v \rangle$ *be connected and reduced (in the category* $\mathbb{K}(\mathrm{Wff})$*). Then* M *is strongly isomorphic to some fragment model* $M(E)$.

Proof. Let $E = \{\varepsilon \in \{\pm\}^n \mid w \models \pi(\varepsilon)$ (in M) for some $w \in M\}$. Since for any $w \in W$ there uniquely exists an $\varepsilon \in E$ such that $w \models \pi(\varepsilon)$, we can define $\phi : W \longrightarrow E$ by $\phi(w) = \varepsilon$. Suppose $\phi(w) = \phi(w') = \varepsilon$. We show by induction that for any $\alpha \in \mathrm{Wff}$, $w \models \alpha$ iff $w' \models \alpha$. The case $\alpha \in \mathrm{Pr} \cup \{\bot\}$ is easily ascertained since $\phi(w) = \phi(w')$. The case $\alpha = \beta \supset \gamma$ is trivial by the definition of $\models$ and by induction hypothesis. Finally, we consider the case $\alpha = \Box\beta$. Then, since M is connected we see $w \models \Box\beta$ iff $w' \models \Box\beta$. Hence, it follows that $\chi_M(w) = \chi_M(w')$. Since M is reduced, we have $w = w'$, by Lemma 4.2. Thus we have proved that ϕ is a bijection. Since both M and $M(E)$ are connected and $v_E(\phi(p)) = v(p)$ for any $p \in \mathrm{Pr} \cup \{\bot\}$, we see that M and $M(E)$ are strongly isomorphic.

Corollary 5.6. *Let the assumptions be as in* Lemma 5.5. *Then the strong isomorphism* $\phi : M \longrightarrow M(E)$ *is unique.*

Proof. Since M is reduced, we have $\mathrm{Aut}(M) = \{1_M\}$, by Lemma 4.7. Since a strong automorphism is an automorphism, we see that ϕ is unique.

Theorem 5.7. *Let* M *be connected and reduced. Suppose* $w \models \chi(E)$ *for some* $w \in M$. *Then* M *is isomorphic to* $M(E)$.

Proof. By Lemma 5.5, we have only to prove:

"If $E \neq E'$ then $(\varepsilon, E) \mathrel{=\!\!|} \chi(E')$ for any $(\varepsilon, E) \in M(E)$."

Suppose $E \neq E'$ and $(\varepsilon, E) \models \chi(E')$ for some $(\varepsilon, E) \in M(E)$. Then we can take a δ such that $\delta \in E\text{-}E'$ or $\delta \in E'\text{-}E$. Suppose $\delta \in E\text{-}E'$. Then $(\varepsilon, E) \models \Diamond\pi(\delta)$. But, since $(\varepsilon, E) \models \chi(E')$ and $\chi(E') \vdash \neg\Diamond\pi(\delta)$, we have a contradiction. The case $\delta \in E'\text{-}E$ may be treated similarly.

Now, let the Universal Model U be expressed as the sum $\sum_{\lambda\in\Lambda} M_\lambda$ of its connected components. Then each M_λ is reduced because $\chi_U = 1_U$. By Lemma 5.5, M_λ is strongly isomorphic to $M(E_\lambda)$ for a suitable E_λ. Let $\phi_\lambda : M_\lambda \longrightarrow M(E_\lambda)$ be the unique strong isomorphism. Define $\phi : U \longrightarrow \sum_{\lambda\in\Lambda} M(E_\lambda)$ by $\phi(w) = \phi_\lambda(w)$ where λ is the unique index such that $w \in M_\lambda$. Since ϕ is a strong isomorphism, we have the following commutative diagram:

$$\begin{tikzcd} U \arrow[rr, "\phi"] \arrow[dr, equal] & & \sum\limits_{\lambda\in\Lambda} M(E_\lambda) \arrow[dl, "\chi_M"] \\ & U & \end{tikzcd}$$

Hence, χ_M is also a strong isomorphism. Suppose $E_\lambda = E_\mu$ for some $\lambda \neq \mu$. Then it is clear that $\mathrm{Aut}(\sum M(E_\lambda)) \supsetneqq \{1\}$. But, by Lemma 4.7, it is contrary to the fact that χ_M is injective. Thus we have:

$$E_\lambda \neq E_\mu \quad \text{if} \quad \lambda \neq \mu.$$

Now, take any $E\ (\neq \emptyset) \subseteq \{\pm\}^n$. By Theorem 4.8, we see $\mathrm{Image}(\chi_{M(E)})$ is connected. Hence it is contained in some M_λ, i.e., $\mathrm{Image}(\chi_{M(E)}) \subseteq M_\lambda$. Take any $(\varepsilon, E) \in M(E)$. Then,

$$(\varepsilon, E) \models \chi(E) \ (\text{in } M(E)).$$

By the definition of $\chi_{M(E)}$,

$$\chi_{M(E)}(\varepsilon, E) \models \chi(E) \ (\text{in } U).$$

Hence,

$$\chi_{M(E)}(\epsilon, E) \models \chi(E) \text{ (in } M_\lambda).$$

By applying ϕ, we have

$$\phi(\chi_{M(E)}(\epsilon, E)) \models \chi(E) \text{ (in } M(E_\lambda)).$$

Therefore by Theorem 5.7, we have $E = E_\lambda$. Thus we have proved the following

Theorem 5.8. U *is strongly isomorphic to* S.

Similarly, we have

Theorem 5.9. *Let* M *be reduced. Then* M *is strongly isomorphic to* $\sum_{E \in \mathbb{E}} M(E)$ *for some* $\mathbb{E} \subseteq 2^{\{\pm\}^n} - \{\emptyset\}$.

Proof. Let $M = \sum_{\lambda \in \Lambda} M_\lambda$. where M_λ $(\lambda \in \Lambda)$ are reduced and connected. Since M is reduced we have that M_λ and M_μ are nonisomorphic if $\lambda \neq \mu$ by considering the automorphism group of M. Hence by Lemma 5.5 we have the desired result.

Corollary 5.10. *An isomorphism* $\phi : M \longrightarrow N$ *between reduced models* M *and* N *is an strong isomorphism.*

On the other hand, it is clear that $\sum_{E \in \mathbb{E}} M(E)$ is reduced for any $\mathbb{E} \subseteq 2^{\{\pm\}^n} - \{\emptyset\}$. Hence we have

Corollary 5.11. *There are* 2^{2^n-1} *nonisomorphic reduced* S5-*models.*

Theorem 5.9 gives a complete classification of reduced models up to (strong) isomorphism. We will further proceed to define for any model M its *characteristic function* $X(M)$.

Let $w = \Gamma \to \Delta \in U$. By the isomorphism $\phi : U \longrightarrow S$ established in Theorem 5.9, we will identify w with $\phi(w)$. Hence w may be written as $w = \Gamma \to \Delta = (\varepsilon, E)$. We define a mapping

$$X_U : U \longrightarrow \mathrm{Wff}$$

by $X_U(w) = \chi(\varepsilon, E)$, where $w = (\varepsilon, E)$. Furthermore, for any model M, we define

$$X_M : M \longrightarrow \mathrm{Wff}$$

by $X_M(w) = X_U(\chi_M(w))$, where χ_M is the characteristic function

$$\chi_M : M \longrightarrow U.$$

Then the following theorem enables us to replace the semantical relation $\models$ by the syntactical one $\vdash$.

Theorem 5.12. *Let* M *be any* S5-*model. Then for any*

$w \in M$ *and* $\alpha \in$ Wff *we have:*

$$w \models \alpha \text{ (in } M) \quad \textit{if and only if} \quad X_M(w) \vdash \alpha.$$

Proof. Since $w \models \alpha$ iff $\chi_M(w) \models \alpha$ (in U), and since $X_M = X_U \circ \chi_M$, it suffices to prove the case $M = U$. So, let $w = \Gamma \to \Delta = (\varepsilon, E)$. We prove by induction on the construction of α that

(a) if $w \models \alpha$ then $X_U(w) \vdash \alpha$

and

(b) if $w \dashv \alpha$ then $X_U(w) \vdash \neg\alpha$.

$\alpha \in \mathrm{Pr} \cup \{\perp\}$: The case $\alpha = \perp$ is trivial. So, suppose $\alpha = p_i \in \mathrm{Pr}$.

(a): Since $(\varepsilon, E) \models p_i$, we have $\varepsilon_i = +$. Hence $\pi(\varepsilon) \vdash p_i$, so that $X_U(w) = \chi(\varepsilon, E) = \pi(\varepsilon) \wedge \chi(E) \vdash p_i$ $(= \alpha)$. The proof of (b) is similar.

$\alpha = \beta \supset \gamma$:

(a): Since $w \models \beta \supset \gamma$, it follows that $w \dashv \beta$ or $w \models \gamma$. Suppose $w \dashv \beta$. Then by induction hypothesis, we have $X_U(w) \vdash \neg\beta$. Since $\neg\beta \vdash \beta \supset \gamma$, we have $X_U(w) \vdash \alpha$. The case $w \models \gamma$ may be treated similarly.

(b): Since $w \dashv \beta \supset \gamma$, it follows that $w \models \beta$ and $w \dashv \gamma$. By induction hypothesis, we have $X_U(w) \vdash \beta$ and $X_U(w) \vdash \neg\gamma$. Hence, $X_U(w) \vdash \beta \wedge \neg\gamma$. Since $\beta \wedge \neg\gamma \vdash \neg(\beta \supset \gamma)$, we have $X_U(w) \vdash \neg\alpha$.

$\alpha = \Box\beta$:

(a): Since $(\varepsilon, E) \models \Box\beta$, we have for any $\delta \in E$, $(\delta, E) \models \beta$. By induction hypothesis, $\pi(\delta) \wedge \chi(E) \vdash \beta$ for any $\delta \in E$. Hence, we have:

$$\vdash \bigvee_{\delta \in E} \pi(\delta),\ \chi(E) \to \beta \tag{1}$$

Now, since $\vdash \to \bigvee_{\delta \in \{\pm\}^n} \pi(\delta)$ and $\vdash \chi(E) \to \neg\pi(\delta)$ for any $\delta \notin E$, we have

$$\vdash \chi(E) \to \bigvee_{\delta \in E} \pi(\delta) \tag{2}$$

Hence, from (1) and (2) we obtain

$$\vdash \chi(E) \to \beta \tag{3}$$

From this, by $(\to\neg)$ and $(\to\Box)$, we have $\chi(E) \vdash \Box\beta$ as desired.

(b) Since $(\varepsilon, E) \dashv \Box\beta$, we have for some $\delta \in E$ $(\delta, E) \dashv \beta$. By induction hypothesis, we have

$$\vdash \pi(\delta),\ \chi(E) \to \neg\beta \tag{4}$$

Let $\chi(E) = \Diamond\pi(\varepsilon_1) \wedge \cdots \wedge \Diamond\pi(\varepsilon_i) \wedge \neg\Diamond\pi(\varepsilon_{i+1}) \wedge \cdots \wedge \neg\Diamond\pi(\varepsilon_j)$. Then from (4) we can construct the following proof figure, which proves (b).

$$\pi(\delta),\ \Diamond\pi(\varepsilon_1),\ \cdots,\ \Diamond\pi(\varepsilon_i),\ \neg\Diamond\pi(\varepsilon_{i+1}),\ \cdots,\ \neg\Diamond\pi(\varepsilon_j) \to \neg\beta$$

$$\beta,\ \pi(\delta),\ \Box\neg\pi(\varepsilon_{i+1}),\ \cdots,\ \Box\neg\pi(\varepsilon_j) \to \Box\neg\pi(\varepsilon_1),\ \cdots,\ \Box\neg\pi(\varepsilon_i) \qquad (\Box\to)$$

$$\Box\beta,\ \pi(\delta),\ \Box\neg\pi(\varepsilon_{i+1}),\ \cdots,\ \Box\neg\pi(\varepsilon_j) \to \Box\neg\pi(\varepsilon_1),\ \cdots,\ \Box\neg\pi(\varepsilon_i) \qquad (\to\neg)$$

$$\Box\beta,\ \Box\neg\pi(\varepsilon_{i+1}),\ \cdots,\ \Box\neg\pi(\varepsilon_j) \to \Box\neg\pi(\varepsilon_1),\ \cdots,\ \Box\neg\pi(\varepsilon_i),\ \neg\pi(\delta) \qquad (\to\Box)$$

$$\Box\beta,\ \Box\neg\pi(\varepsilon_{i+1}),\ \cdots,\ \Box\neg\pi(\varepsilon_j) \to \Box\neg\pi(\varepsilon_1),\ \cdots,\ \Box\neg\pi(\varepsilon_i),\ \Box\neg\pi(\delta) \qquad (\text{extension})$$

$$\Box\beta,\ \Box\neg\pi(\varepsilon_{i+1}),\ \cdots,\ \Box\neg\pi(\varepsilon_j) \to \Box\neg\pi(\varepsilon_1),\ \cdots,\ \Box\neg\pi(\varepsilon_i)$$

$$\chi(E) \to \neg\Box\beta$$

$$\pi(\delta),\ \chi(E) \to \neg\Box\beta$$

In the above proof a double line (═══) means that several trivial applications of rules are omitted.

Now it is clear that (b) implies that if $w \Dashv \alpha$ then $X_U(w) \not\vdash \alpha$. This completes the proof of the theorem.

Corollary 5.13. *Let* $\tilde{X}_U : U \longrightarrow \mathbb{B}$ *be defined by* $\tilde{X}_U(w) = [X_U(w)]$. *Then* $\tilde{X}_U$ *is injective.*

Proof. Take any $w = (\varepsilon, E)$ and $w' = (\varepsilon', E')$ in U. Suppose $\tilde{X}_U(w) = \tilde{X}_U(w')$. Then, by Theorem 5.12, $(\varepsilon, E) \models \pi(\varepsilon') \wedge \chi(E')$. Hence, clearly, $\varepsilon = \varepsilon'$. By Theorem 5.7, we have $E = E'$. Therefore $w = w'$, which means $\tilde{X}_U$ is injective.

In the above proof we have also proved

Corollary 5.14. *Let* $w, w' \in U$. *Then*

(1) $w \models X_U(w')$ *if and only if* $w = w'$.
(2) $X_U(w) \vdash X_U(w')$ *if and only if* $w = w'$.

We extend $X_U : U \longrightarrow \mathrm{Wff}$ to

$$X_U : 2^U \longrightarrow \mathrm{Wff}$$

as follows. Let $P \subseteq W_U$. Then $X_U(P)$ is defined by:

$$X_U(P) = \bigvee_{w \in P} X_U(w).$$

We note that newly defined X_U may be regarded as an extension of the old one by identifying w with $\{w\}$. Now, for any $\alpha \in \mathrm{Wff}$ we can define its *normal form* $\mathrm{norm}(\alpha)$ by

$$\mathrm{norm}(\alpha) = X_U(P_\alpha),$$

where $P_\alpha = \{w \in U \mid w \models \alpha \text{ (in } U)\}$.

Theorem 5.15. *For any* $\alpha \in \mathrm{Wff}$, $\mathrm{norm}(\alpha) \equiv \alpha$.

Proof. Let $w \in P_\alpha$. Then by Theorem 5.12, $\vdash X_U(w) \to \alpha$. Hence we have $\vdash \bigvee_{w \in P_\alpha} X_U(w) \to \alpha$, i.e., $\vdash \mathrm{norm}(\alpha) \to \alpha$. We prove $\vdash \alpha \to \mathrm{norm}(\alpha)$ by means of the Completeness Theorem. Consider any S5-model M and $w \in M$ such that $w \models \alpha$ (in M). Let $w' = \chi_M(w)$. Then $w' \models \alpha$ (in U), i.e., $w' \in P_\alpha$. Since $w' \models X_U(w')$, we have $w' = \chi_M(w) \models \mathrm{norm}(\alpha)$. Hence, by the definition of χ_M, $w \models \mathrm{norm}(\alpha)$. By the Completeness Theorem, we have $\vdash \alpha \to \mathrm{norm}(\alpha)$. Thus, we have proved $\mathrm{norm}(\alpha) \equiv \alpha$.

We are now ready to study the mapping

$$h : 2^U \longrightarrow \mathbb{B}$$

defined by $h(P) = [\![X_U(P)]\!]$. First, we define

$$\Box : 2^U \longrightarrow 2^U$$

by $\Box P = \{w \in U \mid (w, w') \in r(0, 1) \Rightarrow w' \in P\}$. Then 2^U may be considered as an algebra $2^U = \langle 2^{W_U}; \cap, \cup, \Box\rangle$. Furthermore, we consider $\mathbb{B}$ as an algebra $\mathbb{B} = \langle B; \wedge, \vee, \Box\rangle$.

Theorem 5.16. $h : 2^U \longrightarrow \mathbb{B}$ *is an isomorphism.*

Proof. Take any $[\![\alpha]\!] \in B$ and let $P_\alpha = \{w \in U \mid w \models \alpha\}$. Then by Theorem 5.15, we have $h(P_\alpha) = [\![\mathrm{norm}(\alpha)]\!] = [\![\alpha]\!]$. Hence h is injective. Next, take any $P, Q \subseteq U$ and suppose $P \neq Q$. We can take w such that $w \in P-Q$ or $w \in Q-P$. Suppose $w \in P-Q$. Then clearly,

(1) $$X_U(w) \vdash X_U(P).$$

Suppose $X_U(w) \vdash X_U(Q)$. Then by Theorem 5.12, we have $w \models X_U(Q)$. Hence for some $w' \in Q$ we have $w \models X_U(w')$. Then by Corollary 5.14, we see $w = w'$. This is a contradiction since $w \notin Q$ and $w' \in Q$. Thus, we see

(2) $$X_U(w) \not\vdash X_U(Q).$$

By (1) and (2), we have $X_U(P) \neq X_U(Q)$, i.e.,

$$[\![X_U(P)]\!] \neq [\![X_U(Q)]\!].$$

Thus, we see h is injective.

Now, let $P, Q \in 2^U$.

(i) Since $X_U(P \cap Q) \vdash X_U(P)$ and $X_U(P \cap Q) \vdash X_U(Q)$, we have

(3) $$\vdash X_U(P \cap Q) \to X_U(P) \wedge X_U(Q)$$

On the other hand, suppose $w \models X_U(P) \wedge X_U(Q)$, where $w \in U$. Then, by a method similarly as above, we can prove $w \in P \cap Q$. Hence $w \models X_U(P \cap Q)$. Thus we see

(4) $$\vdash X_U(P) \wedge X_U(Q) \to X_U(P \cap Q).$$

By (3) and (4), we have $h(P \cap Q) = h(P) \wedge h(Q)$.

(ii) That $h(P \cup Q) = h(P) \vee h(Q)$ is proved similarly.

(iii) First, take any $w \in U$ such that $w \models X_U(\Box P)$. Then $w \in \Box P$, so that for any $(w, w') \in r(0, 1)$ we have $w' \in P$. Hence $w' \models X_U(P)$. Thus, we have $w \models \Box X_U(P)$. Therefore, we have

(5) $$\vdash X_U(\Box P) \to \Box X_U(P).$$

Next, take any $w \in U$ such that $w \models \Box X_U(P)$. Let w' be such that $(w, w') \in r(0, 1)$. Then we have $w' \models X_U(P)$. Hence $w' \in P$. Then by the definition of $\Box P$, we have $w \in \Box P$. Hence $w \models X_U(\Box P)$. Thus, we have

(6) $$\vdash \Box X_U(P) \to X_U(\Box P).$$

By (5) and (6), we have $h(\Box P) = \Box h(P)$.

Theorems 5.8 and 5.16 determines the structure of the Lindenbaum algebra of S5. Since the cardinality of U

(= S) is easily calculated as

$$|U| = \sum_{i=1}^{2^n} i \cdot \binom{2^n}{i} = 2^n \cdot 2^{2^n - 1},$$

the cardinality of $\mathbb{B}$ is given by

$$|\mathbb{B}| = 2^{|U|} = 2^{2^n \cdot 2^{2^n - 1}}.$$

As an example, we illustrate the structure of U for $n = 2$.

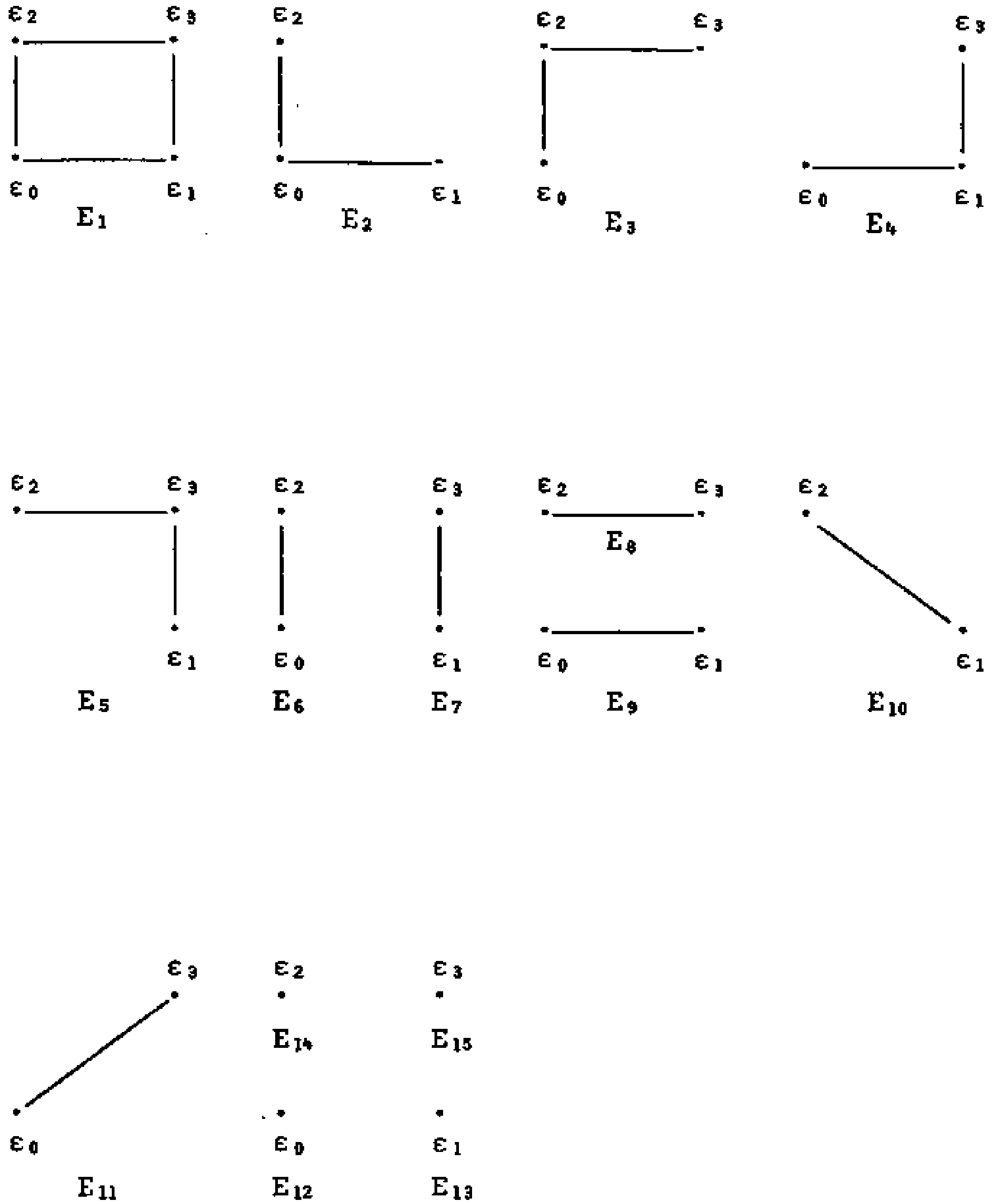

Fig. 5.1. Graphic representation of U(2)[9]

In the above figure, we have put $\varepsilon_0 = \neg p_1 \wedge \neg p_2$, $\varepsilon_1 = p_1 \wedge \neg p_2$, $\varepsilon_2 = \neg p_1 \wedge p_2$ and $\varepsilon_3 = p_1 \wedge p_2$.

Finally, since $\mathbb{B}$ is finite, from Lemma 5.4, we have

Theorem 5.17. $\iota : \mathbb{B} \longrightarrow \mathbb{C}$ *is an anti-order isomorphism.*

Corollary 5.18. *Every theory of* S5 *(over the language* L_n*) is finitely axiomatizable.*

CHAPTER 6

APPLICATIONS

In this chapter we study two puzzles, namely, the puzzle of three wise men and the puzzle of unfaithful wives, by applying the results we have obtained in the preceding chapters.

6.1. *The wise men puzzle*

In this section, as an application of the Completeness Theorem, we give a model theoretic solution to the well-known puzzle of three wise men. We will work on the language $L = (Pr, Sp, T)$, where

$$Pr = \{p_1, p_2, p_3\} ,$$
$$Sp = \{0, S_1, S_2, S_3\} ,$$
$$T = \{1\} .$$

Since T is a singleton set we will write, for example, $[S]\alpha$ in place of $[S1]\alpha$. Now, the puzzle has been modified as follows by McCarthy [21, 22] so that it may be modelled in his knowledge system:

Let S_i $(i = 1, 2, 3)$ denote the 3 wise men, and let p_i be the sentence asserting that S_i has a white spot on his forehead. The following are given as assuptions.

(A1) $p_1 \wedge p_2 \wedge p_3$ --- All spots are white.

(A2) $[O](p_1 \vee p_2 \vee p_3)$ --- They all know that there is at least one white spot.

(A3) $[O](\{S_1\}p_2 \wedge \{S_1\}p_3 \wedge \{S_2\}p_1 \wedge \{S_2\}p_3 \wedge \{S_3\}p_1 \wedge \{S_3\}p_2)$ --- They all know that each can see the spots of the others.

(A4) $[S_3][S_2]\neg[S_1]p_1$ --- S_3 knows that S_2 knows that S_1 doesn't know the color of his spot.

(A5) $[S_3]\neg[S_2]p_2$ --- S_3 knows that S_2 doesn't know the color of his spot.

The problem is to deduce $[S_3]p_3$ (S_3 knows that he has a white spot) from these assumptions.

Let $\alpha = (A1) \wedge (A2) \wedge (A3) \wedge (A4) \wedge (A5)$ and $\pi = \alpha \supset [S_3]p_3$. We will show that $\vdash \pi$ (in K3) by means of the completeness of K3-models. Namely, we show that π is valid in all K3-models. So, by way of contradiction, suppose there is a counter-model $M = \langle W; r, v \rangle$ for π such that $M \dashv \pi$. This means that there is a world $w_0 \in W$ such that

(1) $$w_0 \models \alpha$$

and

(2) $$w_0 \dashv [S_3]p_3.$$

(2) tells the existence of a world w_1 such that

(3) $$w_0 \xrightarrow{S_3} w_1$$

and

(4) $$w_1 \dashv p_3.$$

Since $w_0 \models (A4) \wedge (A5)$, we have, by (3),

$$(5) \qquad w_1 \models [S_2]\neg[S_1]p_1$$

and

$$(6) \qquad w_1 \;=\!\mid\; [S_2]p_2.$$

From (3) we have, by the definition of r,

$$(7) \qquad w_0 \xrightarrow{0} w_1.$$

Hence we have from (1)

$$(8) \qquad w_1 \models \{S_2\}p_3,$$

that is, $w_1 \models [S_2]p_3$ or $w_1 \models [S_2]\neg p_3$. This, together with (4), implies

$$(9) \qquad w_1 \models [S_2]\neg p_3.$$

By (6) we see that there is a world w_2 such that

$$(10) \qquad w_1 \xrightarrow{S_2} w_2$$

and

$$(11) \qquad w_2 \;=\!\mid\; p_2.$$

From (5), (9) and (10) we have

$$(12) \qquad w_2 \;=\!\mid\; [S_1]p_1$$

and

$$(13) \qquad w_2 \;=\!\mid\; p_3.$$

By (10), since $r(S_2, 1) \subseteq r(0, 1)$, we have

(14) $$w_1 \xrightarrow{0} w_2.$$

From (7) and (14), using the transitivity of $r(0, 1)$, we have

(15) $$w_0 \xrightarrow{0} w_2.$$

Since $w_0 \models (A3)$, we have

(16) $$w_2 \models \{S_1\}p_2 \wedge \{S_1\}p_3.$$

From (11), (13) and (16) we have

(17) $$w_2 \models [S_1]\neg p_2$$

and

(18) $$w_2 \models [S_1]\neg p_3.$$

Now, (12) implies the existence of $w_3 \in W$ such that

(19) $$w_2 \xrightarrow{S_1} w_3$$

and

(20) $$w_3 \Dashv p_1.$$

From (17), (18) and (19) we have

(21) $$w_3 \Dashv p_2$$

and

(22) $$w_3 \Dashv p_3.$$

We have

(23) $$w_0 \xrightarrow{0} w_3$$

from (15) and (19). Then, since $w_0 \models$ (A2), we have

(24) $$w_3 \models p_1 \vee p_2 \vee p_3.$$

But, this is contradictory to (20)-(22). Thus, we have proved that π is valid.

Note that we did not use the assumptions (A1) and $[0](\{S_2\}p_1 \wedge \{S_3\}p_1 \wedge \{S_3\}p_2)$. We illustrate the above inference in the following figure.

$$
\begin{tikzcd}
w_0 \arrow[r, "{S_3, O}"] & w_1 \arrow[r, "{S_2, O}"] & w_2 \arrow[r, "{S_1, O}"] & w_3
\end{tikzcd}
$$

w_0	w_1	w_2	w_3
$\neg[S_3]p_3$	$\neg p_3$		
		$\neg p_3$	
$[O]\{S_2\}p_3$	$\{S_2\}p_3$		$\neg p_3$
$[O]\{S_1\}p_3$		$\{S_1\}p_3$	
$[S_3]\neg[S_2]p_2$	$\neg[S_2]p_2$	$\neg p_2$	
			$\neg p_2$
$[O]\{S_1\}p_2$		$\{S_1\}p_2$	
$[S_3][S_2]\neg[S_1]p_1$	$[S_2]\neg[S_1]p_1$	$\neg[S_1]p_1$	$\neg p_1$
$[O](p_1 \vee p_2 \vee p_3)$			$p_1 \vee p_2 \vee p_3$

Fig. 6.1. Proof of the validity of π

For the sake of comparison, we give a formal proof of π in GT3. It may be observed that these two proofs are essentially along the same line.

$$\dfrac{\dfrac{\dfrac{p_2 \rightarrow p_2}{[S_1]p_2 \rightarrow p_2}\;([S_1]\rightarrow)}{[S_1]p_2 \rightarrow p_2,\ [S_1]\neg p_2} \qquad \dfrac{[S_1]\neg p_2 \rightarrow [S_1]\neg p_2}{[S_1]\neg p_2 \rightarrow p_2,\ [S_1]\neg p_2}}{\{S_1\}p_2 \rightarrow p_2,\ [S_1]\neg p_2}\;(\vee\rightarrow)$$

$$\dfrac{\dfrac{\dfrac{p_1 \vee p_2 \vee p_3 \rightarrow p_1,\ p_2,\ p_3}{\neg p_2,\ \neg p_3,\ p_1 \vee p_2 \vee p_3 \rightarrow p_1}\;(\neg\rightarrow)}{\neg p_2,\ \neg p_3,\ [O](p_1 \vee p_2 \vee p_3) \rightarrow p_1}\;([O]\rightarrow)}{[S_1]\neg p_2,\ [S_1]\neg p_3,\ [O](p_1 \vee p_2 \vee p_3) \rightarrow [S_1]p_1}\;(\rightarrow[S_1])_3$$

$$\dfrac{\{S_1\}p_2 \rightarrow p_2,\ [S_1]\neg p_2 \qquad [S_1]\neg p_2,\ [S_1]\neg p_3,\ [O](p_1 \vee p_2 \vee p_3) \rightarrow [S_1]p_1}{[S_1]\neg p_3,\ \{S_1\}p_2,\ [O](p_1 \vee p_2 \vee p_3) \rightarrow p_2,\ [S_1]p_1}\;(\mathrm{cut})$$

$$\dfrac{\{S_1\}p_3 \rightarrow p_3,\ [S_1]\neg p_3 \qquad [S_1]\neg p_3,\ \{S_1\}p_2,\ [O](p_1 \vee p_2 \vee p_3) \rightarrow p_2,\ [S_1]p_1}{\{S_1\}p_3,\ \{S_1\}p_2,\ [O](p_1 \vee p_2 \vee p_3) \rightarrow p_3,\ p_2,\ [S_1]p_1}\;(\mathrm{cut})$$

$$\dfrac{\{S_1\}p_3,\ \{S_1\}p_2,\ [O](p_1 \vee p_2 \vee p_3) \rightarrow p_3,\ p_2,\ [S_1]p_1}{\neg p_3,\ \neg[S_1]p_1,\ \{S_1\}p_3,\ \{S_1\}p_2,\ [O](p_1 \vee p_2 \vee p_3) \rightarrow p_2}\;(\neg\rightarrow)$$

$$\dfrac{\neg p_3,\ \neg[S_1]p_1,\ \{S_1\}p_3,\ \{S_1\}p_2,\ [O](p_1 \vee p_2 \vee p_3) \rightarrow p_2}{\neg p_3,\ \neg[S_1]p_1,\ [O]\{S_1\}p_3,\ [O]\{S_1\}p_2,\ [O](p_1 \vee p_2 \vee p_3) \rightarrow p_2}\;([O]\rightarrow)$$

$$\dfrac{\neg p_3,\ \neg[S_1]p_1,\ [O]\{S_1\}p_3,\ [O]\{S_1\}p_2,\ [O](p_1 \vee p_2 \vee p_3) \rightarrow p_2}{[S_2]\neg p_3,\ [S_2]\neg[S_1]p_1,\ [O]\{S_1\}p_3,\ [O]\{S_1\}p_2,\ [O](p_1 \vee p_2 \vee p_3) \rightarrow [S_2]p_2}\;(\rightarrow[S_2])_3$$

$$\dfrac{\{S_2\}p_3 \rightarrow p_3,\ [S_2]\neg p_3 \qquad [S_2]\neg p_3,\ [S_2]\neg[S_1]p_1,\ [O]\{S_1\}p_3,\ [O]\{S_1\}p_2,\ [O](p_1 \vee p_2 \vee p_3) \rightarrow [S_2]p_2}{\{S_2\}p_3,\ [S_2]\neg[S_1]p_1,\ [O]\{S_1\}p_3,\ [O]\{S_1\}p_2,\ [O](p_1 \vee p_2 \vee p_3) \rightarrow p_3,\ [S_2]p_2}\;(\mathrm{cut})$$

$$\dfrac{\{S_2\}p_3,\ [S_2]\neg[S_1]p_1,\ [O]\{S_1\}p_3,\ [O]\{S_1\}p_2,\ [O](p_1 \vee p_2 \vee p_3) \rightarrow p_3,\ [S_2]p_2}{\neg[S_2]p_2,\ \{S_2\}p_3,\ [S_2]\neg[S_1]p_1,\ [O]\{S_1\}p_3,\ [O]\{S_1\}p_2,\ [O](p_1 \vee p_2 \vee p_3) \rightarrow p_3}\;(\neg\rightarrow)$$

$$\dfrac{\neg[S_2]p_2,\ \{S_2\}p_3,\ [S_2]\neg[S_1]p_1,\ [O]\{S_1\}p_3,\ [O]\{S_1\}p_2,\ [O](p_1 \vee p_2 \vee p_3) \rightarrow p_3}{\neg[S_2]p_2,\ [S_2]\neg[S_1]p_1,\ [O]\{S_2\}p_3,\ [O]\{S_1\}p_3,\ [O]\{S_1\}p_2,\ [O](p_1 \vee p_2 \vee p_3) \rightarrow p_3}\;([O]\rightarrow)$$

$$\dfrac{\neg[S_2]p_2,\ [S_2]\neg[S_1]p_1,\ [O]\{S_2\}p_3,\ [O]\{S_1\}p_3,\ [O]\{S_1\}p_2,\ [O](p_1 \vee p_2 \vee p_3) \rightarrow p_3}{[S_3]\neg[S_2]p_2,\ [S_3][S_2]\neg[S_1]p_1,\ [O]\{S_2\}p_3,\ [O]\{S_1\}p_3,\ [O]\{S_1\}p_2,\ [O](p_1 \vee p_2 \vee p_3) \rightarrow [S_3]p_3}\;(\rightarrow[S_3])_3$$

Fig. 6.2. Proof of π in GT3

6.2. *The puzzle of unfaithful wives*

We begin by explaining the notions of knowledge base and knowledge set, which are fundamental for our formalization of the puzzle of unfaithful wives.

6.2.1 *Knowledge set and knowledge base*

Let L be any language. We consider in KT4 and KT5 over L. We will make the notion of the totality of one's knowledge explicit by the following definitions.

Definition 6.1. $K \subseteq Wff$ is a *knowledge set for* St if K satisfies the following conditions:

(KS1) K is consistent.

(KS2) $K = [St]\bar{K}$.

(KS3) If $K \vdash [St]\alpha_1 \vee \cdots \vee [St]\alpha_n$ then $K \vdash \alpha_i$ for some i $(1 \leq i \leq n)$.

Definition 6.2. $B \subseteq Wff$ is a *knowledge base for* St if B satisfies the following conditions:

(KB1) B is consistent.

(KB2) $B \subseteq [St]\bar{B}$.

(KB3) If $B \vdash [St]\alpha_1 \vee \cdots \vee [St]\alpha_n$ then $B \vdash \alpha_i$ for some i $(1 \leq i \leq n)$.

By (KS2) (or (KB2)) we see that any element in K (or B, resp.) has the form $[St]\alpha$. It is easy to see that if B is a knowledge base for St then $[St]\bar{B}$ is a knowledge set for St. We also note that the above definitions are relative to the logics KT4 and KT5.

Let $\Gamma \subseteq Wff$ be consistent. We compare the following three conditions.

(1) If $\Gamma \not\vdash \alpha$ then $\Gamma \vdash \neg[St]\alpha$.

(2) If $\Gamma \vdash [St]\alpha_1 \vee \cdots \vee [St]\alpha_n$ then $\Gamma \vdash \alpha_i$ for some i $(1 \leq i \leq n)$.

(3) If $\Gamma \vdash \{St\}\alpha$ then $\Gamma \vdash \alpha$ or $\Gamma \vdash \neg\alpha$.

First, we consider in KT4.

Lemma 6.3. *In* KT4, *we have* (1) => (2) => (3) *but* (2) $\not\Rightarrow$ (1).

Proof. (1) => (2): Suppose $\Gamma \vdash [St]\alpha_1 \vee \cdots \vee [St]\alpha_n$ and $\Gamma \not\vdash \alpha_i$ for any i. Then by (1), we have $\Gamma \vdash \neg[St]\alpha_i$ for any i. Then we can prove $\Gamma \vdash \bot$, which is contradictory to the consistency of Γ.

(2) => (3): Trivial.

(2) $\not\Rightarrow$ (1): Since the disjunction property holds in KT4 (Theorem 3.12), the empty set $\emptyset$ is a knowledge base for any St. Let $\Gamma = \emptyset$. Then Γ satisfies (2). Let $p \in Pr$.[10] Then neither p nor $\neg[St]p$ is provable in KT4. Hence, Γ does not satisfy (1).

In KT5, we have the following

Lemma 6.4. *In* KT5, (1), (2), *and* (3) *are equivalent.*

Proof. (1) => (2) => (3) are proved similarly as in Lemma 6.3.

(3) => (1): We prove the contraposition of (1) assuming (3). Suppose $\Gamma \nvdash \neg[St]\alpha$. Since $\vdash [St][St]\alpha \vee [St]\neg[St]\alpha$ in KT5, we have from (3), $\Gamma \vdash [St]\alpha$. Hence $\Gamma \vdash \alpha$.

Note that $\emptyset$ is not a knowledge base in KT5. We now study the semantical characterization of knowledge sets. Let $M = \langle W; r, v \rangle$ be any model (adequate for the logical system we have in mind). For any $w \in W$ and $(S, t) \in Sp \times T$, we define $K_w(St) \subseteq Wff$ by:

$$K_w(St) = \{[St]\alpha \mid w \models [St]\alpha\}.$$

Since, as we will see below, $K_w(St)$ is a knowledge set for St, we call it the *knowledge set for* St *at* w.

Lemma 6.5. $K_w(St)$ *is a knowledge set for* St.

Proof. We only prove (KS2). Let $[St]\alpha \in K_w(St) = K$. Then, we have $K \vdash \alpha$, i.e., $\alpha \in \bar{K}$. Hence $[St]\alpha \in [St]\bar{K}$. Let $[St]\alpha \in [St]\bar{K}$. Then $\alpha \in \bar{K}$, i.e., $K \vdash \alpha$. Since any element in K is of the form $[St]\beta$, and the logical system is KT4 or KT5, we have $K \vdash [St]\alpha$. Since $w \models K$, we have $w \models [St]\alpha$, so that $[St]\alpha \in K$.

Let K be a knowledge set for St. We say $w \in M$ *characterizes* K if $K = K_w(St)$.

Theorem 6.6. *Any knowledge set is characterizable.*

Proof. Let K be a knowledge set. Let $\Delta = \text{Wff-}K_{St}$. We show that the sequent $K \to [St]\Delta$ is consistent. Suppose otherwise, so that $\vdash K \to [St]\Delta$. Then for some finite set $\{\alpha_1, \cdots, \alpha_n\} \subseteq \Delta$ we have, $\vdash K \to [St]\alpha_1, \cdots, [St]\alpha_n$. Hence, by (KS3), there exists an i $(1 \leq i \leq n)$ such that $\vdash K \to \alpha_i$. By (KS2), we have $[St]\alpha_i \in K$. This is a contradiction. Thus, $K \to [St]\Delta$ is consistent. So, by the Generalized Completeness Theorem, we can take a model $M = \langle W; r, v\rangle$ such that $w \dashv K \to [St]\Delta$, for some $w \in W$. Then, clearly, we have $K = K_w(St)$.

6.2.2 *Informal presentation of the puzzle*

The puzzle of unfaithful wives is usually stated like this:

There was a country in which one million married couples inhabited. Among these one million wives, 40 wives were unfaithful. The situation was that each husband knew whether other men's wives are unfaithful but he did not know whether his wife is unfaithful. One day (call it the 1^{st} day), the King of the country publicized the following order:

(i) There is at least one unfaithful wife.

(ii) Each husband knows whether other men's wives are unfaithful or not.

(iii) Every night (from tonight) each man must do his deduction, based on his knowledge so far, and try to prove whether his wife is unfaithful or not.

(iv) Each man, who has succeeded in proving that his wife is unfaithful, must chop off his wife's head next morning.

(v) Every morning each man must see whether somebody chops off his wife's head.

(vi) Each man's knowledge before this order is publicized consists only of the knowledge about other men's wive's unfaithfulness.

The problem is "what will happen under this situation?" The answer is that on the 41^{st} day 40 unfaithful wives will be chopped off their heads. We will treat this puzzle in a formal manner.

6.2.3 *Formal treatment of the puzzle*

We will treat this puzzle by assuming that there are k (≥ 1) married couples in the country. Then the language

$L = (Pr, Sp, T)$ adequate for this puzzle will be:

$$Pr = \{p_1, \cdots, p_k\},$$
$$Sp = \{0, S_1, \cdots, S_k\},$$
$$T = \mathbb{N}^+,$$

where S_i denotes i^{th} husband, p_i means that S_i's wife is unfaithful and $t \in T$ denotes t^{th} day. We employ KT5 over L as our logical system. (Our argument henceforth can be carried out similarly in KT4 except for one point, where an essential use of Lemma 6.4 is necessary. This fact seems to suggest us that the negative introspective character of KT5 is essential for the solution of the puzzle.)

As in §5.2, we define

$$\pi : \{\pm\}^k \longrightarrow \mathrm{Wff}$$

by $\pi(\varepsilon_1 \cdots \varepsilon_k) = \bigwedge_{i=1}^{k} p_i^{\varepsilon_i}$. We put $\Pi = \mathrm{Image}(\pi)$ and $\Pi_0 = \Pi - \{\bigwedge_{i=1}^{k} \bar{p}_i\}$, where $\bar{p}_i = \neg p_i$. We also use π to denote arbitrary element in Π. Now, let Γ denote what the King publicized on the 1^{st} day, and $B_\pi(S_i n)$ $(i = 1, \cdots, k)$ denote a knowledge base for $S_i n$ under the circumstance $\pi = \pi(\varepsilon_1 \cdots \varepsilon_k) \in \Pi_0$. Let us put

$$\lceil B_\pi(S_i n) \vdash \alpha \rceil = \begin{cases} \top & \text{if } B_\pi(S_i n) \vdash \alpha \\ \bot & \text{otherwise} \end{cases}$$

and

$$\ulcorner B_\pi(S_i n) \nvdash \alpha \urcorner = \begin{cases} \top & \text{if } B_\pi(S_i n) \nvdash \alpha \\ \bot & \text{otherwise} \end{cases},$$

where $\alpha \in \mathrm{Wff}$. Then, as a formalization of the puzzle, we postulate the following identities:

$$B_\pi(S_i 1) = [S_i 1]\Gamma \cup \{[S_i 1]p_j^{\varepsilon_j} \mid j \neq i,\ j = 1, \cdots, k\} \qquad \cdots\cdot\ Eq(\pi, i, 1)$$

$$B_\pi(S_i n+1) = [S_i n+1]B_\pi(S_i n) \cup \{[S_i n+1][S_j n]p_j \mid B_\pi(S_j n) \vdash p_j,\ j = 1, \cdots, k\} \cup$$
$$\{[S_i n+1]\neg[S_j n]p_j \mid B_\pi(S_j n) \nvdash p_j,\ j = 1, \cdots, k\} \qquad \cdots\cdot\ Eq(\pi, i, n+1)$$

$$\Gamma = \{[O1] \bigvee_{i=1}^{k} p_i\} \cup \{[O1]\{S_i 1\}p_j \mid j \neq i,\ i = 1, \cdots, k,\ j = 1, \cdots, k\}$$
$$\cup\{[O1](\pi \supset (\ulcorner B_\pi(S_i n) \vdash p_i \urcorner \supset [On+1][S_i n]p_i)) \mid \pi \in \Pi_0,\ i = 1, \cdots, k,\ n \in T\}$$
$$\cup\{[O1](\pi \supset (\ulcorner B_\pi(S_i n) \nvdash p_i \urcorner \supset [On+1]\neg[S_i n]p_i)) \mid \pi \in \Pi_0,\ i = 1, \cdots, k,\ n \in T\}$$
$$\cup\{[O1](\ulcorner B_\pi(S_i n) \vdash \alpha \urcorner \supset [O1](\pi \supset [S_i n]\alpha)) \mid \pi \in \Pi_0,\ i = 1, \cdots, k,\ \alpha \in Wff\} \qquad \cdots\cdot\ Eq(*)$$

The informal meanings of the above equations are as follows:

Eq(π, i, 1): Knowledge base for S_i1 under π consists of the knowledge about what the King says on the 1^{st} day and the knowledge about whether other men's wives are unfaithful.

Eq(π, i, n+1): If S_j could prove p_j in the n^{th} night, then S_i knows on the $n+1^{st}$ morning that $[S_jn]p_j$, since S_i sees that S_j chops off his wife's head in the $n+1^{st}$ morning. If S_j could not prove p_j in the n^{th} night, then S_i knows in the $n+1^{st}$ morning that $\neg[S_jn]p_j$, since S_i sees that S_j does not chop off his wife's head in the $n+1^{st}$ morning.

Eq(*): The meaning of the 1^{st} line of Eq(*) should be clear. The 2^{nd} and 3^{rd} lines mean that FOOL will know every morning whether anybody could prove the unfaithfulness of his wife in the previous night. The last line is an indirect definition of $B_\pi(S_in)$.

Since the meta-notions such as knowledge base and provability ($\vdash$) cannot be expressed directly in our language, we were forced to interpret the King's order into Γ in a somewhat indirect fashion.

Now, if we read Eq(*) as the definition of Γ, then we find that the definition is circular, since in order that Γ

may be definable by (*) it is necessary that $B_\pi(S_i n)$ are already defined, whears $B_\pi(S_i n)$ are defined in terms of Γ in Eqs(π, i, n). So, we will treat these equations as a system $\$ = \{Eq(\pi, i, n) \mid \pi \in \Pi_0, i = 1, \cdots, k, n \in T\} \cup \{Eq(*)\}$ of equations with the unkonwns $\{B_\pi(S_i n) \mid \pi \in \Pi_0, i = 1, \cdots, k, n \in T\}$ and Γ. We will solve $\$$ under the following conditions:

(#) For any $\pi \in \Pi_0$, $\Gamma \cup \{\pi\}$ is consistent.

(##) For any $\pi \in \Pi_0$ and $S_i n$, $B_\pi(S_i n)$ is a knowledge base for $S_i n$.

We think these conditions are natural in view of the intended meanings of Γ and $B_\pi(S_i n)$.

For the sake of notational convenience, we consider $E = \{\pm\}^k$ as a k-fold direct product of the vector space $GF(2) = \{+ (= 1), - (= 0)\}$ with addition $\oplus$. Thus, $\{e_i = - \cdots -\underset{\hat{i}}{+}- \cdots - \mid i = 1, \cdots, k\}$ forms a basis of E. We define a norm on E by $\|\varepsilon\| = |\{i \mid \varepsilon_i = +\}|$, where $\varepsilon = \varepsilon_1 \cdots \varepsilon_k$.[11)] For any $\varepsilon = \varepsilon_1 \cdots \varepsilon_k \in E$ and $i = 1, \cdots, k$, we put

$$\varepsilon(+i) = \varepsilon_1 \cdots \varepsilon_{i-1}{+}\varepsilon_{i+1} \cdots \varepsilon_k,$$

$$\varepsilon(-i) = \varepsilon_1 \cdots \varepsilon_{i-1}{-}\varepsilon_{i+1} \cdots \varepsilon_k,$$

and for any $\pi = \pi(\varepsilon) \in \Pi$, we put

$$\pi(+i) = \pi(\varepsilon(+i)),$$

$$\pi(-i) = \pi(\varepsilon(-i)).$$

We also put $E_0 = E-\{0\} = E-\{- \cdots -\}$.

Now, let us suppose that $\langle\langle B_\pi(S_i n) \mid \pi \in \Pi_0,\ i = 1,\cdots, k,\ n \in T\rangle, \Gamma\rangle$ is a solution of \$ under the conditions (#) and (##). Then the following lemma holds.

Lemma 6.7. *Let* $\pi = \pi(\varepsilon) \in \Pi$ *and* $n \in T$. *Then we have:*

(i) *If* $n \geq \|\varepsilon(+i)\|$ *then*

$$B_{\pi(+i)}(S_i n) \vdash p_i$$

and

$$B_{\pi(-i)}(S_i n) \vdash \bar{p}_i \qquad (\textit{if } \pi(-i) \in \Pi_0).$$

(ii) *If* $n < \|\varepsilon(+i)\|$ *then*

$$B_{\pi(+i)}(S_i n) = B_{\pi(-i)}(S_i n),$$

and hence

$$B_{\pi(+i)}(S_i n) \nvdash p_i$$

and

$$B_{\pi(-i)}(S_i n) \nvdash \bar{p}_i.$$

Proof. We first show that $B_{\pi(+i)}(S_i n) = B_{\pi(-i)}(S_i n)$

implies $B_{\pi(+i)}(S_i n) \nvdash p_i$ and $B_{\pi(-i)}(S_i n) \nvdash \overline{p_i}$. Suppose $B_{\pi(+i)}(S_i n) \vdash p_i$. Then $B_{\pi(-i)}(S_i n) \vdash p_i$. Hence $[01](\pi(-i) \supset (\top \supset [0n+1][S_i n]p_i) \in \Gamma$. So,

(1) $$\Gamma \vdash \pi(-i) \supset p_i.$$

On the other hand,

(2) $$\pi(-i) \vdash \overline{p_i}.$$

From (1) and (2), we have

(3) $$\pi(-i), \Gamma \vdash \bot .$$

This is contradictory to the condition (#). Therefore we have $B_{\pi(+i)}(S_i n) \nvdash p_i$. $B_{\pi(-i)}(S_i n) \nvdash \overline{p_i}$ is proved similarly.

We now prove the lemma by induction on n.

$n = 1$:

Proof of (i). Suppose $\|\varepsilon(+i)\| = 1$. Then, since

$$\vdash \overline{p_1}, \cdots, \overline{p_{i-1}}, \overline{p_{i+1}}, \cdots, \overline{p_k}, \bigvee_{i=1}^{k} p_i \to p_i,$$

$$B_{\pi(+i)}(S_i 1) \vdash [S_i 1]\overline{p_j} \qquad (j \neq i)$$

and

$$B_{\pi(+i)}(S_i 1) \vdash \bigvee_{i=1}^{k} p_i ,$$

we have $B_{\pi(+i)}(S_i 1) \vdash p_i$. The rest of (i) is vacuously true, since $\pi(-i) \in \Pi_0$.

Proof of (ii). Suppose $\|\varepsilon(+i)\| > 1$. Then,

$B_{\pi(+i)}(S_i 1) = B_{\pi(-i)}(S_i 1)$ follows directly from $Eq(\pi(+i), i, 1)$ and $Eq(\pi(-i), i, 1)$.

$n > 1$:

Proof of (i). First we show $B_{\pi(+i)}(S_i n) \vdash p_i$ from the assumption that $n = \|\varepsilon(+i)\|$. Since $n > 1$, we can take $j \neq i$ such that $\varepsilon_j = +$. Then $\pi(+i) = \pi(+i)(+j)$ and $\|\varepsilon(+i)(+j)\| = n > n-1$. By induction hypothesis, we therefore get $B_{\pi(+i)}(S_j n-1) \nvdash p_j$. Hence,

(4) $$[S_i n]\neg[S_j n-1]p_j \in B_{\pi(+i)}(S_j n).$$

On the other hand, since $\pi(-i) = \pi(-i)(+j)$ and $\|\varepsilon(-i)(+j)\| = n-1$, we have by induction hypothesis, $B_{\pi(-i)}(S_j n-1) \vdash p_j$. Hence, by $Eq(*)$

(5) $$[O1](\pi(-i)\supset(\top\supset[On][S_j n-1]p_j)) \in \Gamma.$$

From (4), (5) and $Eq(\pi(+i), i, n)$, we have $B_{\pi(+i)}(S_i n) \vdash \neg\pi(-i)$. Since $B_{\pi(+i)}(S_i 1) \vdash \pi(+i)\vee\pi(-i)$ and $B_{\pi(+i)}(S_i n) \supseteq [S_i n] \cdots [S_i 2]B_{\pi(+i)}(S_i 1)$, we have $B_{\pi(+i)}(S_i n) \vdash \pi(+i)\vee\pi(-i)$. Hence we have $B_{\pi(+i)}(S_i n) \vdash \pi(+i)$. Therefore, $B_{\pi(+i)}(S_i n) \vdash p_i$.

We next show that $B_{\pi(-i)}(S_i n) \vdash \overline{p_i}$ from the assumption that $n = \|\varepsilon(+i)\|$. We can take $j \neq i$ such that $\varepsilon_j = +$. Then $\|\varepsilon(-i)(+j)\| = n-1$. By induction hypothesis, $B_{\pi(-i)}(S_j n-1) \vdash p_j$. Hence,

(6) $$[S_i n][S_j n-1]p_j \in B_{\pi(-i)}(S_i n).$$

Since $\|\varepsilon(+i)(+j)\| = n$, we have by induction hypothesis, $B_{\pi(+i)} \nvdash p_j$. Hence,

$$(7) \qquad [01](\pi(+i)\supset(\tau\supset[0n]\neg[S_j n-1]p_j)) \in \Gamma.$$

From (6) and (7), by an arguement similar as above, we conclude that $B_{\pi(-i)}(S_i n) \vdash \overline{p_i}$.

The case $n > \|\varepsilon(+i)\|$ is now easy, since we have

$$B_\pi(S_i m+1) \supseteq [S_i m+1]B_\pi(S_i m),$$

for any m.

Proof of (ii). We next consider the case $n < \|\varepsilon(+i)\|$. By induction hypothesis, $B_{\pi(+i)}(S_i n-1) = B_{\pi(-i)}(S_i n-1)$. Since $\|\varepsilon(+i)(+j)\| \geq \|\varepsilon(-i)(+j)\| > n-1$ for any j, we have by induction hypothesis,

$$B_{\pi(+i)(+j)}(S_j n-1) = B_{\pi(+i)(-j)}(S_j n-1)$$

and

$$B_{\pi(-i)(+j)}(S_j n-1) = B_{\pi(-i)(-j)}(S_j n-1).$$

Hence $B_{\pi(+i)}(S_j n-1) \nvdash p_j$ and $B_{\pi(-i)}(S_j n-1) \nvdash p_j$. Thus, we have $B_{\pi(+i)}(S_i n) = B_{\pi(-i)}(S_i n)$ by $Eq(\pi(+i), i, n)$ and $Eq(\pi(-i), i, n)$.

Summarizing this lemma, we have:

Corollary 6.8. $B_{\pi(\varepsilon)}(S_i n) \vdash p_i$ *if and only if* $\varepsilon_i = +$ *and* $n \geq \|\varepsilon\|$.

We next prove the following lemma.

Lemma 6.9. *For any* $\pi = \pi(\varepsilon) \in \Pi_0'$, $\{\pi\}\cup\Gamma$ *is complete. I.e., for any* $\alpha \in \mathrm{Wff}$, *either*

$$\vdash \pi, \Gamma \to \alpha$$

or

$$\vdash \alpha, \pi, \Gamma \to \quad .$$

Proof. By induction on the construction of α. First we note that, by condition (#), it is impossible that both $\pi, \Gamma \to \alpha$ and $\alpha, \pi, \Gamma \to$ are provable.

$\alpha \in \mathrm{Pr}\cup\{\perp\}$:

If $\alpha = p_i$ then we have $\pi \vdash p_i^{\varepsilon_i}$. Hence, clearly, $\vdash \pi, \Gamma \to \alpha$ or $\vdash \alpha, \pi, \Gamma \to$. If $\alpha = \perp$ then we have $\vdash \perp, \pi, \Gamma \to$.

$\alpha = \beta\supset\gamma$:

Suppose $\vdash \pi, \Gamma \to \gamma$. Then we have $\vdash \pi, \Gamma \to \alpha$ by the following proof figure:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi, \Gamma \to \gamma\end{array}}{\beta, \pi, \Gamma \to \gamma}}{\pi, \Gamma \to \beta\supset\gamma}$$

Suppose $\vdash \beta, \pi, \Gamma \to$. Then we have $\vdash \pi, \Gamma \to \alpha$, similarly.

By induction hypothesis, we see that the remaining case is $\vdash \pi, \Gamma \to \beta$ and $\vdash \gamma, \pi, \Gamma \to$. Then, we have $\vdash \beta \supset \gamma, \pi, \Gamma \to$ by $(\supset\to)$.

$\alpha = [S_i n]\beta$:

Suppose $\vdash \beta, \pi, \Gamma \to$. Then we can costruct the following proof:

$$\dfrac{\begin{array}{c}\vdots\\ \beta, \pi, \Gamma \to\end{array}}{[S_i n]\beta, \pi, \Gamma \to}$$

Suppose $\vdash \pi, \Gamma \to \beta$.

(A) We first consider the case $n \geq \| \varepsilon(+i) \|$.

(A1) The case $\pi = \pi(+i)$:

In this case, noting that $[01](\pi(+i) \supset (\top \supset [0n+1][S_i n]p_i)) \in \Gamma$ by Lemma 6.7, we first construct the following proof figure.

(1)
$$\dfrac{\dfrac{\pi(+i) \to \pi(+i) \qquad \dfrac{\dfrac{\bot \to}{\to \top} \qquad \dfrac{[S_i n]p_i \to [S_i n]p_i}{[0n+1][S_i n]p_i \to [S_i n]p_i}}{\top \supset [0n+1][S_i n]p_i \to [S_i n]p_i}}{\dfrac{\pi(+i) \supset (\top \supset [0n+1][S_i n]p_i), \ \pi(+i) \to [S_i n]p_i}{[01](\pi(+i) \supset (\top \supset [0n+1][S_i n]p_i)), \ \pi(+i) \to [S_i n]p_i}}}{\pi(+i), \Gamma \to [S_i n]p_i}$$

Let $j \neq i$. Then, since $[01]\{S_i1\}p_j \in \Gamma$, we have the following proof figure.

(2)
$$\dfrac{\dfrac{\dfrac{\dfrac{p_j^{\varepsilon_j} \to p_j^{\varepsilon_j}}{[S_i1]p_j^{\varepsilon_j} \to [S_in]p_j^{\varepsilon_j}} \qquad \dfrac{\dfrac{\dfrac{p_j^{\varepsilon_j} \to p_j^{\varepsilon_j}}{\pi(+i) \to p_j^{\varepsilon_j}}}{\neg p_j^{\varepsilon_j},\ \pi(+i) \to}}{[S_i1]\neg p_j^{\varepsilon_j},\ \pi(+i) \to}}{\pi(+i),\ [S_i1]p_j^{\varepsilon_j} \vee [S_i1]\neg p_j^{\varepsilon_j} \to [S_in]p_j^{\varepsilon_j}}}{\pi(+i),\ [01]\{S_i1\}p_j \to [S_in]p_j^{\varepsilon_j}}}{\pi(+i),\ \Gamma \to [S_in]p_j^{\varepsilon_j}}$$

From (1) and (2) we have

(3) $$\vdash \pi(+i),\ \Gamma \to [S_in]\pi(+i).$$

(A2) The case $\pi = \pi(-i)$:

We treat the critical case of $n = \|\varepsilon(+i)\|$. Then we see $\|\varepsilon(-i)\| = n-1 \geq 1$, since $\pi(-i) = \pi \in \Pi_0$. So, we can take $j \neq i$ such that $\varepsilon_j = +$. Then, since $\|\varepsilon(+i)(+j)\| = n$ and $\|\varepsilon(-i)(+j)\| = n-1$, we have

$$[01](\pi(+i) \supset (\tau \supset [0n]\neg[S_jn-1]p_j) \in \Gamma$$

and

$$[01](\pi(-i) \supset (\tau \supset [0n][S_jn-1]p_j) \in \Gamma.$$

Hence we obtain the following proof figure.

$$\dfrac{\dfrac{\dfrac{\vdots}{\pi(-i),\ \Gamma \to [S_i n][S_j n-1]p_j} \quad \dfrac{\vdots}{\Gamma \to [S_i n](\pi(+i) \supset \neg [S_j n-1]p_j)}}{\pi(-i),\ \Gamma \to [S_i n]\neg\pi(+i)} \qquad \dfrac{\vdots}{\pi(-i),\ \Gamma \to [S_i n](\pi(+i) \vee \pi(-i))}}{\pi(-i),\ \Gamma \to [S_i n]\pi(-i)}$$

From the above proof, for any $n \geq \| \varepsilon(+i) \|$, it follows that

(4) $$\vdash \pi(-i),\ \Gamma \to [S_i n]\pi(-i).$$

Since $\pi = \pi(+i)$ or $\pi = \pi(-i)$, we have from (3) and (4),

(5) $$\vdash \pi,\ \Gamma \to [S_i n]\pi.$$

Using (5), we obtain the desired proof figure:

$$\dfrac{\dfrac{(5)}{\pi,\ \Gamma \to [S_i n]\pi} \qquad \dfrac{\dfrac{\dfrac{\vdots}{\pi,\ \Gamma \to \beta}}{[S_i n]\pi,\ \Gamma \to \beta}}{[S_i n]\pi,\ \Gamma \to [S_i n]\beta}}{\pi,\ \Gamma \to [S_i n]\beta}$$

(B) We next consider the case $n < \| \varepsilon(+i) \|$.

Let $\varepsilon' = \varepsilon \oplus e_i$. Then, by induction hypothesis, we have the following two cases.

(B1) $\vdash \pi(\varepsilon'),\ \Gamma \to \beta$:

The following proof figure takes care of this case.

$$\dfrac{\begin{array}{c}\vdots\\ \pi,\ \Gamma \to [S_in](\pi(\varepsilon)\vee\pi(\varepsilon'))\end{array} \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \pi(\varepsilon),\ \Gamma \to \beta\end{array} \qquad \begin{array}{c}\vdots\\ \pi(\varepsilon'),\ \Gamma \to \beta\end{array}}{\pi(\varepsilon)\vee\pi(\varepsilon'),\ \Gamma \to \beta}}{[S_in](\pi(\varepsilon)\vee\pi(\varepsilon')),\Gamma \to \beta}}{[S_in](\pi(\varepsilon)\vee\pi(\varepsilon')),\Gamma \to [S_in]\beta}}{\pi,\ \Gamma \to [S_in]\beta}$$

(B2) $\vdash\ \beta,\ \pi(\varepsilon'),\ \Gamma \to$:

We first show that

(6) $$\vdash\ \pi,\ \Gamma \to \langle S_in\rangle\pi(\varepsilon').$$

Suppose $\pi = \pi(+i)$. Then, by Lemma 6.7, we have $B_\pi(S_in) \not\vdash p_i$. Since $B_\pi(S_in)$ is a knowledge base by condition (##), we have $B_\pi(S_in) \vdash \neg[S_in]p_i$ by Lemma 6.4. (Note that we are considering in KT5. Here we remark that this is the only point where we use the assumption that our logical system is KT5.) Then by Eq(*), we see that

$$[01](\tau\supset[01](\pi\supset[S_in]\neg[S_in]p_i)) \in \Gamma.$$

Hence we have

(7) $$\vdash\ \pi,\ \Gamma \to \langle S_in\rangle\neg p_i.$$

Now, for any $\sigma,\ \tau \in$ Wff we have

(8) $$\vdash\ \langle S_in\rangle\sigma,\ [S_in]\tau \to \langle S_in\rangle(\sigma\wedge\tau)$$

as can be seen from the following proof figure.

$$\cfrac{\cfrac{\cfrac{\sigma,\ \tau \to \sigma\wedge\tau}{\neg(\sigma\wedge\tau),\ \tau \to \neg\sigma}}{[S_i n]\neg(\sigma\wedge\tau),\ [S_i n]\tau \to [S_i n]\neg\sigma}}{\langle S_i n\rangle\sigma,\ [S_i n]\tau \to \langle S_i n\rangle(\sigma\wedge\tau)}$$

Now we can obtain (6) from (2), (7) and (8) (where we put $\sigma = \neg p_i$ and $\tau = \bigwedge_{j\neq i} p_j^{\varepsilon_j}$). The case $\pi = \pi(-i)$ may be treated similarly.

We can then construct the following proof figure:

$$\cfrac{\begin{array}{c}(6)\\ \pi,\ \Gamma \to \langle S_i n\rangle\pi(\varepsilon')\end{array} \qquad \cfrac{\cfrac{\cfrac{\cfrac{\begin{array}{c}\vdots\\ \beta,\ \pi(\varepsilon'),\ \Gamma \to\end{array}}{\beta,\ \Gamma \to \neg\pi(\varepsilon')}}{[S_i n]\beta,\ \Gamma \to \neg\pi(\varepsilon')}}{[S_i n]\beta,\ \Gamma \to [S_i n]\neg\pi(\varepsilon')}}{\langle S_i n\rangle\pi(\varepsilon'),\ [S_i n]\beta,\ \Gamma \to}}{[S_i n]\beta,\ \pi,\ \Gamma \to}$$

$\alpha = [On]\beta$:

If $\vdash \beta, \pi, \Gamma \to$, then we have $\vdash [On]\beta, \pi, \Gamma \to$ by $([On]\to)$. So, suppose $\vdash \pi, \Gamma \to \beta$. Then we have the following two cases (C) and (D).

(C) The case $n \geq \max\{\|\varepsilon(+i)\| \mid i = 1, \cdots, k\}$.

As in (A2) it is sufficient to prove the critical case of $n = \max\{\|\varepsilon(+i)\| \mid i = 1, \cdots, k\}$. Let us put $I(\varepsilon) = \{i \mid \varepsilon_i = +\}$.

(C1) The case $I(\varepsilon) \neq \{1, 2, \cdots, k\}$:

In this case, we have $n = \|\varepsilon\| + 1$. Consider any i such that $\varepsilon_i = +$. Then we have $\pi = \pi(+i)$, and since $n-1 \geq \|\varepsilon\| = \|\varepsilon(+i)\|$, we have $B_\pi(S_i n-1) \vdash p_i$ by Lemma 6.7. Hence we have

$$[01](\pi \supset (\tau \supset [0n][S_i n-1]p_i)) \in \Gamma.$$

So, we have

$$(9) \qquad \vdash \pi, \Gamma \to [0n][S_i n-1]p_i \quad (\text{if } \varepsilon_i = +)$$

and hence

$$(10) \qquad \vdash \pi, \Gamma \to [0n]p_i \quad (\text{if } \varepsilon_i = +).$$

Let $D = \{\delta \in \{\pm\}^k \mid I(\varepsilon) \subseteq I(\delta)\}$. Then, by (10) we have

$$(11) \qquad \vdash \pi, \Gamma \to [0n] \bigvee_{\delta \in D} \pi(\delta).$$

Now, take any $\delta \in D-\{\varepsilon\}$. Then we have $\|\delta\| > \|\varepsilon\| = n-1$. Since $\pi(\varepsilon) \in \Pi_0$, we can take an i such that $\varepsilon_i = +$. Then we have $\delta = \delta(+i)$. Since $\|\delta\| > n-1$, we have $B_{\pi(\delta)}(S_i n-1) \nvdash p_i$, by Lemma 6.7. Hence, we have

$$[01](\pi(\delta) \supset (\tau \supset [0n] \neg [S_i n-1]p_i)) \in \Gamma.$$

From this, together with (9), we have the following proof figure.

(12)

$$\dfrac{\dfrac{\overset{(9)}{\pi,\ \Gamma \to [On][S_i n-1]p_i} \qquad \overset{\vdots}{\Gamma \to [On](\pi(\delta) \supset \neg [S_i n-1]p_i)}}{\pi,\ \Gamma \to [On]([S_i n-1]p_i \wedge \pi(\delta) \supset \neg [S_i n-1]p_i)}}{\pi,\ \Gamma \to [On]\neg \pi(\delta)}$$

From (11) and (12), we have

(13) $\qquad \vdash \pi,\ \Gamma \to [On]\pi.$

(C2) The case $I(\varepsilon) = \{1, 2, \cdots, k\}$:

In this case, we have $\varepsilon = + \cdots +$ and $n = \|\varepsilon\|$ $(= k)$. Let $\delta \in E_0 - \{\varepsilon\}$. We can find an i such that $\delta_i = +$. Then we have $n-1 \geq \|\delta\| = \|\delta(+i)\|$. Hence, by Lemma 6.7, we have $B_{\pi(\delta)}(S_i n-1) \vdash p_i$. Hence, we have

(14) $\qquad [O1](\pi(\delta) \supset (\tau \supset [On][S_i n-1]p_i)) \in \Gamma.$

On the other hand, since $n-1 < \|\varepsilon\| = \|\varepsilon(+i)\|$, applying Lemma 6.7, we get $B_\pi(S_i n-1) \nvdash p_i$. So, we have

$$[O1](\pi \supset (\tau \supset [On]\neg [S_i n-1]p_i)) \in \Gamma.$$

Hence, we have

(15) $\vdash \pi, \Gamma \to [On]\neg[S_i n-1]p_i.$

From (14) and (15), similarly as in (12), we obtain

(16) $\vdash \pi, \Gamma \to [On]\neg\pi(\delta)$ (if $\delta \in E_0-\{\varepsilon\}$).

By (16), tohether with the fact that $\vdash \Gamma \to [On] \bigvee_{\delta\in E_0} \pi(\delta)$, we have

(17) $\vdash \pi, \Gamma \to [On]\pi.$

Now, by the results of (C1) and (C2), we can construct the following proof figure:

$$\dfrac{\begin{matrix}(13) \text{ or } (17)\\ \pi, \Gamma \to [On]\pi\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ \pi, \Gamma \to \beta\end{matrix}}{[On]\pi, \Gamma \to \beta}}{[On]\pi, \Gamma \to [On]\beta}}{\pi, \Gamma \to [On]\beta}$$

(D) The case $n < \max\{\|\varepsilon(+i)\| \mid i = 1, \cdots, k\}$.

Let $D = \{\delta \in E_0 \mid n < \max\{\|\delta(+i)\| \mid i = 1, \cdots, k\}\}$. Take any $\delta \in E_0-D$ and choose an i such that $\delta_i = +$. Then since $k > n$ by assumption, we have $n \geq \max\{\|\delta(+i)\| \mid i = 1, \cdots, k\} > \|\delta\| = \|\delta(+i)\|$. Hence, we have

$$B_{\pi(\delta)}(S_i n-1) \vdash p_i$$

so that

$$(18) \qquad [01](\pi(\delta)\supset(\tau\supset[0n][S_i n-1]p_i)) \in \Gamma.$$

On the other hand, we have

$$B_\pi(S_i n-1) \not\models p_i$$

regardless of $\pi = \pi(+i)$ or $\pi = \pi(-i)$, so that

$$(19) \qquad [01](\pi\supset(\tau\supset[0n]\neg[S_i n-1]p_i)) \in \Gamma.$$

From (18) and (19), we have

$$(20) \qquad \vdash \pi, \Gamma \to [0n]\neg\pi(\delta) \quad (\text{if } \delta \in E_0-D).$$

From this, we have

$$(21) \qquad \vdash \pi, \Gamma \to [0n]\bigvee_{\delta\in D} \pi(\delta).$$

Next, let $\delta \in D$. Then we can find $\gamma^1, \cdots, \gamma^m \in D$ such that $\gamma^1 = \varepsilon$, $\gamma^m = \delta$ and $\|\gamma^i \oplus \gamma^{i+1}\| = 1$ $(i = 1, \cdots, m-1)$. Now, take any i such that $1 \le i \le m-1$. Let $\gamma^i \oplus \gamma^{i+1} = e_j$. Then we have $\gamma^i = \gamma^i(+j)$ or $\gamma^i = \gamma^i(-j)$. Suppose, first, $\gamma^i = \gamma^i(+j)$. Then $\gamma^{i+1} = \gamma^i \oplus e_j = \gamma^i(-j)$. Since $\gamma^{i+1} \in D$, we have $n < \max\{\|\gamma^{i+1}(+\ell)\| \mid \ell = 1, \cdots, k\} = \|\gamma^{i+1}(+j)\|$. Then we can apply (6) and obtain

$$(22) \qquad \vdash \pi(\gamma^i), \Gamma \to \langle S_j n\rangle\pi(\gamma^{i+1}).$$

We can obtain (22) similarly for the case $\gamma^i = \gamma^i(-j)$. From (22), we get

(23) $$\vdash\ \pi(\gamma^{i}),\ \Gamma \rightarrow \langle On\rangle\pi(\gamma^{i+1}).$$

From (23) we obtain the following proof:

$$\dfrac{\begin{matrix}\vdots\\ \pi(\gamma^1),\ \Gamma \rightarrow \langle On\rangle\pi(\gamma^2)\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \pi(\gamma^2),\ \Gamma \rightarrow \langle On\rangle\pi(\gamma^3)\end{matrix}}{[On]\neg\pi(\gamma^3),\ \Gamma \rightarrow \neg\pi(\gamma^2)}}{[On]\neg\pi(\gamma^3),\ \Gamma \rightarrow [On]\neg\pi(\gamma^2)}}{\langle On\rangle\pi(\gamma^2),\ \Gamma \rightarrow \langle On\rangle\pi(\gamma^3)}}{\pi(\gamma^1),\ \Gamma \rightarrow \langle On\rangle\pi(\gamma^3)}$$

$$\dfrac{\begin{matrix}\vdots\\ \pi(\gamma^1),\ \Gamma \rightarrow \langle On\rangle\pi(\gamma^{m-1})\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ \pi(\gamma^{m-1}),\ \Gamma \rightarrow \langle On\rangle\pi(\gamma^m)\end{matrix}}{\langle On\rangle\pi(\gamma^{m-1}),\ \Gamma \rightarrow \langle On\rangle\pi(\gamma^m)}}{\pi(\gamma^1),\ \Gamma \rightarrow \langle On\rangle\pi(\gamma^m)}$$

Namely, we have

(24) $$\vdash\ \pi,\ \Gamma \rightarrow \langle On\rangle\pi(\delta) \quad (\text{if } \delta \in D).$$

(Though the above proof applies only for $m > 1$, (24) clearly holds even if $m = 1$ (i.e., $\varepsilon = \delta$).)

Now, by induction hypothesis of the lemma, we have the following two cases.

(D1) $\vdash\ \pi(\delta),\ \Gamma \rightarrow \beta$ for any $\delta \in D$:

Let D be enumerated as $D = \{\delta^1, \cdots, \delta^d\}$. Then we have the following proof:

(25)

$$
\cfrac{\pi,\ \Gamma \to [On]\bigvee\limits_{\delta\in D}\pi(\delta) \qquad \cfrac{\cfrac{\begin{matrix}\vdots\\ \pi(\delta^1),\ \Gamma\to\beta\end{matrix} \qquad \begin{matrix}\vdots\\ \pi(\delta^2)\vee\cdots\vee\pi(\delta^d),\ \Gamma\to\beta\end{matrix}}{\bigvee\limits_{\delta\in D}\pi(\delta),\ \Gamma\to\beta}\ (21)}{[On]\bigvee\limits_{\delta\in D}\pi(\delta),\ \Gamma\to[On]\beta}}{\pi,\ \Gamma\to[On]\beta}
$$

$$
\cfrac{\begin{matrix}\vdots\\ \pi(\delta^{d-1}),\ \Gamma\to\beta\end{matrix} \qquad \begin{matrix}\vdots\\ \pi(\delta^{d}),\ \Gamma\to\beta\end{matrix}}{\pi(\delta^{d-1})\vee\pi(\delta^d),\ \Gamma\to\beta}
$$

(D2) $\vdash \beta, \pi(\delta), \Gamma \to$ for some $\delta \in D$:

In this case, we have the following proof figure:

(26)

$$
\cfrac{\pi,\ \Gamma\to\langle On\rangle\pi(\delta) \qquad \cfrac{\cfrac{\cfrac{\begin{matrix}\vdots\\ \beta,\ \pi(\delta),\ \Gamma\to\end{matrix}}{\beta,\ \Gamma\to\neg\pi(\delta)}}{[On]\beta,\ \Gamma\to[On]\neg\pi(\delta)}}{\langle On\rangle\pi(\delta),\ [On]\beta,\ \Gamma\to}\ (24)}{[On]\beta,\ \pi,\ \Gamma\to}
$$

This completes the proof of Lemma 6.9.

Suggested by this lemma, we construct a KT5-model $M = \langle E_0; r, v\rangle$ as follows:

(i) $(\varepsilon, \delta) \in r(S_i, n)$ iff

(a) $\varepsilon \doteq \delta$

or

(b) $\varepsilon \oplus \delta = e_i$ and $n < \|\varepsilon(+i)\| = \|\delta(+i)\|$.

(ii) $(\varepsilon, \delta) \in r(0, n)$ iff

(c) $\varepsilon = \delta$

or

(d) $n < \max\{\|\varepsilon(+i)\| \mid i = 1, \cdots, k\}$ and

$n < \max\{\|\delta(+i)\| \mid i = 1, \cdots, k\}$.

(iii) $\varepsilon \in v(p_i)$ iff $\varepsilon_i = +$.

(iv) $v(\perp) = \emptyset$.

As an example, we illustrate M for $k = 3$

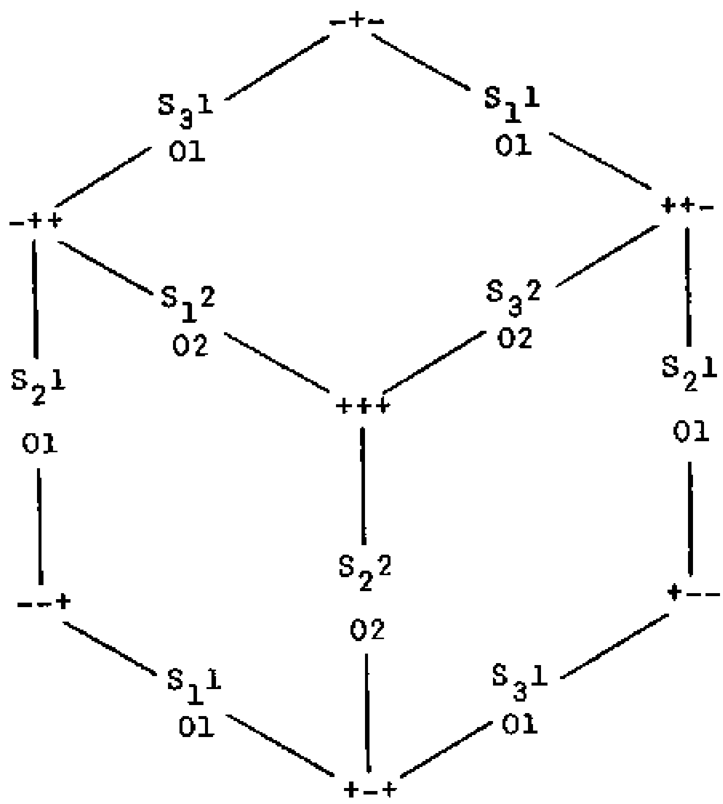

Fig. 6.3. Structure of M for k = 3

The following lemma shows that M is a model of Γ.

Lemma 6.10. *Let* $\varepsilon \in E_0$ *and* $\alpha \in \mathrm{Wff}$. *Then we have* $\vdash \pi(\varepsilon), \Gamma \to \alpha$ *if and only if* $\varepsilon \models \alpha$ (in M).

Proof. The proof is obtained by faithfully tracing the proof of Lemma 6.9. We prove that (a) $\varepsilon \models \alpha$ implies $\vdash \pi(\varepsilon), \Gamma \to \alpha$ and (b) $\varepsilon \mathrel{=\!\!|} \alpha$ implies $\vdash \alpha, \pi(\varepsilon), \Gamma \to$, by induction on the construction of α. However, we only prove the case $\alpha = [On]\beta$ since other cases may be dealt with similarly by referring to the proof of Lemma 6.9.

Proof of (a).

Suppose $\varepsilon \models [On]\beta$. We have two cases.

(A) The case $n \geq \max\{\|\varepsilon(+i)\| \mid i = 1, \cdots, k\}$:

Since $\varepsilon \models \beta$, we have

$$\vdash \pi(\varepsilon), \Gamma \to \beta$$

by induction hypothesis. Together with (13) or (17) in Lemma 6.9, we have:

$$\dfrac{\dfrac{(13)\text{ or }(17)}{\pi(\varepsilon), \Gamma \to [On]\pi(\varepsilon)} \qquad \dfrac{\begin{matrix}\vdots\\ \pi(\varepsilon), \Gamma \to \beta\end{matrix}}{[On]\pi(\varepsilon), \Gamma \to [On]\beta}}{\pi(\varepsilon), \Gamma \to [On]\beta}$$

(B) The case $n < \max\{\|\varepsilon(+i)\| \mid i = 1, \cdots, k\}$:

Let $D_n = \{\delta \in E_0 \mid n < \max\{\|\delta(+i)\| \mid i = 1, \cdots, k\}\}$. By the definition of r, we have $\varepsilon \xrightarrow{On} \delta$ for any $\delta \in D_n$. Then we have $\delta \models \beta$, since $\varepsilon \models [On]\beta$. Hence, by induction hypothesis, we have

$$\vdash \pi(\delta), \Gamma \rightarrow \beta$$

for all $\delta \in D_n$. Then we have

$$\vdash \pi(\varepsilon), \Gamma \rightarrow [On]\beta$$

by (25) in Lemma 6.9.

Proof of (b).

Suppose $\varepsilon \mathrel{=\!|} [On]\beta$. We have some δ such that $\delta \mathrel{=\!|} \beta$ and $\varepsilon \xrightarrow{On} \delta$.

(C) The case $n \geq \max\{\|\varepsilon(+i)\| \mid i = 1, \cdots, k\}$:

In this case, by the definition of r, we have $\delta = \varepsilon$. So, we have

$$\vdash \beta, \pi(\varepsilon), \Gamma \rightarrow$$

by induction hypothesis. Hence we have

$$\vdash [On]\beta, \pi(\varepsilon), \Gamma \rightarrow .$$

(D) The case $n < \max\{\|\varepsilon(+i)\| \mid i = 1, \cdots, k\}$:

By the definition of r, we have $\delta \in D_n$. Then, by (26) in Lemma 6.9, we have

$$\vdash [On]\beta,\ \pi(\varepsilon),\ \Gamma \rightarrow .$$

Lemma 6.11. *Let* $\varepsilon \in E_0$ *and* $\alpha \in \mathrm{Wff}$. *Then we have* $B_{\pi(\varepsilon)}(S_i n) \vdash \alpha$ *if and only if* $\varepsilon \models [S_i n]\alpha$.

Proof. Only if part: Suppose $B_{\pi(\varepsilon)}(S_i n) \vdash \alpha$. Then we have $B_{\pi(\varepsilon)}(S_i n) \vdash [S_i n]\alpha$. Hence, we have

$$[01](\tau\supset[01](\pi(\varepsilon)\supset[S_i n]\alpha)) \in \Gamma.$$

From this we see that

$$\vdash \pi(\varepsilon),\ \Gamma \rightarrow [S_i n]\alpha.$$

Hence, by the above lemma, we have $\varepsilon \models [S_i n]\alpha$.

If part: We have two cases.

(A) $n \geq \|\varepsilon(+i)\|$: Since $[S_i n][S_i n-1] \cdots [S_i 1]p_j^{\varepsilon_j} \in B_{\pi(\varepsilon)}(S_i n)$ for any $j \neq i$, and $B_{\pi(\varepsilon)}(S_i n) \vdash p_i^{\varepsilon_i}$ (Lemma 6.7), we have

$$\vdash B_{\pi(\varepsilon)}(S_i n) \rightarrow \pi(\varepsilon).$$

Since $\varepsilon \models [S_i n]\alpha$, we have

$$\vdash \pi(\varepsilon),\ \Gamma \rightarrow [S_i n]\alpha$$

by Lemma 6.10. Thus we obtain the following proof figure:

$$
\begin{array}{c}
\vdots \qquad\qquad\qquad\qquad \vdots \\
\dfrac{\dfrac{B_{\pi(\varepsilon)}(S_i n) \to \pi(\varepsilon) \qquad \pi(\varepsilon), \Gamma \to [S_i n]\alpha}{B_{\pi(\varepsilon)}(S_i n), \Gamma \to [S_i n]\alpha}}{B_{\pi(\varepsilon)}(S_i n), [S_i 1]\Gamma \to [S_i n]\alpha} \\
\vdots \\
\dfrac{B_{\pi(\varepsilon)}(S_i n), [S_i n] \cdots [S_i 1]\Gamma \to [S_i n]\alpha}{B_{\pi(\varepsilon)}(S_i n) \to [S_i n]\alpha} \quad (\text{extension})
\end{array}
$$

(B) $n < \|\varepsilon(+i)\|$: Let $\delta = \varepsilon \oplus e_i$. Since $\varepsilon \xrightarrow{S_i n} \delta$, we have $\delta \models [S_i n]\alpha$. Hence we have the following proof figure:

$$
\begin{array}{c}
\dfrac{\dfrac{\begin{matrix}\vdots \\ B_{\pi(\varepsilon)}(S_i n) \to \pi(\varepsilon) \vee \pi(\delta)\end{matrix} \qquad \dfrac{\begin{matrix}\vdots \\ \pi(\varepsilon), \Gamma \to [S_i n]\alpha\end{matrix} \qquad \begin{matrix}\vdots \\ \pi(\delta), \Gamma \to [S_i n]\alpha\end{matrix}}{\pi(\varepsilon) \vee \pi(\delta), \Gamma \to [S_i n]\alpha}}{\dfrac{B_{\pi(\varepsilon)}(S_i n), \Gamma \to [S_i n]\alpha}{B_{\pi(\varepsilon)}(S_i n), [S_i n] \cdots [S_i 1]\Gamma \to [S_i n]\alpha}}}{B_{\pi(\varepsilon)}(S_i n) \to [S_i n]\alpha}
\end{array}
$$

Combining the above two lemmas, we have

Corollary 6.12. *Let* $\varepsilon \in E_0$ *and* $\alpha \in$ Wff. *Then we have* $B_{\pi(\varepsilon)}(S_i n) \vdash \alpha$ *if and only if* $\vdash \pi(\varepsilon), \Gamma \to [S_i n]\alpha$.

Let us recall here that we have been arguing by assuming that $\langle\langle B_\pi(S_i n)\rangle, \Gamma\rangle$ is a solution of $\$$ satisfying (#) and (##). By inspecting Eq(*), we see that Γ is uniquely determined by Lemma 6.11 (provided that $\langle\langle B_\pi(S_i n)\rangle, \Gamma\rangle$ is in fact a solution of $\$$ under (#) and (##)). So, let $\tilde{\Gamma} \subseteq Wff$ be defined by:

$$\begin{aligned}\tilde{\Gamma} = &\{[01] \bigvee_{i=1}^{k} p_i\} \cup \{[01](S_i 1)p_j \mid j \neq i,\ i = 1, \cdots, k,\ j = 1, \cdots, k\}\\ &\cup \{[01](\pi \supset (P(\pi, i, n, p_i) \supset [0n+1][S_i n]p_i)) \mid \pi \in \Pi_0,\ i = 1, \cdots, k,\ n \in T\}\\ &\cup \{[01](\pi \supset (\overline{P}(\pi, i, n, p_i) \supset [0n+1]\neg[S_i n]p_i)) \mid \pi \in \Pi_0,\ i=1, \cdots, k,\ n \in T\}\\ &\cup \{[01](P(\pi, i, n, \alpha) \supset [01](\pi \supset [S_i n]\alpha)) \mid \pi \in \Pi_0,\ i = 1, \cdots, k,\ n \in T,\ \alpha \in Wff\}\end{aligned}$$

where P and $\overline{P}$ are defined by

$$P(\pi(\varepsilon), i, n, \alpha) = \begin{cases} \top & \text{if } \varepsilon \models [S_i n]\alpha \\ \bot & \text{otherwise} \end{cases}$$

and

$$\overline{P}(\pi(\varepsilon), i, n, \alpha) = \begin{cases} \top & \text{if } \varepsilon \mathrel{=\!\!|} [S_i n]\alpha \\ \bot & \text{otherwise.} \end{cases}$$

Using this $\tilde{\Gamma}$, we define $\tilde{B}_\pi(S_i n)$ inductively by means of equations:

$$\tilde{B}_\pi(S_i 1) = [S_i 1]\tilde{\Gamma} \cup \{[S_i 1]p_j^{\varepsilon_j} \mid j \neq i,\ j = 1, \cdots, k\},$$

$$\tilde{B}_\pi(S_i n+1) = [S_i n+1]\tilde{B}_\pi(S_i n)$$
$$\cup\{[S_i n+1][S_j n]p_j \mid \tilde{B}_\pi(S_j n) \vdash p_j,\ j = 1, \cdots, k\}$$
$$\cup\{[S_i n+1]\neg[S_j n]p_j \mid \tilde{B}_\pi(S_j n) \nvdash p_j,\ j=1, \cdots, k\},$$

where $\pi = \pi(\varepsilon)$.

In order to show that thus defined $\langle\langle\tilde{B}_\pi(S_i n)\rangle, \tilde{\Gamma}\rangle$ is the unique solution of $\$$ under (#) and (##), we prepare several lemmas.

Lemma 6.13. $\tilde{\Gamma}$ *satisfies* (#), *i.e., for any* $\varepsilon \in E_0$, $\{\pi(\varepsilon)\}\cup\tilde{\Gamma}$ *is consistent.*

Proof. It suffices to prove that $\varepsilon \models \{\pi(\varepsilon)\}\cup\tilde{\Gamma}$ (in M). It is clear that $\varepsilon \models \pi(\varepsilon)$. It remains to show that $\varepsilon \models \tilde{\Gamma}$. However, we only prove (a) $\varepsilon \models [01](\pi\supset(P(\pi, i, n, p_i)\supset[0n+1][S_i n]p_i))$ and (b) $\varepsilon \models [01](\pi\supset(P(\pi, i, n, p_i)\supset[0n+1]\neg[S_i n]p_i))$, and leave the verification of remaining parts to the reader.

Proof of (a).

Take any $\delta \in E_0$ such that $\varepsilon \xrightarrow{01} \delta$ and suppose that $\delta \models \pi$ and $\delta \models P(\pi, i, n, p_i)$. Then we have $\pi = \pi(\delta)$ and $\delta \models [S_i n]p_i$. Suppose, by way of contradiction, that there is a $\gamma \in E_0$ such that $\delta \xrightarrow{0n+1} \gamma$ and $\gamma \dashv [S_i n]p_i$. Then we have $\gamma \neq \delta$ and hence $n+1 < \max\{\|\delta(+\ell)\| \mid \ell = 1, \cdots, k\}$. Hence, $n < \|\delta(+i)\|$. But, since $\delta \models [S_i n]p_i$, we have $n \geq \|\delta(+i)\|$, which is a contradiction.

Proof of (b).

Take any δ such that $\varepsilon \xrightarrow{01} \delta$ and suppose that $\delta \models \pi$ and $\delta \models \overline{F}(\pi, i, n, p_i)$. Then we have $\pi = \pi(\delta)$ and $\delta \dashv [S_i n]p_i$. Suppose further that there is a $\gamma \in E_0$ such that $\delta \xrightarrow{0n+1} \gamma$ and $\gamma \not\models [S_i n]p_i$. Then we have $\gamma \neq \delta$ and hence $n+1 < \max\{\|\gamma(+\ell)\| \mid \ell = 1, \cdots, k\}$. Hence, $n < \|\gamma(+i)\|$. But, since $\gamma \models [S_i n]p_i$, we have $n \geq \|\gamma(+i)\|$. This is a contradiction. Thus, we see $\delta \models [0n+1]\neg[S_i n]p_i$.

Parallel to Lemma 6.9, we have the following lemma.

Lemma 6.14. *Let* $\varepsilon \in E_0$ *and* $\pi = \pi(\varepsilon)$. *Then, for any* $\alpha \in \mathrm{Wff}$, *we have either* $\vdash \pi, \tilde{\Gamma} \to \alpha$ *or* $\vdash \alpha, \pi, \tilde{\Gamma} \to$.

Proof. By a slight modification, the proof goes exactly parallel to that of Lemma 6.9. For example, in place of (6) in Lemma 6.9, we obtain

$$(\tilde{6}) \qquad \vdash \pi, \tilde{\Gamma} \to \langle S_i n\rangle\pi(\varepsilon')$$

by the following reasoning: Suppse $\pi = \pi(+i)$. Then, since $n < \|\varepsilon(+i)\|$, we have $\varepsilon \models [S_i n]\neg[S_i n]p_i$ (by the definition of M). Then, by the definition of $\tilde{\Gamma}$, we see that

$$[01](\tau\supset[01](\pi\supset[S_i n]\neg[S_i n]p_i)) \in \tilde{\Gamma}.$$

Now the proof of $(\tilde{6})$ goes completely parallel to the proof of (6) in Lemma 6.9.

The following lemma may also be proved parallel to Lemma 6.10.

Lemma 6.15. *Let* $\varepsilon \in E_0$ *and* $\alpha \in \mathrm{Wff}$. *Then we have* $\vdash \pi(\varepsilon), \tilde{\Gamma} \to \alpha$ *if and only if* $\varepsilon \models \alpha$.

We next prove the analogue of Lemma 6.11.

Lemma 6.16. *Let* $\varepsilon \in E_0$ *and* $\alpha \in \mathrm{Wff}$. *Then we have* $\tilde{B}_{\pi(\varepsilon)}(S_i n) \vdash \alpha$ *if and only if* $\varepsilon \models [S_i n]\alpha$.

Proof. We prove the following three propositions by induction on n.

(A_n) $\tilde{B}_{\pi(\varepsilon)}(S_i n) \vdash \alpha$ impies $\varepsilon \models [S_i n]\alpha$.

(B_n) $n \geq \|\varepsilon(+i)\|$ implies $\tilde{B}_{\pi(+i)}(S_i n) \vdash p_i$ and $\tilde{B}_{\pi(-i)}(S_i n) \vdash \overline{p_i}$ (if $\pi(-i) \in \Pi_0$).

(C_n) $\varepsilon \models [S_i n]\alpha$ implies $\tilde{B}_{\pi(\varepsilon)}(S_i n) \vdash \alpha$.

We first remark that to prove (A_n) it is sufficient to prove:

(A'_n) $\varepsilon \models \tilde{B}_{\pi(\varepsilon)}(S_i n)$.

For, suppose $\varepsilon \models \tilde{B}_{\pi(\varepsilon)}(S_i n)$ and $\tilde{B}_{\pi(\varepsilon)}(S_i n) \vdash \alpha$. Then we have $\vdash \tilde{B}_{\pi(\varepsilon)}(S_i n) \to \alpha$, and hence $\vdash \tilde{B}_{\pi(\varepsilon)}(S_i n) \to [S_i n]\alpha$ (by $(\to n, [S_i n])$). Since $\varepsilon \models \tilde{B}_{\pi(\varepsilon)}(S_i n)$, we have $\varepsilon \models [S_i n]\alpha$ by the Soundness Theorem.

$n = 1$:

Proof of (A_1'): $\varepsilon \models \tilde{B}_{\pi(\varepsilon)}(S_i 1)$ is easily verified since $\varepsilon \models \tilde{\Gamma}$ and $\vdash \beta \to [S_i 1]\beta$ for any $\beta \in \tilde{\Gamma}$.

Proof of (B_1). This is proved just as in Lemma 6.7.

Proof of (C_1). This is proved similarly as in Lemma 6.11 by means of (B_1) in place of Lemma 6.7 and Lemma 6.15 in place of Lemma 6.10.

$n > 1$:

Proof of (A_n'). That $\varepsilon \models [S_i n]\tilde{B}_{\pi(\varepsilon)}(S_i n-1)$ easily follows from (A_{n-1}'). Next, suppose that $\tilde{B}_{\pi(\varepsilon)}(S_j n-1) \vdash p_j$. By (A_{n-1}) we have

$$(1) \qquad \varepsilon \models [S_j n-1]p_j.$$

Hence, by the definition of M, we have $\varepsilon \models p_j$ and

$$(2) \qquad n-1 \geq \|\varepsilon(+j)\| = \|\varepsilon\|.$$

Suppose $\varepsilon \mathrel{=\!|} [S_i n][S_j n-1]p_j$. Then, for some δ such that $\varepsilon \xrightarrow{S_i n} \delta$, we have

$$(3) \qquad \delta \mathrel{=\!|} [S_j n-1]p_j.$$

From (1) and (3), we see that $\varepsilon \neq \delta$, and hence $n < \|\varepsilon(+i)\|$. This means

$$n-1 < \|\varepsilon\|,$$

which contradicts (2). Thus we have shown that

$$\varepsilon \models [S_i n][S_j n-1]p_j.$$

Suppose now $\tilde{B}_{\pi(\varepsilon)}(S_j n-1) \not\models p_j$. Then we have

(4) $$\varepsilon \mathrel{=\!\!|} [S_j n-1]p_j$$

by (C_{n-1}). By (4) and by the definition of M, we have

(5) $$n-1 < \|\varepsilon(+j)\|.$$

By way of contradiction, let us suppose $\varepsilon \mathrel{=\!\!|} [S_i n]\neg[S_j n-1]p_j$. Then, for some δ such that $\varepsilon \xrightarrow{S_i n} \delta$, we have

(6) $$\delta \models [S_j n-1]p_j.$$

By (4) and (6), we have $\delta = \varepsilon \oplus e_i$. By (6) we see that

(7) $$n-1 \geq \|\delta(+j)\|.$$

By (5) and (7), we have $\|\varepsilon(+j)\| > \|\delta(+j)\|$. Hence we see that $i \neq j$ and $\varepsilon(+i) = \varepsilon$. Now, since $\varepsilon \neq \delta$ and $\varepsilon \xrightarrow{S_i n} \delta$, we have

(8) $$n < \|\varepsilon(+i)\| = \|\varepsilon\|.$$

On the other hand, from (6) we have $n-1 \geq \|\delta(+j)\|$. Hence

$$n \geq \|\delta(+j)(+i)\| = \|\varepsilon(+j)\| = \|\varepsilon\|,$$

which contradicts (8). Therefore we see that $\varepsilon \models [S_i n]\neg [S_j n-1]p_j$ if $\tilde{B}_{\pi(\varepsilon)}(S_j n-1) \not\models p_j$.

Proof of (B_n). First we show that $\tilde{B}_{\pi(+i)}(S_in) \vdash p_i$ from the assumption that $n = \|\varepsilon(+i)\|$. Since $n > 1$, we can take a $j \neq i$ such that $\varepsilon_j = +$. Then $\|\varepsilon(+i)(+j)\| = n > n-1$. Hence we have $\varepsilon(+i) \dashv [S_jn-1]p_j$. So, by (A_{n-1}), we have $\tilde{B}_{\pi(+i)}(S_jn-1) \not\vdash_{,} p_j$. Hence,

$$(9) \qquad [S_in]\neg[S_jn-1]p_j \in \tilde{B}_{\pi(+i)}(S_in).$$

Since $\|\varepsilon(-i)(+j)\| = n-1$, we have $\varepsilon(-i) \models [S_jn-1]p_j$. Hence, by (C_{n-1}), we have $\tilde{B}_{\pi(-i)}(S_jn-1) \vdash p_j$. Hence, we have $P(\pi(-i), j, n-1, p_j) = \tau$, so that

$$(10) \qquad [01](\pi(-i)\supset(\tau\supset[0n][S_jn-1]p_j)) \in \tilde{\Gamma}.$$

From (9) and (10), we have $\tilde{B}_{\pi(+i)}(S_in) \vdash \neg\pi(-i)$. Since $\tilde{B}_{\pi(+i)}(S_in) \vdash \pi(+i)\vee\pi(-i)$, we see, $\tilde{B}_{\pi(+i)}(S_in) \vdash \pi(+i)$. Hence $\tilde{B}_{\pi(+i)}(S_in) \vdash p_i$.

The proof of $\tilde{B}_{\pi(-i)}(S_in) \vdash \overline{p_i}$ from the assumption that $n = \|\varepsilon(+i)\|$ is obtained similarly by modifying the corresponding proof of Lemma 6.7.

The case $n > \|\varepsilon(+i)\|$ is now easy.

Proof of (C_n). Similar to the proof of (C_1).

Corollary 6.17.

$P(\pi, i, n, \alpha) = \tau$ *if and only if* $\tilde{B}_\pi(S_in) \vdash \alpha$.

By Lemma 6.5, we also have the following corollary.

Corollary 6.18. $\tilde{B}_\pi(S_i n)$ *is a knowledge base for* $S_i n$.

By Corollary 6.17, we see that $\langle\langle\tilde{B}_\pi(S_i n)\rangle, \tilde{\Gamma}\rangle$ is indeed a solution of \$. Furthermore, by Lemma 6.13 and Corollary 6.18, we see that $\langle\langle\tilde{B}_\pi(S_i n)\rangle, \tilde{\Gamma}\rangle$ satisfies (#) and (##). Since we already know that \$ has at most one solution under (#) and (##), we have thus established the following theorem.

Theorem 6.19. *Under the conditions* (#) *and* (##), \$ *has the unique solution* $\langle\langle\tilde{B}_\pi(S_i n)\rangle, \tilde{\Gamma}\rangle$.

Thus we have seen that $\tilde{\Gamma}$ may be regarded as the formal counterpart of the King's order in our formal system. The puzzle is then reduced to the problem of showing that:

(P_1) If $\|\varepsilon\| = n$ and $\varepsilon_i = +$, then $\tilde{B}_{\pi(\varepsilon)}(S_i n) \vdash p_i$ and $\tilde{B}_{\pi(\varepsilon)}(S_i n-1) \nvdash p_i$.

We note that we can moreover prove the following:

(P_2) If $\|\varepsilon\| = n$ and $\varepsilon_i = -$, then $\tilde{B}_{\pi(\varepsilon)}(S_i n+1) \vdash \overline{p_i}$ and $\tilde{B}_{\pi(\varepsilon)}(S_i n) \nvdash \overline{p_i}$.

Though Lemma 6.16 gives us a solution to the problems (P_1) and (P_2), we show below a sample proof for the case $k = 3$ and $\varepsilon = ++-$:

We put $\pi = \pi(\epsilon) = p_1 \wedge p_2 \wedge \overline{p_3}$. Noting that $[S_1 2] \neg [S_2 1] p_2 \in \tilde{B}_\pi(S_1 2)$ since $\tilde{B}_\pi(S_2 1) \not\vdash p_2$, and $[01](\pi(-+-) \supset (\tau \supset [02][S_2 1]p_2)) \in \tilde{\Gamma}$ since $\tilde{B}_{\pi(-+-)}(S_2 1) \vdash p_2$, we can costruct a proof of

$$\tilde{B}_\pi(S_1 2) \to p_1$$

as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p_2 \to p_2}{[S_1 1]p_2 \to p_2}}{\tilde{B}_\pi(S_1 1) \to p_2}}{[S_1 2]\tilde{B}_\pi(S_1 1) \to p_2}}{\tilde{B}_\pi(S_1 1) \to p_2} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{p_3} \to \overline{p_3}}{[S_1 1]\overline{p_3} \to \overline{p_3}}}{\tilde{B}_\pi(S_1 1) \to \overline{p_3}}}{[S_1 2]\tilde{B}_\pi(S_1 1) \to \overline{p_3}}}{\tilde{B}_\pi(S_1 1) \to \overline{p_3}}}{\tilde{B}_\pi(S_1 2) \to p_2 \wedge \overline{p_3}}$$

$$\dfrac{\dfrac{\tilde{B}_\pi(S_1 2) \to p_2 \wedge \overline{p_3}}{\tilde{B}_\pi(S_1 2) \to p_1 \wedge p_2 \wedge \overline{p_3},\ \overline{p_1} \wedge p_2 \wedge \overline{p_3}}}{\tilde{B}_\pi(S_1 2) \to \pi \vee \pi(-+-)}$$

$$\dfrac{\dfrac{p_1 \to p_1}{\pi \to p_1}}{\pi,\ \tilde{B}_\pi(S_1 2) \to p_1}$$

$$\dfrac{\dfrac{\bot \to}{\to \top} \qquad \dfrac{[S_2 1]p_2 \to [S_2 1]p_2}{[O2][S_2 1]p_2 \to [S_2 1]p_2}}{\top \supset [O2][S_2 1]p_2 \to [S_2 1]p_2}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\pi(-+-) \to \pi(-+-) \qquad \top \supset [O2][S_2 1]p_2 \to [S_2 1]p_2}{\pi(-+-),\ \pi(-+-) \supset (\top \supset [O2][S_2 1]p_2)) \to [S_2 1]p_2}}{\pi(-+-),\ [O1](\pi(-+-) \supset (\top \supset [O2][S_2 1]p_2)) \to [S_2 1]p_2}}{\pi(-+-),\ \tilde{\Gamma} \to [S_2 1]p_2}}{\pi(-+-),\ [S_1 1]\tilde{\Gamma} \to [S_2 1]p_2}}{\pi(-+-),\ [S_1 2][S_1 1]\tilde{\Gamma} \to [S_2 1]p_2}}{\pi(-+-),\ \tilde{B}_\pi(S_1 2) \to [S_2 1]p_2}$$

$$\dfrac{\dfrac{\dfrac{[S_2 1]p_2 \to [S_2 1]p_2}{[S_2 1]p_2,\ \neg[S_2 1]p_2 \to}}{[S_2 1]p_2,\ [S_1 2]\neg[S_2 1]p_2 \to}}{[S_2 1]p_2,\ \tilde{B}_\pi(S_1 2) \to}$$

$$\dfrac{\dfrac{\pi(-+-),\ \tilde{B}_\pi(S_1 2) \to [S_2 1]p_2 \qquad [S_2 1]p_2,\ \tilde{B}_\pi(S_1 2) \to}{\pi(-+-),\ \tilde{B}_\pi(S_1 2) \to}}{\pi(-+-),\ \tilde{B}_\pi(S_1 2) \to p_1}$$

$$\dfrac{\pi,\ \tilde{B}_\pi(S_1 2) \to p_1 \qquad \pi(-+-),\ \tilde{B}_\pi(S_1 2) \to p_1}{\pi \vee \pi(-+-),\ \tilde{B}_\pi(S_1 2) \to p_1}$$

$$\dfrac{\tilde{B}_\pi(S_1 2) \to \pi \vee \pi(-+-) \qquad \pi \vee \pi(-+-),\ \tilde{B}_\pi(S_1 2) \to p_1}{\tilde{B}_\pi(S_1 2) \to p_1}$$

Fig. 6.4. Proof of $B_\pi(S_1 2) \to p_1$

The model $M = \langle E_0;\ r,\ v\rangle$ has played a crucial role for the solution of \$. We wish to point out that M may be considered as essentially the unique and hence the inherent model of $\tilde{\Gamma}$. Let us consider any KT5-model $N = \langle W_N;\ r_N,\ v_N\rangle$ such that $w_0 \models \tilde{\Gamma}$ (in N) for some $w_0 \in W_N$. Let $W_0 = \{w \in W_N \mid (w_0, w) \in r_N(0, 1)\}$. Then by restricting r_N and v_N to W_0, we obtain a model $N_0 = \langle W_0;\ r_0,\ v_0\rangle$ and still have $w_0 \models \tilde{\Gamma}$ (in N_0). Let $\tilde{N}_0 = \overline{N_0}/\chi_{N_0}$ (where we take relational closure and characteristic function in the category $\mathbb{K}_5(\mathrm{Wff})$). Then by Theorem 4.9, we have that $\tilde{N}_0$ is reduced and $\tilde{w}_0 \models \tilde{\Gamma}$ (in $\tilde{N}_0$). We also have $\tilde{r}_0(0, 1) = \tilde{W}_0 \times \tilde{W}_0$. Hence we have $w \models \tilde{\Gamma}$ (in $\tilde{N}_0$) for all $w \in \tilde{W}_0$. We will prove that $\tilde{N}_0$ is strongly isomorphic to M.

First, we define a function

$$h : \tilde{W}_0 \longrightarrow E_0$$

by letting $h(w)$ be the unique $\varepsilon \in E_0$ such that $w \models \pi(\varepsilon)$ (in $\tilde{N}_0$). Since $w \models \tilde{\Gamma}$ and $[01]\bigvee p_i \in \tilde{\Gamma}$, we see that h is well-defined. Let $w \in \tilde{W}_0$ and $\varepsilon = h(w)$. Take any formula α. Suppose $\varepsilon \models \alpha$ (in M). Then we have $\vdash \pi(\varepsilon), \tilde{\Gamma} \to \alpha$ by Lemma 6.15. From this, since $w \models \tilde{\Gamma}$ and $w \models \pi(\varepsilon)$, we have $w \models \alpha$. Thus, we see that h is a homomorphism (in $\mathbb{K}_5(\mathrm{Wff})$).

Let ε be any element in E_0. Take any $w \in \tilde{W}_0$. Since $\vdash \tilde{\Gamma} \to \langle 01\rangle\pi(\varepsilon)$, we have $w \models \langle 01\rangle\pi(\varepsilon)$. Then there is a w'

$\in \tilde{W}_0$ such that $w' \models \pi(\varepsilon)$. Hence we have $h(w') = \varepsilon$. Thus we see that h is onto.

Since $\tilde{N}_0$ is reduced, $\chi_{\tilde{N}_0} = \chi_M \circ h$ is an injection by Lemmas 4.2 and 4.7. Hence h is also an injection.

Take any $S \in Sp$ and $n \in T$. Let $w, w' \in \tilde{W}_0$. Suppose $w \xrightarrow{Sn} w'$. Then $w \models \langle Sn\rangle\pi(h(w'))$ (in N_0). Hence $h(w) \models \langle Sn\rangle\pi(h(w'))$ (in M). This means $h(w) \xrightarrow{Sn} h(w')$. Next, suppose $h(w) \xrightarrow{Sn} h(w')$. Then $h(w) \models \langle Sn\rangle\pi(h(w'))$ (in M). Since h^{-1} is a homomorphism, we have $w \models \langle Sn\rangle\pi(h(w'))$ (in N_0). Hence there is some w'' such that $w \xrightarrow{Sn} w''$ and $w'' \models \pi(h(w'))$. So, we have $h(w'') = h(w')$. Since h is injective, we have $w'' = w'$, so that $w \xrightarrow{Sn} w'$.

Thus we have proved that $\tilde{N}_0$ is strongly isomorphic to M.

Remark. We can analyze the wise men puzzle furthermore by a method similar to the one we used in this §. We wish to discuss it in a paper to be published jointly with McCarthy et al.

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NOTES

1) Page 13, line ↓5.

$\leq$ denotes the usual ordering of natural numbers.

2) Page 19, line ↑11.

Our definition of GTi are motivated by Ohnishi-Matsumoto [24].

3) Page 32, line ↓2.

We will abbreviate this to $\mathrm{isincons}_i$.

4) Page 56, line ↓1.

For example, the sequent $\rightarrow p, [St]\neg[St]p$ (where $p \in Pr$) is not provable without cut.

5) Page 56, line ↓10.

Using the completeness of KT3, 4-models, Hayashi [9] obtained a model theoretic proof of this theorem by a method due to Kripke [15].

6) Page 59, line ↓1.

Elementary terminology of category theory in this chapter mostly follows Mitchell [23].

7) Page 61, line ↓2.

Mitchell [23] uses the term *null object* instead of *termianal object.*

8) Page 74, line ↓6.

For a finite set A of wffs, we define $\bigwedge_{\alpha \in A} \alpha$ by $\alpha_1 \wedge \cdots \wedge \alpha_n$, where $\alpha_1, \cdots, \alpha_n$ is any enumeration of A.

9) Page 88, line ↑1.

Define a relation R_0 by that $(\varepsilon_i, E_k)\ R_0\ (\varepsilon_j, E_k)$ iff the two points (ε_i, E_k) and (ε_j, E_k) are connected by a line in this figure. Then the reflexive and transitive closure of this relation gives the accessible relation of U.

10) Page 99, line ↑3.

We need to assume that Pr is non-empty. In fact, if $Pr = \emptyset$, we have Lemma 6.4 in place of this lemma, since in this case KT4 is equivalent to KT5.

11) Page 107, line ↑5.

For any $\varepsilon \in E$, we will employ the convention of denoting the i^{th} coordinate of ε by ε_i.