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Noncooperative Game in Cooperation: Reformulation of Correlated Equilibria (II)[1)]

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In our previous paper (Kôno 2008), we introduced two essentially different concepts of correlated equilibria: one is an "exogenous correlated equilibrium relative to Z" of a noncooperative game with a mediator and the other one is an "endogenous correlated equilibrium." In this paper, we will generalize the former to an "exogenous correlated equilibrium relative to Z_1 and Z_2" of a noncooperative game with agents. Under the newly defined framework, we will show that the example given in Fudenberg and Tirole (1991, p. 54) is indeed an equilibrium in the sense of Definition 2 of this paper. We will also investigate Aumann's (1974) examples that were not discussed in the previous paper. We will show some other strategy profiles that are correlated equilibria as per our paper but not found in previous works including Aumann (1974) and Fudenberg and Tirole (1991)[2)].

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1. Introduction

In this paper, in the same way as in our previous paper (Kôno 2008), that is, by using random variables, we will generalize a noncooperative game with a mediator to the one in which, instead of a mediator, each player has his or her own agent who gives a suggestion only to his or her own employer. Under this framework, we can define a new equilibrium concept such as a Nash equilibrium and by using this newly defined framework, we can investigate the examples discussed by Aumann (1974, 1987) and Fudenberg and Tirole (1991) in the context of

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correlated equilibria.

As in our previous paper, we will investigate a noncooperative strategic game Γ using random variables. Our "random variables" are rigorously defined on an abstract and universal probability space $(\Omega, \mathcal{F}, P)$[3], where Ω is an abstract set that is sometimes called the sample space, $\mathcal{F}$ is a σ-field whose element is called an event, and P is a probability measure on $\mathcal{F}$. The sample space Ω can be assumed to be rich enough, if necessary. Fortunately, in this paper, all our calculations on probability are so elementary that any advanced probability theorems are not required.

In this paper, we will discuss a game involving two players, Player 1 and Player 2. Note that with this assumption, a formal extension to an n-person game becomes very easy.

Let $S_i (i=1, 2)$ be a finite set of Player i's pure strategies and let set $S := S_1 \times S_2$. Let $u_i(s_1, s_2)(i=1, 2)$ and $(s_1, s_2) \in S$, denote a real-valued function defined on S, and let us call it Player i's pay-off function. Let $\mathcal{R}(S_i)$ be the set of all S_i-valued random variables and $\mathcal{R}(S)$ be the set of all S-valued random variables, that is, $\mathcal{R}(S)=\mathcal{R}(S_1)\times\mathcal{R}(S_2)$. A strategy of Player i is an element of $\mathcal{R}(S_i)$. An element $(X, Y)\in\mathcal{R}(S)$ is called a strategy profile of the game, that is, a strategy for Player i is the selection of a random variable from $\mathcal{R}(S_i)$. If $P(X=s_1, Y=s_2)$[4]$=1$ for some $s_1 \in S_1$ and $s_2 \in S_2$, then (X, Y) is called a pure strategy profile; otherwise, it is called a mixed strategy profile. Player i's utility is represented by $E[u_i(X, Y)]$, which is the mathematical expectation of a real-valued random variable $u_i(X, Y)$.

2. Formulation of a Noncooperative Game with Agents

In our previous paper, we introduced a noncooperative game with a mediator. However, in this paper, we assume that each player has his or her own agent who will give a suggestion only to his or her own employer.

Let T_i be a finite set of choices for the Player i's agent and let $\mathcal{R}(T_i)$ be a set of all T_i-valued random variables. A T_i-valued random variable could be interpreted as the agent's suggestion to Player i.

Now, we will formulate a framework for a noncooperative game with agents. First, let us fix $Z_i \in \mathcal{R}(T_i)(i=1, 2)$ and let $\mathcal{R}_{Z_1,Z_2}(S)$ be the subset of $\mathcal{R}(S)$ that satisfies Conditions (A-1) and (A-2).

Condition (A-1): $(X, Y)\in\mathcal{R}_{Z_1,Z_2}(S)$ is conditionally independent relative to Z_1 and Z_2, that is,

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[3] Most textbooks on probability theory include the axioms of probability, and therefore, we omit the details.

[4] $(X=s_1, Y=s_2) := \{\omega\in\Omega ; X(\omega)=s_1\}\cap\{\omega\in\Omega ; Y(\omega)=s_2\}$ is an event, that is, an element of σ-field $\mathcal{F}$, by the definiton of random variables and the axiom of $\mathcal{F}$.

$$P(X=i,\ Y=j/Z_1=k,\ Z_2=\ell)=P(X=i/Z_1=k,\ Z_2=\ell)P(Y=j/Z_1=k,\ Z_2=\ell)$$

holds for all $i\in S_1$, $j\in S_2$, $k\in T_1$, and $\ell\in T_2$.

Here, $P(A/B)$ refers to the conditional probability of an event $A\in\mathcal{F}$ relative to an event $B\in\mathcal{F}$ defined by $P(A/B)=\dfrac{P(A,\ B)}{P(B)}$. When $P(B)=0$, then $P(A/B)$ is not defined or is supposed to be zero for convenience.

Condition (A–2): Each player's strategy depends only on his or her own agent's suggestion, that is,
(i) $P(X=i/Z_1=k,\ Z_2=\ell)=P(X=i/Z_1=k)$ and
(ii) $P(Y=j/Z_1=k,\ Z_2=\ell)=P(Y=j/Z_2=\ell)$
hold for all $i\in S_1$, $j\in S_2$, $k\in T_1$, and $\ell\in T_2$ whenever $P(Z_1=k,\ Z_2=\ell)>0$.

Now, we can formulate the equilibrium concept of a noncooperative game with agents.

Definition 1. A game Γ is called a noncooperative game with agents who suggest Z_i to Player i $(i=1,\ 2)$ if and only if all strategy profiles $(X,\ Y)\in\mathcal{R}(S)$ are restricted within the set $\mathcal{R}_{Z_1,Z_2}(S)$.

Definition 2. A strategy profile $(X^*,\ Y^*)\in\mathcal{R}_{Z_1,Z_2}(S)$ is an equilibrium of a noncooperative game with agents who suggest Z_i to Player i $(i=1,\ 2)$ if and only if

$$\text{(i)}\quad E\,[u_1(X^*,\ Y^*)]\geq E\,[u_1(X,\ Y^*)]$$

holds for all $X\in\mathcal{R}(S_1)$ such that $(X,\ Y^*)\in\mathcal{R}_{Z_1,Z_2}(S)$, and

$$\text{(ii)}\quad E\,[u_2(X^*,\ Y^*)]\geq E\,[u_2(X^*,\ Y)]$$

holds for all $Y\in\mathcal{R}(S_2)$ such that $(X^*,\ Y)\in\mathcal{R}_{Z_1,Z_2}(S)$.

We shall call this equilibrium an *exogenous correlated equilibrium* relative to Z_1 and Z_2.

Remark 1. If we assume that $T_1=T_2$ and $P(Z_1=Z_2)=1$, then the notion of the noncooperative game with agents and its equiliblium concept in this paper completely coincide with those of the noncooperative game with a mediator defined in our previous paper (Kôno 2008).

Theorem 1. For any agents' suggestion $Z_1\in\mathcal{R}(T_1)$ and $Z_2\in\mathcal{R}(T_2)$, the set $\mathcal{D}_{Z_1,Z_2}(S)$ of the distributions on S induced by all strategy profiles $(X^*,\ Y^*)\in\mathcal{R}_{Z_1,Z_2}(S)$ that are the exogenous correlated equilibria relative to Z_1

and Z_2 includes the distributions of all the Nash equilibria in the usual sense.

In order to prove Theorem 1, we need to rewrite Definition 2 by using the distributions of $(X^*, Y^*)\in\mathcal{R}_{Z_1,Z_2}(S)$, $(X, Y^*)\in\mathcal{R}_{Z_1,Z_2}(S)$, and $(X^*, Y)\in\mathcal{R}_{Z_1,Z_2}(S)$.

Let $(X, Y)\in\mathcal{R}_{Z_1,Z_2}(S)$. Set $p_{ij}:=P(X=i, Y=j)$ and $z_{k\ell}:=P(Z_1=k, Z_2=\ell)$. Then from our conditions (A-1) and (A-2), we have

$$
\begin{aligned}
p_{ij} &= \sum_{k\in T_1}\sum_{\ell\in T_2} P(X=i, Y=j, Z_1=k, Z_2=\ell)\\
&= \sum_{k\in T_1}\sum_{\ell\in T_2} P(X=i, Y=j/Z_1=k, Z_2=\ell)z_{k\ell}\\
&\quad \text{by using the condition (A-1)}\\
&= \sum_{k\in T_1}\sum_{\ell\in T_2} P(X=i/Z_1=k, Z_2=\ell)P(Y=j/Z_1=k, Z_2=\ell)z_{k\ell}\\
&\quad \text{by using the condition (A-2)}\\
&= \sum_{k\in T_1}\sum_{\ell\in T_2} P(X=i/Z_1=k)P(Y=j/Z_2=\ell)z_{k\ell}\\
&= \sum_{k\in T_1}\sum_{\ell\in T_2} x_{i/k}y_{j/\ell}z_{k\ell},
\end{aligned}
$$

where $x_{i/k}:=P(X=i/Z_1=k)$ and $y_{j/\ell}:=P(Y=j/Z_2=\ell)$.

Since the probability structure of any strategy profile $(X, Y)\in\mathcal{R}_{Z_1,Z_2}(S)$ is determined by a family of probability measures on S_1, $(\{x_{i/k}\}_{i\in S_1}\in\mathcal{P}(S_1)\,;\,k\in T_1)$[5)] and on S_2, $(\{y_{j/\ell}\}_{j\in S_2}\in\mathcal{P}(S_2)\,;\,\ell\in T_2)$, we represent the distribution of $(X, Y)\in\mathcal{R}_{Z_1,Z_2}(S)$ by

$$(X, Y)\approx(\{x_{i/k}\}_{i\in S_1}\,;\,k\in T_1, \{y_{j/\ell}\}_{j\in S_2}\,;\,\ell\in T_2).$$

We note that the distribution of (X, Y) on S alone is not enough to determin the probability structure of $(X, Y)\in\mathcal{R}_{Z_1,Z_2}(S)$ because random variables Z_1 and Z_2 are given in advance.

Now we can rewrite Definition 2 by using these families of probability measure.

Let the distributions of (X^*, Y^*), (X, Y^*), and (X^*, Y) in Definition 2 be represented as follows:

$$
\begin{aligned}
&(X^*, Y^*)\approx(\{x^*_{i/k}\}_{i\in S_1}\,;\,k\in T_1, \{y^*_{j/\ell}\}_{j\in S_2}\,;\,\ell\in T_2)\\
&(X, Y^*)\approx(\{x_{i/k}\}_{i\in S_1}\,;\,k\in T_1, \{y^*_{j/\ell}\}_{j\in S_2}\,;\,\ell\in T_2)\\
&(X^*, Y)\approx(\{x^*_{i/k}\}_{i\in S_1}\,;\,k\in T_1, \{y_{j/\ell}\}_{j\in S_2}\,;\,\ell\in T_2).
\end{aligned}
$$

[5)] As is in our previous paper, we use the term $\mathcal{P}(M)$ to denote all the probability measures on a finite set M.

From simple calculation, conditions (i) and (ii) of Definition 2 can be written as follows.

$$E\,[u_1(X^*,\ Y^*)]-E\,[u_1(X,\ Y^*)]=\sum_{i\in S_1}\sum_{j\in S_2}\sum_{k\in T_1} u_1(i,\ j)(x^*_{i/k}-x_{i/k})(\sum_{\ell\in T_2} y^*_{j/\ell} z_{k\ell})\geq 0$$

holds for all $\{x_{i/k}\}_{i\in S_1}\in\mathcal{P}(S_1)$, $k\in T_1$ and

$$E\,[u_2(X^*,\ Y^*)]-E\,[u_2(X^*,\ Y)]=\sum_{i\in S_1}\sum_{j\in S_2}\sum_{\ell\in T_2} u_2(i,\ j)(y^*_{j/\ell}-y_{j/\ell})(\sum_{k\in T_1} x^*_{i/k} z_{k\ell})\geq 0$$

holds for all $\{y_{j/\ell}\}_{j\in S_2}\in\mathcal{P}(S_2)$, $\ell\in T_2$.

Since the probability measures $\{x_{i/k}\}_{i\in S_1}\in\mathcal{P}(S_1)$ and $\{y_{j/\ell}\}_{j\in S_2}\in\mathcal{P}(S_2)$ can be chosen arbitrarily for each $k\in T_1$ and $\ell\in T_2$, respectively, we have the following lemma.

Lemma 1.

Condition (i) of Definition 2 is equivalent to the following condition.

(i) For each $k\in T_1$ such that $\sum_{\ell\in T_2} z_{k\ell}=:z_{k\bullet}>0$

$$\sum_{i\in S_1}\sum_{j\in S_2} u_1(i,\ j)(x^*_{i/k}-x_i)(\sum_{\ell\in T_2} y^*_{j/\ell} z_{k\ell})\geq 0 \tag{1}$$

holds for all $\{x_i\}_{i\in S_1}\in\mathcal{P}(S_1)$.

Condition (ii) of Definition 2 is equivalent to the following condition.

(ii) For each $\ell\in T_2$ such that $\sum_{k\in T_1} z_{k\ell}=:z_{\bullet\ell}>0$

$$\sum_{i\in S_1}\sum_{j\in S_2} u_2(i,\ j)(y^*_{j/\ell}-y_j)(\sum_{k\in T_1} x^*_{i/k} z_{k\ell})\geq 0 \tag{2}$$

holds for all $\{y_j\}_{j\in S_2}\in\mathcal{P}(S_2)$.

Remark 2. Lemma 1 is equivalent to the definition of "correlated equilibrium" in Rosenthal (1974, p. 119), if one likens Rosenthal's notation

$$(\bar{x}^1/C_1,\ \dots,\ \bar{x}^k/C_k\ ;\ \bar{y}^1/D_1,\ \dots,\ \bar{y}^\ell/D_\ell)$$

to our

$$(\{x_{i/1}\}_{i\in S_1},\ \dots,\ \{x_{i/k}\}_{i\in S_1}\ ;\ \{y_{j/1}\}_{j\in S_2},\ \dots,\ \{y_{j/\ell}\}_{j\in S_2}),$$

where $C_s=\{Z_1=s\}$; $s=1,\ \dots,\ k$; $D_t=\{Z_2=t\}$; $t=1,\ \dots,\ \ell$, and $\bar{x}^i/C_i$(*resp.* $\bar{y}^j/D_j$) is a conditional probability relative to C_i(*resp.* D_j), though he does not mention conditions (A-1) and (A-2) explicitly. The essential difference between our definition of correlated equilibrium retative to Z_1 and Z_2 and his "correlated equilibrium" is that ours is defined for each fixed agents' suggestions Z_1 and Z_2 but his definition seems to define for a possible choice of partions $\{C_i\}_{i=1,\dots,k}$ and $\{D_j\}_{j=1,\dots,\ell}$, because he argues in page 119 that ordinary Nash equilibria are correlated equilibria with trivial partitions. I believe that if he would have stated that ordinary Nash equilibria are always correlated equilibria by identifying a Nash equilibrium $(\bar{x}^*, \bar{y}^*)$ with $\bar{x}^* = \bar{x}^i / C_i (i=1, \dots, k)$ and $\bar{y}^* = \bar{y}^j / D_j (j=1, \dots, \ell)$, then his assertion would coincide with our Theorem 1.

The problem of correlated equilibrium's dependence on the information structure (which is equivalent to a family of partitions) also arises in Aumann's definition. We note that our Theorem 1 clearly states that for any suggestions $Z_1 \in \mathcal{R}(T_1)$ and $Z_2 \in \mathcal{R}(T_2)$, which may or may not be trivial random variables, ordinary Nash equilibria are always exogenous correlated equilibria retative to Z_1 and Z_2. Definition 2 does not imply that a strategy profile (X^*, Y^*) is an equilibrium if and only if for properly chosen Z_1 and Z_2, conditions (i) and (ii) hold.

Now, the proof of Theorem 1 is easy. In Lemma 1, when one sets $\{x^*_{i/k}\}_{i\in S_1} = \{x^*_i\}_{i\in S_1}$ for all $k \in T_1$ and $\{y^*_{j/\ell}\}_{j\in S_2} = \{y^*_j\}_{j\in S_2}$ for all $\ell \in T_2$, then, the inequlities (1) and (2) satisfy the following.

(i) $\sum_{i\in S_1}\sum_{j\in S_2} u_1(i, j) x^*_i y^*_j \geq \sum_{i\in S_1}\sum_{j\in S_2} u_1(i, j) x_i y^*_j$

holds for all $\{x_i\}_{i\in S_1} \in \mathcal{P}(S_1)$, and

(ii) $\sum_{i\in S_1}\sum_{j\in S_2} u_2(i, j) x^*_i y^*_j \geq \sum_{i\in S_1}\sum_{j\in S_2} u_2(i, j) x^*_i y_j.$

holds for all $\{y_j\}_{j\in S_2} \in \mathcal{P}(S_2)$.

These conditions are the same as those that validate that $(\{x^*_i\}_{i\in S_1}, \{y^*_j\}_{j\in S_2})$ is a Nash equilibrium. This completes the proof of Theorem 1.

As is well known, in the case of a usual Nash equilibrium[6], the following lemma arises. This lemma will be usuful in finding a distribution of equilibrium when the pure strategy set S_i has more than two choices.

Lemma 2. For each $k \in T_1$ such that $z_{k\bullet} > 0$, set

$$S_{1k}(Y^*) := \{i \in S_1 \,;\, \max_{s\in S_1} \sum_{j\in S_2} u_1(s, j)(\sum_{\ell\in T_2} y^*_{j/\ell} z_{k\ell}) = \sum_{j\in S_2} u_1(i, j)(\sum_{\ell\in T_2} y^*_{j/\ell} z_{k\ell})\},$$

and for each $\ell \in T_2$ such that $z_{\bullet\ell} > 0$, set

$$S_{2\ell}(X^*) := \{j \in S_2 \,;\, \max_{t\in S_2} \sum_{i\in S_1} u_2(i, t)(\sum_{k\in T_1} x^*_{i/k} z_{k\ell}) = \sum_{i\in S_1} u_2(i, j)(\sum_{k\in T_1} x^*_{i/k} z_{k\ell})\}.$$

Then, Condition (i) of Lemma 1 is equivalent to the following condition.

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[6] cf. Osborne and Rubinstein (1994, p. 33, Lemma 33.2)

(i) For each $k \in T_1$ such that $z_{k\bullet} > 0$

$$\{i \in S_1 \, ; \, x^*_{i/k} > 0\} \subset S_{1k}(Y^*) \tag{3}$$

holds.

We note that the set $\{i \in S_1 \, ; \, x^*_{i/k} > 0\}$ is called the "support" of the probability measure $\{x^*_{i/k}\}_{i \in S_1} \in \mathcal{P}(S_1)$. In other words formula (3) implies that the support of $\{x^*_{i/k}\}_{i \in S_1}$ is included in the set of the best pure strategy response against the given opponent strategy $Y.^*$

Condition (ii) of Lemma 1 is equivalent to the following condition.

(ii) For each $\ell \in T_2$ such that $z_{\bullet \ell} > 0$

$$\{j \in S_2 \, ; \, y^*_{j/\ell} > 0\} \subset S_{2\ell}(X^*) \tag{4}$$

holds.

Proof of Lemma 2. First suppose that (3) holds. Then, from the definition of $S_{1k}(Y^*)$, it follows that

$$\begin{aligned}\max_{s \in S_1} \sum_{j \in S_2} u_1(s, j)\Big(\sum_{\ell \in T_2} y^*_{j/\ell} z_{k\ell}\Big) &= \sum_{i \in S_1} \sum_{j \in S_2} u_1(i, j) x^*_{i/k}\Big(\sum_{\ell \in T_2} y^*_{j/\ell} z_{k\ell}\Big) \\ &\geq \sum_{j \in S_2} u_1(i, j)\Big(\sum_{\ell \in T_2} y^*_{j/\ell} z_{k\ell}\Big)\end{aligned}$$

holds for all $i \in S_1$. Now multiply the both sides of the above inequality by x_i of any $\{x_i\}_{i \in S_1} \in \mathcal{P}(S_1)$ and sum with respect to i. Then, from $\sum_{i \in S_1} x_i = 1$ we have the inequality (1). Conversely, if there exists $i_0 \in S_1$ such that $x_{i_0/k} > 0$ and $i_0 \notin S_{1k}(Y^*)$, then from the definition of $S_{1k}(Y^*)$, it follows that

$$\max_{s \in S_1} \sum_{j \in S_2} u_1(s, j)\Big(\sum_{\ell \in T_2} y^*_{j/\ell} z_{k\ell}\Big) > \sum_{j \in S_2} u_1(i_0, j)\Big(\sum_{\ell \in T_2} y^*_{j/\ell} z_{k\ell}\Big).$$

Therefore, we have

$$\begin{aligned}\sum_{i \in S_1} \sum_{j \in S_2} u_1(i, j) x^*_{i/k}\Big(\sum_{\ell \in T_2} y^*_{j/\ell} z_{k\ell}\Big) &= \sum_{i \in S_1,\, i \neq i_0} \sum_{j \in S_2} u_1(i, j) x^*_{i/k}\Big(\sum_{\ell \in T_2} y^*_{j/\ell} z_{k\ell}\Big) \\ &\quad + \sum_{j \in S_2} u_1(i_0, j) x^*_{i_0/k}\Big(\sum_{\ell \in T_2} y^*_{j/\ell} z_{k\ell}\Big) \\ &< \max_{s \in S_1} \sum_{j \in S_2} u_1(s, j)\Big(\sum_{\ell \in T_2} y^*_{j/\ell} z_{k\ell}\Big) \\ &= \sum_{i \in S_1} \sum_{j \in S_2} u_1(i, j) x_i\Big(\sum_{\ell \in T_2} y^*_{j/\ell} z_{k\ell}\Big)\end{aligned}$$

holds for $\{x_i\}_{i \in S_1} \in \mathcal{P}(S_1)$ such that $\{i \in S_1 \, ; \, x_i > 0\} \subset S_{1k}(Y^*)$, which contradicts the inequality (1). We can check Condition (ii) in a similar manner.

3. Examples

Now, we illustrate our concept using some examples given in Aumann (1974) and Fudenberg and Tirole (1991).

(a) Our first example is referred to as the battle of sexes whose pay-off matrix is given below (Fudenberg and Tirole, 1991, p. 54). Here, Player 1 picks the row, while Player 2 selects the column. The numeral on the left in the parenthesis represents Player 1's pay-off and that on the right denotes Player 2's pay-off.

Table 1 Pay-off matrix of the game

	1	2
1	(5, 1)	(0, 0)
2	(4, 4)	(1, 5)

There are three Nash equilibria whose distributions on $S=S_1\times S_2$ can be represented as follows. Here, (p_{ij}) is a 2×2 matrix given by $p_{ij}:=P(X=i,\ Y=j)$.

$$\sigma_1=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\ \sigma_2=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},\ \sigma_3=\begin{pmatrix}1/4 & 1/4\\ 1/4 & 1/4\end{pmatrix}.$$

By using Aumann's (1974) suggestion, Fudenberg and Tirole (1991, p. 54) show that the players can perform better than in the mixed Nash equilibrium strategy if they use the following distribution for a strategy profile:

$$\sigma_0=\begin{pmatrix}1/3 & 0\\ 1/3 & 1/3\end{pmatrix}.$$

Their method of obtaining the distribution σ_0 induced by a strategy profile $(X^*,\ Y^*)$ by using a random device is as follows.

This random device has three equally likely events A, B, and C. Suppose that if A occurs, then Player 1's choice is $X^*=1$ and if B or C occurs, then $X^*=2$. On the other hand, if A or B occurs, then Player 2's choice is $Y^*=1$ and if C occurs, then $Y^*=2$. That is, two players receive different information from the same random device.

Let us explain the above situation from the perspective of our framework. Set $T_1=\{1, 2\}$; here, we abbreviate the event A to 1 and the event $B\cup C$ to 2. Similarly, set $T_2=\{1, 2\}$; here, we abbreviate the event $A\cup B$ to 1 and the event C to 2. Then, the joint distribution of Z_1 and Z_2, the conditional probabilities of X^* relative to Z_1, and that of Y^* relative to Z_2 can be determined from the above situation as follows.

$$P(Z_1=1,\ Z_2=1)=P(A\cap(A\cup B))=P(A)=1/3,$$
$$P(Z_1=1,\ Z_2=2)=P(A\cap C))=P(\emptyset)=0,$$
$$P(Z_1=2,\ Z_2=1)=P((B\cup C)\cap(A\cup B))=P(B)=1/3,$$
$$P(Z_1=2,\ Z_2=2)=P((B\cup C)\cap C)=P(C)=1/3.$$
$$x^*_{1/1}:=P(X^*=1/Z_1=1)=1,\ x^*_{2/1}:=P(X^*=2/Z_1=1)=0,$$
$$x^*_{1/2}:=P(X^*=1/Z_1=2)=0,\ x^*_{2/2}:=P(X^*=2/Z_1=2)=1,$$
$$y^*_{1/1}:=P(Y^*=1/Z_2=1)=1,\ y^*_{2/1}:=P(Y^*=2/Z_2=1)=0, \text{ and}$$
$$y^*_{1/2}:=P(Y^*=1/Z_2=2)=0,\ y^*_{2/2}:=P(Y^*=2/Z_2=2)=1.$$

Then, from easy calculation we obtain σ_0 as the distribution of $(X^*,\ Y^*)$.

Fudenberg and Tirole intuitively explain that σ_0 is an equilibrium but they do not deduce why σ_0 is an equilibrium.

With our framework, we can analyze this example rigorously and also obtain all the possible equilibria not found in previous literature. We note that the all equilibria are outside the convex hull of the Nash equilibria.

Proposition 1. Assume that $z_{11}=z_{21}=z_{22}=1/3$ and $z_{12}=0$. Then, the possible exogenous correlated equilibria relative to Z_1 and Z_2 excluding the usual Nash equilibria are the following. Since in this case, $x^*_{2/k}=1-x^*_{1/k}$ and $y^*_{2/\ell}=1-y^*_{1/\ell}$, we only indicate $x^*_{1/k}(k=1,\ 2)$ and $y^*_{1/\ell}(\ell=1,\ 2)$.

(1) $x^*_{1/1}=1,\ 0\leq x^*_{1/2}\leq 1/2,\ y^*_{1/1}=1,\ y^*_{1/2}=0,$

$$\sigma_4=(1-x^*_{1/2})\begin{pmatrix}1/3 & 0\\ 1/3 & 1/3\end{pmatrix}+x^*_{1/2}\begin{pmatrix}2/3 & 1/3\\ 0 & 0\end{pmatrix}.$$

Here, $x^*_{1/2}$ can be chosen arbitrarily between 0 and 1/2.

(2) $x^*_{1/1}=0,\ 1/2\leq x^*_{1/2}\leq 1,\ y^*_{1/1}=0,\ y^*_{1/2}=1,$

$$\sigma_5=x^*_{1/2}\begin{pmatrix}1/3 & 1/3\\ 0 & 1/3\end{pmatrix}+(1-x^*_{1/2})\begin{pmatrix}0 & 0\\ 1/3 & 2/3\end{pmatrix}.$$

Here, $x^*_{1/2}$ can be chosen arbitrarily between 1/2 and 1.

(3) $x^*_{1/1}=1,\ x^*_{1/2}=0,\ 1/2\leq y^*_{1/1}\leq 1,\ y^*_{1/2}=0,$

$$\sigma_6=y^*_{1/1}\begin{pmatrix}1/3 & 0\\ 1/3 & 1/3\end{pmatrix}+(1-y^*_{1/1})\begin{pmatrix}0 & 1/3\\ 0 & 2/3\end{pmatrix}.$$

Here, $y^*_{1/1}$ can be chosen arbitrarily between 1/2 and 1.

(4) $x^*_{1/1}=0,\ x^*_{1/2}=1,\ 0\leq y^*_{1/1}\leq 1/2,\ y^*_{1/2}=1,$

$$\sigma_7 = (1-y^*_{1/1})\begin{pmatrix} 1/3 & 1/3 \\ 0 & 1/3 \end{pmatrix} + y^*_{1/1}\begin{pmatrix} 2/3 & 0 \\ 1/3 & 0 \end{pmatrix}.$$

Here, $y^*_{1/1}$ can be chosen arbitrarily between 0 and 1/2.

When we set $x^*_{1/2}=0$ in (1) or $y^*_{1/1}=1$ in (3), we can obtain Fudenberg and Tirole's distribution σ_0.

Remark 3. A necessary and sufficient condition that $X \in \mathcal{R}(S_1)$ and $Y \in \mathcal{R}(S_2)$ are independent is that the matrix of probability measure $\{p_{ij}=P(X=i,\ Y=j)\}_{i\in S_1, j\in S_2} \in \mathcal{P}(S_1 \times S_2)$ has "rank" 1. Since two random variables $X^* \in \mathcal{R}(S_1)$ and $Y^* \in \mathcal{R}(S_2)$ of a strategy profile $(X^*,\ Y^*)$, which is a Nash equilibrium, are independent, from this criterion, we can check whether a distribution of an equilibrium on $S_1 \times S_2$ is a Nash equilibrium or not. In the case of a 2×2 matrix, this condition is equivalent to the state where "determinant" of this matrix is zero[7). Therefore, if the determinant of the matrix $\{p_{ij}=P(X^*=i,\ Y^*=j)\}_{i=1,2; j=1,2}$ for a strategy profile $(X^*,\ Y^*)$ of a correlated equilibrium is zero, then it is a Nash equilibrium; its converse is also true.

(b) The second example is the chicken game discussed in Aumann' (1974) example (2.7).

Table 2 Pay-off matrix of the game

	1	2
1	(6, 6)	(2, 7)
2	(7, 2)	(0, 0)

There are three Nash equilibria whose distributions on $S=S_1\times S_2$ can be represented as follows;

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 4/9 & 2/9 \\ 2/9 & 1/9 \end{pmatrix}.$$

Aumann (1974) shows that the players can perform better than in the mixed Nash equilibrium strategy if they use the following distribution for a strategy profile:

[7) The terminologies "rank" and "determinant" of a matrix can be found in any elementary textbook of linear algebra.

$$\sigma_4=\begin{pmatrix}1/3 & 1/3\\ 1/3 & 0\end{pmatrix}.$$

His method of obtaining the distribution σ_4 induced by a strategy profile (X^*, Y^*) using a random device is as follows.

Consider an objective chance mechanism that chooses one of three events A, B, and C with probability $1/3$ each. Suppose that if A occurs, then Player 1's choice is $X^*=2$ and if B or C occurs, then $X^*=1$. On the other hand, if A or B occurs, then Player 2's choice is $Y^*=1$ and if C occurs, then $Y^*=2$. That is, the two players receive different information from the same chance mechanism.

Let us explain the above situation from the perspective of our framework. Set $T_1=\{1, 2\}$; here, we abbreviate the event A to 2 and the event $B\cup C$ to 1. Similarly, set $T_2=\{1, 2\}$; here, we abbreviate the event $A\cup B$ to 1 and the event C to 2. Then, the joint distribution of Z_1 and Z_2, the conditional probabilities of X^* relative to Z_1, and that of Y^* relative to Z_2 can be determined as follows.

$P(Z_1=1, Z_2=1)=P((B\cup C)\cap(A\cup B))=P(B)=1/3$,
$P(Z_1=1, Z_2=2)=P((B\cup C)\cap C)=P(C)=1/3$,
$P(Z_1=2, Z_2=1)=P(A\cap(A\cup B))=P(A)=1/3$, and
$P(Z_1=2, Z_2=2)=P(A\cap C)=P(\emptyset)=0$.
$x^*_{1/1}:=P(X^*=1/Z_1=1)=1$, $x^*_{2/1}:=P(X^*=2/Z_1=1)=0$,
$x^*_{1/2}:=P(X^*=1/Z_1=2)=0$, $x^*_{2/2}:=P(X^*=2/Z_1=2)=1$,
$y^*_{1/1}:=P(Y^*=1/Z_2=1)=1$, $y^*_{2/1}:=P(Y^*=2/Z_2=1)=0$, and
$y^*_{1/2}:=P(Y^*=1/Z_2=2)=0$, $y^*_{2/2}:=P(Y^*=2/Z_2=2)=1$.

Then, from easy calculation we obtain σ_4 as the distribution of (X^*, Y^*).

Aumann indicates that this distribution is indeed an equilibrium but he does not provide any proof nor any explanation.

By our framework, we can analyze this example rigorously and also obtain all the possible equilibria not found in previous literature.

Proposition 2. Assume that $z_{11}=z_{12}=z_{21}=1/3$ and $z_{22}=0$. Then, the number of possible exogenous correlated equilibria relative to Z_1 and Z_2 excluding the usual Nash equilibria is four. Since in this case, $x^*_{2/k}=1-x^*_{1/k}$ and $y^*_{2/\ell}=1-y^*_{1/\ell}$, we only indicate $x^*_{1/k}(k=1, 2)$ and $y^*_{1/\ell}(\ell=1, 2)$.

(1) $x^*_{1/1}=1$, $x^*_{1/2}=0$, $y^*_{1/1}=1$, $y^*_{1/2}=0$,

$$\sigma_4=\begin{pmatrix}1/3 & 1/3\\ 1/3 & 0\end{pmatrix}.$$

(2) $x^*_{1/1}=1/3$, $x^*_{1/2}=1$, $y^*_{1/1}=1/3$, $y^*_{1/2}=1$,

$$\sigma_5=\begin{pmatrix} 7/27 & 8/27 \\ 8/27 & 4/27 \end{pmatrix}.$$

(3) $x^*_{1/1}=1$, $x^*_{1/2}=1/3$, $y^*_{1/1}=2/3$, $y^*_{1/2}=0$,

$$\sigma_6=\begin{pmatrix} 8/27 & 13/27 \\ 4/27 & 2/27 \end{pmatrix}.$$

(4) $x^*_{1/1}=2/3$, $x^*_{1/2}=0$, $y^*_{1/1}=1$, $y^*_{1/2}=1/3$,

$$\sigma_7=\begin{pmatrix} 8/27 & 4/27 \\ 13/27 & 2/27 \end{pmatrix}.$$

In the next section, we will discuss general 2×2 games — $S_1=S_2=\{1, 2\}$ with agents' suggestion sets $T_1=T_2=\{1, 2\}$ — and give brief proofs of Propositions 1 and 2.

4. 2×2 games with $T_1=T_2=\{1, 2\}$

In this section, we assume that $S_1=S_2=\{1, 2\}$ and $T_1=T_2=\{1, 2\}$. Since, in this case, we have $x_{2/k}=1-x_{1/k}$ for $k=1, 2$ and $y_{2/\ell}=1-y_{1/\ell}$ for $\ell=1, 2$, we use the expression $(x_{1/1}, x_{1/2}, y_{1/1}, y_{1/2})$ to characterize the distribution of $(X, Y)\in\mathcal{R}_{Z_1,Z_2}(S)$ on S. We note that the marginal distritution of X(resp. Y) is determined by $(x_{1/1}, x_{1/2})$(resp. $(y_{1/1}, y_{1/2})$), where $x_{1/1}$ and $x_{1/2}$ (resp. $y_{1/1}$ and $y_{1/2}$) can be chosen arbitrarily between 0 and 1. To avoid a trivial case, hereafter, we assume that

$$z_{k\bullet}:=z_{k1}+z_{k2}>0 \text{ for } k=1, 2 \text{ and } z_{\bullet\ell}:=z_{1\ell}+z_{2\ell}>0 \text{ for } \ell=1, 2.$$

We also use the following abbreviations:

$$A_j=u_1(1, j)-u_1(2, j)\ (j=1, 2) \text{ and } B_i=u_2(i, 1)-u_2(i, 2)\ (i=1, 2).$$

By using these notations and under the framework of this section, the conditions (i) and (ii) of Lemma 1 state the following.

$$\text{(i-1)}\quad (x^*_{1/1}-x)(A_2 z_{1\bullet}+(A_1-A_2)\sum_{\ell=1}^{2} y^*_{1/\ell} z_{1\ell})\geq 0 \tag{5}$$

holds for all $0\leq x\leq 1$,

$$\text{(i-2)}\quad (x^*_{1/2}-x)(A_2 z_{2\bullet}+(A_1-A_2)\sum_{\ell=1}^{2} y^*_{1/\ell} z_{2\ell})\geq 0 \tag{6}$$

holds for all $0 \leq x \leq 1$,

$$\text{(ii-1)} \quad (y^*_{1/1} - y)(B_2 z_{\bullet 1} + (B_1 - B_2)\sum_{k=1}^{2} x^*_{1/k} z_{k1}) \geq 0 \tag{7}$$

holds for all $0 \leq y \leq 1$, and

$$\text{(ii-2)} \quad (y^*_{1/2} - y)(B_2 z_{\bullet 2} + (B_1 - B_2)\sum_{k=1}^{2} x^*_{1/k} z_{k2}) \geq 0 \tag{8}$$

holds for all $0 \leq y \leq 1$.

Now, let us check Table 1. Since in this case, $z_{11} = z_{21} = z_{22} = 1/3$, $z_{12} = 0$, $A_1 = 1$, $A_2 = -1$, $B_1 = 1$, and $B_2 = -1$, conditions (5) to (8) can be written as follows.

$$\text{(i-1)} \quad (x^*_{1/1} - x)(2y^*_{1/1} - 1) \geq 0 \tag{9}$$

holds for all $0 \leq x \leq 1$,

$$\text{(i-2)} \quad (x^*_{1/2} - x)(y^*_{1/1} + y^*_{1/2} - 1) \geq 0 \tag{10}$$

holds for all $0 \leq x \leq 1$,

$$\text{(ii-1)} \quad (y^*_{1/1} - y)(x^*_{1/1} + x^*_{1/2} - 1) \geq 0 \tag{11}$$

holds for all $0 \leq y \leq 1$, and

$$\text{(ii-2)} \quad (y^*_{1/2} - y)(2x^*_{1/2} - 1) \geq 0 \tag{12}$$

holds for all $0 \leq y \leq 1$.

Now, it is easy to check that each case of Proposition 1 satisfies the conditions (9) to (12).

Similarly, in Table 2, $z_{11} = z_{12} = z_{21} = 1/3$ and $z_{22} = 0$. Since $A_1 = -1$, $A_2 = 2$, $B_1 = -1$, and $B_2 = 2$, conditions (5) to (8) can be written as follows.

$$\text{(i-1)} \quad (x^*_{1/1} - x)(-3y^*_{1/1} - 3y^*_{1/2} + 4) \geq 0 \tag{13}$$

holds for all $0 \leq x \leq 1$,

$$\text{(i-2)} \quad (x^*_{1/2} - x)(-3y^*_{1/1} + 2) \geq 0 \tag{14}$$

holds for all $0 \leq x \leq 1$,

$$\text{(ii-1)}\quad (y^*_{1/1}-y)(-3x^*_{1/1}-3x^*_{1/2}+4)\geq 0 \tag{15}$$

holds for all $0\leq y\leq 1$, and

$$\text{(ii-2)}\quad (y^*_{1/2}-y)(-3x^*_{1/1}+2)\geq 0 \tag{16}$$

holds for all $0\leq y\leq 1$.

Now, it is easy to check that each case of Proposition 2 satisfies conditions (13) to (16).

5. Miscellaneous

We can investigate Aumann's (1974) other examples using our framework. Here, we will investigate the following two examples.

(a) First example.

Table 3 Pay-off matrix of the game

	1	2	3
1	(6, 6)	(0, 0)	(2, 7)
2	(0, 0)	(4, 4)	(3, 0)
3	(7, 2)	(0, 3)	(0, 0)

This pay-off matrix is obtained from example (b) in Section 3 by adding a middle row and middle column with appropriate payoffs. We also assume in this game that two players obtain the same agents' suggestions as in the previous game, that is, $T_1=T_2=\{1, 2\}$, $P(Z_1=1, Z_2=1)=1/3$, $P(Z_1=1, Z_2=2)=1/3$, $P(Z_1=2, Z_2=1)=1/3$, and $P(Z_1=2, Z_2=2)=0$.

In this case, the distribution of $(X, Y)\in\mathcal{R}_{Z_1,Z_2}(S)$ is characterized by

$$(\{x_{i/k}\}_{i=1,2,3},\ k=1,\ 2\ ;\ \{y_{j/\ell}\}_{j=1,2,3},\ \ell=1,\ 2),$$

where $\{x_{i/k}\}_{i=1,2,3}$, $k=1, 2$ (resp. $\{y_{j/\ell}\}_{j=1,2,3}$, $\ell=1, 2$) determines the distribution of X(resp. Y), and the joint distribution $p_{ij}:=P(X=i, Y=j)$ of (X, Y) is determined as follows:

$$\begin{aligned} p_{ij} &= \sum_{k=1,\ell=1}^{2} P(X=i/Z_1=k)P(Y=j/Z_2=\ell)P(Z_1=k, Z_2=\ell) \\ &=(x_{i/1}y_{j/1}+x_{i/1}y_{j/2}+x_{i/2}y_{j/1})/3 \qquad i, j=1, 2, 3. \end{aligned}$$

Proposition 3 (Aumann (1974, pp.72-73)). The strategy profile $(X^*, Y^*)\in\mathcal{R}_{Z_1,Z_2}(S)$ having the following distribution is a correlated equilibrium relative to the above suggestions Z_1 and Z_2.

$$x^*_{1/1}=1,\ x^*_{2/1}=0,\ x^*_{3/1}=0,\ x^*_{1/2}=0,\ x^*_{2/2}=0,\ x^*_{3/2}=1 \text{ and}$$
$$y^*_{1/1}=1,\ y^*_{2/1}=0,\ y^*_{3/1}=0,\ y^*_{1/2}=0,\ y^*_{2/2}=0,\ y^*_{3/2}=1.$$

The joint distribution of $P(X^*=i,\ Y^*=j)=:p^*_{ij}$; $i, j=1, 2, 3$ is

$$(p^*_{ij})=\begin{pmatrix} 1/3 & 0 & 1/3 \\ 0 & 0 & 0 \\ 1/3 & 0 & 0 \end{pmatrix}.$$

In fact, we can easily obtain $E[u_1(X^*,\ Y^*)]=E[u_2(X^*,\ Y^*)]=5$. Further, we can easily check that

$$\begin{aligned} E[u_1(X,\ Y^*)] &=(8x_{1/1}+3x_{2/1}+7x_{3/1}+6x_{1/2}+7x_{3/2})/3 \\ &\leq 8(x_{1/1}+x_{2/1}+x_{3/1})/3+7(x_{1/2}+x_{2/2}+x_{3/2})/3 \\ &=5=E[u_1(X^*,\ Y^*)] \end{aligned}$$

holds for any $X\approx(\{x_{i/1}\}_{i=1,2,3}\in\mathcal{P}(S_1),\ \{x_{i/2}\}_{i=1,2,3}\in\mathcal{P}(S_1))$ and

$$\begin{aligned} E[u_2(X^*,\ Y)] &=(8y_{1/1}+3y_{2/1}+7y_{3/1}+6y_{1/2}+7y_{3/2})/3 \\ &\leq 8(y_{1/1}+y_{2/1}+y_{3/1})/3+7(y_{1/2}+y_{2/2}+y_{3/2})/3 \\ &=5=E[u_2(X^*,\ Y^*)] \end{aligned}$$

holds for any $Y\approx(\{y_{j/1}\}_{j=1,2,3}\in\mathcal{P}(S_2),\ \{y_{j/2}\}_{j=1,2,3}\in\mathcal{P}(S_2))$.

(b) Second example. Aumann (1974, p. 71) also investigates a 3-person game. His example (2.5) is as follows.

Table 4 Pay-off matrix of the game

	1	2
1	(0, 0, 3)	(0, 0, 0)
2	(1, 0, 0)	(0, 0, 0)

	1	2
1	(2, 2, 2)	(0, 0, 0)
2	(0, 0, 0)	(2, 2, 2)

	1	2
1	(0, 0, 0)	(0, 0, 0)
2	(0, 1, 0)	(0, 0, 3)

Here, Player 1 picks the row, Player 2, the column, and Player 3, the matrix: $S_3=$ {left, middle, right}. We denote "left" as 1, "middle" as 2 and "right" as 3, respectively, and let X_i be the Player i's strategy, that is, $X_i\in\mathcal{R}(S_i)$.

In this game, there are three pure Nash equilibria and one mixed Nash equilibrium. Furthermore, Aumann pointed out that the following situation allows for a better payoff. Players 1 and 2 get together and toss a fair coin. If the coin falls on heads, Players 1 and 2 choose $1\in S_1$ and $1\in S_2$, respectively; otherwise, they choose $2\in S_1$ and $2\in S_2$, respectively. On the other hand, Player 3 always plays $2\in S_3$. Then, the payoff is 2 for all players. Further, he explains that if Player 3 would know the outcome of the toss, he would be tempted to move away; however, he does not explain why this strategy profile is an equilibrium under the given situation.

By using our framework, we can show that the above strategy profile is an equilibrium in the sense of Definition 2. Moreover, we point out that some other equilibria exist.

First, we note that our framework of a 2-person noncooperative game with agents can be naturally generalized to an n-person noncooperative game with agents. Since we can naturally generalize conditions (A-1) and (A-2) for an n-person game, we can also define $\mathcal{R}_{Z_1,\dots,Z_n}(S)$, where $S=S_1\times\cdots\times S_n$.

Then, Aumann's argument can be restated as follows.

Set $T_1=T_2=\{1, 2\}$ and $T_3=\{1\}$. Let $P(Z_1=Z_2=1)=1/2$, $P(Z_1=Z_2=2)=1/2$, and $P(Z_3=1)=1$ (Z_3 is a trivial random variable). Then, the strategy profile $(X_1^*, X_2^*, X_3^*)\in\mathcal{R}_{Z_1,Z_2,Z_3}(S)$ having the following conditional probabilities

$$
\begin{aligned}
&P(X_1^*=1/Z_1=1)=1,\ P(X_1^*=2/Z_1=2)=1,\\
&P(X_2^*=1/Z_2=1)=1,\ P(X_2^*=2/Z_1=2)=1, \text{ and}\\
&P(X_3^*=2)=1
\end{aligned}
$$

is an equilibrium.

In general, the distribution of $(X_1, X_2, X_3)\in\mathcal{R}_{Z_1,Z_2,Z_3}(S)$ can be determined by the conditional probabilities

$$
\begin{aligned}
&P(X_1=1/Z_1=1)=:p_{1/1},\ P(X_1=2/Z_1=1)=:p_{2/1}=1-p_{1/1},\\
&P(X_1=1/Z_1=2)=:p_{1/2},\ P(X_1=2/Z_1=2)=:p_{2/2}=1-p_{1/2},\\
&P(X_2=1/Z_2=1)=:q_{1/1},\ P(X_2=2/Z_2=1)=:q_{2/1}=1-q_{1/1},\\
&P(X_2=1/Z_2=2)=:q_{1/2},\ P(X_2=2/Z_2=2)=:q_{2/2}=1-q_{1/2},
\end{aligned}
$$

and

$$
\begin{aligned}
&P(X_3=1/Z_3=1)=P(X_3=1)=:r_1,\ P(X_3=2/Z_3=1)=P(X_3=2)=:r_2,\\
&P(X_3=3/Z_3=1)=P(X_3=3)=:r_3.
\end{aligned}
$$

We note that X_3 is independent of (X_1, X_2). Therefore, from conditions (A-1) and (A-2), the joint distribution $p_{ijk}:=P(X_1=i, X_2=j, X_3=k)$ can be expressed as follows.

$$
\begin{aligned}
p_{ijk} &=\sum_{s=1}^{2}P(X_1=i, X_2=j, X_3=k, Z_1=Z_2=s, Z_3=1)\\
&=\sum_{s=1}^{2}P(X_1=i/Z_1=s)P(X_2=i/Z_2=s)P(X_3=k)P(Z_1=Z_2=s)
\end{aligned}
$$

$$=(p_{i/1}q_{j/1}+p_{i/2}q_{j/2})r_k/2.$$

We also note that the distribution of $(X_1, X_2, X_3)\in\mathcal{R}_{Z_1,Z_2,Z_3}(S)$ has 6 parameters $0\leq p_{1/1}\leq 1$, $0\leq p_{1/2}\leq 1$, $0\leq q_{1/1}\leq 1$, $0\leq q_{1/2}\leq 1$, and $0\leq r_1$, $0\leq r_2$ with $r_1+r_2\leq 1$ and that $p_{1/1}$, $p_{1/2}$ determine the distribution of X_1, $q_{1/1}$, $q_{1/2}$ determine the distribution of X_2, and r_1, r_2 determine the distribution of X_3.

Let $(X_1^*, X_2^*, X_3^*)\in\mathcal{R}_{Z_1,Z_2,Z_3}(S)$ be an equilibrium in the sense of Definition 2: we express the distributions by attaching asterisks to the above notations. Then, by using these notations and under the framework of this section, the conditions of Lemma 1 state the following.

$$\text{(i–1)}\quad (p_{1/1}^*-p)((4r_2^*-r_1^*)q_{1/1}^*-2r_2^*)\geq 0 \tag{17}$$

holds for all $0\leq p\leq 1$ and

$$\text{(i–2)}\quad (p_{1/2}^*-p)((4r_2^*-r_1^*)q_{1/2}^*-2r_2^*)\geq 0 \tag{18}$$

holds for all $0\leq p\leq 1$.

$$\text{(ii–1)}\quad (q_{1/1}^*-q)((4r_2^*-r_3^*)p_{1/1}^*-2r_2^*+r_3^*)\geq 0 \tag{19}$$

holds for all $0\leq q\leq 1$ and

$$\text{(ii–2)}\quad (q_{1/2}^*-q)((4r_2^*-r_3^*)p_{1/2}^*-2r_2^*+r_3^*)\geq 0 \tag{20}$$

holds for all $0\leq q\leq 1$.

(iii)

$$3(r_1^*-r_1)\sum_{\ell=1}^{2}p_{1/\ell}^*q_{1/\ell}^*+2(r_2^*-r_2)\sum_{s=1}^{2}\sum_{t=1}^{2}p_{s/t}^*q_{s/t}^*+3(r_3^*-r_3)\sum_{\ell=1}^{2}p_{2/\ell}^*q_{2/\ell}^*\geq 0 \tag{21}$$

holds for all r_1, r_2, $r_3\geq 0$ such that $r_1+r_2+r_3=1$.

Then, Aumann's (1974, p. 71) argument implies that the strategy profile $(X_1^*, X_2^*, X_3^*)\in\mathcal{R}_{Z_1,Z_2,Z_3}(S)$ is an equilibrium in our sense when one sets $p_{1/1}^*=1$, $p_{1/2}^*=0$, $q_{1/1}^*=1$, $q_{1/2}^*=0$ and $r_2^*=1$.

In this case, the joint distribution $p_{ij}^*:=P(X_1^*=i, X_2^*=j)$ is

$$(p_{ij}^*)=\begin{pmatrix}1/2 & 0\\ 0 & 1/2\end{pmatrix}.$$

Further, the strategy profiles with parameters having the following distributions are also equilibria.

$p^*_{1/1}=0$, $p^*_{1/2}=1$, $q^*_{1/1}=1$, $q^*_{1/2}=0$ and $r^*_2=0$, or
$p^*_{1/1}=1$, $p^*_{1/2}=0$, $q^*_{1/1}=0$, $q^*_{1/2}=1$ and $r^*_2=0$,

where r^*_1, $r^*_3 \geq 0$ are arbitrary with the condition $r^*_1+r^*_3=1$.

In this case, the joint distribution $p^*_{ij}:=P(X^*_1=i, X^*_2=j)$ is

$$(p^*_{ij})=\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

The reader can easily check that both cases satisfy conditions (17) to (21).

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