Relation between certain quasi-vortex solutions and solitons of the sine-Gordon equation and other nonlinear equations

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It is shown that the quasi-vortex type solutions recently studied by Hudák of the sine-Gordon equation, $u_{xx}+u_{yy}=\sin u$, can be derived from the known multiple soliton solutions by the proper procedure. This shows, in principle the existence of the multiple quasi-vortex solutions. It also shows that the superposition of usual solitons and quasi-vortex solutions are possible for this equation. Implication of the present results to other soliton equations is briefly discussed.

§1. Introduction

Recently the studies of nonlinear phenomena related to solitons have progressed greatly. For the so-called completely integrable nonlinear partial differential equations, there exist (now well-established) soliton solutions.[1] Besides the ordinary solitons, certain other modes of solutions are known such as resonance solitons,[2] explode-decay mode (ripplon) solutions.[3] Recently Hudák presented certain "quasi-vortex" type solution of the sine-Gordon equation.[4] For the completely integrable equation, if we can obtain new type solution which are different from solitons, it is very interesting. In this sense, the present author was curious about this quasi-vortex solution in that whether it is really basically new type solution or not. Present work is the direct result of the efforts to understand what the quasi-vortex solution studied by Hudák actually is. As will be shown in the following, this quasi-vortex solution turns out to be directly related to the multiple soliton solutions. In this sense, one can say that this solution is mathematically not fundamentally different from the solitons. Still this does not deny the fact that Hudák clarified certain new physical mode of solutions overlooked previously.

§2. Certain quasi-vortex solution of the sine-Gordon equation

First we consider the sine-Gordon equation

$$u_{xx}+u_{yy}=\sin u. \tag{2.1}$$

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Throughout this paper x, y, t in the subscripts represent partial derivatives. This equation is an important model equation for nonlinear phenomena.[5] The quasi-vortex solution considered by Hudák[4] is the exact solution to eq. (2. 1) written as

$$u = \pm \arctan \left\{ \frac{(\sin a_1) \ \sinh \ [(\cos a_1) \ (x + x_1)]}{(\cos a_1) \ \sinh \ [(\sin a_1) \ (y + y_1)]} \right\}, \tag{2. 2}$$

where x_1, y_1 and a_1 are arbitrary constants. In the case of linear version of eq. (2.1), $u_{xx} + u_{yy} = 0$, there exist vortex solution $u = u_0 \arctan [(x + x_1)/(y + y_1)]$ where u_0 is constant. In the velocity variable defined by grad u, the linear solution gives single vortex movement centered at $(-x_1, -y_1)$ as grad $u = (y + y_1, -(x + x_1)) u_0 / [(x + x_1)^2 + (y + y_1)^2]$. As noted by Hudák, in comparison to this linear case, the solution (2.2) of the nonlinear equation (2.1) may be considered as one possible nonlinear counterpart of the linear vortex solution, reduing to the former near $(-x_1, -y_1)$, hence called as quasi-vortex solution. Note however that the solution (2.2) is *not exactly rotationally symmetric*, thus we use the word "quasi-" vortex.

§3. Relation between solitons and quasi-vortex solutions

Is the quasi-vortex solution given by (2. 2) completely new solution? Here we show that multi-soliton generates it. First we review soliton solutions. We use Hirota's bilinear method[6] By the dependent variable transformation

$$u = \pm \arctan [g(x, y) / f(x, y)], \tag{3. 1}$$

the original equation reduces to the bilinear equations for f and g,

$$(D_x{}^2 + D_y{}^2 - 1) f \cdot g = 0, \quad (D_x{}^2 + D_y{}^2)(f \cdot f - g \cdot g) = 0, \tag{3. 2}$$

where operators D_x and D_y are defined as $D_x{}^2 f(x, y) \cdot g(x, y) \equiv (\partial_x - \partial_{x'})^2 f(x, y) g(x', y')|_{x' = x, y' = y} = f_{xx} g - 2 f_x g_x + f g_{xx}$ and similarly for others. One- and two-soliton are vien by $f = 1$, $g = \exp(\eta_1)$, and $f = 1 + \exp(\eta_1 + \eta_2 + A_{12})$, $g = \exp(\eta_1) + \exp(\eta_2)$ respectively where $\eta_i \equiv k_i x + l_i y + \eta_{0i}$, $k_i^2 + l_i^2 = 1$, $\exp(A_{ij}) \equiv [(k_i - k_j)^2 + (l_i - l_j)^2] / [(k_i + k_j)^2 + (l_i + l_j)^2]$, and η_{0i} is arbitrary constant. One can easily check directly that these solutions satisfy eq. (3. 2). In the two-soliton solution, since η_{01} is arbitrary constant, we can shift it by any arbitrary amount. Thus we shift it by

$i\pi$, or rewrite as $\eta_{01} \to \eta_{01} + i\pi$. Then by introducing arbitrary constants x_1 and y_1 instead of η_{01} and η_{02} we have

$$\frac{g}{f} = \sqrt{\frac{(k_1 + k_2)^2 + (l_1 + l_2)^2}{(k_1 - k_2)^2 + (l_1 - l_2)^2}} \; \frac{\sinh\left[\left(\frac{k_1 - k_2}{2}\right)(x + x_1) + \left(\frac{l_1 - l_2}{2}\right)(y + y_1)\right]}{\sinh\left[\left(\frac{k_1 + k_2}{2}\right)(x + x_1) + \left(\frac{l_1 + l_2}{2}\right)(y + y_1)\right]}. \tag{3.3}$$

In accordance with relation $k_i^2 + 1_i^2 = 1$ (i = 1,2), we can parameterize as $(k_1 - k_2)/2 = \cos a_1 \sin b_1$, $(k_1 + k_2)/2 = \sin a_1 \cos b_1$, $(1_1 - 1_2)/2 = -\cos a_1 \cos b_1$, $(1_1 + 1_2)/2 = \sin a_1 \sin b_1$ and eq. (3.3) can be rewritten as

$$\frac{g}{f} = \frac{(\sin a_1)\sinh(\cos a_1)\,[\,(x + x_1)\sin b_1 - (y + y_1)\cos b_1\,]}{(\cos a_1)\sinh(\sin a_1)\,[\,(x + x_1)\cos b_1 + (y + y_1)\sin b_1\,]}. \tag{3.4}$$

Equation (3. 4) for $b_1 = \pi/2$ reproduces eq. (2. 2) via eq. (3.1). Thus we have shown that two-soliton solution generates one-quasi-vortex solution by Hudák. As well-known, exact multi-soliton solution exists for eq. (2. 1).[6] The present result indicates that in the $2N$-soliton solutions, we shift arbitrary parameters as $\eta_{01} \to \eta_{01} + i\pi$, $\eta_{03} \to \eta_{03} + i\pi$, . . . , $\eta_{0,\,2N-1} \to \eta_{0,\,2N-1} + i\pi$ and we obtain N-quasi-vortex solution which should clearly inherit the well-established superposition property of the original multi-soliton solutions since in our derivation the change is introduced simply in the arbitrary constant parameters. Therefore this indicates that the various quasi-vortex solutions of eq. (3. 4) can be superposed with each other with different vortex center $(-x_i, -y_i)$ and different parameters a_i and b_i. It also indicates that arbitrary number of solitons and quasi-vortex solutions can be superposed both with each other and among themselves.

Since we have clarified the general relation between soliton and quasi-vortex, we can extend this argument to other two-space-dimensional soliton systems of similar nature. One example is the one-quasi-vortex solution recently obtained by Takeno for the time dependent sine-Gordon (≡TSG) equation $u_{xx} + u_{yy} - u_{tt} = \sin u$.[7] Here we can look at this from our present viewpoint. By the same transformation (3. 1), TSG reduces to the bilinear form (3. 2) with $D_x^2 + D_y^2$ replaced by $D_x^2 + D_y^2 - D_t^2$. This type of bilinear form is known to admit at least up to two-soliton solutions.[8] Thus from this and the previous argument, we can understand that at least one-quasi-vortex solution exists to TSG. On the other hand, three or higher soliton solution to TSG exist only with restricted parameter condition.[9] Therefore we can expect that two-quasi-vortex solution (corresponding to four-soliton) can exist only for a limited parameter condition.

We notice that for the bilinear equation of the type (3. 2) (including the case of addition of D_t^2 for TSG) it is possible to have two-periodic solution (periodic generalization of two-soliton).[10] Thus it is possible to have periodic generalization of one-quasi-vortex solution to eq. (2. 1) and TSG.

Next we consider the modified KdV equation written in the potential variable, $u_y + 2u_x{}^3 + u_{xxx} = 0$, which is known to have N-soliton solution (usually t appears instead of y).[11] The soliton solution of this equation has the form of eq. (3. 1) where f and g are similar to the sine-Gordon case. Hence similarly as before, we can have one-quasi-vortex from two-soliton and N-quasi-vortex from $2N$-soliton solutions.

We note the limitation of deriving quasi-vortex solution. Among the known completely integrable equations, the ones whose soliton solution has the form $u = \arctan (g/f)$ are rather special. Most frequently the soliton has the form $u = (\log f)_{xx}$. In this case, two-soliton has the form $f = 1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_1 + \eta_2 + A_{12})$ where η_i and A_{12} are similar form appeared previously. The examples belonging to this class are the KdV, Bousinesq, KP, Toda equation and so on.[8] For this case we can easily check that the same previous procedure is possible, but it does not yield vortex-like solution as its result.

§4 Physical Characteristics of the present mode

To grasp the basic physical nature of the given mathematical expression is very important. (This is the reason why we do not go too much into the explicit details of two-quasi-vortex. N-quasi-vortex, periodic quasi-vortex solutions etc, but rather focus attention to the simplest solution.) We study solution (2. 2) more to have the clear feeling of quasi-vortex solution. In eq. (2. 2), we put $x_1 = y_1 = 0$, $a_1 = a$ and take gradient of u

$$\operatorname{grad} u = \frac{\pm 4 \sin a \cos a}{\cos^2 a \sinh^2 (y \sin a) + \sin^2 a \sinh^2 (x \cos a)} \times$$

$$(\cos a \cosh (x \cos a) \sinh (y \sin a), \ - \sin a \sinh (x \cos a) \cosh (y \sin a)). \qquad (4.1)$$

In the limit $x \ll 1, y \ll 1$, we have $\operatorname{grad} u \sim \pm 4\,(y, -x) / (x^2 + y^2)$ which is the previous linear vortex solution with $u_0 = \pm 4$. In the limit $|x| \gg 1, |y| \gg 1$, we have relation $\cosh (x \cos a) \sim \exp |x \cos a|$, $\sinh (y \sin a) \sim (\exp |y \sin a|) \times \operatorname{sgn} (y \sin a), \ldots$, and the asymptotic form

$$\text{grad}\, u \sim \pm 2\,(\,(\cos a)\,\text{sgn}\,(y \sin a), -\,(\sin a)\,\text{sgn}\,(x \cos a)\,)\,/$$
$$\cosh\,[\,|y \sin a| - |x \cos a| + \log\,(\tan a\,)\,]. \tag{4.2}$$

This asymptotic form shows that in the region $|x| \gg 1$, $|y| \gg 1$, the grad u vector has fixed (constant) direction for given sgn x and sgn y. Its absolute value along the line $|y \sin a| - |x \cos a| =$ const. is constant. It has hump along the lines $|y \sin a| - |x \cos a| + \log\,(\tan a\,) = 0$. This feature about the absolute value is nothing but the characteristics of two-soliton solutions which are two lines of one-dimensional humps crossing at certain angle. For the present mode two opposite sides of a one-dimensionally localized hump divided at cross point have opposite direction of grad vector. With other one- dimensionally localized hump arranged in the similar way, totally two of the phase inverted one solitons (across the cross point) create vortex movement which approaches more and more to the rotationally symmetric original linear vortex as the region tends to the cross point or the vortex center. We can roughly characterize the present quasi-vortex mode as two-soliton which is "twisted" properly (phase inverted across the crossing point) such that it carries one-vortex at the intersecting point.

We have seen that present vortex solution is rather special type of vortex directly created from solitons. Totally rotationally symmetric vortex solution to these nonlinear equations belong to the completely different subject. This (rotationally symmetric vortex) topic is perhaps more elementary and important but right now not much is studied about this.

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