STABLE POINTS ON ALGEBRAIC STACKS

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Abstract. This paper is largely concerned with constructing coarse moduli spaces for Artin stacks. The main purpose of this paper is to introduce the notion of stability on an arbitrary Artin stack and construct a coarse moduli space for the open substack of stable points. Also, we present an application to coherent cohomology of Artin stacks.

Key words: algebraic stack, coarse moduli space, stability

Introduction

This paper is largely concerned with constructing coarse moduli spaces for Artin stacks. Roughly speaking, a coarse moduli space for an Artin stack $\mathcal{X}$ is the best approximation of $\mathcal{X}$ by an algebraic space whose underlying space coincides with that of $\mathcal{X}$ (see section 1). Coarse moduli spaces play the role of the bridge between the geometry of Artin stacks and that of algebraic spaces and schemes. Even if we are ultimately interested in stacks, the existence of coarse moduli spaces is useful in various situations. In their influential paper ([22]), Keel and Mori proved the existence of a coarse moduli space for an Artin stack whose objects have finite automorphism groups. In particular, their theorem implies the existence of a coarse moduli space for a Deligne-Mumford stack under a weak assumption. Let us turn our attention to arbitrary Artin stacks. There are many Artin stacks which have positive-dimensional automorphisms. For instance, such examples arise from group actions on algebraic spaces, moduli spaces of vector bundles and complexes on algebraic varieties, affine geometry and so on. Hence it is desired to construct coarse moduli spaces for general Artin stacks. However, we can readily find Artin stacks which do not admit coarse moduli spaces. For example, the quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$ arising from the natural action of the torus $\mathbb{G}_m \subset \mathbb{A}^1$ on the affine line $\mathbb{A}^1$ does not have a coarse moduli space. Thus to construct coarse moduli spaces, we need to impose some condition on Artin stacks.

Let X be an algebraic scheme and G a reductive group acting on X. In his theory of Geometric Invariant Theory ([30]), Mumford defined the notion of (pre-)stable points on X and proved that the quotient of the open subset of (pre-)stable points by G exists as a geometric quotient. In terms of stacks, it says that the open substack of the quotient stack $[X/G]$ associated to the open set of (pre-)stable points has a "coarse moduli scheme".

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The main purpose of this paper is to introduce the notion of stability on an arbitrary Artin stack and to construct a coarse moduli space for the open substack of stable points. By the universality of coarse moduli space its existence depends only upon the local properties of each point. In addition, one of the reasons why the theorem of Keel and Mori is useful, is that the finiteness of automorphisms is a local condition, and so it can be checked pointwisely. Likewise, if p is a point on an Artin stack $\mathcal{X}$, then the stability (of the point p) introduced in this paper is defined by using local data around p. The central notion introduced in this paper is GIT-like p-stability (cf. Definition 5.1). (The letter "p" stands for "pointwise" and "potential".) In its naive form, the first main result of this paper states the following:

Theorem A. *Let $\mathcal{X}$ be an Artin stack locally of finite type over a perfect field. Then GIT-like p-stable points form an open substack $\mathcal{X}^{\mathrm{gs}}$ and there exists a coarse moduli map*

$$\pi : \mathcal{X}^{\mathrm{gs}} \longrightarrow X$$

such that X is an algebraic space locally of finite type. If $X' \to X$ is a flat morphism of algebraic spaces, then the second projection $\mathcal{X}^{\mathrm{gs}} \times_X X' \to X'$ is also a coarse moduli map. (*See Theorem 5.12 for detail.*) *Moreover, the followings hold: for any point $x \in X$, there exists an étale neighborhood $U \to X$ such that $\mathcal{X}^{\mathrm{gs}} \times_X U$ has the form $[W/G]$ of a quotient stack, where W is affine over U and G is a linearly reductive group.* (*cf. Corollary 5.13.*)

Let us recall that in the case of Deligne-Mumford stacks the theorem of Keel and Mori implies a more precise and fruitful correspondence: a Deligne-Mumford stack $\mathcal{X}$ has finite inertia stack, i.e. every object in $\mathcal{X}$ has a finite automorphism group scheme if and only if the following conditions hold: (i) $\mathcal{X}$ has a coarse moduli space X and the formation of coarse moduli space commutes with flat base change $X' \to X$, and (ii) for any point on X there exists an étale neighborhood $U \to X$ such that $\mathcal{X} \times_X U$ has the form $[V/G]$ of a quotient stack, where V is affine over U and G is a finite group acting on V over U. In characteristic zero, our notion of GIT-like p-stability successfully generalizes this correspondence to Artin stacks:

Theorem B. *Let $\mathcal{X}$ be an Artin stack locally of finite type over a field of characteristic zero. Let $\mathcal{X}^{gs}$ be the open substack of GIT-like p-stable points on $\mathcal{X}$. Then $\mathcal{X}^{\mathrm{gs}}$ coincides with $\mathcal{X}$ if and only if the following conditions hold:*

(i) *$\mathcal{X}$ has a coarse moduli space X and the formation of coarse moduli space commutes with flat base change $X' \to X$,*
(ii) *for any point on X there exists an étale neighborhood $U \to X$ such that $\mathcal{X} \times_X U$ has the form $[V/G]$ of a quotient stack, where V is affine over U and G is a reductive group acting on V.*

(*See Theorem 6.9 for the precise statement.*)

It is worth mentioning that in characteristic zero the definition of GIT-like p-stability is a natural generalization of the finiteness of automorphism groups, and it is described by the reductive condition on the automorphism group (cf. (a) in Definition 5.1) and the conditions (a), (b), (c) in Remark 5.2 (iii) (see also Remark 5.16). In the Deligne-Mumford case, the existence of coarse moduli space and the étale local quotient structure have played an important role. For example, Toën's Riemann-Roch theorem for

Deligne-Mumford stacks ([37]) relies on them, and Gromov-Witten theory of Deligne-Mumford stacks ([1]) requires them. Similarly, in our general situation coarse moduli spaces and the étale local quotient structures are quite useful. Indeed we present applications of Theorems A and B to coherent cohomology (see section 7).

As an example of our situation, our existence theorem contains the case of Geometric Invariant Theory. Namely, the relationship with Geometric Invariant Theory is described as follows:

Theorem C. *Let X be a scheme of locally finite type and separated over an algebraically closed field k of characteristic zero. Let G be a reductive group acting on X. Let $X(\mathrm{Pre})$ be the open subset of X consisting of pre-stable points in the sense of Geometric Invariant Theory* ([30])*. Let $\mathcal{S}$ be the maximal open substack of the quotient stack $[X/G]$ such that $\mathcal{S}$ admits a coarse moduli space which is a scheme. Then we have*

$$[X(\mathrm{Pre})/G] = [X/G]^{\mathrm{gs}} \cap \mathcal{S}.$$

(*See Theorem 6.1, Remark 6.2.*)

This paper is organized as follows. In section 1, we recall some basic facts and fix some notation. In sections 2 and 3 we present preliminary notions and preparatory results. We define strong p-stability and prove the existence of coarse moduli spaces in the case of strongly p-stable case. In section 4, we introduce the method of deformations of coarse moduli spaces, which we apply in section 5. In section 5, we then introduce the notion of GIT-like p-stability, which is the main notion of this paper. By studying the local structure of a GIT-like p-stable point, we prove Theorem A. In section 6, we discuss the relationship with Geometric Invariant Theory due to Mumford (in characteristic zero). Namely, we prove Theorem C. From Theorem A and the comparison to Geometric Invariant Theory we deduce Theorem B. Finally, in section 7 we present applications concerning coherent cohomology. In Appendix, for the reader's convenience we collect some results on limit arguments and rigidifications.

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1. Preliminaries

We establish some notation, terminology and basic facts, which we will use in this paper.

We refer to the book [24] as the general reference to the notion of algebraic stacks. We use henceforth the word "algebraic stack" instead of Artin stack. Similarly to this reference, in this paper except Appendix, all schemes, algebraic spaces, and algebraic stacks are assumed to be quasi-separated. Let S be a scheme. For an algebraic stack $p : \mathcal{X} \to S$, the fiber $\mathcal{X}(U)$ over an S-scheme U is the subcategory whose objects is objects that lies over U and whose morphisms are morphisms f where $p(f)$ is the identity of U. If $U = \operatorname{Spec} A$ we often write $\mathcal{X}(A)$ for $\mathcal{X}(U)$. For any S-scheme T and any $\xi, \eta : X \to \mathcal{X}$, let

$$\underline{\mathrm{Isom}}_{\mathcal{X},T}(\xi, \eta) : (X\text{-schemes}) \to (\text{sets})$$

be the functor which to any $f : V \to T$ associates the set of (iso)morphisms from $f^*\xi$ to $f^*\eta$ in $\mathcal{X}(V)$. Here we denote by $f^*\xi \in \mathcal{X}(V)$ (resp. $f^*\eta$) the pullback of ξ (resp. η), that is, the image $\xi(V \to X) \in \mathcal{X}(V)$ (resp. $\eta(V \to T)$). The functor $\underline{\mathrm{Isom}}_{\mathcal{X},T}(\xi, \eta)$ is naturally represented by the fiber product $X \times_{(\xi,\eta),\mathcal{X}\times_S\mathcal{X},\Delta} \mathcal{X}$, where Δ is diagonal. In particular, it is an algebraic space separated and of finite type over T. If $\xi = \eta$, then $\underline{\mathrm{Isom}}_{\mathcal{X},T}(\xi, \eta)$ is a group algebraic space over X. In this case, we write $\underline{\mathrm{Aut}}_{\mathcal{X},T}(\xi)$ for $\underline{\mathrm{Isom}}_{\mathcal{X}}(\xi, \xi)$. If no confusion seems likely to arises, we abuse notation and write $\underline{\mathrm{Isom}}(\xi, \eta)$ or $\underline{\mathrm{Isom}}_{\mathcal{X}}(\xi, \eta)$ or $\underline{\mathrm{Isom}}_X(\xi, \eta)$ for $\underline{\mathrm{Isom}}_{\mathcal{X},T}(\xi, \eta)$. If G is a flat group scheme of finite and separated over a scheme S, then BG denotes the classifying stack of principal G-bundles, that is, the quotient stack $[S/G]$ associated to the trivial action.

Let $\mathcal{X}$ be an algebraic stack over a scheme S. A *coarse moduli map* for $\mathcal{X}$ over S is a morphism $\pi : \mathcal{X} \to X$ from $\mathcal{X}$ to an algebraic space X over S such that the following conditions hold.

1. If K is an algebraically closed S-field, then the map π induces a bijective map between the set of isomorphism classes of objects in $\mathcal{X}(K)$ and $X(K)$.
2. The map π is universal for maps from $\mathcal{X}$ to algebraic spaces over S.

We shall refer to X as a *coarse moduli space* for X.

Let us recall the theorem of Keel and Mori ([22]): Let $\mathcal{X}$ be an algebraic stack $\mathcal{X}$ locally of finite type over a locally noetherian scheme S. Assume that for any object $\xi \in \mathcal{X}(T)$ over an affine S-scheme T the automorphism group $\underline{\mathrm{Aut}}_{\mathcal{X},T}(\xi)$ is finite over T. Then there exists a coarse moduli map

$$\pi : \mathcal{X} \to X,$$

where X is an algebraic space locally of finite type over S. Moreover, the natural homomorphism $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism, and π is quasi-finite and proper. If $\mathcal{X}$ has finite diagonal, then X is separated over S. We shall henceforth refer to this theorem as *Keel-Mori theorem.*

2. Strong p-stability

First of all, recall the notion of effective versal deformations (see [8, section 1], [24, (10.11)]). Let $\mathcal{X}$ be an algebraic stack over a scheme S. Let K be an S-field, and let $\xi_0 \in \mathcal{F}(K)$. An effective versal deformation of ξ_0 is an object $\xi \in \mathcal{X}(A)$, where A is a complete local ring with residue field K, and $\xi : \operatorname{Spec} A \to \mathcal{X}$ which has the following lifting property: For any 2-commutative diagram of solid arrows

$$\begin{array}{ccc} \operatorname{Spec} R/I & \longrightarrow & \operatorname{Spec} A \\ \downarrow & \nearrow & \downarrow \xi \\ \operatorname{Spec} R & \longrightarrow & \mathcal{X} \end{array}$$

where R is an Artin local ring with residue field K, and I is a square zero ideal of R, there exists a dotted arrow filling in the diagram.

Proposition 2.1. *Let S be a locally noetherian scheme and $\mathcal{X}$ an algebraic stack locally of finite type over S. Let p be a closed point on $\mathcal{X}$ in the sense of* [24, (5.2)]. *Then there exist a complete noetherian local S-ring A and an object $\xi \in \mathcal{X}(\operatorname{Spec} A)$ that has properties:*

(1) *The residue field K of A is of finite type over S, and the restriction ξ_K of ξ to K represents the point p.*
(2) *ξ is an effective versal deformation of ξ_K.*

Proof. Let $\mathcal{Y}_p$ denote the reduced closed substack associated to the closed point p. Let $f : X \to \mathcal{X}$ be a smooth surjective morphism from a scheme X. Then $f^{-1}(\mathcal{Y}_p)$ is a closed subscheme of X. Take a closed point $x \in f^{-1}(\mathcal{Y}_p)$. The residue field $K(x)$ is of finite type over S, and $\operatorname{Spec} K(x) \to \mathcal{X}$ represents p. Let $\hat{\mathcal{O}}_{X,x}$ be the completion of the local ring at x. Then the induced morphism $\operatorname{Spec} \hat{\mathcal{O}}_{X,x} \to \mathcal{X}$ is formally smooth at the closed point x. Namely, $\operatorname{Spec} \hat{\mathcal{O}}_{X,x} \to \mathcal{X}$ has the lifting property depicted above (see the definition of effective versal deformations). Thus, it gives rise to a desired effective versal deformation. □

We shall refer to an object $\xi \in \mathcal{X}(\operatorname{Spec} A)$ with properties (1), (2) in Proposition 2.1 as an *effective versal deformation on A* for p.

Lemma 2.2. *Let S be an excellent scheme and $\mathcal{X}$ an algebraic stack locally of finite type over S. Let p be a closed point on $\mathcal{X}$. Let A be a complete noetherian local S-ring with the maximal ideal $\mathfrak{m}$, and let an object $\xi \in \mathcal{X}(\operatorname{Spec} A)$ be an effective versal deformation for p. Then there exist an affine S-scheme U, a closed point u on U, an isomorphism $\hat{\mathcal{O}}_{U,u} \cong A$, and a smooth morphism $\tilde{\xi} : U \to \mathcal{X}$ such that the restriction of $\tilde{\xi}$ to $\operatorname{Spec} \hat{\mathcal{O}}_{U,u}$ is isomorphic to ξ via $\hat{\mathcal{O}}_{U,u} \cong A$.*

Proof. In virtue of Artin's algebraization theorem ([8], [24, (10.10), (10.11)], [5]), there exist an affine S-scheme U, a closed point u on U, an isomorphism $\hat{\mathcal{O}}_{U,u} \cong A$, and a smooth morphism $\tilde{\xi} : U \to \mathcal{X}$ such that the system $\{\tilde{\xi}_n\}_{n \geq 0}$ is isomorphic to $\{\xi_n\}_{n \geq 0}$ in $\lim \mathcal{X}(A/\mathfrak{m}^{n+1})$. Here $\tilde{\xi}_n$ (resp. ξ_n) denotes the restriction of $\tilde{\xi}$ (resp. ξ) to $\operatorname{Spec} A/\mathfrak{m}^{n+1}$. Since $\mathcal{X}$ is an algebraic stack, thus there exists a natural equivalence $\mathcal{X}(A) \cong \lim \mathcal{X}(A/\mathfrak{m}^{n+1})$. Therefore the restriction of $\tilde{\xi}$ to $\operatorname{Spec} \hat{\mathcal{O}}_{U,u}$ is isomorphic to ξ. □

Definition 2.3. Let S be an excellent scheme and $\mathcal{X}$ an algebraic stack locally of finite type over S. A closed point p on $\mathcal{X}$ is strongly p-stable if there exist a complete noetherian local S-ring whose residue field is of finite type over S and an effective versal deformation $\xi \in \mathcal{X}(\operatorname{Spec} A)$ for p, that has the following property (**S**):

> There exists a flat normal closed subgroup $\mathcal{F} \subset \underline{\mathrm{Aut}}_{\mathcal{X},A}(\xi)$ whose quotient $\underline{\mathrm{Aut}}_{\mathcal{X},B}(\xi)/\mathcal{F}$ is finite over $\operatorname{Spec} A$ and the following compatibility condition (C) holds: Let $f, g : \operatorname{Spec} B \to \operatorname{Spec} A$ be two morphisms of S-schemes such that their pullbacks $f^*\xi$, $g^*\xi$ are isomorphic to each other. Let $\underline{\mathrm{Aut}}_{\mathcal{X},B}(f^*\xi) \to \underline{\mathrm{Aut}}_{\mathcal{X},A}(\xi)$ and $\underline{\mathrm{Aut}}_{\mathcal{X},B}(g^*\xi) \to \underline{\mathrm{Aut}}_{\mathcal{X},A}(\xi)$ be the induced morphisms, and let $\mathcal{F}_f \subset \underline{\mathrm{Aut}}_{\mathcal{X},B}(f^*\xi)$ and $\mathcal{F}_g \subset \underline{\mathrm{Aut}}_{\mathcal{X},B}(g^*\xi)$ denote the pullbacks of $\mathcal{F}$ respectively. Then for any isomorphism $f^*\xi \to g^*\xi$ the induced isomorphism $\underline{\mathrm{Aut}}_{\mathcal{X},B}(f^*\xi) \to \underline{\mathrm{Aut}}_{\mathcal{X},B}(g^*\xi)$ gives rise to an isomorphism $\mathcal{F}_f \to \mathcal{F}_g$.

For the sake of simplicity, we shall refer to an effective versal deformation ξ with property (**S**) as a *strongly p-stable effective versal deformation.*

Remark 2.4. (i) To verify the condition (C) in Definition 2.3, it suffices to check that there exists one isomorphism $f^*\xi \to g^*\xi$ inducing an isomorphism $\underline{\mathrm{Aut}}_{\mathcal{X},B}(f^*\xi) \to \underline{\mathrm{Aut}}_{\mathcal{X},B}(g^*\xi)$ that gives rise to $\mathcal{F}_f \to \mathcal{F}_g$. Indeed, the difference of two isomorphisms $a, b : f^*\xi \rightrightarrows g^*\xi$ is $b^{-1} \circ a : f^*\xi \to f^*\xi$. The automorphism $b^{-1} \circ a : f^*\xi \to f^*\xi$ gives rise to an inner automorphism $\underline{\mathrm{Aut}}_{\mathcal{X},B}(f^*\xi) \to \underline{\mathrm{Aut}}_{\mathcal{X},B}(f^*\xi)$ and it induces an isomorphism $\mathcal{F}_f \to \mathcal{F}_f$ because $\mathcal{F}_f$ is *normal* in $\underline{\mathrm{Aut}}_{\mathcal{X},B}(f^*\xi)$.

(ii) We will use the condition (C) also when we define GIT-like stability. Fortunately, in characteristic zero the condition (C) in GIT-like stability is vacant.

Remark 2.5. Let $\mathcal{H} \to \operatorname{Spec} A$ be a *finite* scheme over a complete local ring A. Let $\hat{\mathcal{H}} \to \operatorname{Spf} A$ be the formal scheme obtained by completion with respect to $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is the maximal ideal of A. Then $\hat{\mathcal{H}}$ is finite over $\operatorname{Spf} A$. However, the converse is *not* true. Namely, the finiteness of $\hat{\mathcal{H}} \to \operatorname{Spf} A$ does not imply that $\mathcal{H} \to \operatorname{Spec} A$ is finite. To see this, assume that A is a complete discrete valuation ring with quotient field K. Then $\operatorname{Spec} K \sqcup \operatorname{Spec} A \to \operatorname{Spec} A$ is not finite. On the other hand, the associated formal scheme $\operatorname{Spf} A \to \operatorname{Spf} A$ is finite.

Remark 2.6. As noted in Introduction, the letter "p" in the term "p-stable" stands for *pointwise* and *potential*. It seems natural that a pointwise stability is defined by using the language of deformation theory. In fact, the notion of effective versal deformations plays "the role of the completions of local rings" on algebraic stacks. Put another way, we have Artin's criterion, which provides (only one) powerful and systematic method for verifying algebraicity of stacks (cf. [9]). This criterion is described in terms of deformation theory. Thus, our formulation fits in with Artin's representability criterion. (However, note that in this paper any algebraic stack is assumed to have quasi-compact and separated diagonal.) Hence we make an effort to describe our stability in terms of deformation theory.

Example 2.7. To give a feeling for the stability introduced in Definition 2.3, let us give some typical examples.

(1) Every closed point on schemes and algebraic spaces is strongly p-stable. Let $\mathcal{X}$ be an algebraic stack. Suppose that for any affine scheme $\operatorname{Spec} A$ and any object $\xi \in \mathcal{X}(\operatorname{Spec} A)$ the automorphism group space $\underline{\mathrm{Aut}}_{\mathcal{X},A}(\xi)$ is finite over $\operatorname{Spec} A$. Then every closed point on $\mathcal{X}$ is strongly p-stable. These examples are in the realm of Keel-Mori theorem.
(2) Let X be an algebraic space and $\mathcal{X}$ an fppf gerbe over X, that is, any point $x \in X$ admits an fppf neighborhood $U \to X$ such that the pullback $\mathcal{X} \times_X U$ is isomorphic to the classifying stack $B_U\mathcal{G}$ with some fppf group algebraic space $\mathcal{G}$ over U. Then every closed point on $\mathcal{X}$ is strongly p-stable.
(3) Let X be a *reduced* scheme of finite type and separated over the complex number field $\mathbb{C}$. Let G be a reductive algebraic group over $\mathbb{C}$, that acts on X. Let x be a closed point on X. Suppose that x is pre-stable in the sense of [30, Definition 1.7]. Then the image of x in the quotient stack $[X/G]$ is strongly p-stable (we will see this in section 6 in a more refined and generalized form).

(4) Let X be a projective scheme over a field k. Let G be an algebraic group over k. Let $\mathfrak{Bun}_G$ be a moduli stack of principal G-bundles on X. Let P be a G-bundle on X. This stack $\mathfrak{Bun}_G$ is algebraic (see for example [5, section 1]). Note that for every G-bundle $\mathcal{P}$ over T, the automorphism group scheme $\underline{\mathrm{Aut}}_{\mathfrak{Bun}_G,T}(\mathcal{P})$ contains $\mathrm{Cent}(G) \times_k T$, where $\mathrm{Cent}(G)$ is the center of G. Suppose that there exists an effective versal deformation $\xi \in \mathcal{M}(A)$ for P such that $\underline{\mathrm{Aut}}_{\mathfrak{Bun}_G,A}(\xi)$ is finite modulo $\mathrm{Cent}(G) \times_k A$. Then the point corresponding to P is strongly p-stable.

3. Coarse moduli space (strongly p-stable case)

The main purpose of this section is to prove

Theorem 3.1. *Let S be an excellent scheme. Let $\mathcal{X}$ be an algebraic stack locally of finite type over S. Then strongly p-stable points on $\mathcal{X}$ form an open substack $\mathcal{X}^{st}$, and there exists a coarse moduli map*

$$\pi : \mathcal{X}^{st} \longrightarrow X,$$

such that X is an algebraic space locally of finite type over S. Furthermore, the morphism π is a quasi-finite and universally closed morphism and induces an isomorphism $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$. If $X' \to X$ is a flat morphism of algebraic spaces, then $\mathcal{X}^{st} \times_X X' \to X'$ is also a coarse moduli map.

Let $I\mathcal{X}$ be the inertia stack of $\mathcal{X}$. For an S-scheme Y, $I\mathcal{X}(Y)$ consists of pairs (a, σ), where $a \in \mathcal{X}(Y)$, and σ is an automorphism of a in $\mathcal{X}(Y)$. A morphism $(a, \sigma) \to (a', \sigma')$ in $I\mathcal{X}(Y)$ is a morphism $f : a \to a'$ in $\mathcal{X}(Y)$ such that $\sigma' \circ f = f \circ \sigma$. There is a natural forgetting representable morphism $I\mathcal{X} \to \mathcal{X}$ sending (a, σ) to a. It is isomorphic to the fiber product $\mathcal{X} \times_{\Delta, \mathcal{X} \times_S \mathcal{X}, \Delta} \mathcal{X}$, where $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is a diagonal morphism. The forgetting morphism $I\mathcal{X} \to \mathcal{X}$ is isomorphic to the first projection $\mathrm{pr}_1 : \mathcal{X} \times_{\Delta, \mathcal{X} \times_S \mathcal{X}, \Delta} \mathcal{X} \to \mathcal{X}$. Thus, the morphism $I\mathcal{X} \to \mathcal{X}$ is of finite type and separated.

Let $P : U \to \mathcal{X}$ be a smooth morphism from an affine S-scheme U. Let $\eta \in \mathcal{X}(U)$ be the object corresponding to P. Let $\underline{\mathrm{Aut}}_{\mathcal{X},U}(\eta) \to U$ be the group algebraic space of the automorphism group of η. We have the natural isomorphism $\underline{\mathrm{Aut}}_{\mathcal{X},U}(\eta) \to U \times_{\mathcal{X}} I\mathcal{X}$ over U.

Consider the following contravariant functor

$$F : (\text{affine } U\text{-schemes}) \to (\text{Sets})$$

which to any $f : Y \to U$ associates the set of normal closed subgroup spaces $\mathcal{F} \subset \underline{\mathrm{Aut}}_{\mathcal{X},Y}(f^*\eta)$ over Y, which have following properties (i), (ii), (iii):

(i) $\mathcal{F}$ is flat over Y.
(ii) $\underline{\mathrm{Aut}}_{\mathcal{X},Y}(f^*\eta)/\mathcal{F}$ is finite over Y.
(iii) If $\mathrm{pr}_1, \mathrm{pr}_2 : \underline{\mathrm{Aut}}_{\mathcal{X},Y}(f^*\eta) \times_{I\mathcal{X}} \underline{\mathrm{Aut}}_{\mathcal{X},Y}(f^*\eta) \rightrightarrows \underline{\mathrm{Aut}}_{\mathcal{X},Y}(f^*\eta)$ are the natural projections, then $\mathrm{pr}_1^{-1}(\mathcal{F}) = \mathrm{pr}_2^{-1}(\mathcal{F})$.

Proposition 3.2. *The functor F is locally of finite presentation, that is to say, for any inductive system*

$$A = \varinjlim A_\lambda$$

of finitely presented U-rings (*cf.* [29, Appendix A], [6, section 1]), *the natural map*

$$\Pi : \varprojlim F(\operatorname{Spec} A_\lambda) \to F(\operatorname{Spec} \varinjlim A_\lambda)$$

is an isomorphism.

Proof. It follows from the uniqueness part of Theorem A.1 that Π is injective. Next, we prove that Π is surjective. Let $f_\lambda : \operatorname{Spec} A_\lambda \to U$ be the morphism associated to a U-ring A_λ. Let $\xi \in \mathcal{X}(\operatorname{Spec} A)$ be the object corresponding to the composite $\operatorname{Spec} A \to U \to \mathcal{X}$. Let $\mathcal{F} \to \operatorname{Spec} A$ be a closed group subspace in $\underline{\mathrm{Aut}}_{\mathcal{X},A}(\xi) \to \operatorname{Spec} A$, that has properties (i), (ii) and (iii). By Theorem A.1, there exist $\alpha \in I$ and a closed subspace $\mathcal{F}_\alpha \subset \underline{\mathrm{Aut}}_{\mathcal{X},A_\alpha}(f_\alpha^*\eta)$ which induces $\mathcal{F}$ in $\underline{\mathrm{Aut}}_{\mathcal{X},A}(\xi)$. In addition, by Proposition A.6 and Proposition A.3 we may assume that $\mathcal{F}_\alpha$ is a normal subgroup space that is flat over $\operatorname{Spec} A_\alpha$. Consider the quotient $\underline{\mathrm{Aut}}_{\mathcal{X},A_\alpha}(f_\alpha^*\eta)/\mathcal{F}_\alpha$. It is a (quasi-separated) algebraic space of finite presentation over S_α by [24, (10.13.1)]. Since $\underline{\mathrm{Aut}}_{\mathcal{X},A}(\xi)/\mathcal{F}$ is finite over $\operatorname{Spec} A$, by Proposition A.3 there exists an arrow $\alpha \to \mu$ such that $\underline{\mathrm{Aut}}_{\mathcal{X},A_\mu}(f_\mu^*\eta)/\mathcal{F}_\mu$ is finite over $\operatorname{Spec} A_\mu$, where $\mathcal{F}_\mu = \mathcal{F}_\alpha \times_{A_\alpha} A_\mu$. This means that Π is surjective. □

Proof of Theorem 3.1. Let p be a strongly p-stable closed point. Let A be a complete noetherian local ring A whose residue field is of finite type over S, and let $\xi \in \mathcal{X}(\operatorname{Spec} A)$ be a strongly p-stable effective versal deformation on A for p. By Lemma 2.2, we may assume that there exist an affine S-scheme U, a smooth morphism $P : U \to \mathcal{X}$, and a closed point u such that $\hat{\mathcal{O}}_{U,u} = A$ and the restriction of P to $\operatorname{Spec} \hat{\mathcal{O}}_{U,u}$ is ξ.

Let F be the functor defined above. Observe that $F(\operatorname{Spec} A)$ is nonempty. Indeed, $\xi \in \mathcal{X}(\operatorname{Spec} A)$ has the property (**S**) in Definition 2.3, thus there exists a flat normal closed subgroup $\mathcal{F} \subset \underline{\mathrm{Aut}}_{\mathcal{X},A}(\xi)$ with property (**S**) in Definition 2.3. Clearly, it satisfies the above conditions (i), (ii). It suffices to check (iii). Note that for any affine S-scheme V, the set $\underline{\mathrm{Aut}}_{\mathcal{X},A}(\xi) \times_{I\mathcal{X}} \underline{\mathrm{Aut}}_{\mathcal{X},A}(\xi)(V)$ consists of quintuples

$$(f, g : V \rightrightarrows \operatorname{Spec} A,\ \phi : f^*\xi \xrightarrow{\sim} f^*\xi,\ \psi : g^*\xi \xrightarrow{\sim} g^*\xi,\ \sigma : f^*\xi \xrightarrow{\sim} g^*\xi)$$

where $f^*\xi$ and $g^*\xi$ are the pullbacks of ξ by f and g respectively and $\psi = \sigma\circ\phi\circ\sigma^{-1}$. The first (resp. second) projection $\mathrm{pr}_1, \mathrm{pr}_2 : \underline{\mathrm{Aut}}_{\mathcal{X},A}(\xi) \times_{I\mathcal{X}} \underline{\mathrm{Aut}}_{\mathcal{X},A}(\xi)(V) \to \underline{\mathrm{Aut}}_{\mathcal{X},A}(\xi)(V)$ sends $(f, g, \phi, \psi, \sigma)$ to (f, ϕ) (resp. (g, ψ)). Therefore $\mathrm{pr}_1^{-1}(\mathcal{F})$ (resp. $\mathrm{pr}_2^{-1}(\mathcal{F})$) consists of quintuples

$$(f, g : V \rightrightarrows \operatorname{Spec} A,\ \phi : f^*\xi \xrightarrow{\sim} f^*\xi,\ \psi : g^*\xi \xrightarrow{\sim} g^*\xi,\ \sigma : f^*\xi \xrightarrow{\sim} g^*\xi)$$

such that $\phi \in \mathcal{F}$ (resp. $\psi \in \mathcal{F}$). Thus to prove $\mathrm{pr}_1^{-1}(\mathcal{F}) = \mathrm{pr}_2^{-1}(\mathcal{F})$ it suffices to prove the following: Let $f, g : V \rightrightarrows \operatorname{Spec} A$ be two morphisms, and let $\phi : f^*\xi \xrightarrow{\sim} f^*\xi$ be an automorphism which lies in $\mathcal{F}(f : V \to \operatorname{Spec} A)$. Assume that $f^*\xi$ is isomorphic to $g^*\xi$. Then for any isomorphism $\sigma : f^*\xi \to g^*\xi$, the composite $\sigma \circ \phi \circ \sigma^{-1}$ lies in $\mathcal{F}(g : V \to \operatorname{Spec} A)$. We can check it by using the condition (C) in Definition 2.3. Indeed, for an isomorphism $\sigma : f^*\xi \to g^*\xi$, the induced isomorphism

$$\underline{\mathrm{Aut}}_{\mathcal{X},V}(f^*\xi) \to \underline{\mathrm{Aut}}_{\mathcal{X},V}(g^*\xi)$$

sends ϕ to $\sigma \circ \phi \circ \sigma^{-1}$. Therefore the condition (C) in Definition 2.3 implies $\mathrm{pr}_1^{-1}(\mathcal{F}) = \mathrm{pr}_2^{-1}(\mathcal{F})$. Hence $F(\operatorname{Spec} A)$ is nonempty.

By the algebraic approximation theorem [7, (2.2)] and Proposition 3.2, there exists an étale U-scheme $W \to U$ such that $F(W)$ is nonempty. Then the composite C :

$W \to U \to \mathcal{X}$ is a smooth morphism. Let $\iota \in \mathcal{X}(W)$ be the object corresponding to $W \to \mathcal{X}$. Then there exists the natural cartesian diagram

$$\begin{array}{ccc} \underline{\mathrm{Aut}}_{\mathcal{X},W}(\iota) & \xrightarrow{h} & I\mathcal{X} \\ \downarrow & & \downarrow \\ W & \longrightarrow & \mathcal{X}. \end{array}$$

Since $F(W)$ is nonempty, there exists a normal closed subgroup $\mathcal{F}_\iota \subset \underline{\mathrm{Aut}}_{\mathcal{X},W}(\iota)$ with properties (i), (ii), (iii). Let $\mathcal{Y}$ be the image of $W \to \mathcal{X}$. It is an open substack of $\mathcal{X}$. Then the closed subspace $\mathcal{F}_\iota$ descends to a closed substack $\mathcal{F} \subset I\mathcal{Y}$ that is flat over $\mathcal{Y}$. Here we claim that $\mathcal{F}$ is a subgroup in $I\mathcal{Y}$, more precisely, $\mathcal{F}$ is stable under the multiplication $m : I\mathcal{Y} \times_{\mathcal{Y}} I\mathcal{Y} \to I\mathcal{Y}$, the inverse $i : I\mathcal{Y} \to I\mathcal{Y}$, and the unit section $e : \mathcal{Y} \to I\mathcal{Y}$. To see $m(\mathcal{F} \times_{\mathcal{Y}} \mathcal{F}) \subset \mathcal{F}$, consider the 2-commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{Aut}}_{\mathcal{X},W}(\iota) \times_W \underline{\mathrm{Aut}}_{\mathcal{X},W}(\iota) & \longrightarrow & \underline{\mathrm{Aut}}_{\mathcal{X},W}(\iota) \\ \downarrow{\scriptstyle (h,h)} & & \downarrow{\scriptstyle h} \\ I\mathcal{Y} \times_{\mathcal{Y}} I\mathcal{Y} & \xrightarrow{m} & I\mathcal{Y} \end{array}$$

where the upper horizontal arrow is the multiplication of $\underline{\mathrm{Aut}}_{\mathcal{X},W}(\iota)$. Since the inverse image $(h,h)^{-1}(\mathcal{F} \times_{\mathcal{Y}} \mathcal{F})$ equals to $\mathcal{F}_\iota \times_W \mathcal{F}_\iota$, the morphism $\mathcal{F}_\iota \times_W \mathcal{F}_\iota \to \mathcal{F}_\iota$ descends to $\mathcal{F} \times_{\mathcal{Y}} \mathcal{F} \to \mathcal{F}$. In a similar way, we can easily see that $\mathcal{F}$ is stable under $i : I\mathcal{Y} \to I\mathcal{Y}$. Since the unit section $W \to \underline{\mathrm{Aut}}_{\mathcal{X},W}(\iota)$ factors through $\mathcal{F}_\iota$, thus the composite $L : W \to \underline{\mathrm{Aut}}_{\mathcal{X},W}(\iota) \to I\mathcal{Y}$ factors through $\mathcal{F}$. Note that L is isomorphic to $W \to \mathcal{Y} \xrightarrow{e} I\mathcal{Y}$. Since $W \to \mathcal{Y}$ is essentially surjective, we conclude that $e : \mathcal{Y} \to I\mathcal{Y}$ factors through $\mathcal{F}$. Hence $\mathcal{F}$ is a subgroup of $I\mathcal{Y}$.

By Proposition 2.1, for any closed point y on $\mathcal{Y}$ there exist a complete noetherian local S-ring B whose residue field is of finite type over S and an effective versal deformation $\zeta \in \mathcal{Y}(B)$ for y. Then $\mathcal{F}$ gives rise to a normal closed subgroup space in $\underline{\mathrm{Aut}}_{\mathcal{Y},B}(\zeta)$ with properties (i), (ii), (iii). Thus every closed point on $\mathcal{Y}$ is strongly p-stable.

By Theorem A.7, there exists an fppf gerbe $\rho : \mathcal{Y} \to \mathcal{Y}'$ such that for any affine S-scheme V and any object $a \in \mathcal{Y}(V)$ the homomorphism $\underline{\mathrm{Aut}}_{\mathcal{Y},V}(a) \to \underline{\mathrm{Aut}}_{\mathcal{Y}',V}(\rho(a))$ is surjective and its kernel is $\mathcal{F} \times_{\mathcal{Y},a} V \subset \underline{\mathrm{Aut}}_{\mathcal{Y},V}(a)$. Since $\underline{\mathrm{Aut}}_{\mathcal{X},V}(a)/(\mathcal{F} \times_{\mathcal{Y},a} V)$ is finite for any $a \in \mathcal{X}(V)$, thus the inertia stack $I\mathcal{Y}'$ is finite over $\mathcal{Y}'$. By Keel-Mori theorem and Theorem A.7, there exists a coarse moduli maps $\mathcal{Y}' \to Y'$, and the composite morphism $\mathcal{Y} \to \mathcal{Y}' \to Y'$ is also a coarse moduli map. Also, note that $\mathcal{Y} \to \mathcal{Y}'$ is a quasi-finite and universally closed morphism by (iii) in Theorem A.7. Thus, $\mathcal{Y}' \to Y$ is so since $\mathcal{Y} \to Y$ is proper and quasi-finite. Hence for any strongly p-stable closed point x on $\mathcal{X}$, there exists an open substack $\mathcal{U} \subset \mathcal{X}$ containing x, such that every closed point on $\mathcal{U}$ is strongly p-stable, and it has a coarse moduli map. Therefore strongly p-stable points form an open substack $\mathcal{X}^s$. Using the universality of coarse moduli space, we conclude that $\mathcal{X}^{st}$ has a coarse moduli space X, that is an algebraic space locally of finite type over S. Moreover, the coarse moduli map $\pi : \mathcal{X}^{st} \to X$ is quasi-finite and universally closed. To see that $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism, we may suppose that X is an affine scheme and $\pi : \mathcal{X} \to X$ is a composite morphism $\pi' \circ \rho : \mathcal{X} \to \mathcal{X}' \to X$, where $\mathcal{X} \to \mathcal{X}'$ is an fppf gerbe, $\mathcal{X}' \to X$ is a coarse moduli map such that $\mathcal{O}_X \to \pi'_*\mathcal{O}_{\mathcal{X}'}$ is an isomorphism. Then the natural morphism $\mathcal{O}_{\mathcal{X}'} \to \rho_*\mathcal{O}_{\mathcal{X}}$ induced by the fppf gerbe $\rho : \mathcal{X} \to \mathcal{X}'$ is an isomorphism. Hence $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism.

Finally, we will prove that for any flat morphism $X' \to X$ of algebraic spaces, $\mathcal{X}^{st} \times_X X' \to X'$ is also a coarse moduli space. The claim is Zariski local on X. Thus, we may assume that $\mathcal{X}^{st} \to X$ is divided into $\mathcal{X}^{st} \to \mathcal{X}^{st'} \to X$, where $\mathcal{X}^{st} \to \mathcal{X}^{st'}$ is an fppf gerbe and the inertia stack of $\mathcal{X}^{st'}$ is finite over $\mathcal{X}^{s'}$. Then $\mathcal{X}^{st} \times_X X' \to \mathcal{X}^{st'} \times_X X'$ is an fppf gerbe and by Keel-Mori theorem $\mathcal{X}^{st'} \times_X X' \to X'$ is a coarse moduli map. Hence $\mathcal{X}^{st} \times_X X' \to X'$ is also a coarse moduli space. □

From the proof of Theorem 3.1, we also see:

Corollary 3.3. *If a closed point p on $\mathcal{X}$ is strongly p-stable, then every effective versal deformation for p is a strongly p-stable effective versal deformation.*

We will denote by $\mathcal{X}^{st}$ the open substack consisting of strongly p-stable points on $\mathcal{X}$.

Remark 3.4. In general, the coarse moduli map $\mathcal{X}^{st} \to X$ in Theorem 3.1 is not separated. Let $\mathcal{X} = BG$ be the classifying stack of an affine group scheme G over a field k. Then $BG = BG^{st}$ and the structure morphism $BG \to \operatorname{Spec} k$ is a coarse moduli map. Assume that G is not proper. Then BG is not separated over k.

Proposition 3.5. *Suppose that $\mathcal{X}$ has a reduced open substack. Then $\mathcal{X}^{st}$ is not empty.*

Proof. By [18, 6.11], there exists a nonempty open substack $\mathcal{Y} \subset \mathcal{X}$ such that the natural projection $I\mathcal{X} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$ is flat. Then each closed point on $\mathcal{Y}$ is strongly p-stable. □

4. Deformation of coarse moduli spaces and p-stable points

In this section, we give a construction of deformations of coarse moduli spaces, which is one of key ingredients to a construction of coarse moduli spaces. Let

$$\begin{array}{ccc} \mathcal{X}_0 & \longrightarrow & \mathcal{X} \\ \downarrow{\scriptstyle \pi_0} & & \\ X_0 & & \end{array}$$

be a diagram, where $\mathcal{X}_0 \to \mathcal{X}$ is a nilpotent deformation of an algebraic stack $\mathcal{X}_0$ and π_0 is a coarse moduli map which induces a natural isomorphism $\mathcal{O}_{X_0} \to \pi_*\mathcal{O}_{\mathcal{X}_0}$. We want to construct a coarse moduli space for $\mathcal{X}$ by deforming X_0 (under a certain natural setting). We refer to such a construction as the deformation of a coarse moduli space. Before proceeding into detail, it seems appropriate to begin by observing some motivating examples:

Example 4.1. Let $X = \operatorname{Spec} R$ be an affine scheme of finite type over a field k and $X_{\mathrm{red}} = \operatorname{Spec} R/I$ the associated reduced scheme. Let G be a linearly reductive algebraic group over k. Suppose that G acts on X, and it gives rise to an action on X_{red}. Assume that these actions are closed. (An action is closed if every orbit is closed after some extension of the base field (cf. [30, Definition 0.8].) Namely, every point on X is stable in the sense of Geometric Invariant Theory [30]. Then the geometric quotient for the action on X is $\operatorname{Spec} R^G$. Here R^G denotes the invariant ring. On the other hand, the geometric quotient for the action on X_{red} is $\operatorname{Spec}(R/I)^G$. We can view $\operatorname{Spec} R^G$ and $\operatorname{Spec}(R/I)^G$ as "coarse moduli schemes" for the quotient stacks $[X/G]$ and $[X_{\mathrm{red}}/G]$ respectively. Since G is linearly reductive, the natural map $R^G \to (R/I)^G$ is surjective. Therefore the nilpotent deformation $[X_{\mathrm{red}}/G] \to [X/G]$ induces a deformation $\operatorname{Spec}(R/I)^G \to \operatorname{Spec} R$ of the coarse moduli scheme $\operatorname{Spec}(R/I)^G$, and the coarse moduli scheme for $[X/G]$ can be obtained by deforming that of $[X_{\mathrm{red}}/G]$. This is the reason why the stability in the Geometric Invariant Theory makes reference only to the properties of underlying orbits.

Example 4.2. Let $\mathbb{A}^2 = \operatorname{Spec} k[x,y]$ be an affine space over a field k. Let $G_a = \operatorname{Spec} k[t]$ be an additive group. Consider the action of G_a on $\mathcal{A}^2$, described by $(x,y) \mapsto (x, y+t)$. Set $X_0 = \operatorname{Spec} k[x,y]/(x) \subset \mathbb{A}^2$ and $X_1 = \operatorname{Spec} k[x,y]/(x^2) \subset \mathbb{A}^2$. Then the coarse moduli spaces of the quotient stacks $[X_0/G_a]$ and $[X_1/G_a]$ are

$$\operatorname{Spec}(k[x,y]/(x))^{G_a} = \operatorname{Spec} k$$

and

$$\operatorname{Spec}(k[x,y]/(x^2))^{G_a} = \operatorname{Spec} k[x]/(x^2)$$

respectively. In particular, $(k[x,y]/(x^2))^{G_a} \to (k[x,y]/(x))^{G_a}$ is surjective. Next consider the action on $\mathbb{A}^2$ given by $(x,y) \mapsto (x, tx+y)$. In this case, the closed substack $[X_0/G_a]$ in $[\mathbb{A}^2/G_a]$ is isomorphic to $\operatorname{Spec} k[y] \times_k BG_a$ and its coarse moduli space is $\operatorname{Spec} k[y]$. However, the natural map $(k[x,y]/(x^2))^{G_a} \to (k[x,y]/(x))^{G_a} \cong k[y]$ is *not surjective.*

From the examples in Example 4.1 and 4.2, to construct the deformation of a coarse moduli space, we need to impose a lifting property on invariant rings. The purpose of this section is to axiomatize the property in the framework of algebraic stacks and construct the deformation of a coarse moduli space. Let $\mathcal{X}$ be an algebraic stack over a scheme S. Let $\mathcal{X}_0$ be a closed substack of $\mathcal{X}$, which is determined by a nilpotent coherent ideal sheaf $\mathcal{I}$. Suppose that there exists a coarse moduli map $\pi_0 : \mathcal{X}_0 \to X_0$ such that $\mathcal{O}_{X_0} \to \pi_{0*}\mathcal{O}_{\mathcal{X}_0}$ is isomorphism, and for any flat morphism $X_0' \to X_0$ of algebraic spaces $\mathcal{X}_0 \times_{X_0} X_0' \to X_0$ is also a coarse moduli map.

Proposition 4.3. *For any étale morphism $U_0 \to X_0$ from a scheme U_0, there exists an étale (representable) morphism $\mathcal{U} \to \mathcal{X}$ such that $\mathcal{U} \times_{\mathcal{X}} \mathcal{X}_0 \cong U_0 \times_{X_0} \mathcal{X}_0$. That is, there exists an étale deformation of $\mathrm{pr}_2 : U_0 \times_{X_0} \mathcal{X}_0 \to \mathcal{X}_0$ to $\mathcal{X}$. Moreover, such a deformation is unique up to unique isomorphism.*

Proof. Without loss of generality, we may assume $\mathcal{I}^2 = 0$. Let $\mathsf{L}_{(U_0 \times_{X_0} \mathcal{X}_0)/\mathcal{X}_0}$ and L_{U_0/X_0} denote the cotangent complexes of $\mathrm{pr}_2 : U_0 \times_{X_0} \mathcal{X}_0 \to \mathcal{X}_0$ and $U_0 \to X_0$ respectively (cf. [24, (17.3)], [32]). If $\mathrm{pr}_1 : U_0 \times_{X_0} \mathcal{X}_0 \to U_0$ is the first projection, then by [24, 17.3 (4)], we have

$$\mathsf{L}_{(U_0 \times_{X_0} \mathcal{X}_0)/\mathcal{X}_0} \cong L\mathrm{pr}_1^* \mathsf{L}_{U_0/X_0} \cong L\mathrm{pr}_1^* \Omega_{U_0/X_0} \cong 0.$$

According to Olsson's deformation theory of representation morphisms of algebraic stacks [32, Theorem 1.4], there exists an obstruction $o \in \operatorname{Ext}^2(\mathsf{L}_{(U_0 \times_{X_0} \mathcal{X}_0)/\mathcal{X}_0}, \mathrm{pr}_2^* \mathcal{I})$ whose vanishing is necessary and sufficient for the existence of an étale deformation of

pr_2 to $\mathcal{X}$. Thus, there exists a desired étale morphism $\mathcal{U} \to \mathcal{X}$ since $\mathrm{Ext}^2(\mathsf{L}_{(U_0 \times_{X_0} \mathcal{X}_0)/\mathcal{X}_0}, \mathrm{pr}_2^* \mathcal{I}) = 0$. Furthermore, again by [32, Theorem 1.4], the vanishing $\mathrm{Ext}^0(\mathsf{L}_{(U_0 \times_{X_0} \mathcal{X}_0)/\mathcal{X}_0}, \mathrm{pr}_2^* \mathcal{I}) = \mathrm{Ext}^1(\mathsf{L}_{(U_0 \times_{X_0} \mathcal{X}_0)/\mathcal{X}_0}, \mathrm{pr}_2^* \mathcal{I}) = 0$ imply the uniqueness of such a deformation. □

In the above situation, we will refer to $\mathcal{U} \to \mathcal{X}$ (or simply $\mathcal{U}$) as an étale morphism associated to $U_0 \to X_0$, and write $\mathcal{X}_U$ for $\mathcal{U}$.

Lemma 4.4. *Let $U_0 \to X_0$ and $U_0' \to X_0$ be schemes that are étale over X_0. Let $U_0' \to U_0$ be a morphism over X_0. Then there exists a morphism $\mathcal{X}_{U'} \to \mathcal{X}_U$ which makes the diagram*

$$\begin{array}{ccccc} \mathcal{X}_{U_0'} & \longrightarrow & & \mathcal{X}_{U'} & \\ \downarrow & \searrow & & \downarrow & \searrow \\ & & \mathcal{X}_0 & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow & & \downarrow & \nearrow \\ \mathcal{X}_{U_0} & \longrightarrow & & \mathcal{X}_U & \end{array}$$

2-commutative, where $\mathcal{X}_{U_0'} = U_0' \times_{X_0} \mathcal{X}_0$ and $\mathcal{X}_{U_0} = U_0 \times_{X_0} \mathcal{X}_0$. Such a morphism is unique up to isomorphism.

Proof. Note first that $\mathcal{X}_{U_0} \to U_0$ is a coarse moduli map. Applying Proposition 4.3 to $U_0' \to U_0$ we have an étale morphism $\mathcal{X}_{U'}' \to \mathcal{X}_U$ associated to the étale morphism $U_0' \to U_0$, that is, a unique deformation of $\mathcal{X}_{U_0'} \to \mathcal{X}_{U_0}$. Then the composite $\mathcal{X}_{U_0'}' \to \mathcal{X}_{U_0} \to \mathcal{X}$ is also an étale morphism associated to $U_0' \to X_0$. Finally, the uniqueness of $\mathcal{X}_{U'}$ implies our assertion. □

Consider the following lifting property $(\mathbb{L})$.

$(\mathbb{L})$: For any étale morphism $U_0 \to X_0$ from an affine scheme U_0 the natural map $\Gamma(\mathcal{X}_U, \mathcal{O}_{\mathcal{X}_U}) \to \Gamma(\mathcal{X}_{U_0}, \mathcal{O}_{\mathcal{X}_{U_0}})$ is surjective, where $\mathcal{X}_{U_0}$ denotes $U_0 \times_{X_0} \mathcal{X}_0$.

If $(\mathbb{L})$ is satisfied, we say that $\mathcal{X}$ has the property $(\mathbb{L})$ with respect to $\mathcal{X}_0$.

Proposition 4.5. *Assume that $\mathcal{X}$ has the property $(\mathbb{L})$ with respect to $\mathcal{X}_0$. Then there exists a commutative diagram*

$$\begin{array}{ccc} \mathcal{X}_0 & \longrightarrow & \mathcal{X} \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi} \\ X_0 & \longrightarrow & X, \end{array}$$

where π is a coarse moduli map for $\mathcal{X}$, which induces a natural isomorphism $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$. Furthermore, $X_0 \to X$ is a nilpotent deformation and X is locally of finite type. For any flat morphism $X' \to X$ of algebraic spaces, $\mathrm{pr}_2 : \mathcal{X} \times_X X' \to X'$ *is also a coarse moduli map.*

The proof proceeds in several steps.

Lemma 4.6. *Let $U_0' \to U_0$ be an étale morphism of affine schemes over X_0. Suppose that the natural map $\Gamma(\mathcal{X}_U, \mathcal{O}_{\mathcal{X}_U}) \to \Gamma(\mathcal{X}_{U_0}, \mathcal{O}_{\mathcal{X}_{U_0}}) = \Gamma(U_0, \mathcal{O}_{U_0})$ is surjective, where $\mathcal{X}_{U_0}$ denotes $U_0 \times_{X_0} \mathcal{X}_0$. Let $U := \mathrm{Spec}\, \Gamma(\mathcal{X}_U, \mathcal{O}_{\mathcal{X}_U})$. Let U' be a unique étale deformation of $U_0' \to U_0$ to U (cf.* [29, Theorem 3.23]). *Let $\mathcal{X}_{U'}' = \mathcal{X}_U \times_U U'$. Then there* *exists a unique (up to unique 2-isomorphism) isomorphism* $\mathcal{X}'_{U'} \to \mathcal{X}_{U'}$ *of deformations of* $\mathcal{X}_{U'_0} := \mathcal{X} \times_{X_0} U'_0 \to \mathcal{X}_{U_0}$ *to* $\mathcal{X}_U$*. Furthermore, there exists a natural isomorphism* $\Gamma(\mathcal{X}_{U'}, \mathcal{O}_{\mathcal{X}_{U'}}) = \Gamma(U', \mathcal{O}_{U'})$*. In particular, the natural map* $\Gamma(\mathcal{X}_{U'}, \mathcal{O}_{\mathcal{X}_{U'}}) \to \Gamma(\mathcal{X}_{U'_0}, \mathcal{O}_{\mathcal{X}_{U'_0}}) = \Gamma(U'_0, \mathcal{O}_{U'_0})$ *is surjective.*

Proof. We prove the first claim. Note that by Lemma 4.4 there exists $\mathcal{X}_{U'} \to \mathcal{X}_U$ that can be viewed as a unique étale deformation of $\mathcal{X}_{U'_0} \to \mathcal{X}_{U_0}$ to $\mathcal{X}_U$. Thus by Proposition 4.3 it suffices to prove that $\mathcal{X}'_{U'} \times_{\mathcal{X}_U} \mathcal{X}_{U_0}$ is isomorphic to $\mathcal{X}_{U'_0}$ over $\mathcal{X}_{U_0}$. To see this, notice that there is a natural closed immersion $\mathcal{X}_{U_0} \to \mathcal{X}_U \times_U U_0$ over U_0. This morphism is not necessarily an isomorphism. More generally, we have a diagram

$$\begin{array}{ccccc} \mathcal{X}_{U'_0} & \longrightarrow & \mathcal{X}'_{U'} \times_{U'} U'_0 & \longrightarrow & U'_0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X}_{U_0} & \longrightarrow & \mathcal{X}_U \times_U U_0 & \longrightarrow & U_0 \end{array}$$

where all squares are cartesian, and $\mathcal{X}_{U_0} \to \mathcal{X}_U \times_U U_0$ and $\mathcal{X}_{U'_0} \to \mathcal{X}'_{U'} \times_{U'} U'_0$ are closed immersions. Since $\mathcal{X}_{U_0} \times_{(\mathcal{X}_U \times_U U_0)} (\mathcal{X}'_{U'} \times_{U'} U'_0)$ is naturally isomorphic to $\mathcal{X}_{U_0} \times_{\mathcal{X}_U} \mathcal{X}'_{U'}$, thus we have $\mathcal{X}_{U_0} \times_{\mathcal{X}_U} \mathcal{X}'_{U'} \cong \mathcal{X}_{U'_0}$ over U'_0. This implies the first claim. Next we prove the second claim. Let $p : Z \to \mathcal{X}_U$ be a smooth surjective morphism from an affine scheme Z. Then there exists an exact sequence

$$\Gamma(U, \mathcal{O}_U) = \Gamma(\mathcal{X}_U, \mathcal{O}_{\mathcal{X}_U}) \xrightarrow{p^*} \Gamma(Z, \mathcal{O}_Z) \overset{\mathrm{pr}^*_{1,2}}{\rightrightarrows} \Gamma(Z \times_{\mathcal{X}_U} Z, \mathcal{O}_{Z \times_{\mathcal{X}_U} Z}).$$

Since $U' \to U$ is flat, thus the exactness holds after base changing by $U' \to U$. This implies the second claim because $\mathcal{X}_{U'} \cong \mathcal{X}'_{U'}$. □

Remark 4.7. By the above proof, the property that $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism is stable under flat base changes on X.

Let $X_{0,\text{ét}}$ be the étale site, whose objects are affine schemes over X_0, and whose morphisms are X_0-morphisms. Let $\mathcal{O}$ be a sheaf of rings, which is determined by

$$\mathcal{O}(U_0) := \mathcal{O}(U_0 \to X_0) = \Gamma(\mathcal{X}_U, \mathcal{O}_{\mathcal{X}_U})$$

for any étale morphism $U_0 \to X_0$ in $X_{0,\text{ét}}$. For any $U_0 \to X_0$ the natural morphism $\mathcal{O}(U_0 \to X_0) \to \mathcal{O}_{X_0}(U_0 \to X_0)$ is surjective since $\mathcal{X}$ has the property ($\mathbb{L}$) with respect to $\mathcal{X}_0$. By Lemma 4.6, for any morphism $U'_0 \to U_0$ in $X_{0,\text{ét}}$, there exists a natural isomorphism $\mathcal{O}_{X_0}(U_0) \otimes_{\mathcal{O}(U_0)} \mathcal{O}(U'_0) \cong \mathcal{O}_{X_0}(U'_0)$. Thus, the sheaf $\mathcal{O}$ induces a deformation of the ringed topos associated to $(X_{0,\text{ét}}, \mathcal{O}_{X_0})$. Namely, it gives rise to a deformation $X_0 \hookrightarrow X$ of the algebraic space X_0.

Lemma 4.8. *There exists a morphism* $\pi : \mathcal{X} \to X$ *that has the following properties:*

(a) *The diagram*

$$\begin{array}{ccc} \mathcal{X}_0 & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & X \end{array}$$

is commutative;

(b) *the natural morphism* $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ *is an isomorphism.*

Remark 4.9. The diagram in Lemma 4.8 is not necessarily cartesian.

Proof. Let $Z \to X$ be an étale surjective morphism from an affine scheme Z. Let $\mathcal{X}_Z \to \mathcal{X}$ be an étale morphism associated to $\mathrm{pr}_2 : Z \times_X X_0 \to X_0$. Namely, there exists a natural morphism $\mathcal{X}_Z \to Z$. Let $W \to \mathcal{X}_Z$ be a smooth surjective morphism from a scheme W. To construct a desired morphism $\mathcal{X} \to X$, it suffices to show that there exists a morphism $[W \times_{\mathcal{X}} W \rightrightarrows W] \to [Z \times_X Z \rightrightarrows Z]$ of groupoids (cf. [24, (2.4.3)]). To this end, it is enough to prove that there exists a morphism $\mathcal{X}_Z \times_{\mathcal{X}} \mathcal{X}_Z \to Z \times_X Z$ which makes the diagram

$$\begin{array}{ccc} \mathcal{X}_Z \times_{\mathcal{X}} \mathcal{X}_Z & \rightrightarrows & \mathcal{X}_Z \\ \downarrow & & \downarrow \\ Z \times_X Z & \rightrightarrows & Z \end{array}$$

commute. If $Z_0 \to X_0$ denotes the projection $Z \times_X X_0 \to X_0$, then $\mathrm{pr}_i : \mathcal{X}_Z \times_{\mathcal{X}} \mathcal{X}_Z \to \mathcal{X}_Z$ $(i = 1, 2)$ is an (étale) deformation of projection $\mathrm{pr}_i : \mathcal{X}_0 \times_{X_0} (Z_0 \times_{X_0} Z_0) \to \mathcal{X}_{Z_0} = \mathcal{X}_0 \times_{X_0} Z_0$ $(i = 1, 2)$ respectively. Also, $\mathrm{pr}_i : Z \times_X Z \to Z$ $(i = 1, 2)$ is an (étale) deformation of projection $\mathrm{pr}_i : Z_0 \times_{X_0} Z_0 \to Z_0$ $(i = 1, 2)$ respectively. Thus by Lemma 4.6, the projections $\mathcal{X}_Z \times_{Z,\mathrm{pr}_i} (Z \times_X Z) \to \mathcal{X}_Z$ $(i = 1, 2)$ is naturally isomorphic to $\mathrm{pr}_i : \mathcal{X}_Z \times_{\mathcal{X}} \mathcal{X}_Z \to \mathcal{X}_Z$ $(i = 1, 2)$. Hence we can obtain a desired morphism $\mathcal{X}_Z \times_{\mathcal{X}} \mathcal{X}_Z \to Z \times_X Z$. Finally, by the construction we have $\Gamma(\mathcal{X}_Z, \mathcal{O}_{\mathcal{X}_Z}) = \Gamma(Z, \mathcal{O}_Z)$. This means that $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism. □

Lemma 4.10. *The morphism π in Lemma 4.8 is a coarse moduli map for $\mathcal{X}$. For any flat morphism $X' \to X$ of algebraic spaces,* $\mathrm{pr}_2 : \mathcal{X} \times_X X' \to X'$ *is also a coarse moduli map.*

Proof. First of all, it is clear that for any algebraically closed field K the morphism π induces a bijective map from the set of isomorphism classes of $\mathcal{X}(K)$ to $X(K)$ because π_0 does. If $X' \to X$ is a flat morphism of algebraic spaces, then $\mathrm{pr}_2 : \mathcal{X}_0 \times_{X_0} (X' \times_X X_0) \to X' \times_X X_0$ is a coarse moduli map by our assumption. On the other hand, the underlying map of $\mathrm{pr}_2 : \mathcal{X} \times_X X' \to X'$ can be identified with that of $\mathrm{pr}_2 : \mathcal{X}_0 \times_{X_0} (X' \times_X X_0) \to X' \times_X X_0$. Therefore, pr_2 induces a bijective map from the set of isomorphism classes of $\mathcal{X} \times_X X'(K)$ to $X'(K)$ for any algebraically closed field K. Thus, it remains to prove that for any flat morphism $X' \to X$, $\mathrm{pr}_2 : \mathcal{X} \times_X X' \to X'$ is universal among morphisms to algebraic spaces. Let $f : \mathcal{X} \to W$ be a morphism to an algebraic space W. What we have to prove is that there exists a unique morphism $\phi : X \to W$ such that $f = \phi \circ \pi$. We first prove the case where W is a scheme.

Claim 4.10.1. *If W is a scheme, there exists a unique morphism $\phi : X \to W$ such that $f = \phi \circ \pi$. For any flat morphism $X' \to X$ of algebraic spaces, the projection* $\mathrm{pr}_2 : \mathcal{X} \times_X X' \to X'$ *is also universal among morphisms to schemes.*

Proof of Claim. The composite $\mathcal{X}_0 \to \mathcal{X} \to W$ induces a morphism $X_0 \to W$. It is enough to show that if $U \to X$ is an étale morphism from an affine scheme U, then there exists a unique morphism $U \to W$ extending $U_0 := U \times_X X_0 \to X_0 \to W$, such that $\mathcal{X}_U \cong \mathcal{X} \times_X U \to \mathcal{X} \to W$ is equal to $\mathcal{X}_U \to U \to W$. Since U and U_0 have the same underlying topological space, thus we have a map $U \to W$ as topological spaces. Thus it suffices to construct a morphism of structure sheaves. To this end, we may assume that W is affine. Since we have the morphism $\mathcal{X}_U \to W$, there exists a morphism $\Gamma(W, \mathcal{O}_W) \to \Gamma(\mathcal{X}_U, \mathcal{O}_{\mathcal{X}_U}) = \Gamma(U, \mathcal{O}_U)$. Clearly, it gives rise to a desired morphism. The uniqueness follows from the fact that $U \to W$ should arise from $\Gamma(W, \mathcal{O}_W) \to \Gamma(U, \mathcal{O}_U)$ associated to $\mathcal{X}_U \to \mathcal{X} \to W$. By Remark 4.7, for any flat morphism $X' \to X$, $\mathrm{pr}_2 : \mathcal{X} \times_X X' \to X'$ induces a natural isomorphism $\mathcal{O}_{X'} \to \mathrm{pr}_{2*}\mathcal{O}_{\mathcal{X}\times_X X'}$. Therefore, the above argument can be applied to $\mathrm{pr}_2 : \mathcal{X} \times_X X' \to X'$. □

Next we will show the case when W is an algebraic space by reducing it to Claim 4.10.1. This part is done by a more or less well-known argument. But for the reader's convenience we will present a proof here. Let $V \to \mathcal{X}$ be a smooth surjective morphism from an affine scheme V, which gives rise to

$$R = V \times_{\mathcal{X}} V \overset{\mathrm{pr}_i}{\rightrightarrows} V \to \mathcal{X}.$$

Let $X' \to X$ be a flat morphism of algebraic space. We will prove that the sequence

$$W(X') \to W(X' \times_X V) \rightrightarrows W(X' \times_X R)$$

is exact. Let $\mathcal{X}' = \mathcal{X} \times_X X'$, $V' = X' \times_X V$ and $R' = X' \times_X R$.

We first prove that $W(X') \to W(V')$ is injective. Let $Y \to W$ is an étale surjective morphism from an affine scheme Y. The fiber product $Y_1 := Y \times_W Y$ is quasi-affine (cf. [24, (1.3)]). Let $\xi, \eta : X' \rightrightarrows W$ be two morphisms to W. Assume that ξ, η have the same image in $W(V')$. It suffices to show $\xi = \eta$. Take an étale surjective morphism $X'' \to X'$ so that there exist lifts $\xi_0 : X'' \to Y$ and $\eta_0 : X'' \to Y$ of ξ and η respectively. Since $X'' \to X'$ is étale and surjective, it is enough to show that two composites $X'' \overset{\xi_0, \eta_0}{\rightrightarrows} Y \to W$ coincide. It is equivalent to proving that $(\xi_0, \eta_0) : X'' \to Y \times Y$ factors through the image of $(\mathrm{pr}_1, \mathrm{pr}_2) : Y_1 \to Y \times Y$. Put $R'' = R \times_X X''$, $\mathcal{X}'' = \mathcal{X}' \times_{X'} X''$ and $V'' = V' \times_{X'} X''$. Consider the commutative diagram

$$\begin{array}{ccccccc}
V'' & \longrightarrow & \mathcal{X}'' & \longrightarrow & X'' & \overset{\xi_0}{\underset{\eta_0}{\rightrightarrows}} & Y \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
V' & \longrightarrow & \mathcal{X}' & \longrightarrow & X' & \overset{\xi}{\underset{\eta}{\rightrightarrows}} & W,
\end{array}$$

where the left and middle squares are cartesian diagrams. Notice that the morphism $V'' \to Y \times Y$ induced by (ξ_0, η_0) factors through the image of $(\mathrm{pr}_1, \mathrm{pr}_2) : Y_1 \to Y \times Y$ since $Y \to W$ is étale surjective and ξ, η have the same image in $W(V')$. Denote by $\alpha \in Y_1(V'')$ the image. Since we have the morphism $\mathcal{X}'' \to Y \times Y$ induced by (ξ_0, η_0), thus $\mathrm{pr}_1^*(\alpha)$ and $\mathrm{pr}_2^*(\alpha)$ coincide in $Y_1(R'')$. Since Y_1 is a scheme, by Claim 4.10.1 we see that α comes from $Y_1(X'')$. Therefore, we conclude that $W(X') \to W(V')$ is injective.

Next we will prove the middle exactness in $W(X') \to W(V') \rightrightarrows W(R')$. Let $h : V' \to W$ be a morphism. Suppose that the two composites $h \circ \mathrm{pr}_1, h \circ \mathrm{pr}_2 : R' \to W$ coincide. (Namely, it gives rise to a morphism $\mathcal{X}' \to W$.) It suffices to show that h arises from some $X' \to W$. For ease of notation, we replace $\mathcal{X}'$, X', V' and R' by $\mathcal{X}$, X, V, and R respectively. The morphism $\mathcal{X}_0 \to \mathcal{X} \to W$ induces $X_0 \to W$. Take an étale surjective morphism $X_0' \to X_0$ from an affine scheme X_0' so that there exists a lift $X_0' \to Y$ of $X_0 \to W$. By [29, Theorem 3.23], there exists a unique étale deformation $X' \to X$ of $X_0' \to X_0$. Then it is enough to construct $f : X' \to Y$ and

$g : X'' := X' \times_X X' \to Y_1$ which makes the diagrams (♣)

$$\begin{array}{ccc} X'' & \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} & X' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y_1 & \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} & Y, \end{array} \qquad \begin{array}{ccc} V' & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle h} \\ Y & \longrightarrow & W \end{array}$$

commute, where $V' := V \times_X X'$. We first construct $f : X' \to Y$. To this aim, notice that $V'_0 := X'_0 \times_X V \cong X'_0 \times_{X_0} V_0 \to V_0 := X_0 \times_X V$ factors through $\mathrm{pr}_2 : Y \times_W V_0 \to V_0$. Again by [29, Theorem 3.23], $V' \to V$ factors through $\mathrm{pr}_2 : Y \times_W V \to V$. Thus, we have $f' : V' \to Y \times_W V \to Y$, which fits in with the right diagram of (♣). Next we will observe that $f' \circ \mathrm{pr}_1, f' \circ \mathrm{pr}_2 : R' = R \times_X X' \rightrightarrows Y$ coincide. Clearly, it holds after restricting to $R'_0 = R \times_X X'_0$. This implies that two morphisms $R'_0 \rightrightarrows R_0 \times_W Y$ induced by $R'_0 \rightrightarrows Y$ coincide. The morphisms $R' \rightrightarrows R \times_W Y$ induced by $R' \rightrightarrows Y$ are étale deformations of $R'_0 \rightrightarrows R_0 \times_W Y$ respectively. Thus by [29, Theorem 3.23], two morphisms $R' \rightrightarrows R \times_W Y$ coincide. Hence $f' \circ \mathrm{pr}_1$ and $f' \circ \mathrm{pr}_2$ coincide, and it gives rise to $\mathcal{X}' := \mathcal{X} \times_X X' \to Y$. This induces $\Gamma(Y, \mathcal{O}_Y) \to \Gamma(X', \mathcal{O}_{X'}) = \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$. Let $f : X' \to Y$ be the induced morphism. Next we construct $g : X'' \to Y_1$. By the above construction, we have $\mathcal{X}' \to Y$. It gives rise to $a : \mathcal{X}' \times_{\mathcal{X}} \mathcal{X}' \to Y \times_W Y$ because the right diagram of (♣) commutes. Note that by Lemma 4.6 $(\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}') \times_{\mathcal{X}} \mathcal{X}_0$ is naturally isomorphic to $\mathcal{X}_0 \times_{X_0} (X'_0 \times_{X_0} X'_0)$, whose coarse moduli space is $X'_0 \times_{X_0} X'_0$. Also, notice $X'_0 \times_{X_0} X'_0 \cong (X' \times_X X') \times_X X_0$. Applying the schematic case to $\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}' \to X' \times_X X'$ there exists a unique morphism $g : X' \times_X X' \to Y \times_W Y$ such that $a = g \circ \pi''$, where π'' is the natural morphism $\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}' \to X' \times_X X'$. By the construction of f and g, they make diagrams (♣) commute. Hence we obtain a desired morphism $\phi : X \to W$. □

Proof of Proposition 4.5. We now obtain our Proposition 4.5 from Lemma 4.8 and Lemma 4.10. □

Let $\mathcal{X}$ be an algebraic stack locally of finite type over an excellent scheme S. Let p be a closed point on $\mathcal{X}$. We say that p is p-stable if there exist a quasi-compact open substack $\mathcal{U} \subset \mathcal{X}$ containing p, and a closed substack $\mathcal{U}_0 \subset \mathcal{U}$ determined by a nilpotent ideal sheaf, that has the properties:

(i) p is strongly p-stable over $\mathcal{U}_0$;
(ii) $\mathcal{U}$ has the property $(\mathbb{L})$ with respect to $\mathcal{U}_0$.

According to Theorem 3.1 and Proposition 4.5, we have:

Proposition 4.11. *Let $\mathcal{X}$ be an algebraic stack locally of finite type over an excellent scheme S. Then p-stable points form an open substack $\mathcal{X}^s$, and there exists a coarse moduli space*

$$\pi : \mathcal{X}^s \longrightarrow X,$$

which induces a natural isomorphism $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$. Moreover, it is universally closed and quasi-finite. If $X' \to X$ is a flat morphism of algebraic spaces, then $\mathcal{X}^s \times_X X' \to X'$ is also a coarse moduli space.

Remark 4.12. Ultimately, we shall only be interested in GIT-like p-stability which will be introduced in the next section. The reason, however, that we introduced the ad hoc notion of (strong) p-stability is to prove some properties of GIT-like p-stable points. In addition, we hope that the machinery of p-stability will be useful in the other situations.

The property ($\mathbb{L}$) may seem to be hard to verify. Nonetheless, in the next section we will show that the property ($\mathbb{L}$) is satisfied in the GIT-like stable case which we think of as a reasonable setting.

For the later use, we need:

Lemma 4.13. *Let $\mathcal{X}_0 \to \mathcal{X}$ be a closed immersion defined by a nilpotent ideal $\mathcal{I}$. Suppose that there exists a coarse moduli map $\pi_0 : \mathcal{X}_0 \to U_0$ to an affine scheme, and for any flat morphism $U_0' \to U_0$, the projection $q : \mathcal{X}_{U_0'} := \mathcal{X}_0 \times_{U_0} U_0' \to U_0'$ is also a coarse moduli map. Let $U_0' \to U_0$ be an étale surjective morphism of affine schemes. Let $\mathcal{X}_{U'} \to \mathcal{X}$ be an étale morphism associated to $U_0' \to U_0$, that is, a unique étale deformation $\mathcal{X}_{U'} \to \mathcal{X}$ of $\mathcal{X}_{U_0'} \to \mathcal{X}_0$. Suppose that $\Gamma(\mathcal{X}_{U'}, \mathcal{O}_{\mathcal{X}_{U'}}) \to \Gamma(\mathcal{X}_{U_0'}, \mathcal{O}_{\mathcal{X}_{U_0'}})$ is surjective. Then $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \Gamma(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0})$ is surjective. (In particular, the property* ($\mathbb{L}$) *is étale local property.)*

Proof. By induction, we may and will assume that $\mathcal{I}$ is a square zero ideal. Let $\mathcal{J} := \mathcal{I} \otimes_{\mathcal{O}_{\mathcal{X}_0}} \mathcal{O}_{\mathcal{X}_{U_0}}$. Since $\Gamma(\mathcal{X}_{U'}, \mathcal{O}_{\mathcal{X}_{U'}}) \to \Gamma(\mathcal{X}_{U_0'}, \mathcal{O}_{\mathcal{X}_{U_0'}})$ is surjective and U_0' is affine, we have $\Gamma(U_0', R^1 q_* \mathcal{J}) = 0$. On the other hand, $\Gamma(U_0, R^1\pi_*\mathcal{I}) \otimes_{\Gamma(U_0, \mathcal{O}_{U_0})} \Gamma(U_0', \mathcal{O}_{U_0'}) = \Gamma(U_0', R^1 q_* \mathcal{J})$. Since $U_0' \to U_0$ is étale and surjective, we see $\Gamma(U_0, R^1\pi_*\mathcal{I}) = 0$. This means that $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \Gamma(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0})$ is surjective. □

5. GIT-like p-stability

In this section, we now introduce the notion of *GIT-like p-stable points.* In this section, we will work over a perfect base field k. Let $\mathcal{X}$ be an algebraic stack locally of finite type over k.

Definition 5.1. Let p be a closed point on $\mathcal{X}$. The point p is GIT-like p-stable if there exists an effective versal deformation $\xi \in \mathcal{X}(A)$, which has the following properties:

(a) The special fiber of $\underline{\mathrm{Aut}}_{\mathcal{X},A}(\xi) \to \operatorname{Spec} A$ is linearly reductive, that is, $\underline{\mathrm{Aut}}_{\mathcal{X},A}(\xi) \times_{\operatorname{Spec} A} \operatorname{Spec} A/\mathfrak{m}$ is a linearly reductive algebraic group over $\operatorname{Spec} A/\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of A.
(b) If I denotes the ideal generated by nilpotent elements in A, then there exists a normal subgroup scheme $\mathcal{F}$ in $\underline{\mathrm{Aut}}_{\mathcal{X},A}(\xi) \times_A \operatorname{Spec}(A/I)$ which is smooth and affine over $\operatorname{Spec} A/I$, and whose geometric fibers are connected. Furthermore, the quotient $\underline{\mathrm{Aut}}_{\mathcal{X},A}(\xi) \times_A \operatorname{Spec}(A/I)/\mathcal{F}$ is finite over A/I, and the compatibility condition (C) as in Definition 2.3 holds for $\mathcal{F} \subset \underline{\mathrm{Aut}}_{\mathcal{X},A}(\xi) \times_A \operatorname{Spec}(A/I)$.

Remark 5.2. (i) Let K be a field and $K' \supset K$ an extension of fields. An algebraic group G over K is linearly reductive if and only if so is $G \times_{\operatorname{Spec} K} \operatorname{Spec} K'$ over K'. Therefore, to verify (a) in Definition 5.1, it is enough to show that there exist a field K and a morphism $v : \operatorname{Spec} K \to \mathcal{X}$, such that v represents the point p and $\underline{\mathrm{Aut}}_{\mathcal{X},K}(v)$ is linearly reductive over K.
(ii) According to [14, VI_B Corollaire 4.4], to check that a group scheme $\mathcal{F}$ is smooth over a reduced scheme S, it is enough to prove that every fiber is smooth and $\mathcal{F}$ is Zariski locally equidimensional over S.

(iii) The characteristic zero case is simpler than the general case. If $\mathcal{G} := \underline{\mathrm{Aut}}_{\mathcal{X},A}(\xi) \times_A \mathrm{Spec}(A/I) \to \mathrm{Spec}(A/I)$ is equidimensional, then by [14, VI_B Corollaire 4.4] the identity component $\mathcal{G}^0$ (cf. [14, VI_B Définition 3.1]) is a smooth open normal subgroup, whose geometric fibers are connected. By [14, VI_B Proposition 3.3], the compatibility condition (C) holds for $\mathcal{G}^0$. Hence in characteristic zero, the property (b) in Definition 5.1 is satisfied if and only if the following conditions hold:
  (a) $\mathcal{G} \to \mathrm{Spec}(A/I)$ is equidimensional,
  (b) $\mathcal{G}^0$ is affine over $\mathrm{Spec}(A/I)$,
  (c) $\mathcal{G}/\mathcal{G}^0$ is finite over $\mathrm{Spec}(A/I)$.

(iv) Consider the case when $\mathcal{X}$ has finite inertia stack. Suppose that all closed points on $\mathcal{X}$ are GIT-like p-stable. Then in characteristic zero, $\mathcal{X}$ is a Deligne-Mumford stack ([12]), and in positive characteristic $\mathcal{X}$ is a tame stack introduced by Abramovich, Olsson, and Vistoli ([2]).

Proposition 5.3. *Let $\mathcal{X}_0$ be the reduced stack associated to $\mathcal{X}$. Let p be a GIT-like p-stable point on $\mathcal{X}$ (or equivalently $\mathcal{X}_0$). Then there exists an open substack $\mathcal{Y}_0 \subset \mathcal{X}_0$ containing p, which has a coarse moduli map $\pi_0 : \mathcal{Y}_0 \to Y_0$ such that it induces an isomorphism $\mathcal{O}_{Y_0} \to \pi_{0*}\mathcal{O}_{\mathcal{Y}_0}$, and for any flat morphism $Y_0' \to Y_0$ the pullback $\mathcal{Y}_0 \times_{Y_0} Y_0' \to Y_0'$ is also a coarse moduli map.*

Proof. Let $\xi : \mathrm{Spec}\, A \to \mathcal{X}$ be an effective versal deformation for p, which satisfies the properties (a) and (b) in Definition 5.1. Note that if $\mathcal{I}$ denotes the ideal generated by nilpotent elements in $\mathcal{O}_{\mathcal{X}}$, then the ideal I of nilpotent elements in A is the pullback of $\mathcal{I}$ via $\xi : \mathrm{Spec}\, A \to \mathcal{X}$ because $\mathcal{X}$ is excellent. Then p is a strongly p-stable point on $\mathcal{X}_0$. Thus by Theorem 3.1 we obtain our Proposition. □

Let $\mathcal{X}$ be an algebraic stack locally of finite type over the base field k. According to Proposition 5.3, we will denote by $\mathcal{X}^{\mathrm{gs}}$ the open substack of GIT-like p-stable points. The main purpose of the remainder of this section is to prove Theorem 5.12, that is, to show the existence of a coarse moduli map for $\mathcal{X}^{\mathrm{gs}}$ by applying the results developed in section 4.

Lemma 5.4. *Let $\mathcal{X}$ be an algebraic stack locally of finite type over a perfect field k. Let $\mathcal{X}_0$ be the reduced stack associated to $\mathcal{X}$. Let $\mathcal{X}_0^{\mathrm{gs}}$ be the open substack of GIT-like p-stable points. Let $\mathcal{X}_0^{\mathrm{gs}} \to X_0$ be a coarse moduli map (cf. Proposition 5.3) Let $U_0 \to X_0$ be an étale morphism from an affine scheme U_0. Let $\mathcal{X}_U \to \mathcal{X}^{\mathrm{gs}}$ be an étale morphism associated to $U_0 \to X_0$. (See Proposition 4.3.) Suppose that $\mathcal{X}_{U_0} = \mathcal{X}_0^{\mathrm{gs}} \times_{X_0} U_0$ has the form $[V_0/G]$, where V_0 is an affine U_0-scheme and G is a linearly reductive algebraic group over k, which acts on V_0 over U_0. Then there exist a nilpotent deformation $V_0 \to V$ and an action of G on V, which extends the action on V_0, such that $\mathcal{X}_U$ is isomorphic to $[V/G]$.*

Proof. We may and will assume $\mathcal{I}^2 = 0$. By our assumption, there exists a morphism $f : \mathcal{X}_{U_0} \to BG$ corresponding to the principal G-bundle $V_0 \to [V_0/G] \cong \mathcal{X}_{U_0}$. To prove our claim, we will show that there exists a dotted morphism filling the diagram

$$\begin{array}{ccc} \mathcal{X}_{U_0} & \longrightarrow & \mathcal{X}_U \\ \downarrow f & \swarrow & \\ BG. & & \end{array}$$

Note that $\mathcal{X}_{U_0} \to BG$ is representable because it arises from the G-morphism $V_0 \to \operatorname{Spec} k$. According to [32, Theorem 1.5], an obstruction for the existence of a dotted arrow lies in $\operatorname{Ext}^1(Lf^*\mathsf{L}_{BG/\operatorname{Spec} k}, \mathcal{I}) = \operatorname{Ext}^1(f^*\mathsf{L}_{BG/\operatorname{Spec} k}, \mathcal{I})$, where $\mathsf{L}_{BG/\operatorname{Spec} k}$ denotes the cotangent complex. We claim

$$\operatorname{Ext}^1(Lf^*\mathsf{L}_{BG/\operatorname{Spec} k}, \mathcal{I}) = 0.$$

To prove our claim, we first show that $\mathsf{L}_{BG/\operatorname{Spec} k}$ is of perfect amplitude in $[0,1]$. Consider the composition $\operatorname{Spec} k \xrightarrow{w} [\operatorname{Spec} k/G] = BG \to \operatorname{Spec} k$, where π is the natural projection. Then we have a distinguished triangle

$$Lw^*\mathsf{L}_{BG/\operatorname{Spec} k} \to \mathsf{L}_{\operatorname{Spec} k/\operatorname{Spec} k} \to \mathsf{L}_{\operatorname{Spec} k/BG} \to Lw^*\mathsf{L}_{BG/\operatorname{Spec} k}[1].$$

Since $\mathsf{L}_{\operatorname{Spec} k/\operatorname{Spec} k} = 0$, $\mathsf{L}_{\operatorname{Spec} k/BG} \cong Lw^*\mathsf{L}_{BG/\operatorname{Spec} k}[1]$ (the symbol $\cong$ means the existence of a quasi-isomorphism). The morphism w is flat surjective of finite type, and thus it suffices to show that $\mathsf{L}_{\operatorname{Spec} k/BG}$ is of perfect amplitude in $[-1,0]$. To this end, consider the flat base change $z : \operatorname{Spec} k \times_{BG} \operatorname{Spec} k \cong G \to \operatorname{Spec} k$ of $w : \operatorname{Spec} k \to BG$, we have $z^*\mathsf{L}_{\operatorname{Spec} k/BG} \cong \mathsf{L}_{G/\operatorname{Spec} k}$. Note that the morphism $G \to \operatorname{Spec} k$ is complete intersection in the sense of [15, (19.3.6)], and thus according to [19, Ch. III (3.2.6)] the cotangent complex $\mathsf{L}_{G/\operatorname{Spec} k}$ is of perfect amplitude in $[-1,0]$. Therefore we deduce that $\mathsf{L}_{\operatorname{Spec} k/BG}$ is of perfect amplitude in $[-1,0]$. (If G is smooth, then $\mathsf{L}_{G/\operatorname{Spec} k}$ is of perfect amplitude in $[0]$.) Hence we conclude that $\mathsf{L}_{BG/\operatorname{Spec} k}$ is of perfect amplitude in $[0,1]$. Since f is flat, thus $Lf^*\mathsf{L}_{BG/\operatorname{Spec} k} \cong f^*\mathsf{L}_{BG/\operatorname{Spec} k}$ is of perfect amplitude in $[0,1]$. Thus we have $\mathcal{RH}om(f^*\mathsf{L}_{BG/\operatorname{Spec} k}, \mathcal{I}) \cong \mathcal{RH}om(f^*\mathsf{L}_{BG/\operatorname{Spec} k}, \mathcal{O}_{\mathcal{X}_{U_0}}) \otimes^{\mathbb{L}} \mathcal{I}$, and $\mathcal{RH}om(f^*\mathsf{L}_{BG/\operatorname{Spec} k}, \mathcal{O}_{\mathcal{X}_{U_0}})$ is of perfect amplitude in $[-1,0]$. Note that $[V_0/G] \to BG$ is an affine morphism. In addition, the push forward with respect to $BG \to \operatorname{Spec} k$, is an exact functor from the category of quasi-coherent sheaves on BG to that of quasi-coherent sheaves on $\operatorname{Spec} k$. Therefore the global section functor on $[V_0/G] \cong \mathcal{X}_{U_0}$ is exact. Taking into account local-global spectral sequence for Ext groups, we see that $\operatorname{Ext}^1(Lf^*\mathsf{L}_{BG/\operatorname{Spec} k}, \mathcal{I}) = 0$ and there exists the desired arrow $\mathcal{X}_U \to BG$. The pullback of the natural projection $\operatorname{Spec} k \to BG$ by $\mathcal{X}_U \to BG$ gives rise to a principal G-bundle $V := \operatorname{Spec} k \times_{BG} \mathcal{X}_U \to \mathcal{X}_U$. Notice that $V \to \mathcal{X}_U$ is a flat deformation of $V_0 \to [V_0/G] \cong \mathcal{X}_{U_0}$ to $\mathcal{X}_U$. Thus V is affine. This means that $\mathcal{X}_U$ has the form $[V/G]$ where V is affine, as desired. □

Remark 5.5. (i) From the above proof, we see that if G is a smooth algebraic group over k, then $\mathsf{L}_{BG/\operatorname{Spec} k}$ is of perfect amplitude in $[1]$.

(ii) Also, the same argument shows the following: Let A be a local k-ring with residue field k and maximal ideal $\mathfrak{m}$. Let G be a linearly reductive algebraic group over k. Let $\hat{\mathcal{X}} \to \operatorname{Spf} A$ be a formal algebraic stack, that is, an inductive system of algebraic stacks $\mathcal{X}_n \to \operatorname{Spec} A_n$, where $A_n = A/\mathfrak{m}^{n+1}$. Suppose that $\mathcal{X}_0 = BG = [\operatorname{Spec} k/G]$. Then the system $\mathcal{X}_0 \hookrightarrow \mathcal{X}_1 \hookrightarrow \mathcal{X}_2 \hookrightarrow \cdots \mathcal{X}_n \hookrightarrow$ has the form

$$[\operatorname{Spec} k/G] \hookrightarrow [\operatorname{Spec} B_1/G] \hookrightarrow [\operatorname{Spec} B_2/G] \hookrightarrow \cdots [\operatorname{Spec} B_n/G] \hookrightarrow \cdots,$$

where B_i is an artin local k-ring over A_i for any i. (To see this, replace $f : \mathcal{X}_{U_0} \to BG$ in the proof of Lemma 5.4 by $\mathrm{Id} : \mathcal{X}_0 = BG \to BG$ and apply the same argument.) Furthermore, if G is smooth, we have $\mathrm{Ext}^i(\mathsf{L}_{BG/k}, (\mathfrak{m}^n/\mathfrak{m}^{n+1})|_{BG}) = 0$ for $i = 0, 1$, which deduces, by [32, Theorem 1.5], that the system of G-rings $\{B_i\}_{i \geq 0}$ is unique up to isomorphism.

Proposition 5.6. *Let $\mathcal{X}_0$ be the reduced stack associated to $\mathcal{X}$. Let $\mathcal{X}_0^{\mathrm{gs}}$ be the open substack of GIT-like p-stable points and $\pi_0 : \mathcal{X}_0^{\mathrm{gs}} \to X_0$ the coarse moduli map. Then for any closed point p on X_0 there exists an étale neighborhood $U_0 \to X_0$ and a closed point $u \in U_0$ lying over p, such that (i) if $k(u)$ denotes the residue field of u, then $\operatorname{Spec} k(u) \to X_0$ extends to $\alpha : \operatorname{Spec} k(u) \to \mathcal{X}_0$, (ii) U_0 is an affine $k(u)$-scheme, and (iii) $\mathcal{X}_{U_0} = \mathcal{X}_0^{\mathrm{gs}} \times_{X_0} U_0$ has the form $[V_0/G]$, where V_0 is finite over U_0, and $G \to \operatorname{Spec} k(u)$ is the automorphism group of α.*

The proof of this Proposition proceeds in several steps: Lemma 5.7, 5.8, 5.9, Proposition 5.10 and 5.11.

Lemma 5.7. *Let L be a field. Let $\mathcal{H} \to \operatorname{Spec} A$ be a group scheme that is affine and smooth over A, where A is a complete noetherian local L-ring with residue field L. Suppose that the fiber of $\mathcal{H}$ over the closed point of A is linearly reductive, and all geometric fibres of $\mathcal{H} \to SpecA$ are connected. Then there exists an isomorphism $\mathcal{H} \to H \times_L \operatorname{Spec} A$ of group schemes over $\operatorname{Spec} A$.*

Proof. Let $\hat{\mathcal{H}} \to \operatorname{Spf} A$ be the formal group scheme associated to $\mathcal{H} \to \operatorname{Spec} A$, which we can view as a smooth deformation of H to $\operatorname{Spf} A$. Note that H is linearly reductive and thus higher group cohomology groups are trivial. Thus by the deformation theory of group schemes (cf. [14, Expose III (3.7)]), we see that there exists a unique deformation of H to $\operatorname{Spf} A$, that is, $H \hat{\times}_L \operatorname{Spf} A$, and thus there exists an isomorphism between $\hat{\mathcal{H}}$ and $H \hat{\times}_L \operatorname{Spf} A$. Let $\mathcal{H}om_A(H \times_L A, \mathcal{H})$ (resp. $\mathcal{H}om_A(\mathcal{H}, H \times_L A)$) be a functor which to any $S \to \operatorname{Spec} A$ associates the set of homomorphisms $H \times_L A \times_A S \to \mathcal{H} \times_A S$ (resp. $\mathcal{H} \times_A S \to H \times_L A \times_A S$) of group schemes over S. According to [14, Expose XIX 2.6], $\mathcal{H}$ is a reductive group over $\operatorname{Spec} A$ (cf. [14, Expose XIX 2.7]). Then by [14, Expose XXIV 7.2.3], $\mathcal{H}om_A(H \times_L A, \mathcal{H})$ and $\mathcal{H}om_A(\mathcal{H}, H \times_L A)$ are represented by schemes locally of finite type and separated over A. By the above observation, there exist inductive systems $\{\operatorname{Spec} A/\mathfrak{m}_A^{n+1} \to \mathcal{H}om_A(H \times_L A, \mathcal{H})\}_{n \geq 0}$, $\{\operatorname{Spec} A/\mathfrak{m}_A^{n+1} \to \mathcal{H}om_A(\mathcal{H}, H \times_L A)\}_{n \geq 0}$ arising from the isomorphism $\hat{\mathcal{H}} \cong H \hat{\times}_L \operatorname{Spf} A$. Here $\mathfrak{m}_A$ is the maximal ideal of A. Since $\mathcal{H}om_A(H \times_L A, \mathcal{H})$ and $\mathcal{H}om_A(\mathcal{H}, H \times_L A)$ are schemes, these systems are uniquely extended to $u : \operatorname{Spec} A \to \mathcal{H}om_A(H \times_L A, \mathcal{H})$ and $v : \operatorname{Spec} A \to \mathcal{H}om_A(\mathcal{H}, H \times_L A)$. Let us denote by $u : H \times_L A \to \mathcal{H}$ and $v : \mathcal{H} \to H \times_L A$ the corresponding morphisms respectively (here we abuse notion). The uniqueness also implies that $u \circ v$ is the identity morphism $\mathcal{H}$. Similarly, $v \circ u$ is the identity morphism of $H \times_L A$. This completes the proof. □

Lemma 5.8. *Let p be a GIT-like p-stable closed point. There exists an open substack $\mathcal{U} \subset \mathcal{X}_0$ containing p, that has the property: there exists a closed subgroup $\mathcal{F} \subset I\mathcal{U}$ such that $\mathcal{F}$ is smooth and affine over $\mathcal{U}$, geometric fibers of $\mathcal{F} \to \mathcal{U}$ are connected, and $I\mathcal{U}/\mathcal{F}$ is finite over $\mathcal{U}$.*

Proof. This proof is parallel to Proposition 3.2 and the first, second and third paragraph of the proof of Theorem 3.1. Let $\xi : \operatorname{Spec} A \to \mathcal{X}$ be an effective versal deformation for p, which satisfies the properties (a) and (b) in Definition 5.1. By Lemma 2.2, we extend the versal deformation ξ to a smooth morphism $P : U \to \mathcal{X}$ where U is an affine scheme having a closed point u such that $A \cong \hat{\mathcal{O}}_{U,u}$ and $\xi \cong P|_A$. Consider the following contravariant functor F : (affine U-schemes) $\to$ (Sets) which to any $f : Y \to U$ associates the set of normal closed subgroup spaces $\mathcal{G} \subset \underline{\mathrm{Aut}}_{\mathcal{X},Y}(f^*\eta)$ over Y with following properties (i), (ii), (iii):

(i) $\mathcal{G}$ is smooth and affine over Y, and geometric fibers are connected,
(ii) $\underline{\mathrm{Aut}}_{\mathcal{X},Y}(f^*\eta)/\mathcal{G}$ is finite over Y,
(iii) if $\mathrm{pr}_1, \mathrm{pr}_2 : \underline{\mathrm{Aut}}_{\mathcal{X},Y}(f^*\eta) \times_{I\mathcal{X}} \underline{\mathrm{Aut}}_{\mathcal{X},Y}(f^*\eta) \rightrightarrows \underline{\mathrm{Aut}}_{\mathcal{X},Y}(f^*\eta)$ are the natural projections, then $\mathrm{pr}_1^{-1}(\mathcal{G}) = \mathrm{pr}_2^{-1}(\mathcal{G})$.

As in the proof of Proposition 3.2, using a standard limit argument (cf. Appendix) we see that the functor F is locally of finite presentation. By the approximation theorem, there exists an étale neighborhood $V \to U$ of u such that $F(V)$ is not empty. As in the proof of Theorem 3.1, this implies that there exist an open substack $\mathcal{U} \subset \mathcal{X}$ containing p and a closed subgroup $\mathcal{F} \subset I\mathcal{U}$ that has the desired property. □

(The first setup): For simplicity, we will replace $\mathcal{X}_0^{\mathrm{gs}}$ and X_0 by $\mathcal{X}$ and X respectively. For any $U \to X$, we write $\mathcal{X}_U$ for $\mathcal{X} \times_X U$. Let $p : \mathrm{Spec}\, K \to \mathcal{X}$ be a geometric point and $q : \mathrm{Spec}\, K \to X$ the composite of p and the coarse moduli map, where K is a separable (algebraic) closure of k. Shrinking $\mathcal{X}$ by Lemma 5.8 we assume that there is an closed subgroup $\mathcal{F} \subset I\mathcal{X}$, such that $I\mathcal{X}/\mathcal{F}$ is finite, $\mathcal{F} \to \mathcal{X}$ is smooth and affine, and all geometric fibers of $\mathcal{F} \to \mathcal{X}$ are connected. Fix an étale neighborhood $U \to X$ of $q : \mathrm{Spec}\, K \to X$, where U is an affine scheme. There exists a rigidification $\mathcal{X}_U \to \mathcal{Y}_U$ associated to $\mathcal{F}$ (cf. Theorem A.7). Note that $\mathcal{Y}_U$ has finite inertia stack and each stabilizer is linearly reductive. Then according to [2, Proposition 3.2] (and its proof), there exist an étale neighborhood $U' \to U$ and a closed point $u' \in U'$ (the image of $\mathrm{Spec}\, K \to U'$) such that (i) $\mathrm{Spec}\, k(u') \to U'$ extends to $\alpha : \mathrm{Spec}\, k(u') \to \mathcal{X}_{U'}$ and U' is a $k(u')$-scheme, and (ii) if G, H and $\bar{G}$ denote $I\mathcal{X}_{U'} \times_{\mathcal{X}_{U'},\alpha} k(u') = \underline{\mathrm{Aut}}_{k(u')}(\alpha)$, $\mathcal{F} \times_{\mathcal{X}_{U'},\alpha} k(u')$ and G/H respectively, then the base change $\mathcal{Y}_{U'} = \mathcal{Y}_U \times_U U'$ has the form $[W/\bar{G}]$, where W is an affine scheme which is finite over U' and has the trivial fiber $W \times_{U'} \mathrm{Spec}\, k(u') \cong \mathrm{Spec}\, k(u')$. For ease of notation, we replace U' and u' by U and u respectively. Let $L := k(u)$ be the residue field of u.

Lemma 5.9. *There exists an étale neighborhood $V \to U$ of q, such that (i) the morphism $W_V := W \times_U V \to \mathcal{Y}_U \times_U V$ lifts to $W_V \to \mathcal{X}_V$, and (ii) the group scheme $W_V \times_{\mathcal{X}_V} \mathcal{F} \to W_V$ is isomorphic to the constant group scheme $H \times_L W_V \to W_V$.*

Proof. Clearly, we may assume that $\mathcal{X}$ is quasi-compact. Observe first that it is enough to prove the case when the base field is algebraically closed (in particular, $L = K$). Note that any algebraic extension of k is separable. In addition, X and $\mathcal{X}$ is of finite presentation over k. Thus, standard limit arguments show that an étale morphism $V' \to U \times_L K$ with the desired properties always arises from some étale morphism $V \to U$ with such properties. Therefore we may and will assume that $k = L = K$.

Next we prove that there exists an étale neighborhood $V \to U$ that satisfies (i). Let O be a strict henselization of the local ring of $\mathcal{O}_{U,u}$. Since strict henselization commutes with finite extensions of rings, thus $W' := W \times_U \mathrm{Spec}\, O$ is the disjoint union of spectrums of strict henselian local rings, that is, $W' \cong \amalg_{i=1}^{r} \operatorname{Spec} A_i$, where A_i is a strict henselian local ring with residue field K for all i. Since $u \in U$ has the trivial fiber $\operatorname{Spec} K$, we see $i = 1$. We let $W' := \operatorname{Spec} A$. The rigidifying morphism $\mathcal{X}_U \to \mathcal{Y}_U$ is smooth, and thus there exists a lifting $W' \to \mathcal{X}_U$ of $W' \to \mathcal{Y}_U$. We can write $\operatorname{Spec} O \to U$ as $\operatorname{Spec}(\operatorname{colim}_\lambda B_\lambda)$, where $U_\lambda := \operatorname{Spec} B_\lambda \to U$ are étale neighborhoods of the (geometric) point $u : \operatorname{Spec} K \to U$. Note that $\mathcal{X}_U$ is of finite type over $\mathcal{Y}_U$. Hence there exists some étale neighborhood $U_\mu \to U$ such that $W_\mu := W \times_U U_\mu \to \mathcal{Y}_U$ extends to $W_\mu \to \mathcal{X}_U$.

Next we will show that there exists an étale neighborhood $U_{\mu'} \to U_\mu$ that satisfies (ii). Let $W^\diamond$ denote the spectrum of the completion $\hat{A}$ of A with respect to the maximal ideal. In other words, we also have $W^\diamond = \operatorname{Spec} A\hat{\otimes}_O \hat{O}$, where $\hat{O}$ is the completion of O with respect to the maximal ideal of O. Then by Lemma 5.7, the group scheme $W^\diamond \times_{\mathcal{X}} \mathcal{F}$ is isomorphic to $W^\diamond \times_K H$ over $W^\diamond$. Then considering the category of rings over A and applying Artin's approximation theorem, we see that there exists an isomorphism of group schemes between $W' \times_{\mathcal{X}_{U_\mu}} \mathcal{F}$ and $H \times_K W'$ over W'. Applying Theorem A.2 and A.5 to the system $\{U_{\lambda'}\}_{\lambda' \to \lambda}$, we conclude, by standard limit arguments, that there exists $U_{\mu'} \to U_\mu$ such that the group scheme $W_{\mu'} \times_{\mathcal{X}_{U_{\mu'}}} \mathcal{F} \to W_{\mu'}$ is isomorphic to $H \times_K W_{\mu'} \to W_{\mu'}$. □

Since $\mathcal{Y}_U \cong [W/\bar{G}]$, we see that $W_U \times_{\mathcal{X}} I\mathcal{X}/(W_U \times_{\mathcal{X}} \mathcal{F})$ is embedded in $W_U \times_L \bar{G}$. Let G' be the reduced scheme associated to G and set $\bar{G}' := G'/H$. The schemes G' and $\bar{G}'$ have naturally (smooth) group structures because L is perfect. Next we prove:

Proposition 5.10. *There exists an étale neighborhood $V \to U$ of q such that the group scheme $(W_V \times_{\mathcal{X}} I\mathcal{X}) \times_{(W_V \times_L \bar{G})} (W_V \times_L \bar{G}') \to W_V$ can be embedded into $W_V \times_L G'$ as the inverse image of $(W_V \times_{\mathcal{Y}_U} I\mathcal{Y}_U) \times_{(W_V \times_L \bar{G})} (W_V \times_L \bar{G}') \subset W_V \times_L \bar{G}'$ under $W_V \times_L G' \to W_V \times_L \bar{G}'$.*

Proof. (Step 0) As in the proof of Lemma 5.9, we may assume that the base field is algebraically closed. Thus we will let $k = L = K$. Moreover, by Lemma 5.9 we can take an étale neighborhood $V \to U$ that has the properties (i) and (ii) in Lemma 5.9. For ease of notation, we may replace V by U.

(Step 1) First we observe that it suffices to construct the desired embedding over $W^\diamond$. (For the notation $W^\diamond$, see the proof of Lemma 5.9. We will continue to use notation in the proof of Lemma 5.9.) Assume that the group scheme $\mathsf{G} := (W^\diamond \times_{\mathcal{X}} I\mathcal{X}) \times_{(W^\diamond \times_K \bar{G})} (W^\diamond \times_K \bar{G}') \to W^\diamond$ can be embedded into $W^\diamond \times_K G'$. Consider the natural projection $W' \times_K G' \to W' \times_K \bar{G}'$ and take the inverse image P of $W' \times_{\mathcal{X}} I\mathcal{X}/(W' \times_{\mathcal{X}} \mathcal{F}) \subset W' \times_K \bar{G}$ in $W' \times_K G'$. Let F be the functor which to any $a : Z \to \operatorname{Spec} O$ associates the set of isomorphisms of the group schemes from $(W_Z \times_{\mathcal{X}} I\mathcal{X}) \times_{(W_Z \times_K \bar{G})} (W_Z \times_K \bar{G}')$ to a^*P. Using Theorem A.2 and A.5 we easily see that F is locally of finite presentation. Then we can apply Artin's approximation to conclude that the group scheme $(W' \times_{\mathcal{X}} I\mathcal{X}) \times_{(W' \times_K \bar{G})} (W' \times_K \bar{G}')$ can be embedded into $W' \times_K G'$. By standard limit arguments, we see that there exists an étale neighborhood $V \to U$ of q such that the group scheme $(W_V \times_{\mathcal{X}} I\mathcal{X}) \times_{(W_V \times_K \bar{G})} (W_V \times_K \bar{G}') \to W_V$ can be embedded into $W_V \times_K G'$ in the desired way.

(Step 2) Next we prove that there exists an embedding of the scheme G into $W^\diamond \times_K G'$, that is, an isomorphism between G and $P^\diamond := P \times_{W'} W^\diamond$ as schemes. Note that $W^\diamond \times_{\mathcal{X}} \mathcal{F}$

is isomorphic to $W^\diamond \times_K H$ as group schemes. The quotient $\mathsf{Q} := \mathsf{G}/(W^\diamond \times_{\mathcal{X}} \mathcal{F})$ is a (non-flat) finite group scheme over $W^\diamond$. We have $\mathcal{Y}_U \cong [W_U/\bar{G}]$ and thus Q is a closed subgroup scheme of $W^\diamond \times_K \bar{G}'$. Let $\{\mathrm{Id} = g_1, g_2, \dots, g_n\}$ be the set of (K-valued) points on $\bar{G}'$. (Note that $\bar{G}'$ is finite étale over K. Namely, $\bar{G}'$ can be viewed as the finite group $\{\mathrm{Id} = g_1, g_2, \dots, g_n\}$.) The connected component of Q on which g_i lies, denoted by Q_i, is isomorphic to the pullback of the diagonal $W^\diamond \to W^\diamond \times_K W^\diamond$ by $(\mathrm{Id}_{W^\diamond}, g_i) : W^\diamond \to W^\diamond \times_K W^\diamond$. Therefore, each connected component of Q can be identified with a closed subscheme in $W^\diamond$. Let us identify Q_i with M_i, where M_i is a closed subscheme of $W^\diamond$. Let $b : \mathsf{G} \to \mathsf{Q}$ be the projection. Note that this projection is a principal H-bundle, and M_i is the disjoint union of the spectrums of complete local rings. (H is the identity component of G'.) Also, the principal bundle of a smooth algebraic group over a strict henselian local ring is trivial. Therefore G is isomorphic to $H \times_K \mathsf{Q}$ as a scheme over $W^\diamond$. We will regard G as $H \times_K \mathsf{Q}$. (Here we do not take care of group structures.) Thus, it is enough to prove that the group scheme $\mathsf{G} \to W^\diamond$ is isomorphic to the group scheme $H \times_K \mathsf{Q} \to W^\diamond$ (here $H \times_K \mathsf{Q}$ equips with the group structure arising from that of G' in the natural way).

To this end, we fix some notation. We write $\hat{W}'$ for the formal scheme associated to $W^\diamond$ and the adic topology on $\hat{A}$ arising from the maximal ideal. Let $\hat{\mathsf{G}}$ (resp. $(H \times_K \mathsf{Q})^\wedge$) be the formal scheme obtained from $\mathsf{G} \to W^\diamond$ (resp. $H \times_K \mathsf{Q} \to W^\diamond$) by completion over $\hat{W}'$. Let $\hat{\mathsf{Q}}_i$ be the formal scheme obtained by completing along the closed point of Q_i. Applying Remark 5.5 (ii) to the base change $\hat{\mathcal{X}} \to \operatorname{Spf} \hat{O}$ of $\mathcal{X} \to X$ to $\operatorname{Spf} \hat{O}$, we see that $\hat{\mathsf{G}}$ is a closed formal group subscheme of $G \hat{\times}_K \hat{W}'$. (The special fiber of $\hat{\mathcal{X}}$ is a classifying stack BG (cf. [24, (11.3)]).)

(Step 3) Next we will show that there exists a closed immersion of formal group schemes

$$\phi : \hat{\mathsf{G}} \to G' \hat{\times}_K \hat{W}'$$

over $\hat{W}'$, which identifies $\hat{\mathsf{G}}$ with $(H \times_K \mathsf{Q})^\wedge$. From (Step 2), it is clear that there exists a closed immersion of formal group schemes $\phi' : \hat{\mathsf{G}} \to G \hat{\times}_K \hat{W}'$. Moreover, we know the existence of an isomorphism $\hat{\mathsf{G}} \cong (H \times_K \mathsf{Q})^\wedge$ of formal schemes. Thus, it is enough to show that this immersion factors through $G' \hat{\times}_K \hat{W}' \subset G \hat{\times}_K \hat{W}'$. To this end, observe first that the "identity component" of $\hat{\mathsf{G}}$ is the closed subscheme $H \hat{\times}_K \hat{W}'$ (in $G \hat{\times}_K \hat{W}'$). The fiber of $\hat{\mathsf{G}} \to \hat{W}'$ over the closed point is G' and its identity component is H. Hence it is enough to prove that any smooth deformation of H to $\hat{W}'$ that is embedded in $G^0 \hat{\times}_K \hat{W}'$, is $H \hat{\times}_K \hat{W}'$. Here G^0 is the identity component of G. In characteristic zero, we have $G^0 = H$, thus our assertion is clear since "identity component" of $(H \times_K \mathsf{Q})^\wedge$ is a constant deformation of H to $\hat{W}'$, and any surjective endomorphism of a noetherian local ring is an isomorphism. In positive characteristic, note that by Nagata's classification of linearly reductive groups (see for example [30, page 27]), H is a torus. Moreover, G^0/H is a linearly reductive group, thus by the classification in [2, Proposition 2.13] we see that G^0/H is a diagonalizable group. Let $\hat{\mathcal{H}}$ be a smooth deformation of H to $\hat{W}'$ that is embedded in $G^0 \hat{\times}_K \hat{W}'$. ($\hat{\mathcal{H}}$ is a constant deformation of H.) Consider the composite homomorphism $\hat{\mathcal{H}} \to G^0 \hat{\times}_K \hat{W}' \to (G^0/H) \hat{\times}_K \hat{W}'$, where the second homomorphism is the natural projection. Since $\hat{\mathcal{H}}$ is isomorphic to $H \hat{\times}_K \hat{W}'$ as formal group schemes, the composite $\hat{\mathcal{H}} \to (G^0/H) \hat{\times}_K \hat{W}'$ comes from a homomorphism of abelian groups. Namely, if $H = \operatorname{Spec} K[M]$ and $G^0/H = \operatorname{Spec} K[N]$, then the composite arises from a homomorphism $N \to M$. However, any $N \to M$ is the trivial homomorphism since M is free and $N \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. This means $\hat{\mathcal{H}} \subset H \hat{\times}_K \hat{W}'$. As in characteristic zero case, we deduce that $\hat{\mathcal{H}}$ is equal to $H \hat{\times}_K \hat{W}'$. Using this, we will show that ϕ' factors through $G' \hat{\times}_K \hat{W}' \subset G \hat{\times}_K \hat{W}'$. To see this, we may consider the problem by restricting $\hat{\mathsf{G}}$ to some open and closed subgroup of $\hat{\mathsf{G}} \hat{\times}_{\hat{W}'} \hat{\mathsf{Q}}_i$, that is smooth over $\hat{\mathsf{Q}}_i$ for each i. ($\hat{\mathsf{G}}$ is the union of such subformal schemes.) Namely, we may assume that $\hat{\mathsf{G}}$ is smooth over $\hat{W}'$, that is, we have an isomorphism $\hat{\mathsf{G}} \cong G' \hat{\times}_K \hat{W}'$ of formal group schemes. Consider the composite $\rho \circ \phi' : \hat{\mathsf{G}} \to G \hat{\times}_K \hat{W}' \to (G/G') \hat{\times}_K \hat{W}'$, where ρ is the natural projection. It suffices to prove that the image of $\rho \circ \phi'$ is trivial. In characteristic zero case, it is clear. In positive characteristic, we put $G/G' = \operatorname{Spec} K[N]$, where N is an abelian group whose order is a power of $\operatorname{ch}(k)$. For each connected component $\hat{\mathsf{C}}_i$ of $\hat{\mathsf{G}}$, choose a section $s_i : \hat{W}' \to \hat{\mathsf{C}}_i$. Then the composite $\rho \circ \phi'$ is uniquely determined by the images $\rho \circ \phi'(s_i)$. Note that the number of connected components of $\hat{\mathsf{G}}$ is prime to $\operatorname{ch}(k)$. Moreover, the identity component of $\hat{\mathsf{G}}$ maps to the unit of $(G/G') \hat{\times}_K \hat{W}'$, hence for each i, there exists a positive integer n, such that $\rho \circ \phi'(s_i^n)$ is the unit of $(G/G') \hat{\times}_K \hat{W}'$ and n is prime to $\operatorname{ch}(k)$. Therefore, we see that $\rho \circ \phi'(s_i)$ is the unit. This implies our claim. Hence we have the desired morphism ϕ.

(Step 4) We then claim that ϕ is induced by an isomorphism $\mathsf{G} \to H \times_K \mathsf{Q}$ of group schemes over $W^{\diamond}$. For any i, let $\bar{G}'_i$ be a subgroup of $\bar{G}'$, which is generated by g_i. (Here we regard $\bar{G}'$ as a finite group.) Let G'_i be the preimage of $\bar{G}'_i$ under $G' \to \bar{G}'$. For any i, set $\mathsf{G}_i := G'_i \times_K \mathsf{Q}_i$ via a fixed isomorphism $\mathsf{G} \cong H \times_K \mathsf{Q}$ of schemes. The scheme G_i can naturally be viewed as a closed and open subgroup scheme of $\mathsf{G} \times_{W^{\diamond}} \mathsf{Q}_i$ (over Q_i). (Note that at this point we do not claim that the group structure of G_i comes from that of G'_i.) On the other hand, we denote by $G'_i \hat{\times}_K \hat{\mathsf{Q}}_i$ the formal group scheme over $\hat{\mathsf{Q}}_i \subset W'$, whose group structure arises from that of G'_i. The formal group scheme $G'_i \hat{\times}_K \hat{\mathsf{Q}}_i \to \hat{\mathsf{Q}}_i$ is a closed and open subgroup scheme of $(H \times_K \mathsf{Q})^{\wedge} \hat{\times}_{\hat{W}'} \hat{\mathsf{Q}}_i$. The morphism ϕ identifies the formal completion $\hat{\mathsf{G}}_i$ of G_i over $\hat{\mathsf{Q}}_i$ with $G'_i \hat{\times}_K \hat{\mathsf{Q}}_i$ (as a formal group scheme). We denote by ϕ_i the restriction of ϕ to $\hat{\mathsf{G}}_i$. We will extend ϕ_i to an isomorphism $\mathsf{G}_i \to G'_i \times_K \mathsf{Q}_i$ of group schemes over Q_i. (Note that the target $G'_i \times_K \mathsf{Q}_i$ has the group structure coming from G'_i.) To this aim, consider the functor $\operatorname{Hom}^{\flat}_{\mathsf{Q}_i}(\mathsf{G}_i, G'_i \times_K \mathsf{Q}_i)$ over the category of Q_i-schemes, which to any $T \to \mathsf{Q}_i$ associates the set of morphisms of schemes $f : \mathsf{G}_i \to G'_i \times_K \mathsf{Q}_i$ over Q_i, such that (i) the restriction to the identity component of G_i is a homomorphism of group schemes, and (ii) f commutes with the right actions of identity components of G_i and $G'_i \times_K \mathsf{Q}_i$ on G_i and $G'_i \times_K \mathsf{Q}_i$ respectively. Note that the identity component of G_i is isomorphic to $H \times_K \mathsf{Q}_i$ as a group scheme. In addition, since we have an isomorphism $\mathsf{G}_i \cong G'_i \times_K \mathsf{Q}_i$ of schemes, by choosing a closed point c_j on each connected component of G'_i we can take sections $s_j : \mathsf{Q}_i \to \mathsf{G}_i$. The functor $\operatorname{Hom}^{\flat}_{\mathsf{Q}_i}(\mathsf{G}_i, G'_i \times_K \mathsf{Q}_i)$ is represented by a scheme over Q_i. Indeed, a morphism $f : \mathsf{G}_i \to G'_i \times_K \mathsf{Q}_i$ with the properties (i) and (ii) is uniquely determined by the restriction of f to the connected component of G_i and the image of sections $\{s_j\}_j$ under f. Therefore, if G^0_i denotes the identity component of G_i, then $\operatorname{Hom}_{\mathsf{Q}_i}(\mathsf{G}^0_i, G'_i \times_K \mathsf{Q}_i) \times_{\mathsf{Q}_i} (\mathsf{Q}_i \times_K (G'_i)^r)$ represents $\operatorname{Hom}^{\flat}_{\mathsf{Q}_i}(\mathsf{G}_i, G'_i \times_K \mathsf{Q}_i)$, where $\operatorname{Hom}_{\mathsf{Q}_i}(\mathsf{G}^0_i, G'_i \times_K \mathsf{Q}_i)$ is the hom scheme which to any $T \to \mathsf{Q}_i$ associates the set of group homomorphisms $\mathsf{G}_i^0 \times_{\mathsf{Q}_i} T \to G'_i \times_K T$, and $r+1$ is the number of the connected components of G_i. By [14, Expose XXIV 7.2.3], $\mathrm{Hom}_{\mathsf{Q}_i}(\mathsf{G}_i^0, G'_i \times_K \mathsf{Q}_i)$ is a scheme (locally of finite type and separated) over Q_i. Thus, $\mathrm{Hom}^{\flat}_{\mathsf{Q}_i}(\mathsf{G}_i, G'_i \times_K \mathsf{Q}_i)$ is represented by a scheme over Q_i. The morphism ϕ_i induces an inductive system of sections $\{\mathrm{Spec}\, R_i/\mathfrak{m}_i^{n+1} \to \mathrm{Hom}^{\flat}_{\mathsf{Q}_i}(\mathsf{G}_i, G'_i \times_K \mathsf{Q}_i)\}_{n \geq 0}$, where $\mathrm{Spec}\, R_i = \mathsf{Q}_i$ and $\mathfrak{m}_i$ is the maximal ideal of R_i. Since $\mathrm{Hom}^{\flat}_{\mathsf{Q}_i}(\mathsf{G}_i, G'_i \times_K \mathsf{Q}_i)$ is a scheme, the system is uniquely extended to a section $\mathsf{Q}_i \to \mathrm{Hom}^{\flat}_{\mathsf{Q}_i}(\mathsf{G}_i, G'_i \times_K \mathsf{Q}_i)$. It gives rise to a morphism $\Phi_i : \mathsf{G}_i \to G'_i \times_K \mathsf{Q}_i$ that is an extension of ϕ_i. To check that Φ_i is a group homomorphism, it suffices to show that Φ_i commutes with respect to group structures. It follows from the facts that ϕ_i is a group homomorphism and the natural completion map $\Gamma(\mathsf{G}_i \times_{\mathsf{Q}_i} \mathsf{G}_i, \mathcal{O}_{\mathsf{G}_i \times_{\mathsf{Q}_i} \mathsf{G}_i}) \to \Gamma(\mathsf{G}_i \times_{\mathsf{Q}_i} \mathsf{G}_i, \mathcal{O}_{\mathsf{G}_i \times_{\mathsf{Q}_i} \mathsf{G}_i})^{\wedge}$ is injective (consider the compatibility of Φ_i with group structures in terms of ring homomorphisms). For any i and j, the intersection $(\mathsf{G}_i \times_{\mathsf{Q}_i} (\mathsf{Q}_i \cap \mathsf{Q}_j)) \cap (\mathsf{G}_j \times_{\mathsf{Q}_j} (\mathsf{Q}_i \cap \mathsf{Q}_j))$ is a (reductive) smooth, affine group scheme over $\mathsf{Q}_i \cap \mathsf{Q}_j$. Two morphisms ϕ_i and ϕ_j coincide in the intersection (after associating the formal schemes), and thus taking into account the above argument, we see that Φ_i and Φ_j coincide in $(\mathsf{G}_i \times_{\mathsf{Q}_i} (\mathsf{Q}_i \cap \mathsf{Q}_j)) \cap (\mathsf{G}_j \times_{\mathsf{Q}_j} (\mathsf{Q}_i \cap \mathsf{Q}_j))$. Therefore, Φ_i's are glued together, and thus we have an isomorphism $\mathsf{G} \to H \times_K \mathsf{Q}$ of group schemes over $W^{\diamond}$. □

(The second setup): Taking into account Lemma 5.9 and Proposition 5.10, assume further that there exists a lifting $W \to \mathcal{X}_U$ of $W \to \mathcal{Y}_U \cong [W/\bar{G}]$ such that:

(i) U is an affine scheme of finite type over L,
(ii) the pullback of $\mathcal{F} \to \mathcal{X}$ by the composite $W \to \mathcal{X}_U \to \mathcal{X}$ is isomorphic to $W \times_L H$,
(iii) $(W \times_{\mathcal{X}} I\mathcal{X}) \times_{(W \times_L \bar{G})} (W \times_L \bar{G}')$ is embedded into $W \times_L G'$ as the inverse image of $(W \times_{\mathcal{Y}_U} I\mathcal{Y}_U) \times_{(W \times_L \bar{G})} (W \times_L \bar{G}') \subset W \times_L \bar{G}'$ under $W \times_L G' \to W \times_L \bar{G}'$.

Proposition 5.11. *With the notation as above, $\mathcal{X}_U$ has the form $[W/G]$.*

Proof. Note first that we have the lifting $\xi : W \to \mathcal{X}_U$ and thus by Theorem A.7 (iii) there exists a natural isomorphism $\alpha : W \times_{\mathcal{Y}_U} \mathcal{X}_U \to W \times_L BH$ over W. This isomorphism is described as follows. Let $\rho : \mathcal{X}_U \to \mathcal{Y}_U$ be the rigidifying morphism. For any $\omega : T \to W$, an object which belongs to $W \times_{\mathcal{Y}_U} \mathcal{X}_U(T)$ amounts to data

$$\{\omega : T \to W, \eta : T \to \mathcal{X}_U,\ \theta : T \to \underline{\mathrm{Isom}}_{\mathcal{Y}_U,T}(\rho(\eta), \bar{\xi}(\omega)) \cong \underline{\mathrm{Isom}}_{\mathcal{X}_U,T}(\eta, \xi(\omega))/H\}$$

where $\bar{\xi}$ is the composite $W \to \mathcal{X}_U \to \mathcal{Y}_U$. Then consider the principal H-bundle

$$\underline{\mathrm{Isom}}_{\mathcal{X}_U,T}(\eta, \xi(\omega)) \times_{\underline{\mathrm{Isom}}_{\mathcal{X}_U,T}(\eta,\xi(\omega))/H,\theta} T \to T$$

and denote by $\phi_\theta : T \to BH$ the corresponding morphism. Then α sends (ω, η, θ) to (ω, ϕ_θ). There exists the diagram

$$\begin{array}{ccccc} W \times_L \bar{G} & \longrightarrow & W \times_L BH & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ W & \longrightarrow & \mathcal{X}_U & \longrightarrow & \mathcal{Y}_U \end{array}$$

where the right vertical arrow is determined by $W \to [W/\bar{G}]$ and all squares are cartesian. Note that $W \times_L \bar{G} \to W \times_L BH$ is a morphism over W. We will prove the claim:

Claim 5.11.1. *The composite* $W \times_L \bar{G} \to W \times_L BH \to BH$ *factors through second projection* $W \times_L \bar{G} \to \bar{G}$.

Before the proof of the above claim, assuming that the claim holds, we will complete the proof of our Proposition. By the claim, $W \times_L \bar{G} \to W \times_L BH$ arises from $\bar{G} \to BH$. The fiber of $W \times_L \bar{G} \to W \times_L BH$ over $\operatorname{Spec} L \to W \times_L \operatorname{Spec} L \to W \times_L BH$ is $G \to \operatorname{Spec} L$. Here the first morphism $\operatorname{Spec} L \to W \times_L \operatorname{Spec} L$ is determined by the unique point on W lying over $u \in U$, and the second morphism $W \times_L \operatorname{Spec} L \to W \times_L BH$ is defined by the identity of W and the natural projection $\operatorname{Spec} L \to BH$. Thus we obtain the diagram

$$\begin{array}{ccccc} W \times_L G & \longrightarrow & W \times_L \bar{G} & \longrightarrow & W \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow & & \downarrow \\ W & \longrightarrow & W \times_L BH & \longrightarrow & \mathcal{X}_U \end{array}$$

where all squares are cartesian. This means that $\mathcal{X}_U$ is isomorphic to $[W/G]$ where the action of G on W is an extension of that of $\bar{G}$. □

Proof of Claim 5.11.1. To prove our Claim, it suffices to prove that the restriction of $W \times_L \bar{G} \to W \times_L BH \to BH$ to $W \times_L \bar{G}'$ factors through the second projection $W \times_L \bar{G}' \to \bar{G}'$. To see this reduction, it is enough to show that the morphism $\phi : W \times_L \bar{G}' \to BH$ uniquely extends to a morphism $W \times_L \bar{G} \to BH$, and any morphism $\psi : \bar{G}' \to BH$ uniquely extends to a morphism $\bar{G} \to BH$. To see this, set $\mathcal{I}$ and $\mathcal{J}$ are the ideals of $\mathcal{O}_{W \times_L \bar{G}}$ and $\mathcal{O}_{\bar{G}}$ determined by $W \times_L \bar{G}'$ and $\bar{G}'$ respectively, and assume that $\mathcal{I}^2 = 0$ and $\mathcal{J}^2 = 0$. By the deformation theory [32, Theorem 1.5], it suffices to check $\operatorname{Ext}^0(L\phi^*\mathsf{L}_{BH/\operatorname{Spec} L}, \mathcal{I}) = \operatorname{Ext}^1(L\phi^*\mathsf{L}_{BH/\operatorname{Spec} L}, \mathcal{I}) = 0$ and $\operatorname{Ext}^0(L\psi^*\mathsf{L}_{BH/\operatorname{Spec} L}, \mathcal{J}) = \operatorname{Ext}^1(L\psi^*\mathsf{L}_{BH/\operatorname{Spec} L}, \mathcal{J}) = 0$. Notice that ϕ and ψ are flat. (Any morphism $X \to BH$ factors through the natural projection $\operatorname{Spec} L \to BH$ after replacing X with some fppf cover $X' \to X$.) Thus by the same argument in the proof of Lemma 5.4, it suffices to see that $\mathsf{L}_{BH/\operatorname{Spec} L}$ is of perfect amplitude in [1]. It follows from Remark 5.5 (i).

Next observe the morphism $W \times_L \bar{G} \to W \times_{\mathcal{Y}_U} \mathcal{X}_U \cong W \times_L BH$. Let (ω, g) be a T-valued point of $W \times_L \bar{G}$, where $\omega : T \to W$ and $g : T \to \bar{G}$. Considering the cartesian diagram

$$\begin{array}{ccc} W \times_L \bar{G} & \longrightarrow & W \\ \downarrow & & \downarrow \\ W \times_{\mathcal{Y}_U} \mathcal{X}_U & \longrightarrow & \mathcal{X}_U \end{array}$$

where the top horizontal arrow is the action of $\bar{G}$, we see that the image of (ω, g) is described by

$$\{\omega \in W(T), g\omega \in \mathcal{X}_U(T), g \in \underline{\operatorname{Isom}}_{\mathcal{Y}_U,T}(\bar{\omega}, \bar{g\omega})(T)\}$$

where $\bar{\omega}$ and $\bar{g\omega}$ denote the images of ω and $g\omega$ in $\mathcal{Y}_U$ respectively, and for any $f : T' \to T$ we identify $\underline{\operatorname{Isom}}_{\mathcal{Y}_U,T}(\bar{\omega}, \bar{g\omega})(T')$ with $\{g' \in \bar{G}(T') |\ g'(f^*\omega) = f^*(g\omega)\}$. Thus we have the diagram

$$\begin{array}{ccccc} \mathcal{P} & \longrightarrow & \underline{\mathrm{Isom}}_{\mathcal{X}_U,T}(\omega, g\omega) & & \\ \downarrow & & \downarrow & & \\ T & \longrightarrow & \underline{\mathrm{Isom}}_{\mathcal{Y}_U,T}(\bar{\omega}, g\bar{\omega}) & \longrightarrow & \bar{G}\times_L T \end{array}$$

where the square is cartesian, and the lower right horizontal arrow is a closed immersion. The principal H-bundle $\mathcal{P}$ corresponds to the image of $W\times_L \bar{G}\to W\times_L BH\to BH$. Suppose $g\in \bar{G}'(T)$. Let $\underline{\mathrm{Isom}}'_{\mathcal{Y}_U,T}(\bar{\omega}, g\bar{\omega}) := \underline{\mathrm{Isom}}_{\mathcal{Y}_U,T}(\bar{\omega}, g\bar{\omega}) \times_{\bar{G}\times_L T} (\bar{G}'\times_L T)$ and $\underline{\mathrm{Isom}}'_{\mathcal{X}_U,T}(\omega, g\omega) := \underline{\mathrm{Isom}}_{\mathcal{X}_U,T}(\omega, g\omega) \times_{\underline{\mathrm{Isom}}_{\mathcal{Y}_U,T}(\bar{\omega},g\bar{\omega})} \underline{\mathrm{Isom}}'_{\mathcal{Y}_U,T}(\bar{\omega}, g\bar{\omega})$. Here we claim that $\underline{\mathrm{Isom}}'_{\mathcal{X}_U,T}(\omega, g\omega)\to \underline{\mathrm{Isom}}'_{\mathcal{Y}_U,T}(\bar{\omega}, g\bar{\omega})$ can be identified with the principal H-bundle $\underline{\mathrm{Isom}}'_{\mathcal{Y}_U,T}(\bar{\omega}, g\bar{\omega}) \times_{(\bar{G}'\times_L T)} (G'\times_L T) \to \underline{\mathrm{Isom}}'_{\mathcal{Y}_U,T}(\bar{\omega}, g\bar{\omega})$. In particular, $\mathcal{P}$ depends only on $g\in \bar{G}'(T)$. To this end, we first consider the case when $g: T\to \bar{G}'$ extends to $\tilde{g}: T\to G'$ and $\omega\cong g\omega$ in $\mathcal{X}_U$. If $\mathrm{Stab}'(\omega) := \underline{\mathrm{Isom}}'_{\mathcal{Y}_U,T}(\bar{\omega},\bar{\omega}) \subset \bar{G}'\times_L T$, then $\underline{\mathrm{Isom}}_{\mathcal{Y}_U,T}(\bar{\omega}, g\bar{\omega})$ is $g\cdot \mathrm{Stab}'(\omega)$ in $\bar{G}'\times_L T$. Thus if G' acts on W via $G'\to \bar{G}'$ and $\widetilde{\mathrm{Stab}'(\omega)}$ denotes the stabilizer group scheme of ω with respect to the action of G', then the inverse image of $\underline{\mathrm{Isom}}'_{\mathcal{Y}_U,T}(\bar{\omega}, g\bar{\omega})$ in $G'\times_L T$ is $\tilde{g}\cdot\widetilde{\mathrm{Stab}'(\omega)}$. By our assumption and the second setup (iii), we have $\widetilde{\mathrm{Stab}'(\omega)} \cong \underline{\mathrm{Aut}}_{\mathcal{X}_U,T}(\omega)\times_{\underline{\mathrm{Isom}}_{\mathcal{Y}_U,T}(\bar{\omega},\bar{\omega})} \underline{\mathrm{Isom}}'_{\mathcal{Y}_U,T}(\bar{\omega},\bar{\omega})$ over T. In the general case, take an étale surjective morphism $T'\to T$ so that $T'\to T\to \bar{G}'$ extends to $\tilde{g}: T'\to G'$ and $w|_{T'}\cong gw|_{T'}$ in $\mathcal{X}_U$. Then by the previous case and the descent theory, there exists a dotted closed immersion

$$\begin{array}{ccccc} \underline{\mathrm{Isom}}'_{\mathcal{X}_U,T'\times_T T'}(\omega|_{T'\times_T T'}, \tilde{g}\omega|_{T'\times_T T'}) & \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} & \underline{\mathrm{Isom}}'_{\mathcal{X}_U,T'}(\omega|_{T'}, \tilde{g}\omega|_{T'}) & \longrightarrow & \underline{\mathrm{Isom}}'_{\mathcal{X}_U,T}(\omega, \tilde{g}\omega) \\ \downarrow & & \downarrow & & \vdots \\ G'\times_L (T'\times_T T') & \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} & G'\times_L T' & \longrightarrow & G'\times_L T, \end{array}$$

where the left and central vertical arrows are pullbacks of

$$\underline{\mathrm{Isom}}'_{\mathcal{Y}_U,T'\times_T T'}(\bar{\omega}|_{T'\times_T T'}, g\bar{\omega}|_{T'\times_T T'}) \subset \bar{G}'\times_L (T'\times_T T')$$

and $\underline{\mathrm{Isom}}'_{\mathcal{Y}_U,T'}(\bar{\omega}|_{T'}, g\bar{\omega}|_{T'}) \subset \bar{G}'\times_L T'$ respectively. This implies our Claim. □

Theorem 5.12. *Let $\mathcal{X}$ be an algebraic stack locally of finite type over a perfect field. Then the open substack $\mathcal{X}^{\mathrm{gs}}$ of GIT-like p-stable points has a coarse moduli map $\pi: \mathcal{X}^{\mathrm{gs}}\to X$, such that X is locally of finite type. Moreover, π is universally closed and quasi-finite, which induces an isomorphism $\mathcal{O}_X\to \pi_*\mathcal{O}_{\mathcal{X}}$. If $X'\to X$ is a flat morphism, then the second projection $\mathcal{X}^{\mathrm{gs}}\times_X X'\to X'$ is a coarse moduli map.*

Proof. Let $\mathcal{X}_0^{\mathrm{gs}}$ be the reduced algebraic stack associated to $\mathcal{X}^{\mathrm{gs}}$. By Proposition 5.3 there exists a coarse moduli map $\pi_0: \mathcal{X}_0^{\mathrm{gs}}\to X_0$. Then by Proposition 5.11, any point on X_0 has an étale neighborhood $U_0\to X_0$, such that $\mathcal{X}_0^{\mathrm{gs}}\times_{X_0} U_0$ has the form $[W_0/G]$, where W_0 is an affine scheme which is finite over the affine scheme U_0, and G is a linearly reductive group. By Lemma 5.4, the étale deformation $\mathcal{X}_U\to\mathcal{X}^{\mathrm{gs}}$ of $\mathcal{X}_0^{\mathrm{gs}}\times_{X_0} U_0\to \mathcal{X}_0^{\mathrm{gs}}$ to $\mathcal{X}^{\mathrm{gs}}$ has the form $[W/G]$. Since G is linearly reductive, thus

$$\Gamma(\mathcal{X}_U, \mathcal{O}_{\mathcal{X}_U}) = \Gamma(W, \mathcal{O}_W)^G \to \Gamma(\mathcal{X}_0^{\mathrm{gs}}\times_{X_0} U_0, \mathcal{O}_{\mathcal{X}_0^{\mathrm{gs}}\times_{X_0} U_0}) = \Gamma(W_0, \mathcal{O}_{W_0})^G$$

is surjective. Applying Lemma 4.13 we conclude that the nilpotent deformation $\mathcal{X}_0^{\mathrm{gs}} \to \mathcal{X}^{\mathrm{gs}}$ has the property ($\mathbb{L}$) in section 4. Therefore according to Proposition 4.5, $\mathcal{X}^{\mathrm{gs}}$ has a coarse moduli map. By the construction, it is clear that π is universally closed and quasi-finite. The last assertion also follows from Proposition 4.5. □

From Proposition 4.5, Theorem 5.12, Lemma 5.4, Proposition 5.6, we deduce

Corollary 5.13. *Let $\mathcal{X}$ be an algebraic stack locally of finite type over a perfect field. Let $\pi : \mathcal{X}^{\mathrm{gs}} \to X$ be a coarse moduli map. Then every point on X admits an étale neighborhood $U \to X$ such that $\mathcal{X}^{\mathrm{gs}} \times_X U \to U$ has the form $[W/G]$ where W is finite over U, and G is a linearly reductive group acting on W over U. More precisely, U is an affine scheme over a finite (separable) extention $k' \supset k$ of fields and G is a linearly reductive group over k', which acts on W over U so that the quotient stack $[W/G]$ is isomorphic to $\mathcal{X}^{\mathrm{gs}} \times_X U$ over U.*

Remark 5.14 (Isovariant étale)**.** Corollary 5.13 says that GIT-like stable point is approximated by a quotient stack via an étale morphism $[W/G] \to \mathcal{X}$ (with the above notation). Moreover, $[W/G] \to \mathcal{X}$ preserves the structures of automorphisms. More precisely, if $I[W/G] \to [W/G]$ and $I\mathcal{X} \to \mathcal{X}$ denote the inertia stacks of $[W/G]$ and $\mathcal{X}$ respectively, then the natural morphism $I[W/G] \to I\mathcal{X} \times_{\mathcal{X}} [W/G]$ is an isomorphism. It is quite useful. For instance, generalizing Thomason's descent theory ([36]) Joshua developed the *isovariant étale descent theory* for G-theory on algebraic stacks ([21, Section 5]). Informally, an isovariant étale morphism of algebraic stacks is an étale morphism which preserves the structures of automorphisms (cf. [21, DEFINITIONS 3.1 (iii)]), and the notion of isovariant étaleness is crucial for the descent theory of G-theory. Note that the above morphism $[W/G] \to \mathcal{X}$ is isovariant étale. Moreover, under the stable condition (see Definition 5.15 , Remark 5.16), one can reduce G-theory of algebraic stacks to equivariant G-theory, that is, the case of an affine scheme provided with the action of a reductive group. Since this topic is beyond the scope of this paper, we will discuss these issues and applications in another paper.

Motivated by Theorem 5.12 and Corollary 5.13, we propose a class of Artin stacks.

Definition 5.15. Let $\mathcal{X}$ be an algebraic stack locally of finite type over a perfect field k. We say that $\mathcal{X}$ is *of GIT-like stable type* over k (or simply *stable algebraic stack* over k) if $\mathcal{X}^{\mathrm{gs}} = \mathcal{X}$. In other words, $\mathcal{X}$ is of GIT-like stable type if the automorphism groups of all closed points are linearly reductive and any closed point admits an open neighborhood $\mathcal{U} \subset \mathcal{X}$, such that the inertia stack $I\mathcal{U}_0$ of the reduced stack associated to $\mathcal{U}$ has a closed subgroup $\mathcal{F}_0$ that satisfies:

(i) $\mathcal{F}_0$ is smooth and affine over $\mathcal{U}_0$, and all geometric fibers are connected,
(ii) the quotient $I\mathcal{U}_0/\mathcal{F}_0$ is finite.

Remark 5.16. By Remark 5.2, in the case of characteristic zero $\mathcal{X}$ is of GIT-like stable type if and only if the followings hold: (i) the automorphism group of every geometric point on $\mathcal{X}$ is reductive, (ii) if $\mathcal{X}_{\mathrm{red}}$ denotes the reduced stack associated to $\mathcal{X}$ then the inertia stack $I\mathcal{X}_{\mathrm{red}}$ is Zariski locally equidimensional over $\mathcal{X}_{\mathrm{red}}$, (iii) the identity component of $I\mathcal{X}_{\mathrm{red}}$ is affine over $\mathcal{X}_{\mathrm{red}}$, and (iv) the quotient of $I\mathcal{X}_{\mathrm{red}}$ by its identity component is finite over $\mathcal{X}_{\mathrm{red}}$. (See Theorem 6.9.)

In virtue of the works of Inaba [20] Lieblich [25], Toën and Vaquié [38], we have the moduli Artin stack $\mathfrak{D}_{\mathrm{p}}^b(X)$ of objects in the derived category of perfect complexes (satisfying a certain condition) on a proper flat scheme X. Also, the general result of [38] yields the moduli stack of complexes of representations of a finite quiver. Recent developments on derived category reveal the importance of these moduli stacks. On the other hand, we would like to call attention to the fact: one cannot interpret these stacks as quotient stacks (at least a priori since abstract approaches are applied). It would be interesting to apply our GIT-like p-stability to these stacks. We hope to come back to this topic in a future work.

6. Comparing with Mumford's Geometric Invariant Theory

By an algebraic scheme over a field k we mean a scheme locally of finite type and separated over a field k. In this section, we assume that the base field k is algebraically closed of characteristic zero except Theorem 6.9. In this section we discuss the relationship between our GIT-like p-stability and Geometric Invariant Theory due to Mumford ([30]). Moreover, we prove Theorem B.

Let X be an algebraic scheme over k. Let G be a linearly reductive group scheme over k. (An algebraic group over k is linearly reductive if and only if it is reductive.) Let $\sigma : G \times_k X \to X$ be an action on X. Let $X(\mathrm{Pre}) \subset X$ be the open subset of X, consisting of pre-stable points in the sense of [30, Definition 1.7]. The main purpose of this section is to prove:

Theorem 6.1. *Let* $[X(\mathrm{Pre})/G]$ *be the open substack of* $[X/G]$*, associated to* $X(\mathrm{Pre})$*. Let* $[X/G]^{\mathrm{gs}}$ *be the open substack consisting of GIT-like p-stable points on* $[X/G]$*. Let* $\mathcal{S}$ *be the maximal open substack of* $[X/G]$*, admitting a coarse moduli space that is a scheme. Then*

$$[X(\mathrm{Pre})/G] = [X/G]^{\mathrm{gs}} \cap \mathcal{S}.$$

Remark 6.2. We would like to invite your attention to some of the advantages of our approach. First of all, as in Keel-Mori theorem coarse moduli are allowed to be algebraic spaces (rather than schemes). It makes the framework more amenable. Moreover, contrary to Geometric Invariant Theory our approach does not rely on global quotient structures (cf. Remark 5.16). It is important for several reasons. First, in practice it is hard to prove that a give algebraic stack is a quotient stack. In addition, algebraic stacks do not necessarily have such structures (see [17, section 2]). Secondly, as mentioned at the close of section 5, nowadays we often use abstract methods for constructing algebraic moduli stacks, such as Artin's representability theorem, and Geometric Invariant Theory is not applicable to stacks constructed by such abstract methods. For example, Lieblich applied the Artin's theorem to the constructions of moduli stacks of twisted sheaves ([26]) and complexes ([25]), and Olsson used the theorem in the work on Hom stacks ([34]). In addition, Lurie proved an amazing generalization of Artin's representability theorem to derived algebraic geometry ([28]).

Remark 6.3. According to Theorem 6.1, we may say that in a sense the notion of GIT-like p-stability is an intrinsic generalization of pre-stability, that is, the "local part" of Geometric Invariant Theory.

By the theory of good moduli spaces [3, Theorem 6.6], the geometric quotient of $X(\mathrm{Pre})$ by G in the sense of Definition 0.6 of [30] is a coarse moduli space for $[X(\mathrm{Pre})/G]$. (We will recall the notion of good moduli spaces introduced by Alper in section 7.) Thus $[X(\mathrm{Pre})/G] \subset \mathcal{S}$. Namely, $[X(\mathrm{Pre})/G] = [X(\mathrm{Pre})/G] \cap \mathcal{S}$.

We first show $[X(\mathrm{Pre})/G] \subset [X/G]^{\mathrm{gs}}$. Note that if $[X(\mathrm{Pre})/G]$ is contained in $[X/G]^{\mathrm{gs}}$, then $[X(\mathrm{Pre})/G] \subset [X/G]^{\mathrm{gs}} \cap \mathcal{S}$.

Proposition 6.4. *Any closed point on $[X(\mathrm{Pre})/G]$ is a GIT-like p-stable point.*

To prove this Proposition, we may and will assume that X is affine, and the action of G on X is closed (cf. [30, Definition 1.7]).

Lemma 6.5. *Let p be a closed point on $X(\mathrm{Pre})$. Then the stabilizer group is (linearly) reductive.*

Proof. According to Matsushima's theorem (cf. [27, page 84]), an orbit is affine if and only if the stabilizer is reductive. Since every orbit in $X(\mathrm{Pre})$ is a closed set in an affine open set, thus our claim follows. □

Let X_0 be the reduced affine scheme associated to X. The base field is perfect, and thus $G \times_k X_0$ is also reduced. Therefore, the action on X induces a (closed) action $\sigma_0 : G \times_k X_0 \to X_0$. Let $\mathsf{Stab} \to X_0$ be the stabilizer group, which is defined to be the second projection $(G \times_k X_0) \times_{X_0 \times_k X_0} X_0 \to X_0$, where $(\sigma_0, \mathrm{pr}_2) : G \times_k X_0 \to X_0 \times_k X_0$, and $X_0 \to X_0 \times_k X_0$ is the diagonal map. Let $\mathcal{F}$ be the identity component of Stab. Note that $\mathcal{F} \to X_0$ is Zariski locally equidimensional (cf. [30, page 10]) and geometric fibers are connected and smooth because the base field is characteristic zero. Then by [14, VI_B Corollaire 4.4], $\mathcal{F} \to X_0$ is a smooth open normal subgroup of Stab. Note that the inertia stack $I[X_0/G] \to [X_0/G]$ is described by

$$[\mathsf{Stab}/G] \to [X_0/G],$$

where G acts on $\mathsf{Stab} \subset G \times_k X_0$ by $(h, x) \mapsto (ghg^{-1}, gx)$ for any $(h, x) \in \mathsf{Stab} \subset G \times_k X_0$ and any $g \in G$. Thus $\mathcal{F}$ descends to an open subgroup stack $[\mathcal{F}/G] \to [X_0/G]$ of the inertia stack. Moreover, the affineness of Stab over X_0 together with the following Lemma 6.6 implies that $\mathcal{F}$ is affine over X_0.

Lemma 6.6. *The quotient $\mathsf{Stab}/\mathcal{F}$ is finite over X_0.*

Proof. Let $x \in X_0$ be a closed point on X. Let G_x be the stabilizer group of x. The étale slice theorem of Luna ([32, page 96]) says that there exist a locally closed affine G_x-invariant subscheme $V \subset X_0$ containing x, and the commutative diagram

$$\begin{array}{ccccc} (V \times G)/G_x & \longrightarrow & [V/G_x] & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X_0 & \longrightarrow & [X_0/G] & \longrightarrow & Y, \end{array}$$

where all squares are cartesian, all vertical arrows are affine and étale, and Z (resp. Y) are geometric quotients of V (resp. X_0) by G_x (resp. G). Here if $V = \operatorname{Spec} A$ and $X_0 = \operatorname{Spec} B$, then $Z = \operatorname{Spec} A^{G_x}$ and $Y = \operatorname{Spec} B^G$, and $(V \times_k G)/G_x$ is the quotient of $V \times_k G$ by the action of G_x determined by $h \cdot (v, g) = (hv, gh^{-1})$. Using the diagram, we want to reduce the problem to $[V/G_x]$. Let $\mathsf{Stab}_V \to V$ be the stabilizer group scheme of the action of G_x on V. Since Stab_V is equidimensional, thus Stab_V contains $G_x^0 \times_k V$, where G_x^0 is the identity component of G_x. To see that $\mathsf{Stab}/\mathcal{F}$ is finite over X_0, it suffices to show that $\mathsf{Stab}_V/(G_x^0 \times_k V)$ is finite over V. We can consider the action of G_x on V to be the action of the finite group $\bar{G}_x := G_x/G_x^0$. Then it follows from [30, Proposition 0.8] that the action of $\bar{G}_x$ on V is proper. In particular, $\mathsf{Stab}_V/(G_x^0 \times_k V)$ is finite over V. This completes the proof. □

Proof of Proposition 6.4. By Lemma 6.5, every closed point on $[X(\mathrm{Pre})/G]$ has a (linearly) reductive automorphism group. Let $x \in [X/G] = [X(\mathrm{Pre})/G]$ be a closed point. Take a closed point $x' \in X$ lying over x. Let $\hat{\mathcal{O}}_{X,x'}$ be the completion of local ring $\mathcal{O}_{X,x'}$ at x'. Then $\operatorname{Spec}\hat{\mathcal{O}}_{X,x'} \to [X/G]$ is an effective versal deformation for x. And by Lemma 6.6, the restriction of the subgroup scheme $\mathcal{F}$ to $\operatorname{Spec}\hat{\mathcal{O}}_{X,x'}$ yields (b) in Definition 5.1. □

Next we will show the converse:

Proposition 6.7. *We have* $[X(\mathrm{Pre})/G] \supset [X/G]^{\mathrm{gs}} \cap \mathcal{S}$.

Proof. We say that a point $x \in X$ is regular if it admits an open neighborhood on which stabilizers have constant dimension. Let X^{reg} denote the open set of regular points. Clearly, $[X/G]^{\mathrm{gs}} \subset [X^{\mathrm{reg}}/G]$. Let $V \subset X$ be the open subset consisting of points lying over $[X/G]^{\mathrm{gs}}$. Let $v : V \to [X/G]^{\mathrm{gs}}$ be the natural projection. Let $p : [X/G]^{\mathrm{gs}} \to Z$ be a coarse moduli map. Let $z' \in [X/G]^{\mathrm{gs}}$ be a closed point and z the image in Z. Suppose that z has a Zariski open neighborhood which is a scheme. We will show that z has an affine neighborhood T such that $v^{-1}p^{-1}(T)$ is an affine scheme. By Corollary 5.13, there exists an étale neighborhood $U \to Z$ of z, such that $[X/G]^{\mathrm{gs}} \times_Z U \to U$ has the form $[W/H] \to U$ where W is affine over an affine scheme U, and H is a linearly reductive group over k. Then $[W/H] \to BH \times_k U$ is affine, and $BH \times_k U \to U$ gives rise to an exact push forward functor from the category of quasi-coherent sheaves on $BH \times_k U$ to that of quasi-coherent sheaves on U. If T' denotes the image of $U \to Z$, then we can conclude that $(p \circ v)_*$ is an exact functor from the category of quasi-coherent sheaves on $v^{-1}p^{-1}(T')$ to that of quasi-coherent sheaves on T'. Thus $v^{-1}p^{-1}(T') \to p^{-1}(T') \to T'$ is an affine morphism (notice that $v : V \to [X/G]^{\mathrm{gs}}$ is a principal G-bundle, in particular an affine morphism) . Take an affine neighborhood $z \in T \subset T'$. Then $v^{-1}p^{-1}(T)$ is affine. Since $[X/G]^{\mathrm{gs}} \subset [X^{\mathrm{reg}}/G]$, the action on $v^{-1}p^{-1}(T)$ is closed by [30, page 10, line 19-20]. This implies our claim. □

Proof of Theorem 6.1. Proposition 6.4 and 6.7 imply our claim. □

Remark 6.8. In the proof of Proposition 6.4, we can find the filtration as in Definition 5.1 (b).

In the proof of Lemma 6.5 and 6.6, we observed the closed action of a (linearly) reductive group on an affine scheme. From the observation, we have the following characterization:

Theorem 6.9. *Let $\mathcal{X}$ be an algebraic stack locally of finite type over a field k of characteristic zero. Then $\mathcal{X}$ is stable algebraic stack (Definition 5.15 and Remark 5.16) if and only if the following conditions hold:*

(i) *There exists a coarse moduli map $\pi : \mathcal{X} \to X$, such that X is locally of finite type. For any étale morphism $X' \to X$ of algebraic spaces, the second projection $\mathcal{X} \times_X X' \to X'$ is a coarse moduli map.*

(ii) *For any point $x \in X$, there exists an étale neighborhood $U \to X$, a finite extension of $k' \supset k$ fields, and a (linearly) reductive group G over k', such that U is an affine k'-scheme and $\mathcal{X} \times_X U$ has the form $[W/G]$, where W is a scheme that is affine over U and G acts on W over U.*

Proof. The "only if" direction follows from Theorem 5.12 and Corollary 5.13. We next prove the "if" direction. Taking into account Remark 5.2 (iii) we may and will assume that the base field is algebraically closed. As noted above, by Lemma 6.5 and 6.6 it suffices to show that if $[W/G] \to U$ is a quotient stack as in (ii), then the action of G is closed. Since $[W/G] \to U$ is a coarse moduli map, the action of G on W is closed, that is, every orbit is closed. This completes the proof. □

Remark 6.10. Perhaps one wish to have a necessary and sufficient condition for the existence of a coarse moduli space. However, if one removes the condition (ii) Theorem 6.9, then there are pathological examples even in the case of Deligne-Mumford stacks (that have only quasi-finite inertia stacks). For instance, it happens that a Deligne-Mumford stack which does not have finite inertia stack, has a coarse moduli space. Let $\mathbb{A}^1_{\mathbb{C}}$ be a complex affine line and $f : \mathbb{A}^1_{\mathbb{C}} \sqcup \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ the fold map. Removing one point q from $f^{-1}(0)$, we obtain a flat group scheme $g = f|_G : G = \{\mathbb{A}^1_{\mathbb{C}} \sqcup \mathbb{A}^1_{\mathbb{C}} \setminus q\} \to \mathbb{A}^1_{\mathbb{C}}$ over $\mathbb{A}^1_{\mathbb{C}}$, such that $g^{-1}(p)$ consists of two points (resp. one point) if $p \neq 0$ (resp. $p = 0$). Let BG be the classifying stack of G over $\mathbb{A}^1$. Then the inertia stack is quasi-finite over BG but not finite over BG. However, $\mathbb{A}^1_{\mathbb{C}}$ is a coarse moduli space for BG. We can not apply Keel-Mori theorem to BG, and in particular BG^{gs} does not coincide with BG. The point is that BG does not satisfies (ii) in Theorem 6.9.

Furthermore there exists a Deligne-Mumford stack which does not admit a coarse moduli space. Such an example can be found in Example 7.15 of [35].

Remark 6.11. Strong p-stability can differ from Mumford's theory. We will present the simple example which illustrates it. Let $\mathbb{P}^1_{\mathbb{C}}$ be the projective sphere over the complex number field $\mathbb{C}$. Let $G = \mathrm{PGL}(2, \mathbb{C})$ be the algebraic group that is the automorphism group of $\mathbb{P}^1_{\mathbb{C}}$ over $\mathbb{C}$. The algebraic group G acts on $\mathbb{P}^1_{\mathbb{C}}$ in the natural manner. Every affine open set U on $\mathbb{P}^1_{\mathbb{C}}$ is not G-invariant, that is, $G(U) \neq U$. Thus every point on $\mathbb{P}^1_{\mathbb{C}}$ is not pre-stable in the sense of Geometric Invariant Theory. In fact, the stabilizer group is unipotent. Let $\mathsf{Stab} \to \mathbb{P}^1_{\mathbb{C}}$ denote the stabilizer group scheme over $\mathbb{P}^1_{\mathbb{C}}$. Then we can easily see that $\mathsf{Stab} \to \mathbb{P}^1_{\mathbb{C}}$ is flat. Therefore, every (and only one) closed point on $[\mathbb{P}^1_{\mathbb{C}}/G]$ is strong p-stable in our sense. In this case, the coarse moduli space is $\operatorname{Spec} \mathbb{C}$. In general, algebraic stacks which have non-reductive (and positive dimensional) automorphisms are complicated. For instance, by Nagata's example the finite generation of invariant rings does not hold.

7. Application

In this section, we discuss the finiteness of coherent cohomology as an application.

Proposition 7.1. *Let $\mathcal{X}$ be a stable algebraic stack locally of finite type over a perfect field. Let $\pi : \mathcal{X} \to X$ be a coarse moduli map. Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$. Then $R\pi^0_*\mathcal{F}$ is a coherent sheaf on $\mathcal{X}$, and $R\pi^i_*\mathcal{F} = 0$ for $i > 0$.*

Proof. Our claim is étale local on X. Thus by Corollary 5.13 we may assume that $\mathcal{X} \to X$ is of the form $[W/G] \to U$, where $W = \operatorname{Spec} B$ is finite over an affine scheme $U = \operatorname{Spec} A$, and G is a linearly reductive group. A coherent sheaf $\mathcal{F}$ on $[W/G]$ amounts to a finite B-module M which is equipped with a G-action commuting with the action on B. The direct image $\pi_*\mathcal{F}$ corresponds to M^G. Thus the rule $\mathcal{F} \mapsto \pi_*\mathcal{F}$ is an exact functor because G is linearly reductive. Hence $R\pi_*^i\mathcal{F} = 0$ for $i > 0$. It remains to prove that $R\pi_*^0\mathcal{F}$ is a coherent sheaf on $\mathcal{X}$. It is enough to show that M^G is a finite A-module. Note that M is a finite A-module, and M^G is a sub A-module. Since A is noetherian, M^G is a finite A-module. □

Theorem 7.2. *Let $\mathcal{X}$ be an algebraic stack locally of finite type over a field k. Let $\bar{k} \supset k$ be an algebraic closure, and suppose that $\mathcal{X} \times_k \bar{k}$ is a stable algebraic stack over $\bar{k}$ and its coarse moduli space X is proper over $\bar{k}$. Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$. Then the cohomology group $H^i(\mathcal{X}, \mathcal{F})$ is finite-dimensional for $i \geq 0$.*

Proof. We may assume that the base field is algebraically closed. Consider the composition $\mathcal{X} \xrightarrow{\pi} X \to \operatorname{Spec} k$, where π is a coarse moduli map. Taking into account the finiteness theorem for algebraic space [23, IV Theorem 4.1] and Leray spectral sequence for the composition, it suffices to prove that $R^i\pi_*\mathcal{F}$ is coherent for all i. Thus our claim follows from Proposition 7.1. □

For example, it has the following direct corollary.

Corollary 7.3. *Let $\mathcal{X}$ be an algebraic stack of finite type over a field k. Suppose that $\mathcal{X} \times_k \bar{k}$ is a stable algebraic stack, where $\bar{k}$ is an algebraic closure of k. If a coarse moduli space for $\mathcal{X} \times_k \bar{k}$ is proper over $\bar{k}$, then $\mathcal{X}$ has a versal deformation (cf.* [4, Definition 4.1.1]*).*

Proof. By the proof of [4, section 4.2], it suffices to show the finiteness of coherent cohomology. Thus, our claim follows from Theorem 7.2. □

Remark 7.4. If $\mathcal{X}$ is proper over k, the above corollary is due to Aoki (cf. [4, Theorem 1.3]).

Remark 7.5. The finiteness theorem of coherent cohomology for proper algebraic stacks has been proved by Laumon and Moret-Bailly under some hypotheses (cf. [24, (15.6)]). Later, Faltings proved the finiteness theorem for general proper stacks via a surprising method of rigid geometry (cf. [16]). Recently, Olsson-Gabber proved Chow's lemma for algebraic stacks and reproved the finiteness theorem (cf. [33]).

We would like to stress that our theorem can be applied to algebraic stacks whose stabilizers are linearly reductive and positive dimensional. To compare our result with the previous theorems, consider a proper algebraic stack $\mathcal{X} \to \operatorname{Spec} A$. Then the diagonal $\mathcal{X} \to \mathcal{X} \times_{\operatorname{Spec} A} \mathcal{X}$ is proper. Practically, in many cases, the proper diagonal is a *finite* morphism. (In characteristic zero, it also implies that $\mathcal{X}$ is Deligne-Mumford.) Then if $\mathcal{X}$ has finite diagonal, then by Keel-Mori theorem $\mathcal{X}$ has a proper coarse moduli space X. Therefore we may summarize the above as follows: The finiteness theorem for proper stacks practically tells us that if an algebraic stack has finite diagonal and a proper coarse moduli space, then it has the finiteness of coherent cohomology. The main advantage of our finiteness result is that it is applicable to a certain class of algebraic stacks whose stabilizers are positive-dimensional.

Remark 7.6. Here we would like to relate our results with the notion of good moduli spaces introduced by Alper ([3]). Let us recall the definition of good moduli space: Let $\mathrm{QCoh}(\mathcal{X})$ denote the abelian category of quasi-coherent sheaves on an algebraic stack $\mathcal{X}$. Let $f:\mathcal{X}\to X$ be a morphism to an algebraic space X. The morphism $f:\mathcal{X}\to X$ is said to be a good moduli space for $\mathcal{X}$ if the followings hold:

(a) f is a quasi-compact morphism and the pushforward functor

$$f_*:\mathrm{QCoh}(\mathcal{X})\longrightarrow \mathrm{QCoh}(X)$$

is exact,

(b) the natural morphism $\mathcal{O}_X\to f_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism.

In [3], many properties of good moduli spaces are systematically studied. It is worth remarking that by Theorem 5.12 and Proposition 7.1 a stable algebraic stack admits a good moduli space. Namely, a coarse moduli space for a stable algebraic stack is a good moduli space. (Notice that the proof of Proposition 7.1 shows the condition (a).)

Let $\mathcal{X}$ be an algebraic stack of finite type over a perfect field k. Suppose that $\mathcal{X}$ is of GIT-like stable type over k. Let $\pi:\mathcal{X}\to X$ be a coarse moduli map (cf. Theorem 5.12). The coarse moduli space X is an algebraic space of finite type over k. The remainder of this section is devoted to giving a criterion for the properness of X over k, which is described in terms of $\mathcal{X}$ without making reference to X. We begin by considering the condition which assures that X is locally separated.

Let p be a closed point on $\mathcal{X}$. Since p is a GIT-like p-stable point, there exists an effective versal deformation $\xi:\operatorname{Spec}A\to\mathcal{X}$ such that if I denotes the ideal generated by nilpotent elements of A, then there exists a normal subgroup scheme $\mathcal{F}_\xi$ of $\underline{\mathrm{Aut}}_{\mathcal{X},\operatorname{Spec}A/I}(\xi|_{\operatorname{Spec}A/I})\to\operatorname{Spec}A/I$ satisfying (b) in Definition 5.1. Let F be a functor on the category of $\operatorname{Spec}A/I\times_k\operatorname{Spec}A/I$-schemes which to any $(f,g):T\to\operatorname{Spec}A/I\times_k\operatorname{Spec}A/I$ associates the set of sections $\underline{\mathrm{Isom}}_{\mathcal{X},T}(f^*\xi,g^*\xi)/\mathcal{F}_\xi(T)$. Let $\mathcal{X}_0$ be the reduced stack associated to $\mathcal{X}$. Let $\mathcal{X}_0\to\mathcal{X}_0^{\mathrm{rig}}$ be the rigidification associated to an algebraization of $\mathcal{F}_\xi$ after replacing $\mathcal{X}$ by a neighborhood of p (see Proposition 5.3). Let p' be the image of p in $\mathcal{X}\to\mathcal{X}^{\mathrm{rig}}$. Then the composite morphism $\operatorname{Spec}A/I\to\mathcal{X}_0\to\mathcal{X}_0^{\mathrm{rig}}$ is an effective versal deformation for p' since $\mathcal{X}_0\to\mathcal{X}_0^{\mathrm{rig}}$ is smooth. By the construction of the rigidification (cf. Remark A.8), the functor F is represented by the natural morphism

$$\operatorname{Spec}A/I\times_{\mathcal{X}_0^{\mathrm{rig}}}\operatorname{Spec}A/I\to\operatorname{Spec}A/I\times_k\operatorname{Spec}A/I.$$

Now we prove:

Proposition 7.7. *Suppose that the functor F is proper over* $\operatorname{Spec}A/I\times_k\operatorname{Spec}A/I$. *Then $\mathcal{X}^{\mathrm{rig}}$ is separated in a neighborhood of p'.*

Proof. Without loss of generality, we may assume that $\mathcal{X}$ is reduced. Namely, $\mathcal{X}_0=\mathcal{X}$ and $I=0$. It suffices to prove that there exists a smooth neighborhood $U\to\mathcal{X}^{\mathrm{rig}}$ of p', such that U is an affine scheme and $U\times_{\mathcal{X}^{\mathrm{rig}}}U\to U\times_k U$ is proper. To show this, we need the following Lemma.

Lemma 7.8. *Let X and Y be schemes locally finite presentation over a quasi-compact excellent scheme S. Let $X\to Y$ be a morphism over S. Let s be a closed point, and let $R:=\hat{\mathcal{O}}_{S,s}$ be the completion of the local ring at s. Suppose that the morphism*

$X \times_S \operatorname{Spec} R \to Y \times_S \operatorname{Spec} R$ induced by $\operatorname{Spec} R \to S$ is proper. Then there exists a neighborhood $W \subset S$ of s such that the induced morphism $X \times_S W \to Y \times_S W$ is proper.

Proof. Let P be a functor on the category of S-schemes which to $T \to S$ associates the set consisting of one element if $X \times_S T \to Y \times_S T$ is proper, and associates the empty set if otherwise. By Theorem A.2, we see that this functor is locally of finite presentation. Then applying Artin's approximation ([7]) we conclude that there exists an étale neighborhood $U \to S$ of s, such that $X \times_S U \to Y \times_S U$ is proper. Let W be the image of U in S. It is an open set. By the descent theory, $X \times_S W \to Y \times_S W$ is proper. □

We continue the proof of Proposition 7.7. By Lemma 2.2, there exist an affine scheme U, a smooth morphism $w : U \to \mathcal{X}^{\mathrm{rig}}$, and a closed point $u \in U$ such that $A \cong \hat{\mathcal{O}}_{U,u}$ and $\xi|_{\operatorname{Spec} A} \cong w|_{\operatorname{Spec} A}$ Consider the natural morphism $U \times_{\mathcal{X}^{\mathrm{rig}}} \operatorname{Spec} A \to U \times_k \operatorname{Spec} A$ over U. Then applying Lemma 7.8 to this diagram, we see that there exists a neighborhood V of u such that the restriction $V \times_{\mathcal{X}^{\mathrm{rig}}} \operatorname{Spec} A \to V \times_k \operatorname{Spec} A$ is proper. (The algebraic space $U \times_{\mathcal{X}^{\mathrm{rig}}} \operatorname{Spec} A$ is a scheme since it is quasi-finite and separated over $U \times_k \operatorname{Spec} A$ (cf. [24, (A.2)]). Applying Lemma 7.8 to the morphism $V \times_{\mathcal{X}^{\mathrm{rig}}} U \to V \times_k U$ over U again, we conclude that after shrinking the neighborhood V, the morphism $V \times_{\mathcal{X}^{\mathrm{rig}}} V \to V \times_k V$ is proper. □

Corollary 7.9. *Let q be the image of p on the coarse moduli space. Under the assumption of Proposition 7.7, X is separated in a neighborhood of q.*

Proof. It follows from Keel-Mori theorem since by Proposition 7.7 $\mathcal{X}^{\mathrm{rig}}$ has finite diagonal after shrinking $\mathcal{X}^{\mathrm{rig}}$. □

We say that p satisfies the locally separated property if there exist an effective versal deformation $\xi : \operatorname{Spec} A \to \mathcal{X}$ for p and a closed subgroup scheme $\mathcal{F}_\xi$ as above, such that F (see Proposition 7.7) is proper over $\operatorname{Spec} A/I \times_k \operatorname{Spec} A/I$, where I is the ideal generated by nilpotent elements of A. From now on, we suppose that every point satisfies the locally separated property. Namely, according to Corollary 7.9, the coarse moduli space X is *locally separated.* Next we consider a criterion for X to be universally closed over k.

Proposition 7.10. *The coarse moduli space X is universally closed over k if and only if $\mathcal{X}$ is universally closed over k, i.e., $\mathcal{X}$ satisfies a valuative criterion in* [24, Théorèm 7.3].

Proof. The "only if" direction is clear since π is a universally closed map. The "if" direction follows from the easy fact: If $\mathcal{X}$ is universally closed over k, and $\mathcal{X} \to X$ is surjective, then $X \to \operatorname{Spec} k$ is a universally closed map. □

Finally, we consider a valuative criterion for the separatedness of X.

Lemma 7.11. *The coarse moduli space X is separated if and only if the image of diagonal map $\mathcal{X} \to \mathcal{X} \times_k \mathcal{X}$ is closed.*

Proof. The diagonal $X \to X \times_k X$ is a quasi-compact immersion since X is locally separated. Thus, X is separated if and only if the image of $X \to X \times_k X$ is closed. Suppose that the image $\mathcal{Z}$ of diagonal map $\mathcal{X} \to \mathcal{X} \times_k \mathcal{X}$ is closed. Then the image of

$\mathcal{Z}$ in $X \times_k X$ is closed. Indeed, the composite map $\mathcal{X} \times_k \mathcal{X} \to \mathcal{X} \times_k X \to X \times_k X$ is closed since $\mathcal{X} \to X$ is universally closed. Conversely, assume that X is separated. Let Z be the image of diagonal $X \to X \times_k X$. Since $\mathcal{X} \to X$ is a coarse moduli map, thus the image of diagonal map $\mathcal{X} \to \mathcal{X} \times_k \mathcal{X}$ set-theoretically coincides with the preimage of Z under $\mathcal{X} \times_k \mathcal{X} \to X \times_k X$. It follows that the image of diagonal map $\mathcal{X} \to \mathcal{X} \times_k \mathcal{X}$ is closed. □

Proposition 7.12. *The following conditions are equivalent:*

(i) *X is separated over k.*

(ii) *Let R be a valuation ring with quotient field K, and let α, β be objects in $\mathcal{X}(R)$ such that there is an isomorphism $\alpha|_K \cong \beta|_K$. Then the fiber of* $\mathrm{Isom}_R(\alpha, \beta)$ *over the closed point of R is nonempty.*

Proof. We first show that (ii) implies (i). Since X is locally separated, thus by Lemma 7.11 it is enough to prove that the image $\mathcal{Z}$ of the diagonal map $\Delta : \mathcal{X} \to \mathcal{X} \times_k \mathcal{X}$ is closed. By [24, (5.9.4)], the underlying set of $\mathcal{Z}$ is a constructible set. It suffices to show that $\mathcal{Z}$ is stable under specialization. Thus it is enough to prove that if v is a generic point on $\mathcal{X}$, and y' is a specialization of $y = \Delta(v)$, then there exists a point $v' \in \mathcal{X}$ lying over y'. According to [24, (7.2)], there exist a valuation ring R with quotient field K, $\operatorname{Spec} K \to \mathcal{X}$ lying over v, and $\operatorname{Spec} R \to \mathcal{X} \times_k \mathcal{X}$ such that the diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \Delta \\ \operatorname{Spec} R & \longrightarrow & \mathcal{X} \times_k \mathcal{X} \end{array}$$

commutes, and the closed point of $\operatorname{Spec} R$ maps to y' under $\operatorname{Spec} R \to \mathcal{X} \times_k \mathcal{X}$. Then applying the condition (ii) to $\mathrm{Isom}_R(\alpha, \beta) \cong \operatorname{Spec} R \times_{(\mathcal{X} \times_k \mathcal{X})} \mathcal{X} \to \operatorname{Spec} R$ we see that y' belongs to $\mathcal{Z}$.

We next show that (i) implies (ii). The image of $(\alpha, \beta) : \operatorname{Spec} R \to \mathcal{X} \times_k \mathcal{X}$ set-theoretically belongs to $\mathcal{Z}$ since $\mathcal{Z}$ is closed by Lemma 7.11. Hence there exists a point $s \in \mathrm{Isom}_R(\alpha, \beta)$ lying over the closed point of $\operatorname{Spec} R$. □

Remark 7.13. For stable algebraic stacks, it seems that three conditions: the locally separated condition, "universally closed condition" (cf. Proposition 7.10) and (ii) in Proposition 7.12 are important. I like to think of three conditions as "*virtual properness*" of $\mathcal{X}$. In other words, under these conditions, $\mathcal{X}$ behaves like proper in some contexts. For example, it is hopeful that the class of such properness provides a good setting in which one has a good theory of Riemann-Roch.

Appendix

Limit argument. Let S_0 be a scheme (resp. quasi-separated algebraic space). Let I be a filtered inductive system in the sense of [6, section 1] or [29, Appendix A]. Let us consider functor

$$I \to (\text{quasi-coherent } \mathcal{O}_{S_0}\text{-algebras})$$

sending α to $\mathcal{A}_\alpha$. Let

$$S = \varprojlim_{\alpha \in I} S_\alpha$$

be the associated projective system of schemes (resp. quasi-separated algebraic spaces) that are affine over S_0. (By [29, Appecdix A, Corollary 2] it is represented by a scheme (resp. quasi-separated algebraic space).) We would like to recall the some results of limits arguments in [15, IV (8.2)]. For our purpose, we need some assertions in the case of algebraic spaces, though the generalizations are straightforward. However, we could not find the appropriate literature, thus we decided that it is best to collect them here. All algebraic spaces are assumed to be quasi-separated.

Theorem A.1 (cf. EGA IV (8.6.3))**.** *Assume that S_0 is a quasi-compact algebraic space. Let $Y \subset S$ be a closed subspace. Then there exist $\lambda \in I$ and a closed subspace $Y_\lambda \subset S_\lambda$ such that $S \times_{S_\lambda} Y_\lambda = Y$. Moreover, $Y_\lambda \subset S_\lambda$ is unique in the following sense: If there exists another closed subscheme $Y_{\lambda'} \subset S_{\lambda'}$ such that $S \times_{S_{\lambda'}} Y_{\lambda'} = Y$, then there exist $\lambda \to \mu$ and $\lambda' \to \mu$ such that $Y_\lambda \times_{S_\lambda} S_\mu = Y_{\lambda'} \times_{S_{\lambda'}} S_\mu$.*

Proof. Let $\tilde{S}_0 \to S_0$ be an étale surjective morphism from a quasi-compact and separated scheme $\tilde{S}_0$. Considering the base changes $\{\tilde{S}_0 \times_{S_0} S_\alpha\}_{\alpha \in I}$ the uniqueness follows from the case when S_0 is a scheme ([15, IV (8.6.3)]). Next we will prove the existence. Let $X_\alpha := \tilde{S}_0 \times_{S_0} S_\alpha$ and $X = \tilde{S}_0 \times_{S_0} S$. Let $p_\alpha : X_\alpha \to S_\alpha$ and $p : X \to S$ be natural projections. Then by the case of schemes there exist $\lambda \in I$ and a closed subscheme $W_\lambda \subset X_\lambda$ such that $X \times_{X_\lambda} W_\lambda = p^{-1}(Y)$. Let $R_\alpha := X_\alpha \times_{S_\alpha} X_\alpha$ and $R := X \times_S X$. Since X_α is a quasi-compact and quasi-separated scheme, thus R_α is also a quasi-compact and quasi-separated scheme. Let $\mathrm{pr}_1, \mathrm{pr}_2 : R_\alpha \rightrightarrows X_\alpha$ be the first and second projection respectively. (Here we abuse notation and omit the index "α".) It suffices to show that there exists some arrow $\lambda \to \mu$ such that $\mathrm{pr}_1^{-1}(W_\mu) = \mathrm{pr}_2^{-1}(W_\mu)$, where $W_\mu = W_\lambda \times_{X_\lambda} X_\mu$. Since $\mathrm{pr}_1^{-1}(p^{-1}(Y)) = \mathrm{pr}_2^{-1}(p^{-1}(Y))$, the uniqueness $\mathrm{pr}_1^{-1}(W_\lambda)$ and $\mathrm{pr}_2^{-1}(W_\lambda)$ in the system $\{R_\alpha\}_{\alpha \in I}$ implies that there exists $\lambda \to \mu$ such that $\mathrm{pr}_1^{-1}(W_\mu) = \mathrm{pr}_2^{-1}(W_\mu)$. Thus W_μ descends to a closed subspace $Y_\mu \subset S_\mu$ such that $S \times_{S_\mu} Y_\mu = Y$. □

Theorem A.2 (cf. EGA IV (8.10.5) (17,7.8))**.** *Let S_0 be a quasi-compact scheme. Let X_λ and Y_λ be schemes of finite presentation over S_λ. Let $X_\lambda \to Y_\lambda$ be an S_λ-morphism. Consider the following properties: (i) an isomorphism, (ii) a closed immersion, (iii) quasi-finite, (iv) finite, (v) affine, (vi) flat, (vii) smooth, (viii) proper, (ix) separated. If $X_\lambda \times_{S_\lambda} S \to Y_\lambda \times_{S_\lambda} S$ has one or more properties (i)–(ix), then there exists an arrow $\lambda \to \mu$ such that $X_\lambda \times_{S_\lambda} S_\mu \to Y_\lambda \times_{S_\lambda} S_\mu$ has the same properties.*

Proposition A.3. *Suppose that S_α is a quasi-compact and quasi-separated scheme for any $\alpha \in I$. Let $f : X_\lambda \to S_\lambda$ be an algebraic space of finite presentation over S_λ. Consider the following properties: (i) affine, (ii) flat, (iii) finite, (iv) smooth, (v) separated. If $X := X_\lambda \times_{S_\lambda} S \to S$ has one or more properties (i)–(v), then there exists an arrow $\lambda \to \mu$ such that $X_\mu := X_\lambda \times_{S_\lambda} S_\mu \to S_\mu$ has the same properties.*

Proof. We first prove (ii). Let $W_\lambda \to X_\lambda$ be an étale surjective morphism from a quasi-compact scheme W_λ. Note that $W_\lambda \to X_\lambda$ is a quasi-compact morphism since X_λ is quasi-separated. By the descent theory, it is enough to show that there exists an arrow $\lambda \to \mu$ such that $W_\lambda \times_{S_\lambda} S_\mu$ is flat over S_μ. Applying Theorem A.2 to the projective system $\{W_\lambda \times_{S_\lambda} S_{\lambda'}\}_{\lambda' \in I/\lambda}$, we obtain (i). (Here I/λ is the inductive system over λ.)

Next we prove (iv). The proof is similar to the proof of (ii).

Next we prove (v). It suffices to show that $X_\mu \to X_\mu \times_{S_\mu} X_\mu$ is a closed immersion for some $\lambda \to \mu$. Note that $X_\lambda \to X_\lambda \times_{S_\lambda} X_\lambda$ is a schematic morphism and $X_\lambda \times_{S_\lambda} X_\lambda$ is of finite presentation over S_λ. Thus our claim follows from Theorem A.2 (ii) and descent theory.

Next we prove (iii). According to Theorem A.2 (iii) we may assume that f is quasi-finite. There exists some $\lambda \to \mu$ such that f_μ is separated by (v). Then by [24, Thèoréme (A.2)] X_μ is a scheme. Therefore the schematic case implies (iii).

Finally, we prove (i). Since $X = X_\lambda \times_{S_\lambda} S$ is affine of finite presentation over S, there exists a closed immersion

$$\iota : X \to S \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Z}[t_1, \dots, t_n] =: \operatorname{Spec} S[t_1, \dots, t_n].$$

It is enough to show that there is a closed immersion $\iota_\mu : X_\mu \to \operatorname{Spec} S_\mu[t_1, \dots, t_n]$ which induces ι. Let

$$[R_\lambda = W_\lambda \times_{X_\lambda} W_\lambda \rightrightarrows W_\lambda]$$

denote an étale equivalence relation for X_λ with a quasi-compact scheme X_λ. Using Proposition A.5 and the étale equivalence relation, we have a morphism $X_\mu \to \operatorname{Spec} S_\mu[t_1, \dots, t_n]$ for some μ, which induces ι. Then by the above (iii), we may suppose that

$$X_\mu \to \operatorname{Spec} S_\mu[t_1, \dots, t_n]$$

is finite. In particular, we can assume that X_μ is a scheme. Now our claim follows from Theorem A.2 (ii). □

Proposition A.4. *Suppose that S_α is a noetherian scheme for any $\alpha \in I$. Let $f_\lambda : X_\lambda \to S_\lambda$ is a scheme of finite type over S_λ. Suppose that all geometric fibers of $X_\lambda \times_{S_\lambda} S \to S$ are connected. Then there exists an arrow $\lambda \to \mu$ such that all geometric fibers of $X_\lambda \times_{S_\lambda} S_\mu \to S_\mu$ are connected.*

Proof. Let E be the set consisting of points $s \in S_\alpha$, such that $f^{-1}(s)$ is geometrically connected. By [15, IV, 9.7.7], the set E is a constructible set in S_λ. Moreover, if a point $s \in S_\lambda$ does not lie in the image of $S \to S_\lambda$, then there exists $\lambda \to \lambda'$ such that s does not lie in the image of $S_{\lambda'} \to S_\lambda$. On the other hand, the image of $S \to S_\lambda$ is contained in E. Therefore, using noetherian induction argument, we easily see that there exists $\lambda \to \mu$ such that the image of $S_\mu \to S_\lambda$ is contained in E. This completes the proof. □

Theorem A.5 (cf. EGA IV (8.8.2))**.** *Assume that S_0 be a quasi-compact and quasi-separated scheme. Let X_α be a quasi-compact scheme and let Y_α be a scheme of finite presentation over S_α. Let $X_\mu = X_\alpha \times_{S_\alpha} S_\mu$ and $Y_\mu = Y_\alpha \times_{S_\alpha} S_\mu$. Let $X = X_\alpha \times_{S_\alpha} S$ and $Y = Y_\alpha \times_{S_\alpha} S$. Then the natural map*

$$A = \varinjlim_{\lambda} \operatorname{Hom}_{S_\lambda}(X_\lambda, Y_\lambda) \to \operatorname{Hom}_S(X, Y)$$

is bijective.

Proposition A.6. *Assume that S_0 be a quasi-compact and quasi-separated scheme. Let $G_\alpha \to S_\alpha$ be a separated group algebraic space of finite presentation over S_α. Let $H_\alpha \subset G_\alpha$ be a closed subspace such that $H = H_\alpha \times_{S_\alpha} S$ is a subgroup space of $G = G_\alpha \times_{S_\alpha} S$. Then there exists an arrow $\alpha \to \mu$ such that $H_\mu := H_\alpha \times_{S_\alpha} S_\mu$ is a subgroup space of $G_\mu = G_\alpha \times_{S_\alpha} S_\mu$. If H is normal in G, then $H_\alpha \times_{S_\alpha} S_\mu$ can be chosen to be normal.*

Proof. It suffices to show that there exists an arrow $\alpha \to \mu$ such that H_μ has the following properties: (i) the multiplication $m : G_\mu \times_{S_\mu} G_\mu \to G_\mu$ induces $H_\mu \times_{S_\mu} H_\mu \to H_\mu$, (ii) the inverse $i : G_\mu \to G_\mu$ carries H_μ to H_μ, (iii) the unit section $e : S_\mu \to G_\mu$ factors through H_μ. Let us consider the property (i). Let

$$\begin{array}{ccc} U_\alpha & \xrightarrow{\tilde{m}} & V_\alpha \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ G_\alpha \times_{S_\alpha} G_\alpha & \xrightarrow{m} & G_\alpha \end{array}$$

be a commutative diagram where U_α and V_α are quasi-compact schemes and two vertical arrows are étale surjective morphisms. (Such a diagram exists.) It suffices to prove that after the base change by some arrow $\alpha \to \mu$ the pullback $p^{-1}(H_\mu \times_{S_\mu} H_\mu)$ maps to $q^{-1}(H_\mu)$. Since H is a subgroup of G, in particular $G \times_S G \to G$ induces $H \times_S H \to H$, thus applying Theorem A.5 to $p^{-1}(H_\alpha \times_{S_\alpha} H_\alpha) \to V_\alpha \leftarrow q^{-1}(H_\alpha)$ we see that there exists an arrow $\alpha \to \mu$ such that $p^{-1}(H_\mu \times_{S_\mu} H_\mu) \to V_\mu$ factors through $q^{-1}(H_\mu) \subset G_\mu$. By a similar argument, we may assume that H_μ has also the property (ii).

Next, consider the property (iii). Let

$$\begin{array}{ccc} W_\alpha & \xrightarrow{\tilde{e}} & V_\alpha \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle q} \\ S_\alpha & \xrightarrow{e} & G_\alpha \end{array}$$

be a commutative diagram where W_α and V_α are quasi-compact schemes and two vertical arrows are étale surjective morphisms. It suffices to prove that after the base change by some arrow $\alpha \to \mu$ the unit section $W_\mu \to V_\mu$ factors through $q^{-1}(H_\mu)$. Again by applying Theorem A.5 to $W_\alpha \to V_\alpha \leftarrow q^{-1}(H_\alpha)$, we conclude that there exists an arrow $\alpha \to \mu$ such that $W_\mu \to V_\mu$ factors through $q^{-1}(H_\mu) \subset G_\mu$. By what we have proven, we conclude that there exists an arrow $\alpha \to \mu$ such that H_μ is a subgroup space of G_μ.

Finally, we will prove the last assertion. Consider the morphism

$$\phi : G_\alpha \times_{S_\alpha} G_\alpha \to G_\alpha$$

which sends (g_1, g_2) to $g_1 g_2 g_1^{-1}$. A group subspace H_α is normal in G_α if and only if $\phi(G_\alpha \times_{S_\alpha} H_\alpha) \subset H_\alpha$ (scheme-theoretically). Thus by a similar argument using Theorem A.5 to the above we easily see the last assertion. □

Rigidification. For the convenience, we would like to recall the rigidifications of algebraic stacks, that has been discussed by many authors (see for example [24], [1], [2],[31]).

Let $\mathcal{X}$ be a quasi-separated algebraic stack locally of finite presentation over a quasi-separated scheme S. Let $\mathcal{G}$ be a closed subgroup stack in the inertia stack $I\mathcal{X} \to \mathcal{X}$, that is flat and of finite presentation over $\mathcal{X}$. Namely, the multiplication $I\mathcal{X} \times_{\mathcal{X}} I\mathcal{X} \to I\mathcal{X}$ induces $\mathcal{G} \times_{\mathcal{X}} \mathcal{G} \to \mathcal{G}$, the inverse $I\mathcal{X} \to I\mathcal{X}$ sends $\mathcal{G}$ to $\mathcal{G}$, and in addition the unit section $\mathcal{X} \to I\mathcal{X}$ factors through $\mathcal{G}$. Let V be an affine S-scheme. Let $\xi \in \mathcal{X}(V)$ be an object and $V \to \mathcal{X}$ the corresponding morphism. It gives rise to a natural morphism $h : \underline{\mathrm{Aut}}_{\mathcal{X},V}(\xi) \to I\mathcal{X}$. The inverse image $\mathcal{G}(\xi) := h^{-1}(\mathcal{G}) \subset \underline{\mathrm{Aut}}_{\mathcal{X},V}(\xi)$ is a normal closed subgroup. (Every automorphism $\sigma : \xi \to \xi$ gives rise to an inertia automorphism $\underline{\mathrm{Aut}}_{\mathcal{X},V}(\xi) \to \underline{\mathrm{Aut}}_{\mathcal{X},V}(\xi)$ and it sends $h^{-1}(\mathcal{G})$ to $h^{-1}(\mathcal{G})$. Thus $\mathcal{G}(\xi)$ is normal.)

Theorem A.7. *There exist an algebraic stack $\mathcal{Y}$ locally of finite presentation and a morphism $f : \mathcal{X} \to \mathcal{Y}$ such that:*

(i) *$f : \mathcal{X} \to \mathcal{Y}$ is an fppf gerbe.*

(ii) *For any affine S-scheme V and any object $\xi \in \mathcal{X}(V)$, the homomorphism of group algebraic spaces*

$$\underline{\mathrm{Aut}}_{\mathcal{X},V}(\xi) \to \underline{\mathrm{Aut}}_{\mathcal{Y},V}(f(\xi))$$

is surjective and its kernel is $\mathcal{G}(\xi)$.

(iii) *Let $V \to \mathcal{Y}$ be a morphism from an affine scheme V and let $\xi \in \mathcal{X}(V)$ be the corresponding object. Then there exists a natural isomorphism $V \times_{\mathcal{Y}} \mathcal{X} \cong B_V\mathcal{G}(\xi)$ over V, where $B_V\mathcal{G}(\xi)$ is the classifying stack of $\mathcal{G}(\xi)$.*

(iv) *Let $g : \mathcal{X} \to \mathcal{W}$ be a morphism of algebraic stacks such that for any object $\xi \in \mathcal{X}$, $\mathcal{G}(\xi)$ lies in the kernel of $\underline{\mathrm{Aut}}_{\mathcal{X},V}(\xi) \to \underline{\mathrm{Aut}}_{\mathcal{W},V}(g(\xi))$. Then there exists a morphism $h : \mathcal{Y} \to \mathcal{W}$ such that $h \circ f$ is isomorphic to g. It is unique up to isomorphism.*

(v) *$\mathcal{Y}$ admits a coarse moduli space if and only if $\mathcal{X}$ has one. If they have, both coarse moduli spaces coincide.*

(vi) *If $\mathcal{X}$ is a Deligne-Mumford stack, then $\mathcal{Y}$ is so.*

Proof. Assertions (i), (ii) (iii) are proved in [2, (A.1)]. To see (iv), (v) and (vi), we should go back to the construction of $\mathcal{Y}$. Let $\mathcal{Y}^p$ be the fibered category defined as follows. The objects of $\mathcal{Y}$ are those of $\mathcal{X}$. Let $\phi : V' \to V$ be a morphism of affine S-schemes. For any $\xi' \in \mathcal{X}(V')$ and any $\xi \in \mathcal{X}(V)$, we define the set of morphisms $\mathrm{Hom}_{\mathcal{Y}^p}(\xi', \xi)$ over $V' \to V$ to be the set of sections of the quotient

$$\underline{\mathrm{Isom}}_{\mathcal{X},V'}(\xi', \phi^*\xi)/\mathcal{G}(\xi')$$

where the action of $\mathcal{G}(\xi')$ is the natural faithful right action defined by the composition. Then it is not hard to show that $\mathcal{Y}^p$ is a prestack. Let $\mathcal{Y}$ be a stack associated to the prestack $\mathcal{Y}^p$. It is showed in [2, (A.1)] that $\mathcal{Y}$ is an algebraic stack locally of finite presentation over S with properties (i), (ii) and (iii). By the construction of $\mathcal{Y}^p$, clearly, the natural morphism $\mathcal{X} \to \mathcal{Y}^p$ is universal for maps from $\mathcal{X}$ to prestacks, that kill $\mathcal{G}(\xi)$ for any object $\xi \in \mathcal{X}$. Since $\mathcal{Y}$ is the associated stack, thus (iv) follows. The "if part" of (v) follows from (iv) and (i). (Note that for any algebraically closed S-field K, $\mathcal{X} \to \mathcal{Y}$ induces a bijective map $\mathcal{X}(K)/\sim \to \mathcal{Y}(K)/\sim$. Here $\sim$ means "up to isomorphism".) The "only if" part also follows from (iv). The last assertion in (v) is clear. To see (vi), notice that for each object $\xi \in \mathcal{X}(V)$ the group space $\underline{\mathrm{Aut}}_{\mathcal{X},V}(\xi) \to V$ is unramified because $\mathcal{X}$ is a Deligne-Mumford stack (cf. [24, (4.2)]).

For each object $\xi \in \mathcal{X}(V)$, $\mathcal{G}(\xi)$ is flat and thus $\mathcal{G}(\xi)$ is étale over V. By (iii), we see that $\mathcal{X} \to \mathcal{Y}$ is étale. Hence $\mathcal{Y}$ is a Deligne-Mumford stack. □

Remark A.8. The morphism $\mathcal{X} \to \mathcal{Y}$ is characterized by the universal property in (iv). We refer to this morphism as the rigidification of $\mathcal{X}$ (or rigidifying morphism) with respect to $\mathcal{G}$. If objects $\xi, \eta \in \mathcal{Y}(V)$ arise from $\mathcal{X}(V)$, then the above proof reveals that there exists an isomorphism

$$\underline{\mathrm{Isom}}_{\mathcal{Y},V}(\xi,\eta) \cong \underline{\mathrm{Isom}}_{\mathcal{X},V}(\xi,\eta)/\mathcal{G}(\xi).$$

Also, the algebraic space on the right side is isomorphic to $\mathcal{G}(\eta) \setminus \underline{\mathrm{Isom}}_{\mathcal{X},V}(\xi,\eta)$.

References

[1] D. Abramovich, T. Graber and A. Vistoli, Gromov-Witten theory of Deligne-Mumford stacks, Amer. J. Math. 130 (2008), 1337–1398.
[2] D. Abramovich, M. Olsson and A. Vistoli, Tame stacks in positive characteristic, Anneles de l'Institut Fourier 58 (2008), 1057-1091.
[3] J. Alper, Good moduli spaces for Artin stacks, arXiv:0804.2242.
[4] M. Aoki, Deformation theory of algebraic stacks, Compos. Math. 141 (2005) 19–34.
[5] D. Arapura and A. Dhillon, The motives of the moduli stack of G-bundles over the universal curve, arXiv:math/0609505.
[6] M. Artin, Grothendieck topologies, Harvard Univ. (1962).
[7] M. Artin, Algebraic approximation of structures over complete local rings, Publ. Math. I.H.E.S., tome 36 (1969), 23–58.
[8] M. Artin, Algebraization of formal moduli: I, Global Analysis (Papers in honor of K. Kodaira), Univ. Tokyo Press, Tokyo (1969), 21–71.
[9] M. Artin, Versal deformations and algebraic stacks, Invent. Math. 27 (1974) 165–189.
[10] P. Berthelot, A. Grothendieck, and L. Illusie, Théorie des intersections et th'eor'em de Riemann-Roch, (SGA6), Lecture Notes in Math. 225 Springer-Verlag (1971).
[11] B. Conrad and J. de Jong, Approximation of versal deformations, J. Algebra, 255 no.2 (2002).
[12] P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus, Publ. I.H.E.S., tome 36. (1969), 75–109.
[13] M. Demazure and P. Gabriel, Groupes Algébriques I, Masson et Cie, (1971).
[14] M. Demazure and A. Grothendieck, Schéma en groupes I, II, III, Lecture Notes in Math. 151, 152, 153, Springer-Verlag (1970).
[15] J. Dioudonné and A. Grothendieck, Élements de Géométrie Algébriques I, II, III, IV, Publ. Math. I.H.E.S. 4, 8, 11, 17, 20, 24, 28, 32 (1960–1967)
[16] G. Faltings, Finiteness of coherent cohomology for proper fppf stacks, J. Alg. Geom. 12 (2003), pp357–366.
[17] D. Edidin, B. Hassett, A. Kresch, and A. Vistoli, Brauer groups and quotient stacks, Amer. J. Math. 123 (2001), 761-777.
[18] A. Grothendieck et all, Revêtements étale et Groupe Fondemental (SGA 1), Lecture Notes in Math. 224, Springer (1971).
[19] L. Illusie, Complexe Cotangent et Déformations I, II, Lecture Notes in Math. 239 (1971), 283 (1972).
[20] M. Inaba, Toward a definition of moduli of complexes of coherent sheaves on a projective scheme, J. Math. Kyoto Univ. 42 (2002), 317–329.
[21] R. Joshua, Riemann-Roch for algebraic stacks:I, Compositio Math. 136 (2003), 117–169.
[22] S. Keel and S. Mori, Quotients by groupoids, Ann. Math. 145 (1997), 193–213.
[23] D. Knutson, Algebraic spaces, Lecture Notes in Math. 203 Springer (1971)
[24] G. Laumon and L. Moret-Bailly, Champs Algébriques, Springer-Verlag (2000).
[25] M. Lieblich, Moduli of complexes on a proper morphism, J. Alg. Geom. 15 (2006), 175–206.
[26] M. Lieblich, Moduli of twisted sheaves, Duke Math. J. 138 (2007), 23–118.

[27] D. Luna, Slices étalés, Mém. S. M. F., tome 33 (1973), 81–105.
[28] J. Lurie, Derived Algebraic Geometry I, PhD thesis, M.I.T. (2004).
[29] J. S. Milne, Étale Cohomology, Princeton Univ. Press (1980).
[30] D. Mumford, J. Fogarty, and F. Kirwan, Geometric Invariant Theory, Third Enlarged Edition, Springer-Verlag (1994).
[31] M. Romagny, Group actions on stacks and applications, Michigan Math. J. 53 (2005), no. 1, 209–236.
[32] M. Olsson, Deformation theory of representable morphisms of algebraic stacks, Math. Z. 253 (2006), 25–62.
[33] M. Olsson, On proper coverings of Artin stacks, Adv. Math. 198 (2005), 93–106.
[34] M. Olsson, Hom-stacks and restriction of scalars, Duke Math. J. 134 (2006), 139–164.
[35] D. Rydh, Existence of quotients by finite groups and coarse moduli spaces, arxiv:0708.3333
[36] R. Thomason, Algebraic K-theory and étale cohomology, Ann. Sci. de l'ENS 18 (1985), 437–552.
[37] B. Toën, Théorèm de Riemann-Roch pour les Champs de Deligne-Mumford, K-theory, 18 (1999), 33–76.
[38] B. Toën and M. Vaquié, Moduli of objects in dg-categories, Ann. Sci. de l'ENS 40 (2007) 387–444