THEORY OF IRRADIANCE DISTRIBUTION FUNCTION IN TURBULENT MEDIA

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The irradiance distribution function of light–waves propagated through turbulent media has been investigated both experimentally and theoretically by many authors, but no successful theoretical attempt has been made to account for the experimental fact [1–3] that the irradiance distribution is close to the Gaussian distribution with respect to the logarithm of irradiance even in the region where the variance of irradiance scintillation is saturated. In this paper, a cluster approximation is applied to the solutions of moment equation of 4th and higher orders to express those moments in terms of the lower order ones and the irradiance distribution function is analytically derived based on the obtained expression of moment of irradiance for various orders.

It is physically expected that, if ψ denotes the (scalar) wave function, the distribution function P(I) with respect to the irradiance $I = \psi^*\psi$ will be uniquely determined when the moments $< I^\nu >$, $\nu = 1, 2, \ldots$ are given for all orders. Mathematically, it is true when $< I^\nu >$ has the asymptotic form

$$< I^\nu > \lesssim \exp(\nu E_c), \qquad \nu \sim \infty, \tag{1}$$

and is analytic on the right half–plane of ν–complex plane, and P(I) is then given by the integral [4]

$$P(I) = \frac{1}{2\pi i}\int_{-i\infty-\epsilon}^{i\infty-\epsilon} d\nu\ I^{-\nu-1} < I^\nu >, \qquad \epsilon = +0. \tag{2}$$

It is however practically impossible to evaluate $< I^{\nu} >$ for all orders of ν except for the special case where the irradiance scintillation is caused purely by the wondering of narrow wave beam [3–5]. On the other hand, according to the cluster expansion [6] based on the moment equations of the wave function, the normalized moment of irradiance $m_{\nu} \equiv < I^{\nu} > / < I >^{\nu}$, $\nu = 1, 2, 3, \ldots$ can be expressed in terms of the lower order ones as follows:

$$m_{\nu} = \exp \left[\binom{\nu}{2} \ell og\, m_2 + \binom{\nu}{3} \ell og\, (m_3 / m_2{}^3) + \binom{\nu}{4} \ell og\, (m_4\, m_2{}^6 / m_3{}^4) + \cdots \right]. \qquad (3)$$

Here, if the convergnece of series is good enough to keep only the first two terms of the series (3), we obtain the approximate expression

$$m_{\nu} \sim \exp \left[\tfrac{1}{2} \nu (\nu - 1) k_1 \{ 1 - (\nu - 2) \Delta' \} \right],$$
$$k_1 = \ell og\, m_2 , \qquad \Delta' = 1 - \tfrac{1}{3} \ell og\, m_3 / \ell og\, m_2 . \qquad (4)$$

Here, if $| (\nu - 2) \Delta' | \leqslant 1$, as is assumed, Eq. (4) is also expressed by

$$m_{\nu} \sim \exp \left[\tfrac{1}{2} \nu (\nu - 1) k_1 \{ 1 + (\nu - 2) \Delta \}^{-1} \right], \quad \Delta = \Delta' (1 - \Delta')^{-1} \sim \Delta' . \qquad (5)$$

Both expressions (4) and (5) give the correct values for the orders of $\nu = 1, 2, 3$ and, to the first order of approximation of Δ , also for the higher orders.

In Table 1, the recent experimental values of m_{ν} observed by Gracheva et al. [2] are shown along

Table 1

ν \ m_{ν}	Experimental	Theoretical & Error (%) $\Delta = 0.0262$	Theoretical & Error (%) $\Delta = 0.0346$	Log–normal ($\Delta = 0$)	Experimental /Log–normal
2	2.18	2.18	2.18	2.18	1.0
3	9.76	9.76	9.58 (1.8%)	10.36	0.94
4	79.32	85.0 (7.2%)	79.32	107.32	0.74
5	XXXX	1374	1165	2424	(0.57 ; 0.48)

with the corresponding theoretical values by (5). The parameter Δ critically depends on the value of m_3, and therefore two values of Δ are prepared, one being determined by the value of m_3 and the other by m_4. The first value gives rise to the error of 7.2 % for m_4 whereas the second one to the error of only 1.8 % for m_3.

Note that $\Delta > 0$ and $\Delta \ll 1$. The first point can be confirmed also theoretically to the first order of approximation which is valid when the distance of wave propagation is short enough. It is also worthwhile to check the third term in the series (3): according to the experimental values given in Table 1,

$$\log (m_4 m_2{}^6 / m_3{}^4) = 4.3735 + 6 \times 0.7793 - 4 \times 2.2783 = -0.064,$$

which is just 0.7 % of the cancelled values. Therefore the above value will be certainly within the range of experimental error, and the large cancellation will mean that the third term is very small to be negiected or $m_4 \sim m_3{}^4 / m_2{}^6$ which is exactly the expression obtained by the first two terms of the series or by (4). Thus, we may conclude, to the extent of the existing experimental data, that the agreement of the experimental and theoretical values is very good.

The essential difference between the expressions (4) and (5) for $< I^{\nu} >$ is that, as $\nu \to \infty$, (4) tends to the unphysical expression $\exp [-\frac{1}{2} \nu^3 k_1 \Delta'] \to 0$ whereas (5) tends to $\exp [\frac{1}{2} \nu k_1 / \Delta]$ and therefore satisfies the condition of applicability (1) for the integral representation (2) of P(I). Therefore, if we use the expression (5) for $< I^{\nu} >$ in the integral (2), the distribution function P(E) with respect to the log–irradiance $E = \log (I / < I >)$ is given by

$$P(E) = P(I)\, dI / dE = \frac{1}{2\pi i} \int_{-i\infty - \epsilon}^{i\infty - \epsilon} d\nu \, \exp [\tfrac{1}{2} \nu (\nu - 1) k_1 \{ 1 + (\nu - 2) \Delta \}^{-1} - \nu E] \tag{6}$$

$$= \exp (-\tfrac{1}{2} a^2) \delta (E - E_c) + \begin{cases} \nu_0 (a / b) \exp [-\frac{1}{2} (a^2 + b^2)] \, I_1 (ab), & E < E_c \\ 0, & E > E_c . \end{cases}$$

Here $I_1 (X)$ is the modified Bessel function of the 1st order and

$$\nu_0 = \Delta^{-1} - 2\ ,\ E_c = k_1/(2\Delta)\ ,\ a = [(1+\nu_0)k_1/\Delta]^{1/2},\ b = [2\nu_0(E_c - E)]^{1/2}. \quad (7)$$

In the practically important range of E where $a \sim b \gg 1$ and $E/E_c \ll 1$,

$$P(E) \sim (2\pi k_1)^{-1/2} \exp(-\tfrac{1}{2}F^2),$$
$$F = a - b \sim k_1^{1/2}[E/k_1 + \tfrac{1}{2} + \tfrac{1}{2}\Delta(E/k_1 - \tfrac{1}{2})(E/k_1 - \tfrac{3}{2})], \quad (8)$$

which tends to the log–normal distribution $P_0(E)$ as $\Delta \to 0$. In Fig. 1 is shown the ratio

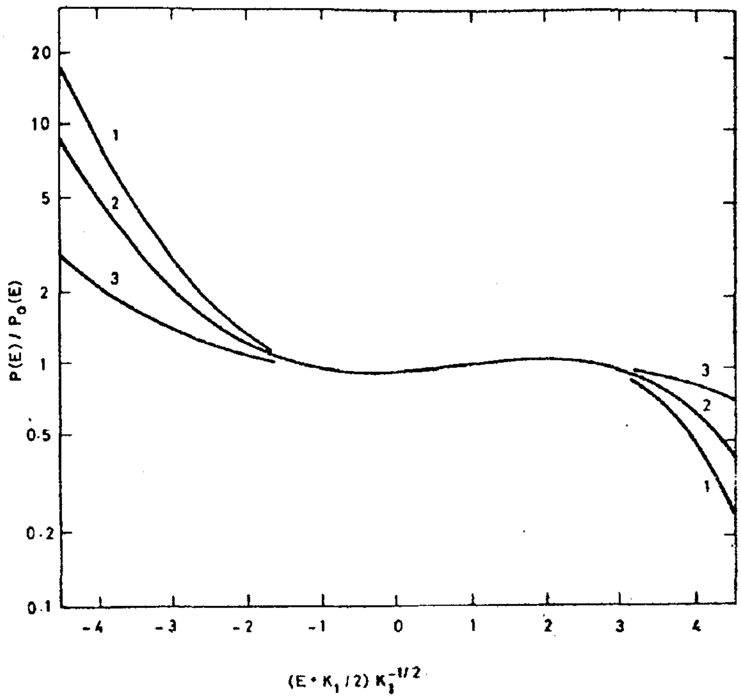

FIG. 1. Ratio of the distribution function of log–irradiance P (E) to the log–normal distribution function P_0 (E). K_1 = 0.7793 and 1, Δ = 0.05;2, Δ = 0.0346;3, Δ = 0.015.

$P(E)/P_0(E)$ against the variable $F|_{\Delta=0} = (E/K_1 + \frac{1}{2})K_1^{i/2}$, and in Fig. 2 is shown the corresponding cumulative distribution function $P_c(E)$ for the same values of the parameters. The details of the theory were published elsewhere [7].

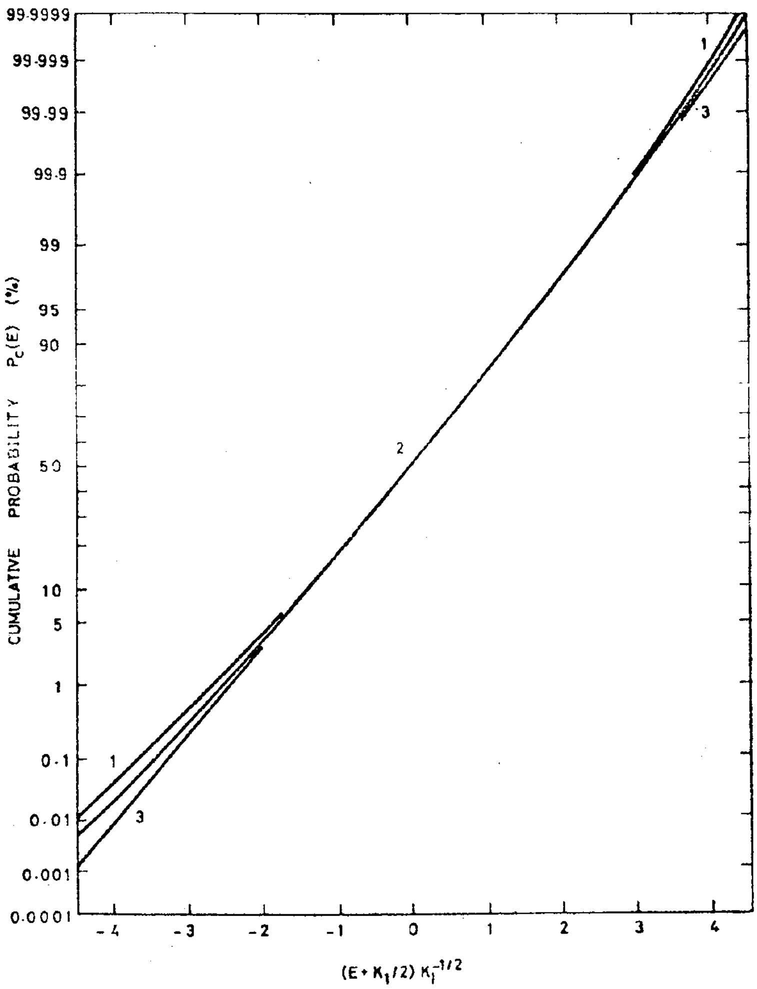

FIG. 2. Cumulative–probability distribution function $P_C(E)$. The values of the parameters are the same as those in Fig. 1 for each curve.

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液体 ^{4}He の動的構造因子

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δ－関数相互作用をもつ１次元Bose粒子系の高密度の極限における最低エネルギーおよび構造因子 S_k は，Lieb とLinigerが与えた厳密解から直接または間接的に求められることがわかっている。この結果は集団変数の理論を用いて導いたものと一致していることから，集団変数の理論は高密度 Bose 粒子系に対して正しい取扱い方を与えるものと考えられる。

また、集団変数の理論に関する疑問点として，ながらく残されていた発散項の処理，および密度の正準共役量の存在については，最近 Tsujii と著者の一人 (N) によって，一応解決されている。[1] 他方 ^4He の問題に広く応用されている Feenberg の相関基底関数法（*CBF*）は実験とかなりよく一致した結果を与えるという点で高く評価されているが，１次元系および charged Bose gas に適用したとき，構造因子の値が厳密な結果と一致しないことが指摘されている。[2] Berdahl [2] は *CBF* と集団変数の理論の結果との相違が，基準状態における密度のゆらぎの３体相関積分 $\langle \rho_{\vec{k}}\ \rho_{\vec{l}}\ \rho_{\vec{m}} \rangle_0$ にあることをつきとめ，*CBF* で使用されている相関積分を，集団変数の理論から求めたもので置き換えることによって正しい結果が導かれることを示した。この事実は構造因子に限らず，*CBF* ですでに得られている運動量分布や動的構造因子についても集団変数の理論を用いて再検討する必要があることを示している。

ここでは１次元のLieb-Liniger模型に対する動的構造因子の長波長の極限形を求め，それが厳密解と矛盾していないことを示す。[3] 次に *model independent* の方法を用いて ^4He に適用した結果について述べる。

まず高密度長波長の極限では，