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Jacobian Elliptic Function Calculation to the Nonlinear Equations Related to Lumped LC Networks

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(Synopsis)

A Self-dual and other network equations are derived through the transformations of the nonlinear difference-differential equations. The soliton train solutions of the former are obtained from the elliptic function solutions of the latter by these transformations. The soliton train of the self-dual network equation is found to be characterized by an arbitrary parameter θ in addition to the wave number κ.

§1. Introduction

A discretized equation $\dot{N}_n = (N_{n-1} - N_{n+1})(a + bN_n + cN_n^2)$, considered as modelling the corresponding continuum equation, is reduced through simple transformation to the following four types.

$$\dot{A}_n = (A_{n-1} - A_{n+1}) A_n \quad , \cdots\cdots(1) \qquad \dot{B}_n = (B_{n-1} - B_{n+1}) B_n^2 \quad , \cdots\cdots(2)$$

$$\dot{C}_n = (C_{n-1} - C_{n+1})(1 - C_n^2) \, , \cdots\cdots(3) \qquad \dot{D}_n = (D_{n-1} - D_{n+1})(1 + D_n^2) \, . \cdots\cdots(4)$$

Equation (1), or the one-dimensional chain of Volterra competition equations finds its application in a model of energy transfer kinetics of Langmuir waves in plasma[1)], in addition to its role as a model of a nonlinear electric network[2)]. Hirota's direct method was employed[2)] for

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finding the N-soliton solution of this equation.

Equation (2) is one of the discrete analogue of KdV equation[3] and its application is found in a nonlinear lumped network[3]. Hirota's direct method converts it to a single bilinear form[4].

Ablowitz and Ladik[5] succeeded in deducing eq. (3) and eq. (4) by discretizing the generalized Zakharov-Shabat formalism of inverse scattering, and showed their solution under the rapidly decaying initial condition at the infinity. The above restraint for the eqs. (3) and (4) was removed by Noguchi, Watanabe and Sakai[6], who showed their solutions for the nonvanishing initial condition at the infinity. Hirota and Satsuma[3] obtained an expression of the N-soliton solution to an extended ladder type network, which is reduced to the solutions of eq. (1) and eq. (3) with nonvanishing condition at the infinity.

Equation (4), first solved by Hirota[7], has more desirable property as a model of electric network as compared to eq. (3), while eq. (3) has its significance in relation to the equalities

$$A_n = B_{n-\frac{1}{2}} B_{n+\frac{1}{2}} = (1 - C_{n-\frac{1}{2}})(1 + C_{n+\frac{1}{2}}) \ . \qquad \cdots\cdots (5)$$

We will explain in the next section the additional transformation relations with respect to the new quantities connecting eq. (2) and eq. (3).

Finally we refer to the Jacobian elliptic function solutions to eq. (1), eq. (2) and eq. (3) here, the derivation of which from a unified point of view is the main concern of this study. They are,

$$A_n = -\frac{\omega}{\operatorname{sn} 2\kappa}[1 - k^2 \operatorname{sn}\kappa \operatorname{sn} 2\kappa \operatorname{sn}(x + \frac{\kappa}{2}) \operatorname{sn}(x - \frac{\kappa}{2})] , \qquad \cdots\cdots (6)$$

$$B_n = \sqrt{-\frac{\omega}{2 \operatorname{sn}\kappa \operatorname{cn}\kappa \operatorname{dn}\kappa}[1 - k^2 \operatorname{sn}^2\kappa \operatorname{sn}^2 x]} , \qquad \cdots\cdots (7)$$

$$C_n = \frac{\operatorname{sn}\kappa}{\operatorname{cn}\kappa \operatorname{dn}\kappa \operatorname{sn}\theta}[\operatorname{cn}\theta \operatorname{dn}\theta + k^2(\operatorname{sn}^2\theta - \operatorname{sn}^2\kappa) \operatorname{sn} x \operatorname{sn}(x + \theta)] , \qquad \cdots\cdots (8)$$

in which $x = \kappa n + \omega t$. In. (6) and (7) $\kappa\ (0 < \kappa \leq K(k))$ and ω are arbitrary, and ω in a function of arbitrary κ and $\theta\ (0 < \theta \leq K(k))$, given by

$$\omega = \frac{2 \operatorname{sn}\kappa}{\operatorname{cn}\kappa \operatorname{dn}\kappa}(\frac{\operatorname{sn}^2\kappa}{\operatorname{sn}^2\theta} - 1) \ . \qquad \cdots\cdots (9)$$

§ 2. Transformations

(I) We start with a differential-difference equation

$$\dot{\varphi}_n = \coth(\varphi_{n-1}+\varphi_n) - \coth(\varphi_n+\varphi_{n+1}) \ . \qquad \cdots\cdots (10)$$

Introducing P_n defined by $P_n = \tanh\varphi_n$, we can express eq. (10) alternatively in the form

$$\dot{P}_n = \left(\frac{1}{P_{n-1}+P_n} - \frac{1}{P_n+P_{n+1}}\right)(1-P_n^2)^2 \ . \qquad \cdots\cdots (11)$$

The transformation to the variables B_n and C_n is found to be given by

$$B_n = \frac{2i}{\exp(-2\varphi_{n+\frac{1}{2}}) - \exp 2\varphi_{n-\frac{1}{2}}}, \cdots\cdots (12) \text{ and } C_n = \coth(\varphi_{n-\frac{1}{2}}+\varphi_{n+\frac{1}{2}}) . \qquad \cdots\cdots (13)$$

We will find the elliptic function solution to eq. (10) in § 3, where we use another expression of eq. (10) given by

$$\dot{u}_n = 2\frac{P_{n+\frac{1}{2}} - P_{n-\frac{1}{2}}}{P_{n+\frac{1}{2}} + P_{n-\frac{1}{2}}} \ , \qquad \cdots\cdots (14) \text{ and } u_{n-\frac{1}{2}} + u_{n+\frac{1}{2}} = \cosh 2\varphi_n . \qquad \cdots\cdots (15)$$

The direct substitution shows that u_n satisfies

$$\dot{u}_n = (u_{n-1} - u_{n+1})(1-C_n^2) \ . \qquad \cdots\cdots (16)$$

(II) The next differential-difference equation that we investigate in § 3 is given by

$$\frac{d}{dt}\tanh\psi_n = 4\left[\coth(\psi_{n-1}+\psi_n) - \coth(\psi_n+\psi_{n+1})\right] \ . \qquad \cdots\cdots (17)$$

Introducing Q_n defined by $Q_n = \tanh\psi_n$ we can express eq. (17) alternatively in the form

$$\dot{Q}_n = 4\left(\frac{1}{Q_{n-1}+Q_n} - \frac{1}{Q_n+Q_{n+1}}\right)(1-Q_n^2) \ . \qquad \cdots\cdots (18)$$

In § 3 we seek the elliptic function solution of (18) using another representation

$$\frac{\dot{v}_n}{v_n} = 4\frac{Q_{n-\frac{1}{2}} - Q_{n+\frac{1}{2}}}{Q_{n-\frac{1}{2}} + Q_{n+\frac{1}{2}}} \ , \qquad \cdots\cdots (19) \quad \text{and} \quad v_{n-\frac{1}{2}}\, v_{n+\frac{1}{2}} = \operatorname{sech}^2\psi_n . \qquad \cdots\cdots (20)$$

The transformation to the variables B_n and C_n is found to be given by

$$B_n = \frac{2iv_n}{Q_{n-\frac{1}{2}} + Q_{n+\frac{1}{2}}}, \qquad \cdots\cdots (21) \quad \text{and} \quad C_n = \frac{2+Q_{n-\frac{1}{2}} - Q_{n+\frac{1}{2}}}{Q_{n-\frac{1}{2}} + Q_{n+\frac{1}{2}}} . \qquad \cdots\cdots (22)$$

The substitution shows that v_n and Q_n satisfy

$$\dot{v}_n = (v_{n-1} - v_{n+1}) B_n^2 \ , \quad \cdots\cdots (23) \quad \text{and} \quad \dot{Q}_n = (Q_{n-1} - Q_{n+1}) A_n \ . \quad \cdots\cdots (24)$$

(III) We consider the elliptic function solution to an equation

$$\dot{R}_n = \frac{1}{R_{n+1} + R_n} - \frac{1}{R_n + R_{n-1}} \ , \quad \cdots\cdots (25)$$

in the final part of §3, where we use another representation

$$\dot{w}_n = \frac{R_{n-\frac{1}{2}} - R_{n+\frac{1}{2}}}{R_{n-\frac{1}{2}} + R_{n+\frac{1}{2}}} \ , \quad \cdots\cdots (26) \quad \text{and} \quad w_{n-\frac{1}{2}} + w_{n+\frac{1}{2}} = R_n^2 \ . \quad \cdots\cdots (27)$$

The transformation to B_n is given by

$$B_n = \frac{1}{R_{n+\frac{1}{2}} + R_{n-\frac{1}{2}}} \quad \cdots\cdots (28)$$

and the substitution shows that w_n and R_n satisfy

$$\dot{w}_n = (w_{n-1} - w_{n+1}) B_n^2 \ , \quad \cdots\cdots (29) \quad \text{and} \quad \dot{R}_n = (R_{n-1} - R_{n+1}) A_n \ . \quad \cdots\cdots (30)$$

§ 3. Elliptic Function Solutions

In this section we investigate the elliptic function solutions to the precedingly presented differential-difference equations.

In (i) we see that a postulated elliptic function solution to $\exp 2\varphi_n$ and u_n, proved to satisfy (15) in Appendix 1, can be transformed through (13) to the solution (8) to eq. (3). (9) is proved there by the comparison of the both sides of (3). Next we see that this solution is transformed to the elliptic function solutions of eqs. (2) and (1), formally equivalent to (7) and (6).

In (ii) we start with an assumption of an elliptic function solution to v_n and transform it to Q_n using relation (20). The substitution of them into the both sides of eq. (19) proves (9) again. We see that B_n and C_n obtained from v_n and Q_n coincide with the results found in (i).

In (iii) we assume an elliptic function solution to w_n and transform it to R_n using (27). The substitution of them to the both sides of (26) proves the validity of the assumption. R_n is transformed through (28) into the solution (7) for eq. (2).

(i) We postulate an elliptic function solution to eqs. (14) and (15) as follows,

$$u_n = \frac{\mathrm{sn}\kappa}{2\sqrt{\mathrm{sn}^2\kappa - \mathrm{sn}^2\theta}} [\, \mathrm{cn}\theta \, \mathrm{dn}\theta + k^2 \mathrm{sn}^2\theta \, \mathrm{sn}x \, \mathrm{sn} (x + \theta) \quad \cdots\cdots (31)$$

$$=\frac{1}{2}\sqrt{\frac{1-\mathrm{dn}\kappa_0}{2(1+\mathrm{dn}\theta_0)(\mathrm{dn}\theta_0-\mathrm{dn}\kappa_0)}}\left[2\mathrm{cn}\theta_0-(1-\mathrm{dn}\theta_0)\frac{\mathrm{dn}(x_0+\frac{\theta_0}{2})-\mathrm{dn}\frac{\theta_0}{2}}{\mathrm{dn}(x_0+\frac{\theta_0}{2})+\mathrm{dn}\frac{\theta_0}{2}}\right], \quad \cdots\cdots (32)$$

and

$$\exp 2\varphi_n=\sqrt{\frac{\mathrm{sn}\frac{1}{2}(\kappa_0+\theta_0)\,\mathrm{cn}\frac{1}{2}(\kappa_0+\theta_0)}{\mathrm{sn}\frac{1}{2}(\kappa_0-\theta_0)\,\mathrm{cn}\frac{1}{2}(\kappa_0-\theta_0)}\cdot\frac{\mathrm{dn}(x_0+\frac{\theta_0}{2})+\mathrm{dn}\frac{1}{2}(\theta_0-\kappa_0)}{\mathrm{dn}(x_0+\frac{\theta_0}{2})+\mathrm{dn}\frac{1}{2}(\theta_0+\kappa_0)}} \quad \cdots\cdots (33)$$

where $x_0=\kappa_0 n+\omega_0 t$ and θ and θ_0 are arbitrary. k, θ, κ, and ω are related to k_0, θ_0, κ_0 and ω_0 through a transformation of variable for elliptic functions

$$k=\frac{1-k_0'}{1+k_0'},\quad \theta=\frac{1}{2}(1+k_0')\theta_0,\quad \kappa=\frac{1}{2}(1+k_0')\kappa_0,\ \text{and}\ \omega=\frac{1}{2}(1+k_0')\omega_0\ (k_0'=\sqrt{1-k_0^2}). \quad \cdots\cdots (34)$$

The proof that (32) and (33) satisfy eq. (15) is given in Appendix 1. (Equation (14) can be unused.) P_n is obtained from (33). Using an equality (A1–1) in Appendix 1, we find from (33).

$$\exp 2(\varphi_{n-\frac{1}{2}}+\varphi_{n+\frac{1}{2}})=\frac{[\mathrm{dn}(\frac{\theta_0}{2}+\kappa_0)-\mathrm{dn}\frac{\theta_0}{2}][\mathrm{dn}(x_0+\frac{\theta_0}{2})+\mathrm{dn}(\frac{\theta_0}{2}-\kappa_0)]}{[\mathrm{dn}(\frac{\theta_0}{2}-\kappa_0)-\mathrm{dn}\frac{\theta_0}{2}][\mathrm{dn}(x_0+\frac{\theta_0}{2})+\mathrm{dn}(\frac{\theta_0}{2}+\kappa_0)]}\ . \quad \cdots\cdots (35)$$

Substituting (35) to (13) we find that C_n is given as

$$C_n=\frac{1}{\mathrm{dn}\frac{\theta_0}{2}[\mathrm{dn}(\frac{\theta_0}{2}+\kappa_0)-\mathrm{dn}(\frac{\theta_0}{2}-\kappa_0)]}\cdot\Bigg[\mathrm{dn}(\frac{\theta_0}{2}+\kappa_0)\,\mathrm{dn}(\frac{\theta_0}{2}-\kappa_0)-\mathrm{dn}^2\frac{\theta_0}{2}$$

$$-(\mathrm{dn}(\frac{\theta_0}{2}+\kappa_0)-\mathrm{dn}\frac{\theta_0}{2})(\mathrm{dn}(\frac{\theta_0}{2}-\kappa_0)-\mathrm{dn}\frac{\theta_0}{2})\cdot\frac{\mathrm{dn}(x_0+\frac{\theta_0}{2})-\mathrm{dn}\frac{\theta_0}{2}}{\mathrm{dn}(x_0+\frac{\theta_0}{2})+\mathrm{dn}\frac{\theta_0}{2}}\Bigg] \quad \cdots\cdots (36)$$

$$=\frac{\mathrm{sn}\kappa_0}{\mathrm{cn}\kappa_0\,\mathrm{sn}\theta_0}\left[\mathrm{cn}\theta_0+\frac{\mathrm{dn}\theta_0-\mathrm{dn}\kappa_0}{1+\mathrm{dn}\kappa_0}\cdot\frac{\mathrm{dn}(x_0+\frac{\theta_0}{2})-\mathrm{dn}\frac{\theta_0}{2}}{\mathrm{dn}(x_0+\frac{\theta_0}{2})+\mathrm{dn}\frac{\theta_0}{2}}\right], \quad \cdots\cdots (37)$$

where (A2–3) in Appendix 2 can be used for the deformation (37) from (36). The transformation of variable (34) applied to (37) leads to the expression (8) for the elliptic function solution of (3).

Now the substitution of (36) to the both sides of eq. (3) is found to require that the following condition is satisfied.

$$\omega_0 = 8\,\mathrm{sn}\,\kappa_0 \mathrm{cn}\,\kappa_0 \cdot \frac{[\mathrm{dn}(\frac{\theta_0}{2}+\kappa_0)-\mathrm{dn}\frac{\theta_0}{2}][\mathrm{dn}(\frac{\theta_0}{2}-\kappa_0)-\mathrm{dn}\frac{\theta_0}{2}]}{\Delta(\frac{\theta_0}{2},\,\kappa_0)[\mathrm{dn}(\frac{\theta_0}{2}+\kappa_0)-\mathrm{dn}(\frac{\theta_0}{2}-\kappa_0)]^2} \qquad \cdots\cdots (38)$$

$$= 4\frac{\mathrm{sn}\,\kappa_0}{\mathrm{cn}\,\kappa_0} \cdot \frac{\mathrm{dn}\,\theta_0-\mathrm{dn}\,\kappa_0}{(1+\mathrm{dn}\,\kappa_0)(1-\mathrm{dn}\,\theta_0)}\ , \qquad \cdots\cdots (39)$$

where the definition of $\Delta(\alpha,\beta)$ in (38) is given by (A1–2) in Appendix 1, and (A2–3) is used for the deformation (39) from (38). The transformation of variable (34) for the expression (39) leads to the formula (9).

Meanwhile the solution (33) for $\exp 2\varphi_n$ is transformed to B_n as seen below.

$$B_n = 2\sqrt{\frac{\mathrm{sn}\frac{1}{2}(\theta_0+\kappa_0)\,\mathrm{cn}\frac{1}{2}(\theta_0+\kappa_0)}{\mathrm{sn}\frac{1}{2}(\theta_0-\kappa_0)\,\mathrm{cn}\frac{1}{2}(\theta_0-\kappa_0)}} \cdot \frac{\mathrm{dn}\frac{\theta_0}{2}-\mathrm{dn}(\frac{\theta_0}{2}-\kappa_0)}{\mathrm{dn}(\frac{\theta_0}{2}+\kappa_0)-\mathrm{dn}(\frac{\theta_0}{2}-\kappa_0)}$$

$$\cdot \frac{\mathrm{dn}(x_0+\frac{\theta_0}{2}+\frac{\kappa_0}{2})+\mathrm{dn}\frac{1}{2}(\theta_0-\kappa_0)}{\mathrm{dn}(x_0+\frac{\theta_0}{2}+\frac{\kappa_0}{2})+\mathrm{dn}\frac{1}{2}(\theta_0+\kappa_0)} \cdot \frac{\mathrm{dn}(x_0+\frac{\theta_0}{2})+\mathrm{dn}(\frac{\theta_0}{2}+\kappa_0)}{\mathrm{dn}(x_0+\frac{\theta_0}{2})+\mathrm{dn}\frac{\theta_0}{2}}$$

$$= 2\frac{\sqrt{\vartheta_1(\frac{1}{2}(\theta_0'+\kappa_0'))\vartheta_2(\frac{1}{2}(\theta_0'+\kappa_0'))\vartheta_1(\frac{1}{2}(\theta_0'-\kappa_0'))\vartheta_2(\frac{1}{2}(\theta_0'-\kappa_0'))}\cdot\vartheta_1(\frac{\kappa_0'}{2})\vartheta_2(\frac{\kappa_0'}{2})}{\vartheta_1(\frac{\theta_0'}{2})\vartheta_2(\frac{\theta_0'}{2})\vartheta_1(\kappa_0')\vartheta_2(\kappa_0')}$$

$$\cdot \frac{\vartheta_3(\frac{1}{2}(x_0'+\kappa_0'))\vartheta_4(\frac{1}{2}(x_0'+\kappa_0'))\vartheta_3(\frac{1}{2}(x_0'-\kappa_0'))\vartheta_4(\frac{1}{2}(x_0'-\kappa_0'))}{[\vartheta_3(\frac{x_0'}{2})\vartheta_4(\frac{x_0'}{2})]^2} \qquad \cdots\cdots (40)$$

$$= \frac{1}{\mathrm{cn}\,\kappa_0}\sqrt{\frac{\mathrm{dn}\,\kappa_0-\mathrm{dn}\,\theta_0}{2(1+\mathrm{dn}\,\kappa_0)(1-\mathrm{dn}\,\theta_0)}}\left[1+\mathrm{dn}\kappa_0+(1-\mathrm{dn}\,\kappa_0)\frac{\mathrm{dn}x_0-1}{\mathrm{dn}x_0+1}\right], \qquad \cdots\cdots (41)$$

where ′ in (40) means the multiplication by $\frac{1}{2K(k)}$. Further transformations (5) from B_n in (41) and C_n in (37) to A_n coincide with each other and give

$$A_n = 4\frac{\vartheta_1(\frac{1}{2}(\theta_0'+\kappa_0'))\vartheta_2(\frac{1}{2}(\theta_0'+\kappa_0'))\vartheta_1(\frac{1}{2}(\theta_0'-\kappa_0'))\vartheta_2(\frac{1}{2}(\theta_0'-\kappa_0'))\vartheta_1^2(\frac{\kappa_0'}{2})\vartheta_2^2(\frac{\kappa_0'}{2})}{\vartheta_1^2(\frac{\theta_0'}{2})\vartheta_2^2(\frac{\theta_0'}{2})\vartheta_1^2(\kappa_0')\vartheta_2^2(\kappa_0')}$$

$$\cdot\frac{\vartheta_3(\frac{x_0'}{2}+\frac{3}{4}\kappa_0')\vartheta_4(\frac{x_0'}{2}+\frac{3}{4}\kappa_0')\vartheta_3(\frac{x_0'}{2}-\frac{3}{4}\kappa_0')\vartheta_4(\frac{x_0'}{2}-\frac{3}{4}\kappa_0')}{\vartheta_3(\frac{x_0'}{2}+\frac{\kappa_0'}{4})\vartheta_4(\frac{x_0'}{2}+\frac{\kappa_0'}{4})\vartheta_3(\frac{x_0'}{2}-\frac{\kappa_0'}{4})\vartheta_4(\frac{x_0'}{2}-\frac{\kappa_0'}{4})}$$

$$=\frac{2(\mathrm{dn}\,\kappa_0-\mathrm{dn}\,\theta_0)}{\mathrm{cn}^2\kappa_0(1+\mathrm{dn}\kappa_0)(1-\mathrm{dn}\theta_0)}\left[\mathrm{dn}\,\kappa_0+\mathrm{cn}\,\kappa_0(1-\mathrm{dn}\kappa_0)\frac{\mathrm{dn}x_0-\mathrm{dn}\frac{\kappa_0}{2}}{\mathrm{dn}x_0+\mathrm{dn}\frac{\kappa_0}{2}}\right]. \quad \cdots\cdots (42)$$

The transformation of variable (34) for (41) and (42) gives expressions equivalent to (7) and (6), in which ω is replaced by a function of κ and θ given by (9).

For the later convenience we remark here that the solution (37) for eq. (3) may be expressed as a sum of two functions of x_0 given by

$$C_n=\frac{\mathrm{sn}\kappa_0\mathrm{cn}\theta_0(1+\mathrm{dn}\theta_0)(\mathrm{dn}x_0+\mathrm{dn}\,\kappa_0)}{\mathrm{sn}\theta_0\mathrm{cn}\kappa_0(1+\mathrm{dn}\,\kappa_0)(\mathrm{dn}x_0+\mathrm{dn}\theta_0)}+\frac{k_0^2\mathrm{sn}\,\kappa_0(\mathrm{dn}\,\kappa_0-\mathrm{dn}\,\theta_0)\,\mathrm{sn}x_0\mathrm{cn}x_0}{\mathrm{cn}\,\kappa_0(1+\mathrm{dn}\,\kappa_0)(\mathrm{dn}x_0+1)(\mathrm{dn}x_0+\mathrm{dn}\theta_0)}. \quad \cdots (43)$$

(43) is verified by the substitution of an equality

$$\frac{\mathrm{dn}(x_0+\frac{\theta_0}{2})-\mathrm{dn}\frac{\theta_0}{2}}{\mathrm{dn}(x_0+\frac{\theta_0}{2})+\mathrm{dn}\frac{\theta_0}{2}}=\frac{\mathrm{cn}\theta_0(\mathrm{dn}x_0-1)}{\mathrm{dn}x_0+\mathrm{dn}\theta_0}-\frac{k_0^2\mathrm{sn}\theta_0\,\mathrm{sn}x_0\,\mathrm{cn}x_0}{(\mathrm{dn}x_0+1)(\mathrm{dn}x_0+\mathrm{dn}\theta_0)} \quad \cdots\cdots (44)$$

to (37).

(ii) We first assume that the elliptic function solution v_n satisfying eqs. (19) and (20) has a form

$$v_n=\frac{\sqrt{\mathrm{sn}^2\kappa-\mathrm{sn}^2\theta}}{\mathrm{sn}\,\kappa\,\mathrm{cn}\,\theta\,\mathrm{dn}\,\theta}[1-k^2\mathrm{sn}^2\theta\,\mathrm{sn}^2x] \quad \cdots\cdots (45)$$

$$=\frac{1}{\mathrm{cn}\,\theta_0}\sqrt{\frac{2(\mathrm{dn}\,\theta_0-\mathrm{dn}\,\kappa_0)}{(1+\mathrm{dn}\,\theta_0)(1-\mathrm{dn}\,\kappa_0)}}\cdot\frac{\mathrm{dn}x_0+\mathrm{dn}\,\theta_0}{\mathrm{dn}x_0+1}. \quad \cdots\cdots (46)$$

Then the use of equality (Al−1) leads to

$$v_{n+\frac{1}{2}}v_{n-\frac{1}{2}}$$

$$=-4\frac{[\mathrm{dn}(\theta_0+\frac{\kappa_0}{2})-\mathrm{dn}\frac{\kappa_0}{2}][\mathrm{dn}(\theta_0-\frac{\kappa_0}{2})-\mathrm{dn}\frac{\kappa_0}{2}]}{[\mathrm{dn}(\theta_0+\frac{\kappa_0}{2})-\mathrm{dn}(\theta_0-\frac{\kappa_0}{2})]^2}\cdot$$

$$\frac{[\mathrm{dn}x_0+\mathrm{dn}(\theta_0+\frac{\kappa_0}{2})][\mathrm{dn}x_0+\mathrm{dn}(\theta_0-\frac{\kappa_0}{2})]}{[\mathrm{dn}x_0+\mathrm{dn}\frac{\kappa_0}{2}]^2}, \qquad \cdots\cdots (47)$$

where (A2−3) is available for proving the deformation. (47) leads to the expression of Q_n as given below.

$$Q_n=\frac{1}{\mathrm{dn}\frac{\kappa_0}{2}[\mathrm{dn}(\frac{\kappa_0}{2}+\theta_0)-\mathrm{dn}(\frac{\kappa_0}{2}-\theta_0)]}\cdot\Bigg[\mathrm{dn}(\frac{\kappa_0}{2}+\theta_0)\mathrm{dn}(\frac{\kappa_0}{2}-\theta_0)-\mathrm{dn}^2\frac{\kappa_0}{2}$$

$$-(\mathrm{dn}(\frac{\kappa_0}{2}+\theta_0)-\mathrm{dn}\frac{\kappa_0}{2})(\mathrm{dn}(\frac{\kappa_0}{2}-\theta_0)-\mathrm{dn}\frac{\kappa_0}{2})\cdot\frac{\mathrm{dn}x_0-\mathrm{dn}\frac{\kappa_0}{2}}{\mathrm{dn}x_0+\mathrm{dn}\frac{\kappa_0}{2}}\Bigg] \qquad \cdots\cdots (48)$$

$$=\frac{\mathrm{sn}\,\theta_0}{\mathrm{cn}\,\theta_0\mathrm{sn}\,\kappa_0}\left[\mathrm{cn}\,\kappa_0+\frac{\mathrm{dn}\,\kappa_0-\mathrm{dn}\,\theta_0}{1+\mathrm{dn}\,\theta_0}\cdot\frac{\mathrm{dn}x_0-\mathrm{dn}\frac{\kappa_0}{2}}{\mathrm{dn}x_0+\mathrm{dn}\frac{\kappa_0}{2}}\right] \qquad \cdots\cdots (49)$$

$$=\frac{\mathrm{sn}\theta}{\mathrm{cn}\theta\,\mathrm{dn}\theta\,\mathrm{sn}\,\kappa}[\mathrm{cn}\kappa\mathrm{dn}\kappa+k^2(\mathrm{sn}^2\kappa-\mathrm{sn}^2\theta)\,\mathrm{sn}(x+\frac{\kappa}{2})\,\mathrm{sn}(x-\frac{\kappa}{2})], \qquad \cdots\cdots (50)$$

where (A2−3) with an exchange $\kappa_0\leftrightarrow\theta_0$ is used in the deformation (49) from (48).

From (49) we obtain

$$Q_{n-\frac{1}{2}}-Q_{n+\frac{1}{2}}=\frac{2k_0^2\,\mathrm{sn}\theta_0(\mathrm{dn}\,\kappa_0-\mathrm{dn}\theta_0)\mathrm{sn}x_0\mathrm{cn}x_0}{\mathrm{cn}\,\theta_0(1+\mathrm{dn}\theta_0)(\mathrm{dn}x_0+1)(\mathrm{dn}x_0+\mathrm{dn}\,\kappa_0)}, \qquad \cdots\cdots (51)$$

and

$$Q_{n-\frac{1}{2}}+Q_{n+\frac{1}{2}}=2\frac{\mathrm{sn}\theta_0\mathrm{cn}\,\kappa_0(1+\mathrm{dn}\kappa_0)(\mathrm{dn}x_0+\mathrm{dn}\theta_0)}{\mathrm{sn}\,\kappa_0\mathrm{cn}\theta_0(1+\mathrm{dn}\,\theta_0)(\mathrm{dn}x_0+\mathrm{dn}\,\kappa_0)}. \qquad \cdots\cdots (52)$$

(51) can be proved by the use of (A1−1), and the derivation of (52) is given in Appendix 2. The substitution of (46), (51) and (52) to the both sides of eq. (19) proves that the assumption of the solution (46) is valid under the condition that (9) is satisfied. Also (46) and (52), and (51) and (52) are found to reproduce the expression (41) for B_n and the expression (43) for C_n respectively.

(iii) We assume that the elliptic function solution satisfying eqs. (26) and (27) is given as

$$w_n = \frac{\mathrm{sn}\,\kappa}{4\omega\,\mathrm{cn}\,\kappa\,\mathrm{dn}\,\kappa}\left[\,\mathrm{sn}^2\kappa - \mathrm{dn}^2\kappa - 2k^2\mathrm{sn}^2\kappa\,\mathrm{sn}^2 x\,\right] \qquad \cdots\cdots (53)$$

$$= \frac{\mathrm{sn}\,\kappa_0}{4k_0^2\omega_0\,\mathrm{cn}\,\kappa_0(1+\mathrm{dn}\,\kappa_0)}\left[4-3k_0^2+(k_0^2-4)\mathrm{dn}\,\kappa_0+2k_0^2(1-\mathrm{dn}\,\kappa_0)\frac{\mathrm{dn}\,x_0-1}{\mathrm{dn}\,x_0+1}\right]. \qquad \cdots\cdots (54)$$

Then we find that the relation (27) leads to

$$R_n = \frac{1}{1+\mathrm{dn}\,\kappa_0}\sqrt{-\frac{\mathrm{sn}\,\kappa_0}{2\omega_0\,\mathrm{cn}\,\kappa_0}}\left[2\,\mathrm{cn}\,\kappa_0-(1-\mathrm{dn}\,\kappa_0)\frac{\mathrm{dn}\,x_0-\mathrm{dn}\frac{\kappa_0}{2}}{\mathrm{dn}\,x_0+\mathrm{dn}\frac{\kappa_0}{2}}\right] \qquad \cdots\cdots (55)$$

$$= \sqrt{-\frac{\mathrm{sn}\,\kappa}{2\omega\,\mathrm{cn}\,\kappa\,\mathrm{dn}\,\kappa}}\left[\mathrm{cn}\,\kappa\,\mathrm{dn}\,\kappa + k^2\mathrm{sn}^2\kappa\,\mathrm{sn}\left(x+\frac{\kappa}{2}\right)\mathrm{sn}\left(x-\frac{\kappa}{2}\right)\right]. \qquad \cdots\cdots (56)$$

(A2−4) applied to the expression (55) leads to

$$R_{n+\frac{1}{2}} + R_{n-\frac{1}{2}} = \sqrt{-\frac{2\,\mathrm{sn}\,\kappa_0\,\mathrm{cn}\,\kappa_0}{\omega_0}}\cdot\frac{\mathrm{dn}\,x_0+1}{\mathrm{dn}\,x_0+\mathrm{dn}\,\kappa_0}\,, \qquad \cdots\cdots (57)$$

and the substitution of (57) and the time differential of (54) to the both sides of (26) proves the validity of the assumption (54). The transformation (28) and the transformation of variable (34) to the result leads to an elliptic function solution B_n given by (7).

§4. Discussion

Since θ in the solution (8) is related to the mean level, the propagation velocity is a function of not only κ but arbitrary θ, as shown in (9). Also the amplitude and wave form is related to θ, and the propagation velocity and the amplitude change their sign according as $\theta < \kappa$ or $\theta > \kappa$. Here we examine the extreme case in which $\theta = K(k)$. We find in this case

$$C_n = k_1\mathrm{sn}(\kappa_1, k_1)\,\mathrm{sn}(x_1, k_1)\,, \qquad \cdots\cdots (58)$$

and

$$\omega = \omega_0 \equiv -2(1+k_1)\mathrm{sn}(\kappa_1, k_1)\,, \qquad \cdots\cdots (59)$$

where $k_1=\dfrac{1-k'}{1+k'}$, $\kappa_1=(1+k')\kappa$ and $x_1=(1+k')x$.

ω always satisfies the inequality $\omega\geq\omega_0$ in the solution (8) and in the related solutions such as B_n obtained from (41). The restriction seen in the latter is absent if we represent the solutions as (7) and (6).

Appendix 1

Using the following two equalities

$$\Delta(u,w)[\mathrm{dn}(u+w)+\mathrm{dn}v][\mathrm{dn}(u-w)+\mathrm{dn}v]$$

$$=\Delta(v,w)[\mathrm{dn}(v+w)+\mathrm{dn}u][\mathrm{dn}(v-w)+\mathrm{dn}u]\ , \qquad \cdots\cdots(\mathrm{A}\,1-1)$$

in which

$$\Delta(\alpha,\beta)=1-k_0^2\mathrm{sn}^2\alpha\,\mathrm{sn}^2\beta\ , \qquad \cdots\cdots(\mathrm{A}1-2)$$

and

$$\mathrm{dn}\left(x_0+\frac{\theta_0}{2}+\frac{\kappa_0}{2}\right)\mathrm{dn}\left(x_0+\frac{\theta_0}{2}-\frac{\kappa_0}{2}\right)-\mathrm{dn}^2\frac{\theta_0}{2}=\frac{e_1\mathrm{dn}^2\left(x_0+\frac{\theta_0}{2}\right)+e_2}{\Delta\left(x_0+\frac{\theta_0}{2},\ \frac{\kappa_0}{2}\right)}\ , \qquad \cdots\cdots(\mathrm{A}\,1-3)$$

in which

$$e_1=\mathrm{cn}^2\frac{\kappa_0}{2}-\mathrm{dn}^2\frac{\theta_0}{2}\,\mathrm{sn}^2\frac{\kappa_0}{2}\ , \qquad \cdots\cdots(\mathrm{A}\,1-4) \qquad\qquad e_2=k_0^2\,\mathrm{sn}^2\frac{\kappa_0}{2}-\mathrm{dn}^2\frac{\theta_0}{2}\,\mathrm{cn}^2\frac{\kappa_0}{2}\ , \qquad \cdots\cdots(\mathrm{A}\,1-5)$$

we find that the assumption (32) leads to

$$u_{n-\frac{1}{2}}+u_{n+\frac{1}{2}}$$

$$=\sqrt{\frac{1-\mathrm{dn}\,\kappa_0}{2(1+\mathrm{dn}\,\theta_0)(\mathrm{dn}\,\theta_0-\mathrm{dn}\,\kappa_0)}}$$

$$\left[2\mathrm{cn}\,\theta_0-\frac{(1-\mathrm{dn}\,\theta_0)\left(e_1\mathrm{dn}^2\left(x_0+\frac{\theta_0}{2}\right)+e_2\right)}{\Delta\left(\frac{\theta_0}{2},\frac{\kappa_0}{2}\right)\left(\mathrm{dn}\left(x_0+\frac{\theta_0}{2}\right)+\mathrm{dn}\frac{1}{2}(\theta_0+\kappa_0)\right)\left(\mathrm{dn}\left(x_0+\frac{\theta_0}{2}\right)+\mathrm{dn}\frac{1}{2}(\theta_0-\kappa_0)\right)}\right].$$

$$\cdots\cdots(\mathrm{A}\,1-6)$$

On the other hand, using an equality

$$\mathrm{sn}\frac{1}{2}(\kappa_0\pm\theta_0)\mathrm{cn}\frac{1}{2}(\kappa_0\pm\theta_0)=-\frac{e_1\mathrm{dn}^2\frac{1}{2}(\kappa_0\pm\theta_0)+e_2}{2k_0^2\mathrm{sn}\frac{\kappa_0}{2}\mathrm{cn}\frac{\kappa_0}{2}\mathrm{dn}\frac{\theta_0}{2}}, \qquad \cdots\cdots (\mathrm{A}1-7)$$

we find that the assumption (33) leads to

$$\cosh 2\varphi_n=-\frac{1}{4k_0^2\mathrm{sn}\frac{\kappa_0}{2}\mathrm{cn}\frac{\kappa_0}{2}\mathrm{dn}\frac{\theta_0}{2}\sqrt{\mathrm{sn}\frac{1}{2}(\kappa_0+\theta_0)\mathrm{cn}\frac{1}{2}(\kappa_0+\theta_0)\mathrm{sn}\frac{1}{2}(\kappa_0-\theta_0)\mathrm{cn}\frac{1}{2}(\kappa_0-\theta_0)}}$$

$$\cdot\left[2\left(e_1\mathrm{dn}\frac{1}{2}(\kappa_0+\theta_0)\mathrm{dn}\frac{1}{2}(\kappa_0-\theta_0)+e_2\right)\right.$$

$$\left.+\frac{\left(\mathrm{dn}\frac{1}{2}(\kappa_0+\theta_0)-\mathrm{dn}\frac{1}{2}(\kappa_0-\theta_0)\right)^2\left(e_1\mathrm{dn}^2(x_0+\frac{\theta_0}{2})+e_2\right)}{\left(\mathrm{dn}(x_0+\frac{\theta_0}{2})+\mathrm{dn}\frac{1}{2}(\kappa_0+\theta_0)\right)\left(\mathrm{dn}(x_0+\frac{\theta_0}{2})+\mathrm{dn}\frac{1}{2}(\kappa_0-\theta_0)\right)}\right]. \qquad \cdots\cdots (\mathrm{A}1-8)$$

Substituting two equalities

$$\mathrm{sn}\frac{1}{2}(\kappa_0+\theta_0)\mathrm{cn}\frac{1}{2}(\kappa_0+\theta_0)\mathrm{sn}\frac{1}{2}(\kappa_0-\theta_0)\mathrm{cn}\frac{1}{2}(\kappa_0-\theta_0)$$

$$=\frac{2k_0^2\mathrm{sn}^2\frac{\kappa_0}{2}\mathrm{cn}^2\frac{\kappa_0}{2}\mathrm{sn}^2\frac{\theta_0}{2}\mathrm{cn}^2\frac{\theta_0}{2}}{[\Delta(\frac{\theta_0}{2},\frac{\kappa_0}{2})]^2}\cdot\frac{\mathrm{dn}\theta_0-\mathrm{dn}\kappa_0}{(1-\mathrm{dn}\kappa_0)(1-\mathrm{dn}\theta_0)}, \qquad \cdots\cdots (\mathrm{A}1-9)$$

$$e_1\mathrm{dn}\frac{1}{2}(\kappa_0+\theta_0)\mathrm{dn}\frac{1}{2}(\kappa_0-\theta_0)+e_2$$

$$=-\frac{4k_0^2\mathrm{cn}\theta_0\mathrm{sn}\frac{\theta_0}{2}\mathrm{cn}\frac{\theta_0}{2}\mathrm{dn}\frac{\theta_0}{2}\mathrm{sn}^2\frac{\kappa_0}{2}\mathrm{cn}^2\frac{\kappa_0}{2}}{\mathrm{sn}\theta_0\cdot\Delta(\frac{\theta_0}{2},\frac{\kappa_0}{2})}, \qquad \cdots\cdots (\mathrm{A}1-10)$$

and the difference formula of dn to (A1–8), we can verify the coincidence of two expressions (A1–6) and (A1–8).

Appendix 2

From (48) we find

$$Q_{n-\frac{1}{2}}+Q_{n+\frac{1}{2}}=\frac{1}{\mathrm{dn}(\frac{\kappa_0}{2}+\theta_0)-\mathrm{dn}(\frac{\kappa_0}{2}-\theta_0)}\left[\mathrm{dn}(\frac{\kappa_0}{2}+\theta_0)+\mathrm{dn}(\frac{\kappa_0}{2}-\theta_0)-2\mathrm{dn}\frac{\kappa_0}{2}\right.$$

$$+\left(\mathrm{dn}\left(\frac{\kappa_0}{2}+\theta_0\right)-\mathrm{dn}\frac{\kappa_0}{2}\right)\left(\mathrm{dn}\left(\frac{\kappa_0}{2}-\theta_0\right)-\mathrm{dn}\frac{\kappa_0}{2}\right)$$

$$\left.\cdot\frac{\mathrm{dn}\left(x_0+\frac{\kappa_0}{2}\right)+\mathrm{dn}\left(x_0-\frac{\kappa_0}{2}\right)+2\,\mathrm{dn}\frac{\kappa_0}{2}}{\left(\mathrm{dn}\left(x_0+\frac{\kappa_0}{2}\right)+\mathrm{dn}\frac{\kappa_0}{2}\right)\left(\mathrm{dn}\left(x_0-\frac{\kappa_0}{2}\right)+\mathrm{dn}\frac{\kappa_0}{2}\right)}\right] . \qquad \cdots\cdots (\mathrm{A}\,2-1)$$

We obtain (50) by substituting the following three equalities to (A2–1), i.e.,

$$\frac{\mathrm{dn}\left(\frac{\kappa_0}{2}+\theta_0\right)+\mathrm{dn}\left(\frac{\kappa_0}{2}-\theta_0\right)-2\,\mathrm{dn}\frac{\kappa_0}{2}}{\mathrm{dn}\left(\frac{\kappa_0}{2}+\theta_0\right)-\mathrm{dn}\left(\frac{\kappa_0}{2}-\theta_0\right)}=\frac{\mathrm{sn}\,\theta_0\left[(1+\mathrm{dn}\,\theta_0)\,\mathrm{cn}\,\kappa_0+\mathrm{dn}\,\kappa_0-\mathrm{dn}\,\theta_0\right]}{\mathrm{sn}\,\kappa_0\,\mathrm{cn}\,\theta_0\,(1+\mathrm{dn}\,\theta_0)}, \qquad \cdots\cdots (\mathrm{A}\,2-2)$$

$$\frac{\left(\mathrm{dn}\left(\frac{\kappa_0}{2}+\theta_0\right)-\mathrm{dn}\frac{\kappa_0}{2}\right)\left(\mathrm{dn}\left(\frac{\kappa_0}{2}-\theta_0\right)-\mathrm{dn}\frac{\kappa_0}{2}\right)}{\mathrm{dn}\left(\frac{\kappa_0}{2}+\theta_0\right)-\mathrm{dn}\left(\frac{\kappa_0}{2}-\theta_0\right)}=\frac{\mathrm{dn}\frac{\kappa_0}{2}\,\mathrm{sn}\,\theta_0\,(\mathrm{dn}\,\theta_0-\mathrm{dn}\,\kappa_0)}{\mathrm{sn}\,\kappa_0\,\mathrm{cn}\,\theta_0\,(1+\mathrm{dn}\,\theta_0)}, \qquad \cdots\cdots (\mathrm{A}\,2-3)$$

$$\frac{\mathrm{dn}\left(x_0+\frac{\kappa_0}{2}\right)+\mathrm{dn}\left(x_0-\frac{\kappa_0}{2}\right)+2\,\mathrm{dn}\frac{\kappa_0}{2}}{\left(\mathrm{dn}\left(x_0+\frac{\kappa_0}{2}\right)+\mathrm{dn}\frac{\kappa_0}{2}\right)\left(\mathrm{dn}\left(x_0-\frac{\kappa_0}{2}\right)+\mathrm{dn}\frac{\kappa_0}{2}\right)}=\frac{\mathrm{sn}\,\kappa_0\left[(1-\mathrm{cn}\,\kappa_0)\,\mathrm{dn}\,x_0+\mathrm{dn}\,\kappa_0+\mathrm{cn}\,\kappa_0\right]}{\mathrm{sn}\frac{\kappa_0}{2}\,\mathrm{cn}\frac{\kappa_0}{2}\,(1+\mathrm{dn}\,\kappa_0)(\mathrm{dn}\,x_0+\mathrm{dn}\,\kappa_0)} .$$

$$\cdots\cdots (\mathrm{A}\,2-4)$$

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