Scale and Special Conformal Covariance in Statistical Mechanics

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A short review is given of the postulates of scale and special conformal covariance for the critical point cumulant correlation functions. In particular a summary is presented of recent work which shows that the dilatation and special conformal Ward-Takahashi identities together with the operator expansion hypothesis of Kadanoff leads to such covariance.

I. Introduction

In this review I want to give a brief summary of some relatively recent developments in the statistical mechanical theory of critical phenomena. These concern the ideas of covariance of the cumulant correlation functions at the critical point under scale and special conformal transformations. As we will see, these cumulant functions are in general not covariant under such transformations, so that the critical point of a system is quite an exceptional thermodynamic point, where significant symmetries come into play. It is probably fair to say that we still have only a limited understanding of the consequences of the particularly stringent requirements of special conformal covariance. It is also worth nothing that from one point of view the proof of such covariances is intimately linked to the ideas of the operator algebra, originally introduced into the theory of critical phenomena by L. Kadanoff, whose ideas have been of such fundamental importance in this field. Thus a study of the response of the correlation functions to infinitesimal scale and special conformal transformations involve many of the elegant ideas underlying the present theory of critical phenomena and therefore provides a particularly nice area for analysis.

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Although I will shortly give a precise definition of these transformations it is perhaps worth first describing them in words. The scale (or dilation) transformation is well known and simply corresponds to a change of the length scale of a system. In its infinitesimal form it can be interpreted as a uniform, isotropic infinitesimal stretching of the distance between any two points. The special conformal transformation, on the other hand, is perhaps less well known, but in its infinitesimal form can also be interpreted as an isotropic but spatially *non-uniform* stretching of the distance between any two points. By "covariance" I mean, loosely speaking, that the infinitesimal change of a cumulant correlation function is itself proportional to the correlation function. Thus the covariance under special conformal transformations (being a spatially non-uniform stretching) leads to stronger restrictions on the form of the correlation functions than does scale covariance. It is therefore a more interesting and presumably more powerful symmetry principle than that of dilatation covariance.

The outline of my review is as follows. In part II I will give a survey of the phenomenological approach to these covariances, which historically is the way the subject developed. In part III I will discuss a more microscopic approach to the problem based on certain Ward-Takahashi identities and the ideas of the operator algebra. Finally in part IV I will outline what I think are possible areas for future investigation, as well as summarizing what is "rigorously known" at the moment.

I should note that my treatment of part II draws heavily from the paper of Polyakov while that of part III follows closely the paper by Wolsky and Green. By necessity my discussion in general is somewhat schematic. I apologize in advance for my own inadequate treatment of the various ideas involved, but hope that this brief review might serve as a stimulus for further research. Finally I should state that the subject of conformal covariance has been studied far more extensively in quantum field theory than in statistical mechanics. I have given a very incomplete list of references to work in this field at the end.

II. Phenomenological Approach

In statistical mechanics we are often interested in the behavior of correlation functions of various local variables $\theta_1(\vec{r}_1), \theta_2(\vec{r}_2), \cdots \theta_n(\vec{r}_n), \cdots$. This is particularly true near a critical point, at which various thermodynamc quantities such as the specific heat or the magnetic susceptibility $\chi = (\partial M/\partial H)_T$ diverge. These divergences arise from the cumulant correlation functions decaying much more slowly with spatial separation than they do at other thermodynamic points. For example, the magnetic susceptibility $\chi = \left(\frac{\partial M}{\partial H}\right)_T$ can be shown to be given by

$$\chi = \int <M(\vec{r})M(O)>_c \, d\vec{r} \tag{1}$$

where $M(\vec{r})$ is the local magnetization, $< \quad >$ denotes the statistical mechanics expectation value and c denotes the cumulant. Now since $<M(\vec{r})M(O)>_c$ can be shown to be a bounded function, the fact that at a critical point $\chi^{\to\infty}$ implies that $<M(\vec{r}) M(O)>_c$ has a long tail, which is usually assumed to be of an inverse power law behavior, i.e. at the critical point

$$<M(\vec{r})(M)(O)>_c \sim \frac{A}{r^{2\omega_M}}. \tag{2}$$

It is of interest to calculate such "critical exponents" as ω_M which characterize the diverging thermodynamic properties. Here I want to outline for you two interesting hypotheses that have been put forward concerning the behaviour of such cumulant functions, say *)

$$K_{\theta_1 \cdots \theta_n} \theta_n (\vec{r} \cdots \vec{r}_n) \equiv <\theta_1(\vec{r}_1) \cdots \theta_n(\vec{r}_n)>_c \tag{3}$$

Although these hypotheses in themselves do not suffice to solve the critical behavior of the K's (i.e. the critical exponents) they do tell us a great deal (if true) about the K's. Indeed, it is even possible that such hypotheses can be used as part of a formal procedure for calculating the critical exponents, as I will discuss later. Before going on, let me give you an idea of the types of operators we have in mind, for, say, a magnet or a fluid. For a fluid, these operators include the local density $\rho(\vec{r})$, mixed products of these variables, their spatial gradients, etc. For a magnet they include the local magnetization, local energy density, $\cdots$. They might even include less obvious variables associated with "subtle symmetries" of the system, such as the spinor variables for the two dimensional Ising model (see Kadanoff, reference 9 and 10). They always include the local order parameter and local energy density for the system.

*) In this article I will keep my notation as simple as possible for convenience. A better notation to include the possibility of an operator appearing more than once in the cumulant is given by Fisher (see references).

Finally, let us also define what we mean by the dilatation and special conformal transformations. The dilatation is simply

$$\vec{X} \to \vec{X}' = e^{\lambda} \vec{X} \tag{4}$$

for constant λ. If λ is infinitesimal such that $e^{\delta\lambda} = 1 + \delta\lambda$, $\delta\lambda \ll 1$, then this transformation can be interpreted as a uniform isotropic, infinitesimal stretching of the distance between any two points, with $\vec{X}_{ik} \to \vec{X}'_{ik}$, where

$$\delta \ell n |\vec{X}_{ik}| = \delta\lambda. \tag{5}$$

The special conformal transormation involves an inversion with respect to a unit sphere, I, with

$$\vec{X} \to \vec{X}' = \vec{X}/X^2 \tag{6}$$

and a translation P in an arbitrary direction $\hat{e}$.

$$\vec{X} \to \vec{X}' = \vec{X} + \lambda\hat{e}. \tag{7}$$

The special conformal transformation (S. C. T.) is then given by C = IPI, i. e. an inversion, followed by a translation and a second inversion. The net effect of this is

$$\vec{X} \to \vec{X}' = \frac{\vec{X} + \lambda\hat{e}X^2}{[1 + 2\lambda\vec{X}\cdot\hat{e} + \lambda^2 X^2]} . \tag{8}$$

If one now considers λ as infinitesimal, then to first order in $\delta\lambda$

$$\delta\vec{X} \equiv \vec{X}' - \vec{X} = [X^2 \hat{e} - 2(\vec{X}\cdot\hat{e})\vec{X}] \delta\lambda. \tag{9}$$

The corresponding change in the distance between two points, $\vec{X}_{ik}$, is $\vec{X}_{ik} \to \vec{X}'_{ik}$, with

$$\delta \ell n |\vec{X}_{ik}| = -(\vec{X}_i + \vec{X}_k) \cdot \hat{e}\delta\lambda. \tag{10}$$

If we let $\vec{X}_i \to \vec{X}_k \equiv \vec{X}$ then we find from the above (upon comparing it with the scale transformation) that this infinitesimal S. C. T. can be interpreted as an isotropic but *spatially nonuniform stretching* by a factor

$$\delta\lambda(\vec{X}) = -2\lambda\vec{X}\cdot\hat{e}. \tag{11}$$

with these preliminary remarks let us now consider the two postulates of covariance which we have mentioned earlier.

A. *Scaling Covariance*

In the middle 1960's Kadanoff and others proposed a static scaling hypothesis for the thermodynamic free energy and cumulant correlation functions. We will summarize this hypothesis in the form it takes at the critical point, in a language suitable for our present discussion. Namely consider a dilation transformation which simultaneously changes the length scale by

$$\vec{X} \to \vec{X}' = e^{\lambda} \vec{X} \tag{4}$$

and transforms the critical operators by

$$\theta_j(\vec{X}) \to \theta_j'(\vec{X}') = e^{-\lambda\omega_j}\,\theta_j(\vec{X}) \tag{12}$$

where ω_j is called the critical dimension of the operator θ_j. We then postulate that under this scale transformation the cumulant transforms as

$$K_{\theta_1\cdots\theta_m}(\{\vec{X}_k\}) \to K'_{\theta_1\cdots\theta_m} = K_{\theta_1\cdots\theta_m}(\{\vec{X}_k'\}) \tag{13}$$

and that

$$<\prod_j \theta_j(\vec{X}_k)>_c \to <\Pi\theta_j(\vec{X}_j)>'_c = <\prod_j \theta_j'(\vec{X}_j')>_c \tag{14}$$

If we then equate these two expressions for the transformed cumulants, assuming covariance, then we obtain from (12), (13) and (14)

$$\begin{aligned} K_{\theta_1\cdots\theta_m}(\{e^{\lambda}\vec{X}_k\}) &= \exp(-\lambda\sum_{j=1}^{m}\omega_j) <\prod_{j=1}^{m}\theta_j(\vec{X}_k)>_c \\ &= \exp(-\lambda\sum_{j=1}^{m}\omega_j)\, K_{\theta_1\cdots\theta_m}(\vec{X}_1\cdots\vec{X}_m) \end{aligned} \tag{15}$$

i. e. the Kadanoff scaling hypothesis at the critical point. This assumption of homogeneity and its consequences has been discussed extensively by Kadanoff, Fisher and many others.

It will be quite useful for our later discussion to restate the above for an infinitesimal stretching. For this case we consider the scale change (5) along with the corresponding change in $\theta_j(\vec{X})$

$$\delta\theta_j(\vec{X}) = -\omega_j\,\theta_j(\vec{X})\,\delta\lambda. \tag{16}$$

Then from the corresponding form of (13) we obtain for the two-point cumulant, say,

$$\begin{aligned} K'_{\theta_1\theta_2} &= K_{\theta_1\theta_2}(X'_{12}) \\ &= K_{\theta_1\theta_2}(X_{12}(1+\delta\lambda)) \end{aligned} \tag{17}$$

the infinitesimal change

$$\delta K_{\theta_1\theta_2}(X_{12}) \simeq \frac{\partial K_{\theta_1\theta_2}}{\partial \ell n X_{12}}\,\delta\lambda. \tag{18}$$

But from the corresponding form of (14) we obtain for $\delta < \theta_1(\vec{X}_1)\,\theta_2(\vec{X}_2) >_c$ the result

$$\delta < \theta_1(\vec{X}_1)\,\theta_2(\vec{X}_2) > = -(\omega_1 + \omega_2)\,K_{\theta_1\theta_2}(X_{12})\,\delta\lambda. \tag{19}$$

since from (16) each operator contributes an amont $\delta\theta_j = \omega_j\theta_j\delta\lambda$ to the change in the cumulant. If we thus equate (18) and (19) we obtain the differential equation

$$\frac{\partial K_{\theta_1\theta_2}}{\partial \ell n X_{12}} = -(\omega_1 + \omega_2)\,K_{\theta_1\theta_2} \tag{20}$$

whose solution is

$$K_{\theta_1\theta_2}(X_{12}) \simeq \frac{A_{12}}{X_{12}^{\omega_1+\omega_2}} \tag{21}$$

where A_{12} is some constant amplitude. The above is an example of the static scaling hypothesis. Its derivation illustrates the method used in deriving conclusions in both cases of scale and S. C. covariance. Finally let us note that for both postulates (scale and S. C. covariance) the arguments are only meant to be valid for large spatial separation of any two distance variables X_{12} involved in the cumulant : i. e. the postulates only concern "asymptotic covariance" (See Fisher's Nobel Symposium article for further discussion.)

B. *Special Conformal Covariance*

We now turn to the postulate first made by Polyakov, who suggested that cumulant correlation functions might be covariant at the critical point under a S. C. T. Since this corresponds to a spatially non-uniform stretching, it is not at all intuitively obvious why this should be so (even if it is intuitively obvious why scale covariance should hold!). Polyakov was motivated by a bootstrap approach which he and also Migdal had developed, but in the absence of suitable space for discussion, we will simply baldly assert his postulate as follows. Following our discussion of scale covariance let us cosider an infinitesimal S. C. T. (9) which generates a change in the critical operator

$$\delta\theta_j(\vec{X}) = -\omega_j\theta_j(\vec{X})\,[-2\vec{X}\cdot\hat{e}\,\delta\lambda\,], \tag{22}$$

a natural generalization of (16) upon noting (11). We then postulate that the changes in the cumulant averages are given by (13) and (14). Further, as before we assume that we can equate (13) and (14). If we now follow the procedure used above for scale covariance we will obtain partial differential equations, but whose solutions yield much stronger constraints on the cumulants than did the requirement of scale covariance. For example, consider the simple case of the two point cumulant. As before we obtain

$$\begin{aligned}\delta K_{\theta_1\theta_2} &= \frac{\partial K_{\theta_1\theta_2}}{\partial \ell n X_{12}}\,\delta\lambda n X_{12} \\ &= \frac{\partial K_{\theta_1\theta_2}}{\partial \ell n X_{12}}\,[-\lambda(\vec{X}_1+\vec{X}_2)\cdot\hat{e}\,].\end{aligned} \tag{23}$$

We also find that

$$\delta < \theta_1(\vec{X}_1)\,\theta_2(\vec{X}_2) >_c = [\,\omega_1\vec{X}_1\cdot\hat{e} + \omega_2\vec{X}_2\cdot\hat{e}\,]\;K_{\theta_1\theta_2}\cdot\delta\lambda \tag{24}$$

since each θ_j contributes a change $\delta\theta_j(\vec{X}_j) = 2(\vec{X}_j\cdot\hat{e})\,\theta_j(\vec{X}_j)\,\delta\lambda$ to this infinitesimal change, from (22). Thus if we equate (23) and (24) we obtain *two* equations (rather than one, as in the scale case)

$$\begin{aligned}\frac{\partial K_{\theta_1\theta_2}}{\partial \ell n X_{12}} &= -2\omega_1 K_{\theta_1\theta_2} \\ \frac{\partial K_{\theta_1\theta_2}}{\partial \ell n X_{12}} &= -2\omega_2 K_{\theta_1\theta_2}\end{aligned} \tag{25}$$

whose solutions are

$$\begin{aligned}K_{\theta_1\theta_2}(K_{12}) &= A_{12}/X_{12}^{2\omega_1} && \text{if } \omega_1 = \omega_2 \\ K_{\theta_1\theta_2}(X_{12}) &= 0 && \text{if } \omega_1 \neq \omega_2.\end{aligned} \tag{26}$$

Thus in addition to the prediction of scale covariance, S. C. covariance requires that opeators with *different* critical dimensions be asymptotically *orthogonal*. An even more striking result is obtained for the three-point function. Here an extension of the above analysis leads to three simultaneous differential equations, whose solutions is

$$K_{\theta_1\theta_2\theta_3}(X_{12},X_{13},X_{23}) = \frac{A_{123}}{X_{12}^{\omega_1+\omega_2-\omega_3}X_{13}^{\omega_1+\omega_3-\omega_2}X_{23}^{\omega_2+\omega_3-\omega_1}} \tag{27}$$

i.e. the requirement of S. C. covariance *completely determines* the asymptotic behavior of the three point function. One also finds restrictions on the higher order cumulants although less severe than (27) as the number of equations is less than the number of independent distances. Nevertheless considerable restrictions result. For example, one finds that

$$K_{\theta_1\theta_2\theta_3\theta_4}(X_{12}, X_{13}, X_{14}, X_{23}, X_{24}, X_{34})$$
$$= \frac{X_{12}^{\omega_3+\omega_4}\ X_{34}^{\omega_1+\omega_2}}{X_{13}^{\omega_1+\omega_3}\ X_{24}^{\omega_2+\omega_4}\ X_{14}^{\omega_1+\omega_4}\ X_{23}^{\omega_2+\omega_3}} \cdot F_{1234}\left(\frac{X_{12}X_{34}}{X_{13}X_{24}}, \frac{X_{12}X_{34}}{X_{14}X_{23}}\right). \qquad (28)$$

Thus we see that S. C. covariance implies considerable constraints on the functional form of the cumulants. The first question is, how valid are the predictions? One finds for the two dimensional Ising model that its predictions for two point cumulants and for the 3-point energy, magnetization, magnetization cumulant are valid (see Polyakov.) Thus one clearly should take the postulate seriously and I therefore turn to a discussion of some recent work which gives further insight on both these covariance postulates.

III. Ward Identities, Operator Algebra and Scale and Special Conformal Covariance

We now turn to the problem of obtaining a more microscopic understanding of the possible origin of these covariances. The obvious approach is to study the infinitesimal response of a given cumulant to the infinitesimal change in its spatial arguments produced by a group transformation*) $\vec{X} \to \vec{g}(\vec{X}, \lambda)$. Now this infinitesimal change corresponding to $K_{\theta_1\theta_2\cdots\theta_n}(\{\vec{X}_k\}) \to K'_{\theta_1\cdots\theta_m} = K_{\theta_1\cdots\theta_m}(\{\vec{g}(\vec{X}_k, \lambda)\})$ can be expressed by a first order expansion as

$$\delta K_{\theta_1\cdots\theta_m} = \sum_{i=1}^{m} \vec{g}\,'(\vec{X}_i) \cdot \nabla_i K_{\theta_1\cdots\theta_m}(\vec{X}_1 \cdots \vec{X}_i \cdots \vec{X}_m)\,\delta\lambda \qquad (29)$$

where the prime $(\vec{g}\,')$ denotes $d/d\lambda$ evaluated at $\lambda = 0$, and where $\delta\lambda$ is infinitesimal. In this section we will summarize the derivation of identities for such a response to either a dilatation or special conformal transformation. Following the original work of

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*) At this point we introduce $g(X, \lambda)$ to define the new position variable induced by the group operation where g takes different forms depending on the nature of the group, e.g., dilatation or S. C. T.

Wolsky and Green, we will derive these Ward-Takahashi identities for the case of a fluid, although they also can be derived for other systems, such as magnets. We will then use these identities together with the postulates of the operator algebra to show that at a critical point the cumulant correlation functions are covariant.

A. *Ward-Takahashi Identities for Fluids*

Thus we are led to an evaluation of

$$\sum_{i=1}^{m} \vec{g}\,(\vec{X}_i) \cdot \nabla_i K_{\theta_i \cdots \theta_n} (\vec{X}_1 \cdots \vec{X}_n) \tag{30}$$

i. e. to the infinitesimal response of a cumulant average of local fluctuating products to changes in its argument caused by a dilatation or special conformal transformation. Here $\vec{g}(\vec{X}, \lambda)$ denotes the transformation of $\vec{X}$ under the group operation, which is parametrized by λ. In particular, for

$$\underline{\text{Dilatation}} \qquad \vec{g}\,(\vec{X}, \lambda) = e^{\lambda}\vec{X}, \tag{31}$$

$$\underline{\text{Special Conformal}} \qquad \vec{g}\,(\vec{X}, \lambda\hat{e}) = \frac{\vec{X}+\lambda\hat{e}X^2}{1+2\lambda\hat{e}\cdot\vec{X}+\lambda^2 X^2} \tag{32}$$

I will now give a brief outline of the steps involved in evaluating the above infinitesimal response for the case of simple fluids.

1. Construction of a generating functional

The procedure is straightforward – We add a source term to our usual Hamiltonian which includes the coupling of an external field E_j to the operator θ_j (or operators) of interest. This is the standard way to obtain a generating functional for the usual cumulants. For example, if one were interested in the usual multi-density cumulant $<\rho\,(\vec{X}_1)\,\rho(\vec{X}_2)\cdots\rho(\vec{X}_m)>_c$ one would construct the functional

$$Z\,(E) = \sum_{N=0}^{\infty} \int d\Gamma \exp\left\{-\beta H^N(p_1 q) + \alpha N + \int \rho(\vec{X})E(\vec{X})d\vec{X}\right\} \tag{33}$$

where H^N is the usual fluid Hamiltonian, $\alpha = -\beta\mu$ and $E\,(\vec{X})$ is a local external field which couples to the density. Then if one took functional derivatives $\delta/\delta E$ of ℓnZ one would get

$$\frac{\delta \ell n Z(E)}{\delta E(\vec{X}_1)} = <\rho(\vec{X}_1)>_c$$

$$\frac{\delta^2 \ell n Z(E)}{\delta E(\vec{X}_2)\,\delta E(\vec{X}_1)} = <\rho(\vec{X}_1)\,\rho(\vec{X}_2)> - <\rho(\vec{X}_1)><\rho(\vec{X}_2)> = K_{\rho\rho}(\vec{X}_1, \vec{X}_2) \tag{34}$$

etc.

Now the trick necessary to construct the Ward identity is to do the same as above except that the source term is modified in such a way that the generating functional in this new ensemble yields the cumulants $<\theta_1(\vec{g}(\vec{X}_1, \lambda))\,\theta_2(\vec{g}(\vec{X}_2, \lambda))\cdots\theta_m(\vec{g}(\vec{X}_m, \lambda))>_c$. One can then evaluated $d/d\lambda$ of the above and get the Ward identity. So

Step 1 : Consider the generating functional†

$$Z(E, \lambda) = \sum_{N=0}^{\infty} \int d\Gamma\, \mathrm{esp}\{-\beta H^N(q, p)\} + \alpha N + \sum_i \int d\vec{X}\theta_i(\vec{g}(\vec{X}, \lambda))\, E_i(\vec{X}). \tag{35}$$

Then as before

$$\prod_{i=1}^{m} \frac{\delta}{\delta E_j(\vec{X}_j)} \ell n Z(E, \lambda) = < \prod_{j=1}^{m} \theta_j(\vec{g}(\vec{X}_j, \lambda))>_c \tag{36}$$

so $Z(E, \lambda)$ is the generating functional for the transformed cumulants. Furthermore

$$\left[\frac{d}{d\lambda} \prod_{j=1}^{m} \frac{\delta}{\delta E_j(\vec{X}_j)} \ell n Z(E, \lambda)\right]_{\lambda=E=0} = \sum_{i=1}^{m} \vec{g}(\vec{X}_i)\cdot\nabla_i < \sum_{j=1}^{i-1} \theta_j(\vec{X}_j)\,\theta_i(\vec{X}_i) \prod_{j=i+1}^{m} \theta_j(\vec{X}_j)>_c \tag{37}$$

Thus we conclude that $\ell n Z(E, \lambda)$ is the appropriate generating functional to use to study the infinitesimal response of the cumulant to the change produced by the group transformation.

Step 2 : Explicit evaluation of

$$\left[\frac{d}{d\lambda} \prod_{j=1} \frac{\delta}{\delta E_j(\vec{X}_j)} \ell n Z(E, \lambda)\right]_{\lambda=E=0} = \left[\prod_{j=1}^{m} \frac{\delta}{\delta E_j}(\vec{X}_j)\, \frac{Z'(E)}{Z(E)}\right]_{E=0} \tag{38}$$

† In this section I am being somewhat schematic. The reader should see the paper by Wolsky and Green for a more explicit treatment.

where we assume the order of performing derivatives can be interchanged. To evaluate $dZ(E,\lambda)/d\lambda$ we change variables from the original $\vec{q}$'s and $\vec{p}$'s to $\{\vec{Q},\vec{P}\}$ by using the canonical transformation induced by $\vec{g}(\vec{q},\lambda)$, i. e.

$$\vec{g}(\vec{q},-\lambda)\equiv\vec{Q}, \qquad P_\alpha = P_\beta \frac{\partial g_\alpha(\vec{g}(\vec{q},\lambda),-\lambda)}{\partial g_\alpha} \equiv G_\alpha(\vec{q},\vec{p},-\lambda). \tag{39}$$

Then

$$\begin{aligned} &H^N(p,q) \quad \rightarrow H^N(g(Q,\lambda),G(Q,p,\lambda)) \\ &d\Gamma(p,q) \quad \rightarrow d\Gamma(P,Q,\lambda) \end{aligned} \tag{40}$$

and the volume in which the system is contained is also transformed. Then one can explicitly evaluate $Z'(E)$ and then perform the $\delta/\delta E_j$. The final result of this (after some algebraic manipulation) is

$$\begin{aligned} &\sum_{i=1}^{m} \vec{g}'(\vec{X}_i)\cdot\nabla_i < \prod_{j=1}^{i-1} \theta_j(\vec{X}_j)\,\theta_i(\vec{X}_i) \prod_{j=i+1}^{m} \theta_j(\vec{X}_j)>_c \\ &\quad = -\beta < H' \prod_{j=1}^{m} \theta_j(\vec{X}_j)>_c \\ &\quad + \sum_{i=1}^{m} < \prod_{j=1}^{i-1} \theta_j(\vec{X}_j)\,[\,\theta_i'(\vec{X}_i) - J'(\vec{X}_i)\,\theta_i(\vec{X}_i)\,] \prod_{j=i+1}^{m} \theta_j(\vec{X}_j)>_c \end{aligned} \tag{41}$$

where J is the Jacobian determinant of $\vec{g}(\vec{X},\lambda)$ and $J'=3$ and $-6\hat{e}\cdot\vec{X}$ for dilatation S. C. T. respectively. Thus the W. T. identity equates the infinitesimal response of the cumulants due to changes in its arguments produced by a group transformation to the sum if two terms : the first explicitly expresses the change in H^N produced by the transformation while the second reflects the manner in which the fluctuating quantities themselves transform under the group action. For our later purposes I will write these W. T. identities explicitly for the density cumulants, for a fluid with two body forces :

$$\begin{aligned} &\sum_{i=1}^{m} \vec{X}_i\cdot\nabla_i < \prod_{j=1}^{i-1} \rho(\vec{X}_j)\,\rho(\vec{X}_i) \prod_{j=1+1}^{m} \rho(\vec{X}_j)>_c \\ &=\beta\int d^3z\{3<g(\vec{z})\prod_{j=1}^{m}\rho(\vec{X}_j)>_c\} - 3m < \prod_{j=1}^{m} \rho(\vec{X}_j)>_c \end{aligned} \tag{42}$$

$$\sum_{i=1}^{m} (X_i^2 \hat{e} - 2(\vec{X}_i \cdot \hat{e})\vec{X}_i) \cdot \nabla_i < \prod_{j=1}^{i-1} \rho(\vec{X}_j)\rho(\vec{X}_i) \prod_{j=i+1}^{m} \rho(\vec{X}_j) >_c$$
$$= \beta \int d^3 z\,(-6\hat{e} \cdot \vec{z}) < P(\vec{z}) \prod_{j=1}^{m} \rho(\vec{X}_j) >_c - \sum_{j=1}^{m} (-6\hat{e} \cdot \vec{X}_j) < \prod_{j=1}^{m} \rho(\vec{X}_j) >_c \qquad (43)$$

where $P(\vec{z})$ is the local fluctuating pressure, i. e. in terms of the discrete variables

$$P = \frac{1}{3} \left\{ 2\sum_i \frac{\vec{p}_i^{\,2}}{2m} - \frac{1}{2} \sum_{i \neq j} R_{ij} \frac{dV}{dR_{ij}} \right\} \qquad (44)$$

where V is the two body potential.

It is well worth noting at this point that it is already clear that there is an intimate connection between the dilatation and special conformal transofrmations in that the same quantity, namely the local pressure, arises from both H'_C and H'_D. Thus the right-hand sides of (42) and (43) are quite similar.

B. *Operator Algebra*

I want to digress briefly at this point to discuss a concept originally introduced by Kadanoff in critical phenomena (and independently by K. G. Wilson in field theory) which is of great importance. According to this postulate, at the critical point the product of two local fluctuating quantities (operators) which are "near" each other can be regarded as a single fluctuating quantity which itself can be expressed as a linear combination of certain fundamental fluctuating variables. That is, if $A(\vec{X}_1)$ and $B(\vec{X}_2)$ are two such fluctuating quantities, then

$$A(\vec{X}_1)B(\vec{X}_2) = X(\frac{1}{2}(\vec{X}_1 + \vec{X}_2), \vec{X}_1 - \vec{X}_2) \qquad (45)$$

and the expansion is

$$X(\vec{R}, \vec{r}) = \sum_i c_i(\vec{r})\,\theta_i(\vec{R}). \qquad (46)$$

The interpretation of this expansion is that for any $Y(\vec{R}')$

$$< X(\vec{R}, \vec{r})\, Y(\vec{R}') > = \sum_i c_i(\vec{r}) < \theta_i(\vec{R})\, Y(\vec{R}') >. \qquad (47)$$

Now if one takes $Y(\vec{R}')$ to be the unit operator we can conclude that

$$c_i = \frac{\partial \langle X \rangle}{\partial \langle \theta_i \rangle} \qquad (48)$$

unless θ_i is the unit operator, where $c_i = \langle X \rangle$. This form of the operator algebra was developed independently for fluids by Green and myself. The most striking success to date of this notion of the operator algebra was L. P. Kadanoff's determination of all the critical exponents for the two dimensionla Ising model, from the knowledge of all the relevant operators and of which of the coefficients c_i are identically zero. Let's now return to the W. T. identities and analyze them using the above ideas.

C. *W. T. Identities, Operator Algebra, Scale and Conformal Covariance*

Recall that according to the postulates the effect of the scale transformation should be simply to produce a change in the density cumulant

$$\delta_D \langle \rho(\vec{X}_1) \cdots \rho(\vec{X}_m) \rangle_c = -m\omega_\rho \langle \rho(\vec{X}_1) \cdots \rho(\vec{X}_m) \rangle_c \delta\lambda. \tag{49}$$

Similarly the conformal transformation should produce

$$\delta_c \langle \rho(\vec{X}_1) \cdots \rho(\vec{X}_m) \rangle_c = \omega_\rho \sum_{i=1}^{m} (2\hat{e} \cdot \vec{X}_i) \langle \rho(\vec{X}_1) \cdots \rho(\vec{X}_m) \rangle_c \,\delta\lambda \tag{50}$$

from (22). Now we know what this change of the cumulant is from our Ward identity, namely.

$$\delta \langle \rho(\vec{X}_1) \cdots \rho(\vec{X}_m) \rangle_c = \sum_{i=1}^{m} \vec{g}(\vec{X}_i) \cdot \nabla_i \langle \rho(\vec{X}_1) \cdots \rho(\vec{X}_i) \cdots \rho(\vec{X}_n) \rangle_c \delta\lambda \tag{51}$$

So it remains to be seen if the right-hand side of our W. T. identities reduce to (40) and (50) above. To answer this we apply the algebra. From inspection of both Ward identities we see that the central object in the theory is

$$\langle P(\vec{z}) \prod_{j=1}^{m} \rho(\vec{X}_j) \rangle_c.$$

We try to analyze this cumulant when all the $\vec{X}_i$ are far from each other (the condition under which we expect covariance to hold) and consider two different situations, recalling that

$$A(\vec{X}) = \Sigma c_i \theta_i = \langle A(\vec{X}) \rangle + \sum_i{}^1 \frac{\partial \langle A \rangle}{\partial \langle \theta_i \rangle} \theta_i.$$

Case A : The position $\vec{z}$ is far from any of the $\vec{X}_j$. In this case we can regard $P(\vec{z})$ as the fluctuating variable and expand it in terms of the fluctuating quantities θ_i, of which the density is the dominant variable. Thus we have

$$P(\vec{z}) \simeq \langle P(\vec{z}) \rangle + \left(\frac{\partial P}{\partial \rho}\right)_T [\rho(\vec{z}) - \langle \rho(\vec{z}) \rangle] + \cdots. \tag{52}$$

In this case

$$\langle P(\vec{z}) \prod_{j=1}^{m} \rho(\vec{X}_j) \rangle_c = \left(\frac{\partial P}{\partial \rho}\right) \langle \rho(\vec{z}) \prod_{j=1}^{m} \rho(\vec{X}_j) \rangle_c \tag{53}$$

since the term involving $\langle P \rangle$ gives zero. Now if we are *not* at the critical point the existence of this term (53) would *destroy* scale or conformal covariance. However, *at* $T = T_c$, $\left(\frac{\partial P}{\partial \rho}\right) \to 0$!

Case B : $\vec{z}$ near one of the $\vec{X}_j$. Then we must regard $P(\vec{z})\, \rho(\vec{X}_j)$ as a single fluctuating variable and from the algebra expand it as

$$P(\vec{z})\rho(\vec{X}_j) \simeq \langle P(\vec{z})\rho(\vec{X}_j)\rangle + \frac{\partial}{\partial \rho} \langle P(\vec{z})\rho(\vec{X}_j)\rangle\, \delta\rho(y) + \cdots \tag{54}$$

with $y = \frac{1}{2}(\vec{z} + \vec{X}_j) \simeq \vec{X}_j$. Again the first term drops out and so we are left with (for dilatations)

$$\int d^3z \left\{ 3\beta_c \frac{\partial}{\partial \rho} \langle P(\vec{z})\, \rho(\vec{X}_j) \rangle \right\} \langle \prod_{i=1}^{m} \rho(\vec{X}_i) \rangle_c$$

summed over j. If we define

$$X = 3\beta_c \int d^3z \frac{\partial}{\partial \rho} \langle P(\vec{z})\, \rho(\vec{X}_j) \rangle_{T_c}. \tag{55}$$

we then find that for dilatations

$$\sum_{i=1}^{m} \vec{X}_i \cdot \nabla_i \langle \rho(\vec{X}_1) \cdots \rho(\vec{X}_i) \cdots \rho(\vec{X}_m) \rangle_c \equiv -m\omega_\rho \langle \rho(\vec{X}_1) \cdots \rho(\vec{X}_m) \rangle_c \tag{56}$$

with $\omega_\rho = 3 - x = (1+\eta)/2$ (in the usual notation). Thus we find scale covariance at the critical point. In the case of S. C. T. we have

$$\beta \int d^3z\, (-6\hat{e} \cdot \vec{z}) \frac{\partial}{\partial \rho} \langle P(\vec{z})\, \rho(\vec{X}_j) \rangle_c \simeq -2X\hat{e} \cdot \vec{X}_j \tag{57}$$

since $\vec{z}$ is near $\vec{X}_j$. Therefore we get

$$\begin{aligned} &\sum_{i=1}^{m} [X_i^2\, \hat{e} - 2(\vec{X}_i \cdot \hat{e})\, \vec{X}_i] \cdot \nabla_i \langle \rho(\vec{X}_1) \cdots \rho(\vec{X}_m) \rangle_c \\ &\quad = -\omega_\rho \left(-\sum_{j=1}^{m} 2\hat{e} \cdot \vec{X}_j \right) \times \langle \rho(\vec{X}_1) \cdots \rho(\vec{X}_m) \rangle_c \end{aligned} \tag{58}$$

i. e. the conformal covariance as predicted by Polyakov. The avove arguments, due to Wolsky and Green, were generalized by Wolsky. Green and Gunton to include the energy fluctuations, with similar results.

III. Conclusions

I would like to end this discussion by summarizing the present state of knowledge for these covariances as well as mentioning outstanding problems of interest.

A. *Present Knowledge*

To begin with, we have considerable detailed confirmation now of the hypothesis of scale covariance. For example it has been verified explicitly in many instances for the two dimensional Ising model. As well, Brezin and Coworkers (and other people) have established it via the Callen-Symanzik renormalization group equation near four dimensions for the n-Vector model (where $n = 1$ is Ising, $n = 2$ is x-y, $n = 3$ is Heisenberg, and $n \rightarrow \infty$ is the spherical model) of magnetism. The stronger hypothesis of conformal covariance has not been studied in such detail yet, although as we noted earlier it has been verified in same cases for the two dimensional Ising model. Nevertheless the work of Wolsky and Green and Wolsky, Green and Gunton has shown the intimate relationship that exists between these two covariances. Further studies of this will doutless be forthcoming.

B. *Future Problems*

There are in fact many things that come to mind, the basic point being that when one has a symmetry or invariance principle in physics, particularly one as interesting as the special conformal symmetry, one might expect many significant consequences. As little has been explored yet, one might hope that future investigations will yield unexpected results. I will end by briefly outlining a few possible topics of interest.

1. Determination of Critical Exponents

This would of course be the ultimate goal to attain and preliminary attempts have already been made in this direction which exploit covariance within the framework of the bootstrap approach. As I have little space to discuss these attempts I can only give you a brief summary of the essential ideas. The pioneer work in this field was done by Migdal and Polyakov who proposed the bootstrap as the dynamical mechanism of critical phenomena. The name bootstrap simply means that in writing equations for certain cumulant functions, inhomogeneous terms representing bare couplings are ignored. One then attempts to determine the renormalized interactions selfconsistently from the resulting nonlinear integral equations. As these equations are both scale and conformal covariant at the critical point, they seem a natural starting point to exploit these symmetries. However so far not too much progress has been made in solving these equations. Mack originally suggested solving them by an expansion in powers of $\epsilon = 4 - d$, but I am unaware of what has been actually achieved so far. It is worth noting that Parisi and Peliti were able to obtain the correct critical exponents for the spherical model in arbitrary dimensions from such conformal covariant selfconsistency conditions. For further discussions of these points I must refer the reader to the original literature, including a nice discussion by Brout of the basic ideas of the bootstrap and conformal covariance.

On the same subject of calculating critical exponents I should also note that Brout (a firm believer in the power of conformal covariance) has used the notion of conformal covariance to give a heuristic argument that the critical isotherm exponent $\delta = (d+2)/(d-2)$, where $H \sim M^{\delta}$ $(T = T_c)$. However, as this result is incorrect for the two dimensional Ising Model it seems necessary to be somewhat cautious about this conclusion for $d = 3$. In particular it disagrees with the length scaling result $\delta = \frac{d+2-\eta}{d-2+\eta}$ (for $\eta \neq 0$) which is correct for the two dimensional Ising model for which $\eta = 1/4$ and $\delta = 15$. (Brout doubts the validity of length scaling for $d = 3$).

2. Rigorous Proofs of the Validity Scale and Conformal Covariance.

Finally I would like to make the obvious point that from the point of view of mathematical rigor no rigorous proof of scale or conformal covariance yet exists. I invite those mathematically inclined to improve (or disprove !) the arguments given so far. In particular I might note that the problem of the validity of hyperscaling might be examined from the framework of the scale and conformal Ward identities.

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