ものは存在しそうにないが，上記の結果は，少くともこの γ' 値に近い質量比を持つ分子の高い振動励起状態が，古典極限として，〝カオス的〟であることを示唆しているように思われる。現実の分子は量子力学的系であり，上記の古典的解析結果が，ただちにいわゆる〝量子的カオス〟（現時点においては，この概念とその定義は確立されているとはとうてい考えられないが[8]）の存在を断言するものではない。しかし，いわゆる Wogner 分布関数のふるまいと，古典的な Poincaré map の間には近似的な平行関係があることが指摘されており，[9] 古典極限における解析も，量子力学的解析への一つの参考材料になるものと思われる。

References

1) T. Matsushita, A. Narita and T. Terasaka, Chem. Phys. Letters (to be published).
2) A. Narita, T. Terasaka and T. Matsushita, (in preparation).
3) M. Hénon and C. Heiles, Astron. J. **69** (1964) 73.
4) R. J. Wolf and W. L. Hase, J. Chem. Phys. **73** (1980) 3779.
5) R. T. Lawton and M. S. Child, Mol. Phys. **44** (1981) 709.
6) J. M. Greene, R. S. Mackay, E. Vivaldi and M. J. Feigenbaum, Physica **D3** (1981) 468.
7) V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, New York, 1978) App. 7 and 8.
8) For example, M. Shapiro and M. S. Child, J. Chem. Phys. **76** (1982) 6176 and references cited therein.
9) J. S. Hutchinson and R. E. Wyatt, Chem. Phys. Letters 72 (1980) 378; Phys. Rev. **A23** (1981) 1567.

Chaos around hyperbolic points

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Here we want to illustrate some generic properties of two-dimensional nonlinear, area-preserving mappings, such as on the Poincaré surface of Hamiltonian systems with two degrees of freedom, by means of a simple model of a billiard between non-concentric circles. Most features of the mapping could be well understood from the knowledge of fixed points on the plane. As regards fixed points, their properties are summarized in the following theorem, if we could quote it from the letter of J. Ford addressed to G. Casati and M. Berry;

Theorem: In a sufficiently small neighborhood of each type fixed point,
elliptic fixed points are stable (in general) but non-integrable;
hyperbolic fixed points are unstable but integrable.

Specifically he noticed the paradoxical nature that homolinic structure destroys integrability about an elliptic fixed points but this same structure does not destroy integrability about a hyperbolic fixed point. In the letter, he intended to take a poll among workers in nonlinear dynamics seeking to determine how many are familiar with the theorem and, among those who are familiar with it, how they resolve the paradoxes. In the following an attempt to resolve this paradox is presented. To do this, first we will illustrate the theorem on elliptic points and secondly mention to the chaotic behavior around hyperbolic points and its coin toss mechanism. We could say that the homoclinic structure around the point turns out itself to be a resolver of the paradox.

I. Around elliptic points

We consider a billiard moving in the annular region bounded by two non-concentric circles as shown in Fig. 1, whose details is omitted here (see Physica D (1982) to appear). The dynamics of our billiard can be described by an area-preserving mapping on a coordinate plane (ℓ,s) where $2\pi\ell$ is the length of arc from P and s is the sine of the reflection angle. Its typical example is shown in Fig. 2. In the case, the origin (0,0) is an elliptic fixed point and surrounded by invariant curves proven by the KAM theorem and in the sense elliptic points are stable.

The enlargement of KAM near-integrable region (A-region in Fig. 2) is shown in Fig. 3. The numbers of each curve indicate n of the mapping T^n. In the case the limiting rotation number of the fixed point is slightly larger than 1/8. The rotation numbers of the invariant curves slowly decrease from that value as one moves away from the origin and the invariant curve with the rotation number 1/8 would consists purely of period-eight fixed points. Moreover, further away from the origin, one finds two interlaced chains of nine islands. Between them we also might expect period-nine hyperbolic fixed points, whose separatrices

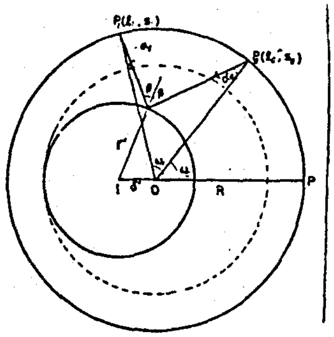

Fig. 1 This figure shows the annular region of billiard motion bounded by the inner circle of radius r' and the outer circle of radius R. The centers of the two circles are separated by the distance δ'. Meaning is given to the other symbols in this drawing in the accompanying textual narrative.

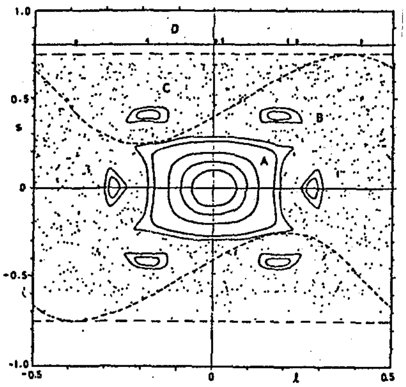

Fig. 2

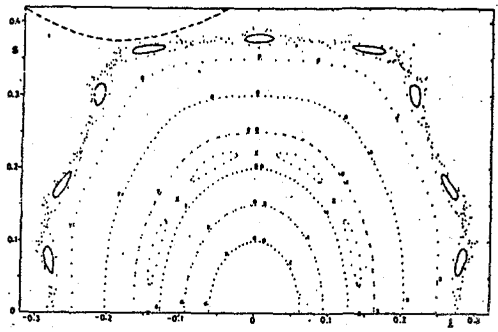

Fig. 3

intersect transversally and contain homoclinic points. We will later mention that chaotic behaviors appear around them. This complicated situation is found in any small neighborhood of the origin, because there exist higher periodic points, which illustrates non-integrability of elliptic fixed points established by the Zehnder theorem (see Moser's Stable and Random Motions in Dynamical Systems, p. 105).

II. Around hyperbolic fixed points

Hyperbolic fixed points are unstable by their definition, but Moser proved integrability in its neighborhood by establishing the convergence of the Birkhoff series transformation to normal form (Commun. Pure and Appl. Math. **9**, 673 (1956)). Fig. 4 has a hyperbolic fixed point at $(\pm 1/2, 0)$. We note that the coordinate plane is periodic in the direction of ℓ. The magnified view

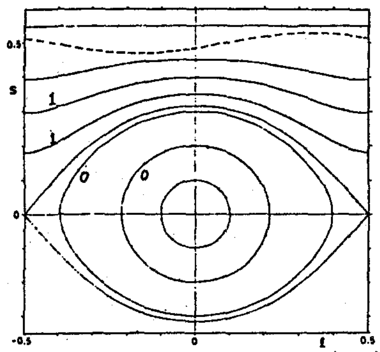

Fig. 4 Closed curves represent type 0 motion and lines represent typeII motion.

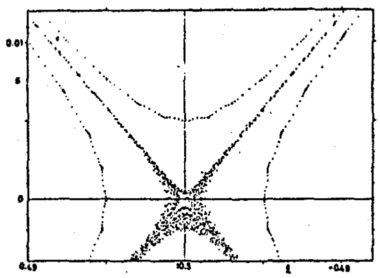

Fig. 5

of the mapping points from an initial point near the fixed point is shown in Fig. 5, which gives very chaotic figure. Indeed, a coin toss mechanism is hidden in this chaotic property. The orbits are distinguished into two types, denoted by 0 and 1 in Fig. 4. In a sense, the motion 0 corresponds to a libration and 1 to a rotation in celestial mechanics. Then the orbit can be expressed by a sequence of 0 and 1. By way of examples, we have

(1) 010100000.....

for the initial (0.008, 0)

and

(2) 000010100111.........

for (0.004, 0)

in the case $r = 0.5$, $\delta = -0.05$. How can the chaotic behavior be produced without a contradiction of Moser's theorem? In fact the mapping points are transfered along the invariant curves (almost hyperbola) shown by the dotted lines in Fig. 6, although the curves are written merely by linear approximation. This implies the integrability around the hyperbolic point. Fig. 6 also shows the homoclinic structure; the forward mapping of the straight unstable invariant curve W_1^- on the right of the figure returns to the left with waving W_2^- and intersects transversally with the stable invariant W_1^+, and similarly the backward integration of W_1^+ gives the waving invariant curve W_2^+ on the right. This reveals the homoclinic points of intersection between the invariant curves. But we note here that we can distinguish two region 0 and 1 where the motions of type 0 and 1 are taking place. Consider the mapping, for example, from (0.008, 0). This mapping must lie on the almost hyperbola in accordance with Moser's integrability theorem. On the other hand, (0.008, 0) lies on a shaded region between two invariant curves and takes successively the neighboring shaded regions as indicated by arrows. However by doing this the mapping points occupy the regions 010...... successively just like a coin toss. The same is true for the initial point (0.004, 0) which gives rise a series 000010...... Therefore the homoclinic structure gives rise to the coin toss mechanism and chaos in the integrable region around the hyperbolic point. It would require more investigations for the details, but the *coin toss mechanism of homoclinic structure seems to be important in the origin of chaos* in various systems.

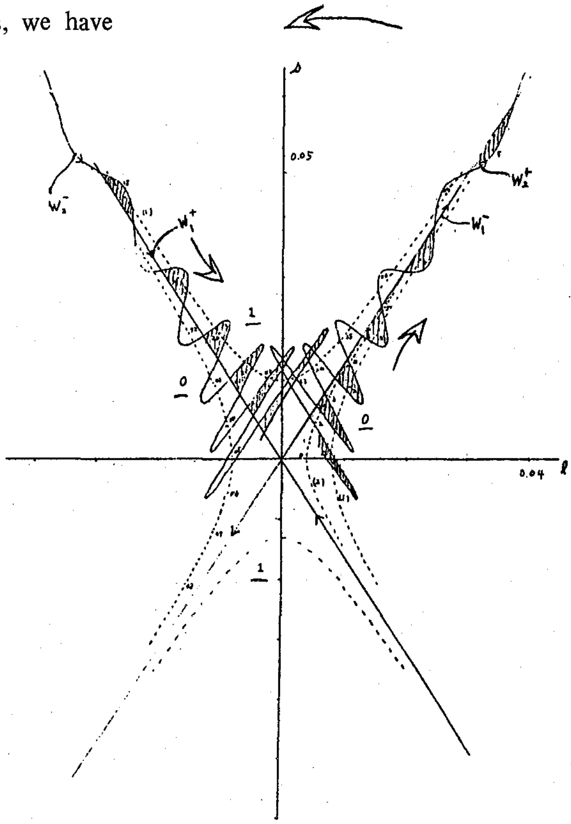

Fig. 6 The motions of type 0 and 1 take place in the regions denoted by 0 and 1 respectively.