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Rigorous Deduction of the Dynamical Equations[†] for the Persistent Current Electron with $g = -2(1 + \alpha/2\pi)$ by the New Frame in Physics

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Abstract

After proposing a new refined physico-mathematical frame for the covariant application of the principle of least action, the principle with the new frame has been applied to the persistent current classical model of the electron, deriving all the basic kinematical equations of the electron. By adopting the quantization procedure of the new frame, these c-number equations transit directly to the Dirac type q-number equations of the electron with $g = -2(1 + \alpha/2\pi)$. The derived equations are delicately different from the currently accepted equations in their highest order accuracy, with definite physical origin for the difference. The Thomas precession and the spin-orbit coupling are analyzed in detail, clarifying the origin of the difference between the expressions in classical physics and in the Dirac Hamiltonian. It has been found that the equation for the precession is almost independent of the contribution of the external vector potencial to the spin angular momentum, $S^{\mu\nu}$, and, if it contributes with a factor, 1/2, the induced change in the model makes the electron spin perfectly diamagnetic, indicating the coexistence of the angular momentum and flux quantizations. The principle of the factor two is proposed at the interface between classical and quantal physics.

§1. Introduction

In 1974, we have proposed a persistent Vortex Ring model of the electron[1]. We refer it as VR hereafter. Attractive feature of this semi-classical model is that it has no adjustable parameter and has almost all

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The author believe that this paper and Ref. 3) should be printed in more well-distributed journal. However, there are serious publication difficulties in such journals, for which the author is not responsible.

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non-wave characters of the electron, in complete conformity with the classical electromagnetism. An application of this model for the understanding of the hyperfine field in magnetic oxides was presented in 1981 [2]. In 1983, we have proposed a new frame in physics[3]. The guiding principle of the new frame is that, there must be a general and analytical interconnection between the classical and quantum physics, because the two frames are the different side views of the same object, having the accuracy of 10^{-8} or more. The proposed interconnection is that the classical physics described by c-number equations transit directly to the quantum equations, when regarded them as q-number equations with very simple, mostly well-known ways of the quantization. The new frame has a great advantage for dealing with complicated quantum systems, because we can utilize the classical concept up to its limit, and, with the use of the concept of classical ensemble adequately, many of the quantum characters of the system can be obtained classically.

Now in the new frame, in order to establish analytical continuation between classical and quantal physics, we needed to have the best classical model of the electron for the consistent description of the classical Maxwell-Lorentz electromagnetism. We found that the VR model satisfies this requirement.

We know that there is an old concept of point charge electron, in which the classical size of the electron is assumed to be less than 10^{-15} m. Since we take VR as the classical electron, the classical size is in the order of 10^{-12} m = 10^{-2} Å, which necessitates to abandon the point electron concept. This must be due in the new frame, because we have needed to have a strictly self-consistent Maxwell-Lorentz classical electromagnetism with electrons, whereas the electrostatic energy associated with the point charge electron in the size of less than 10^{-15} m reaches to more than 10 times of the rest energy of the electron. Therefore, this old model is self-inconsistent by itself, as the basic entities in the classical electromagnetism of the new frame. More detailed reasons will be given in Appendix A.

Now, there has been, however, a lack in the knowledge about the relativistic kinematical equations expected from this model. We have found that the least action principle and principles of the new frame applied to VR can deduce strictly and precisely the desired relativistic kinematical equations of the electron, which are identical to the currently accepted equations in the usual accuracy, but contain a few higher order terms with reasonable physical structures. Quantized wave equations have also been obtained by regarding these c-number equations as the q-number equations. This paper presents its outline, together with many newly found suggestive ideas for the electron.

It is to be noted that, since electrons and electromagnetic fields are the two principal entities, on which our classical frame of physics has been built up, the new frame of physics has a strong emphasis on these two entities. It is further noted that the present analyses are all effective for the muon. But, we are not in a position to challenge the present day Q.E.D. By the new frame, we propose another useful approximation of physics, which is complementary to Q.E.D. and is effective for the fields, where the Q.E.D. is not easy to apply. The accuracy is less but is sufficient for them. An example is the thermo-statistics of the orbital diamagnetism of many electrons, where, we believe, the new frame has corrected the previous misunderstanding[3]. Complicated macroscopic quantum systems, which are important in solid state physics and devices, are generally such systems.

We use the MKSP system[1], or, the MKS rationalized Gauss unit system, throughout. The difference in the formula from the CGS Gauss unit system is only in the factor of 4π or $1/4\pi$.

§2. A brief review and the basic data of the VR model

The essential feature of the model will be summarised as follows.

1). It is a tiny ring current with the ring radius of

$$^{\circ}R = |g|\lambda\!\!\!^{-}_e = 2(1 + \frac{\alpha}{2\pi})\frac{\hbar}{mc} = 7.73185 \times 10^{-13}\mathrm{m} \tag{2-1}$$

and with a very small cross sectional radius, $^{\circ}\eta$ of the ring segment. In this paper, the quantities in a proper frame will be indicated by a left superscript °. The values of $^{\circ}\eta$ are $(1.241 \text{ or } 0.967) \times 10^{-386}\,\mathrm{m}$, for the uniform or surface charge and current distributions†. In the following calculations, in order to simplify the situation, we assume a nearly surface charge and current distributions. Although we do not believe that the Maxwell electromagnetism can simply be extended to such a tiny region, the attractive feature of VR model is that its essential characters are independent of the details of these super-microscopic structures.

2). The total energy without external fields consists of electric and magnetic energies $^{\circ}U_{E,0}$ and $^{\circ}U_{M,0}$,

$$^{\circ}U_{E,0} = \frac{^{\circ}mc^2}{2} + \frac{e^2}{8\pi^2\,^{\circ}R} = \frac{^{\circ}mc^2}{2}\left(1 + \frac{\frac{\alpha}{2\pi}}{1 + \frac{\alpha}{2\pi}}\right) = \frac{e^2}{8\pi^2\,^{\circ}R}\left[\ln\frac{8\,^{\circ}R}{^{\circ}\eta}\right] , \tag{2-2}$$

$$^{\circ}U_{M,0} = \frac{^{\circ}mc^2}{2} - \frac{e^2}{8\pi^2\,^{\circ}R} = \frac{^{\circ}mc^2}{2}\left(\frac{1}{1 + \frac{\alpha}{2\pi}}\right) = \frac{e^2}{8\pi^2\,^{\circ}R}\left[\ln\frac{8\,^{\circ}R}{^{\circ}\eta} - 2\right] . \tag{2-3}$$

† We had a small mistake for the values of $^{\circ}\eta = {^{\circ}r}$ in the original paper.

Please note the subtle relation between $°R$, α, and $\ln 8°R/°\eta$ of the electron.

3). The Magnitudes of the angular momentum and the magnetic moment in each single classical VR are $\hbar$ and $2(1+\alpha/2\pi)\mu_B$, respectively, so that an ensemble concept is used for bridging the model with the real quantal characters of the electron. The spin state, α, or, the state having the spin directed along +z axis, is represented by an ensemble, in which the top of the angular momentum vectors distribute uniformly over the upper hemisphere. Hence, the averaged charge distribution becomes spherical with the angular momentum and magnetic moment components of

$$\frac{\hbar}{2} \quad \text{and} \quad (1+\frac{\alpha}{2\pi})\mu_B \quad , \tag{2-4}$$

respectively. Therefore, VR has an intrinsic anomalous magnetic moment, being in agreement with the result of Q.E.D., down to the second order perturbation, i.e., less than 2 ppm. This figure indicates the accuracy of our approximation for the magnetic moment.

4). A new momentum-energy density four vector[1] is employed for the convenience of the calculation of its self-produced electromagnetic momentum and energy. Therefore, the self-produced momentum-energy density four vector in the proper frame is

$$\{\frac{°\rho°\mathbf{a}}{2c}\ ,\ \underline{i}\ \frac{1}{c}[\frac{°\rho°\phi}{2}+\frac{(°\mathbf{j}\cdot°\mathbf{a})}{2c}]\} = \frac{°\rho}{2c}[\{°\mathbf{a},\ \underline{i}\ °\phi\} + \{0,\ \underline{i}\ \frac{(°\mathbf{j}\cdot°\mathbf{a})}{°\rho c}\}] \tag{2-5}$$

where $°\rho$, $°\mathbf{j}$, $°\phi$, and $°\mathbf{a}$ are the Maxwell-Lorentz charge and current densities, and electric and vector potentials, respectively. A self-factor 1/2 has been introduced and, the imaginary number expression is used here for the four space, and $\underline{i}$ is the i for this purpose[4]. The concrete representation of Eq.(2-5) will be shown soon.

5). VR is electromagnetically stable. In the Maxwell-Lorentz electromagnetism, the Lorentz electric force of repulsion is almost exactly cancelled by the Lorentz magnetic force of attraction and the magnitude of the next term is extremely small, such as 10^{-365} of the main term. Since the gravitational force expected is quite large, such as 10^{-43} of the main term, we expect that the model is stable in terms of general relativity.

6). The model keeps a quantized flux, hc/e, which is twice of the fluxoid of superconducting circuits.

In the new frame of physics, since the quantal equations are directly derivable from the classical equations, it is important to obtain the self-consistent detailed frame of classical physics.

In Fig. 1, we show the relativistic geometry for the numerical calculations. The coordinate system, $°K$, is an instantaneous proper frame of VR, which is moving with the velocity **V** and has its origin at the center of VR. K is the fixed original laboratory frame and *K is an instantaneous laboratory frame for which the Lorentz transformation between °K and *K does not need translational adjustment. Although the VR may make precessional motion, the velocity, $°\mathbf{u} = °\omega \times °\mathbf{r}$ ($°\omega = |\gamma|°H$), expected is in the range of

$$°u \sim °\omega\, °R \;=\; °R\frac{|g|e}{2mc}°H \sim 10^{-(2\sim0)}\,\mathrm{m/sec} \quad , \tag{2-6}$$

for the magnetic field $°H$ of $10^{3\sim5}$ Oe . Therefore, although this velocity itself has to be taken into account, since $(°u/c) \sim 3 \times 10^{-(11\sim9)}$, we can put

$$\gamma_u = \frac{1}{\sqrt{1 - (\frac{°u}{c})^2}} = 1 \quad , \tag{2-7}$$

so that no general relativistic consideration will be made.

The four currents and four potentials in °K are[1)]

$$\{°\mathrm{j},\ \underline{i}c°\rho\} = \{°\rho°\mathbf{v},\ \underline{i}\ c°\rho\} = \{°\rho\frac{\mathbf{v}_R(°\mathbf{r}) + °\boldsymbol{\omega} \times °\mathbf{r}}{(1 - \frac{°R°\omega}{c}\cos\theta\ )}\ ,\ \underline{i}c°\rho\} \tag{2-8}$$

$$|°\mathbf{v}| = \left|\frac{\mathbf{v}_R(°\mathbf{r}) + °\boldsymbol{\omega} \times °\mathbf{r}}{(1 - \frac{°R°\omega}{c}\cos\theta\ )}\right| = c \quad , \tag{2-9}$$

$$\{°\mathbf{a}(°\mathbf{r}),\ \underline{i}°\phi(°\mathbf{r})\} = \{-\frac{e}{4\pi^2°R}[\ln\frac{8°R}{°\eta} - 2]\frac{°\mathbf{v}}{c}\ ,\ \underline{i}(-\frac{e}{4\pi^2°R}\ln\frac{8°R}{°\eta})\}$$

$$= \frac{2}{(-e)}°U_{M,0}\ \frac{1}{°\rho c}\{°\mathrm{j},\ \underline{i}c°\rho\} + \frac{2}{(-e)}(°U_{E,0} - °U_{M,0})\{0,\ \underline{i}\}\ ,\ (e > 0) \tag{2-10}$$

and, in K, they are

$$\{\mathrm{j},\ \underline{i}c\rho\} = \{\rho\mathbf{v}(\mathbf{r}),\ \underline{i}c\rho\} = \{°\mathrm{j}_\perp + \gamma_V(°\mathrm{j}_\parallel + °\rho V),\ \gamma_V(°\rho + \frac{°\mathrm{j}_\parallel V}{c^2})\} \tag{2-11}$$

$$\{\mathbf{a}(\mathbf{r}),\ \underline{i}\phi(\mathbf{r})\} = \{°\mathbf{a}_\perp + \gamma_V(°\mathbf{a}_\parallel + °\phi\frac{V}{c}),\ \underline{i}\gamma_V(°\phi + \frac{°a_\parallel V}{c})\} \tag{2-12}$$

$$\gamma_V = 1/\sqrt{1 - (V/c)^2}$$

Here, $\mathbf{v}_R(°\mathbf{r})$ is the velocity of the charge in the rotating frame $^R K$, in which VR is at rest. For °K , $\mathbf{v}_R$ is transformed by the Lorentz transformation and θ is the angle between $°\boldsymbol{\omega}$ and the magnetic moment axis of VR. In the major part of the calculations, we assume VR rigid. The reason will be explained in §6, where non-rigid VR will be introduced. It is noted that $°\mathrm{a}(\mathbf{r})$ has been derived to be parallel to the current $°\mathrm{j}(\mathbf{r})$. It is further noted that, in K , the self-produced momentum-energy density four vector of Eq.(2-5) becomes

$$\{\frac{{}^\circ\rho \mathbf{a}}{2c} + \gamma_V \frac{({}^\circ\mathbf{j}\cdot{}^\circ\mathbf{a})}{2c^2}\frac{\mathbf{V}}{c}\,,\ \underline{i}\,\frac{1}{c}[\frac{{}^\circ\rho\dot{\phi}}{2} + \gamma_V\frac{({}^\circ\mathbf{j}\cdot{}^\circ\mathbf{a})}{2\,{}^\circ\rho c}]\}$$

$$= \frac{{}^\circ\rho}{2c}[\{\mathbf{a},\ \underline{i}\phi\} + \{\gamma_V\frac{({}^\circ\mathbf{j}\cdot{}^\circ\mathbf{a})}{{}^\circ\rho c}\frac{\mathbf{V}}{c}\,,\ \underline{i}\ \gamma_V\frac{({}^\circ\mathbf{j}\cdot{}^\circ\mathbf{a})}{{}^\circ\rho c}\}] = \frac{{}^\circ\rho}{2c}\{\mathbf{b},\ \underline{i}\phi_b\}$$

$$\rightarrow \frac{{}^\circ\rho}{2c}\{b^\mu\} = \{q^\mu\} = \frac{{}^\circ\rho}{(-e)c}[{}^\circ U_{M,0}\{\dot{x}^\mu\} + {}^\circ U_{E,0}\{\dot{\bar{x}}^\mu\}] \quad . \tag{2-13}$$

Here, we have defined a pseudo vector potential four vector $\{b^\mu\}$, and momentum-energy density four vector $\{q^\mu\}$, for the convenience of the following analyses. $\{q^\mu\}$ becomes the total momentum energy four vector of the electron, $\{p^\mu\}$, when integrated by the proper volume element $d^\circ V$. We have introduced the Minkowski notation. The four coordinates of the charge element are indicated by x^μ, and those of the center of VR by $\bar{x}^\mu$. They will be explained hereafter.

§3. Refined physico-mathematicl frame for the covariant application of the principle of least action to VR

We believe that the physico-mathematical structure of the covariant variation problem, which is crucial for the real application of the principle of least action to VR has not been well-understood[5,6,7]. Therefore, we begin with the explanation of a few general characters of the problem.

Let us take the orbital motion of a point charge electron as the example. The action integral is

$$I = \int_{\text{World Point (1)}}^{\text{World Point (2)}} L_s \frac{ds}{c} \tag{3-1}$$

Here, $ds/c = d\tau = \gamma_V^{-1}dt$ is the proper time, and L_s is the covariant Lagrangian, which is a Lorentz scalar having the dimension of energy. According to the guiding principle of the new frame, L_s is to be chosen so as to transit directly to the popular non-covariant Lagrangian L_t by[7])

$$L_s \frac{ds}{c} = L_t\ dt\ , \qquad L_t = L_s\,\gamma_V^{-1} \quad . \tag{3-2}$$

As the real examples of L_s , we consider $-{}^\circ mc^2$ and $-{}^\circ mc^2 - qA^\mu\dot{x}_\mu$, i.e., the most simple Lagrangian for the free electron, and that in the electromagnetic fields, $A^\mu(\bar{x}^\sigma)$. Here, the four coordinates, $\bar{x}^\mu$, indicate the location of the center of the electron. Now, in the covariant variation problem, presence of the relativistic constraints is essential. We propose that the Lagrangian multiplier method is most appropriate for this purpose[5]), because, first, by this method, we can make the variation free, and, second, since the original Lagrangian itself has not been given apriori, by manipulating the Lagrangian multiplier term adequately, we can choose the most appropriate Lagrangian, L_s , which can make the amount of necessary brain work minimum. For the analysis of VR, since there are quite a few terms and constraints, simplification is a crucial factor.

Then, our initial Lagrangian L_s transit to $\overline{L}_s$ with the Lagrangian multiplier term[7),

$$L_s(\overline{x}^\mu,\dot{\overline{x}}^\mu) \rightarrow \overline{L}_s(\overline{x}^\mu,\dot{\overline{x}}^\mu) = L_s + \frac{{}_1\Lambda}{2}(\dot{\overline{x}}^\mu\dot{\overline{x}}_\mu - 1). \tag{3-3}$$

The variation of the action is

$$\delta I = \int_{(1)}^{(2)} \delta\overline{L}_s \frac{ds}{c} + \int_{(1)}^{(2)} \overline{L}_s \delta(\frac{ds}{c}) \tag{3-4}$$

$$\delta(\frac{ds}{c}) = \frac{1}{c}\delta\sqrt{d\overline{x}^\mu d\overline{x}_\mu} = \frac{1}{cds} d\overline{x}^\mu \delta d\overline{x}_\mu = \dot{\overline{x}}^\mu \delta\dot{\overline{x}}_\mu \frac{ds}{c} \quad . \tag{3-5}$$

Therefore,

$$\begin{aligned}\delta I &= \int_{(1)}^{(2)} [\frac{\partial\overline{L}_s}{\partial\overline{x}_\mu}\delta\overline{x}_\mu + \frac{\partial\overline{L}_s}{\partial\dot{\overline{x}}_\mu}\delta\dot{\overline{x}}_\mu + \overline{L}_s\dot{\overline{x}}^\mu\delta\dot{\overline{x}}_\mu]\frac{ds}{c} \\ &= \int_{(1)}^{(2)} \{\frac{d}{ds}[\frac{\partial\overline{L}_s}{\partial\dot{\overline{x}}_\mu}\delta\overline{x}_\mu + \overline{L}_s\dot{\overline{x}}^\mu\delta\overline{x}_\mu] + [\frac{\partial\overline{L}_s}{\partial\overline{x}_\mu} - \frac{d}{ds}(\frac{\partial\overline{L}_s}{\partial\dot{\overline{x}}_\mu} + \overline{L}_s\dot{\overline{x}}^\mu)]\delta\overline{x}_\mu\}\frac{ds}{c} \quad .\end{aligned} \tag{3-6}$$

Then, adopting the postulates of the principle of least action, we get the Euler-Lagrange equation of motion as

$$\frac{d}{ds}[-\overline{L}_s\dot{\overline{x}}^\mu - \frac{\partial\overline{L}_s}{\partial\dot{\overline{x}}_\mu}] + \frac{\partial L_s}{\partial\overline{x}_\mu} = 0 \quad . \tag{3-7}$$

Inserting Eq.(3-3) into Eq.(3-7), we have

$$\frac{d}{ds}[-(L_s + {}_1\Lambda)\dot{\overline{x}}^\mu - \frac{\partial L_s}{\partial\dot{\overline{x}}_\mu}] + \frac{\partial L_s}{\partial\overline{x}_\mu} = 0 \quad , \tag{3-8}$$

$$-\dot{L}_s\dot{\overline{x}}^\mu - L_s\ddot{\overline{x}}^\mu - \frac{d}{ds}(\frac{\partial L_s}{\partial\dot{\overline{x}}_\mu}) - {}_1\dot{\Lambda}\dot{\overline{x}}^\mu - {}_1\Lambda\ddot{\overline{x}}^\mu + \frac{\partial L_s}{\partial\overline{x}_\mu} = 0 \quad . \tag{3-9}$$

For ${}_1\Lambda$, by multiplying $\dot{\overline{x}}_\mu$, we get

$${}_1\dot{\Lambda} = -\dot{L}_s - \frac{d}{ds}(\dot{\overline{x}}_\mu\frac{\partial L_s}{\partial\dot{\overline{x}}_\mu}) + \ddot{\overline{x}}_\mu\frac{\partial L_s}{\partial\dot{\overline{x}}_\mu} + \dot{\overline{x}}_\mu\frac{\partial L_s}{\partial\overline{x}_\mu} = -\frac{d}{ds}(\dot{\overline{x}}_\mu\frac{\partial L_s}{\partial\dot{\overline{x}}_\mu}) \quad . \tag{3-10}$$

Therefore,

$${}_1\Lambda = -\frac{\partial L_s}{\partial\dot{\overline{x}}_\lambda}\dot{\overline{x}}_\lambda + C \quad , \tag{3-11}$$

so that the Euler-Lagrange equation of motion becomes

$$\frac{d}{ds}\left[\left(-L_s - C + \frac{\partial L_s}{\partial \dot{\bar{x}}_\lambda}\dot{\bar{x}}_\lambda\right)\dot{\bar{x}}^\mu - \frac{\partial L_s}{\partial \dot{\bar{x}}_\mu}\right] + \frac{\partial L_s}{\partial \bar{x}_\mu} = 0 \quad . \tag{3-12}$$

Brehme has shown[7] that the exact solution of the least action from L_t of Eq.(3-2) requests $C = 0$ in Eqs.(3-11) and (3-12). There is a very subtle point in physical mathematics. If the original L_s is replaced with $L_s' = L_s + C$, then, Brehme's analysis has shown that Eq.(3-12) is a consistent equation for this case. Therefore, mathematically, Eqs.(3-12), (3-8) and (3-7) has no inconsistency, so that C is undetermined from these equations alone. Physically, however, C is an absolute Lorentz scalar having the dimension of energy, which should be represented solely by L_s, $\bar{x}^\lambda$ and $\dot{\bar{x}}^\lambda$ in not very unusual form. The only possible representation that we can conclude is

$$C = 0 \quad . \tag{3-13}$$

Therefore, C = 0 is not a mathematical conclusion but a physical conclusion, supported perfectly by the Brehme's analysis for L_t .

Now, it is well-known that correct kinematical equation of Eq.(3-12) is obtainable by neglecting the last term of Eq.(3-4). This is because, if we assume that $d[\delta\bar{x}(s)^\mu]/ds$ is so small that each short segment of the route, ds/c or ds'/c , can be regarded as a straight line, then, ds/c and ds'/c have no difference within the first order in $\delta\bar{x}^\mu$, because they are the two geodesic lines just mutually displaced. Therefore, in this condition, $ds' \to ds$ in the final variation, so that this variational calculation should afford at least necessary conditions. In this case, we have[5]

$$\bar{L}_s' = L_s + \frac{{}_1\Lambda}{2}(\dot{\bar{x}}^\mu\dot{\bar{x}}_\mu - 1) \tag{3-14}$$

$$\delta' I = \int_{(1)}^{(2)} \delta\bar{L}_s' \frac{ds}{c} = \int_{(1)}^{(2)}\left\{\frac{d}{ds}\left(\frac{\partial L_s}{\partial \dot{\bar{x}}^\mu}\delta\bar{x}_\mu\right) + \left[\frac{\partial L_s'}{\partial \bar{x}_\mu} - \frac{d}{ds}\left(\frac{\partial \bar{L}_s'}{\partial \dot{\bar{x}}_\mu}\right)\right]\delta\bar{x}_\mu\right\}\frac{ds}{c} \tag{3-15}$$

$$\frac{\partial \bar{L}_s'}{\partial \bar{x}_\mu} - \frac{d}{ds}\left(\frac{\partial \bar{L}_s'}{\partial \dot{\bar{x}}_\mu}\right) = 0 \tag{3-16}$$

$$\frac{d}{ds}\left[-{}_1\Lambda'\dot{\bar{x}}^\mu - \frac{\partial L_s}{\partial \dot{\bar{x}}^\mu}\right] + \frac{\partial L_s}{\partial \bar{x}_\mu} = 0 \quad , \tag{3-17}$$

so that we get[5]

$${}_1\Lambda' = L_s - \frac{\partial L_s}{\partial \dot{\bar{x}}_\lambda}\dot{\bar{x}}_\lambda + C \quad . \tag{3-18}$$

Here, again we have the delicate problem of the integral constant C . In this case, different from Eq.(3-8), the L_s in Eq.(3-16) seems to accept any additional constant, without spoiling the equation. This difference, however, is not essential, because we can get the correct equation, if we put $C = 0$ in Eq.(3-18) from the similar physical considerations. In other words, the two variational procedures give essentially the same correct equation of motion, if we put the integral constant $C = 0$. We must be careful in this case, because, different from Eq. (3-11), if L_s has a constant term, it will be dropped in Eqs.(3-16) and (3-17), so that the integral constant C in Eq.(3-18) is an additional constant than this intrinsic constant term of L_s. Since the second method is simpler, we shall employ this form of variation extensively in this paper, so that the superscript, " ' ", will be omitted hereafter.

Now, in Eq.(3-14), if we use $\bar{L}'_s$ as the original L_s in such a way as

$$\bar{L}_s = L_s + \frac{L_s - \frac{\partial L_s}{\partial \dot{\bar{x}}_\lambda}\dot{\bar{x}}_\lambda}{2}(\dot{\bar{x}}^\mu \dot{\bar{x}}_\mu - 1) + \frac{{}_1\Lambda}{2}(\dot{\bar{x}}^\mu \dot{\bar{x}}_\mu - 1) \quad , \tag{3-19}$$

we get easily

$${}_1\Lambda = 0 \quad , \tag{3-20}$$

so that the Lagrangian multiplier term becomes a dummy term, while keeping the variation still free. When the original $L_s = -{}^\circ mc^2$, we get the new L_s as

$$L_s = -\frac{{}^\circ mc^2}{2}(\dot{\bar{x}}^\mu \dot{\bar{x}}_\mu + 1) \quad . \tag{3-21}$$

We regard that this modified expression is important for the covariant Lagrangian analyses. In addition to the essential reduction of the effect of the Lagrangian multiplier term, Eq.(3-21) allows to have a simple physical understanding that the mechanical momentum, P^μ, can be derived directly as

$$P^\mu = -\frac{\partial L_s}{\partial \dot{\bar{x}}^\mu} = {}^\circ mc^2 \dot{\bar{x}}^\mu \quad , \tag{3-22}$$

which obviously originates from the fact of Eq.(3-20).

In the process of the investigation for the most appropriate Lagrangian to VR, Eq.(3-22) had to be used as a guide line for assuring the right track of the trial. Therefore, we propose that the form of L_s of Eq.(3-21) is essentially important for the research as well as for the final frame. It is noted that, because of Eq.(3-16), $-{}^\circ mc^2/2$ in Eq.(3-21) looks to have no meaning at the initial stage of the anlysis, however, as have been explained in detail, it had a serious meaning in Eq. (3-18), but, adopting the present frame, again, it becomes not important, because ${}_1\Lambda'$ of Eq.(3-18), as a whole, becomes zero, at least for this constant. It is further noted that the situation is more complicated in the actual analysis of VR, but, according to the physical reasons explained, we put mostly the integral constant as zero and look for whether the solution is in agreement with the experiences or not. If the result is in agreement with the experiences, we shall not argue further about its mathematical reasoning.

By accepting these frames, now, we have justified to make most of the variation free, regardless of whether constraints are there or not. If the constraints are present, we have to add the Lagrangian multiplier terms, but they might be dummy terms, or, at least, we look for the Lagrangian, L_s , for which the effect of the Lagrangian multiplier terms may become minimum, under the free variation procedure.

We may add that, when the original $L_s = -{}^{\circ}mc^2 - qA^{\mu}\dot{x}_{\mu}$, the new Lagrangian, L_s , becomes

$$L_s = -\frac{{}^{\circ}mc^2}{2}(\dot{x}^{\mu}\dot{x}_{\mu} + 1) - qA^{\mu}\dot{x}_{\mu} \quad , \tag{3-23}$$

which gives

$${}_1\Lambda' = \frac{{}^{\circ}mc^2}{2}(\dot{x}^{\mu}\dot{x}_{\mu} - 1) = 0 \quad . \tag{3-24}$$

It is noted that, by accepting this principle, we have to distinguish clearly between the algebraical expressions and their constrained values. Even if the constrained value is zero, if the zero is in the first order, it can have a great significance in its variation. This principle also requests to disregard the distinction between the terms in the original Lagrangian and the terms coming from the Lagrangian multiplier terms. We have to remark that no one knows the original Lagrangian of VR. We are in a position to look for the best Lagrangian for our purpose, and, in order to make the analysis simpler, we have set up the explained prescription of the mathematical procedure. According to the principle of the new frame in physics, the analytical continuation to the established classical physics should be strictly maintained, but, besides this, we don't know the results, because they are in an unknown field where no well-established investigation has been present. It will be mentioned in advance, that, actually, we have obtained our best Lagrangian, $\bar{L}_s$, at the analytical extension of the Lagrangian, L_s , of Eq.(3-23).

Let us start the explanation of the proposed Lagrangian of VR. We focus our attention to an invariant part of the electric charge

$${}^{\circ}\rho_{(\alpha)}d^{\circ}V_{(\alpha)} \tag{3-25}$$

with its four coordinates $x^{\mu}_{(\alpha)}$, where (α) indicates a specific portion of the charge[†5)].

The original action integral that we propose is

$$I = \int_{\text{World Point (1)}}^{\text{World Point (2)}} L'_s \frac{ds}{c} = \int_{(1)}^{(2)}\iiint \left(-\frac{j^\mu b_\mu}{4c} - \frac{j^\mu A_\mu}{c}\right) d^\circ V \frac{ds}{c}$$
$$= \int_{(1)}^{(2)}\iiint \left(-\frac{c\dot{x}^\mu q_\mu}{2} - \frac{j^\mu A_\mu}{c}\right) d^\circ V \frac{ds}{c} \qquad (3\text{-}26)$$

in which j^μ and A^μ are the four currents and the externally given four potentials, respectively, so that the last term indicates the well-known mutual interaction between VR and the external electromagnetic fields. Here, L'_s indicates the effective parts of L_s without the additive constant of $-{}^\circ mc^2/2$ of Eq.(3-23) and the Lagrangian multiplier terms. We shall show soon that the orbital self-energy part of Eq.(3-26) is identical to $-({}^\circ mc^2/2)\dot{x}^\mu\dot{x}_\mu$ of Eq.(3-23). Since $j^\mu d^\circ V$ can be understood as

$$j^\mu d^\circ V = {}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)} c\dot{x}^\mu_{(\alpha)} \quad , \quad \left(\dot{x}^\mu_{(\alpha)} = \frac{dx^\mu_{(\alpha)}}{ds}\right) \qquad (3\text{-}27)$$

the integral can be rewritten as

$$I = \int_{(1)}^{(2)} \sum_{(\alpha)} \left[-{}^\circ\rho_{(\alpha)}\dot{x}^\mu_{(\alpha)}(s)\left\{\frac{1}{4} b_\mu[x^\lambda_{(\alpha)}(s),\ \dot{x}^\lambda_{(\alpha)}(s); x^\rho_{(\alpha)}(s)] + A_\mu(x^\rho_{(\alpha)}(s))\right\}\right]$$
$$d^\circ V_{(\alpha)}\frac{ds}{c} = \int_{(1)}^{(2)} L'_s \frac{ds}{c} \quad . \qquad (3\text{-}28)$$

Here, $x^\mu_{(\alpha)}(s)$ and $\dot{x}^\mu_{(\alpha)}(s)$ represent the trajectory of ${}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)}$, and $b_\mu[x^\lambda_{(\beta)}(s),\ \dot{x}^\lambda_{(\beta)}(s);\ x^\rho_{(\alpha)}(s)]$ indicates that b_μ at $x^\rho_{(\alpha)}(s)$ is a functional of the whole functions of $x^\lambda_{(\beta)}(s)$'s and $\dot{x}^\lambda_{(\beta)}(s)$'s. It is to be noted that, by assumption, the velocity of ${}^\circ\rho\, d^\circ V$ is always c , or, the light velocity. Therefore,

$$\dot{x}^\mu_{(\alpha)} = \left\{\gamma_V\left(1 + \frac{{}^\circ v^{(\alpha)}_{\parallel} V}{c^2}\right),\ \frac{{}^\circ \mathbf{v}^{(\alpha)}_{\perp}}{c} + \gamma_V\left(\frac{{}^\circ \mathbf{v}^{(\alpha)}_{\parallel}}{c} + \frac{\mathbf{V}}{c}\right)\right\} \quad , \quad \dot{x}^\mu_{(\alpha)}\dot{x}^{(\alpha)}_\mu = 0. \qquad (3\text{-}29)$$

By using Eq.(3-28), the invariant four space integration form of Eq.(3-26) becomes implicit, but, this form is necessary in order to trace $\dot{x}^\mu_{(\alpha)}(s)$ in the integral.

By assumption, at points (1) and (2), $x^\mu_{(\alpha)}$'s and $\dot{x}^\mu_{(\alpha)}$'s are all fixed. Assuming that the solution $x^\mu_{(\alpha)}$'s for the least action integral have been obtained, the general expression of $x^\mu_{(\alpha)}(s)$ after the variation is

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† Here we refer the work by Barut[6)]. Although it was not possible to understand the details of his pioneering analyses, the notation (α) comes from this earlier work.

$$x'^{\mu}_{(\alpha)}(s') = x^{\mu}_{(\alpha)}(s) + \varepsilon^{\mu\nu}(s)x^{(\alpha)}_{\nu}(s) + \delta x^{\mu}(s) \quad , \tag{3-30}$$

where $\varepsilon^{\mu\nu}(s)$ defines general Lorentz transformation including pure rotation. Putting the coordinates of the center of VR as $\bar{x}^{\mu}$, we get

$$x^{\mu}_{(\alpha)}(s) = \bar{x}^{\mu}(s) + \Delta x^{\mu}_{(\alpha)}(s) \tag{3-31}$$

so that

$$x'^{\mu}_{(\alpha)}(s') = \bar{x}^{\mu}(s) + \delta\bar{x}^{\mu}(s) + \Delta x^{\mu}_{(\alpha)}(s) + \delta\Delta x^{\mu}_{(\alpha)}(s) \tag{3-32}$$

$$\delta\bar{x}^{\mu}(s) = \delta x^{\mu}(s) + \varepsilon^{\mu\nu}(s)\bar{x}_{\nu}(s) \tag{3-33}$$

$$\delta\Delta x^{\mu}_{(\alpha)}(s) = \varepsilon^{\mu\nu}(s)\Delta x^{(\alpha)}_{\nu}(s) \qquad (\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}) \quad , \tag{3-34}$$

where $\Delta x^{\mu}_{(\alpha)}(s)$'s are the radial four vectors of VR. $\delta\bar{x}^{\mu}(s)$ and $\varepsilon^{\mu\nu}(s)$ are the proposed two kinds of variations. Since $\varepsilon^{\mu\nu}(s)$ includes Lorentz boosts[8] which are related to $\delta\dot{\bar{x}}^{\mu}(s)$, $\varepsilon^{\mu\nu}(s)$ and $\delta\bar{x}^{\mu}(s)$ are not completely independent.

Since we have

$$*\bar{x}^{\mu} = \{\gamma_V c^{\circ}t,\ \gamma_V \mathbf{V}^{\circ}t\} \quad , \qquad \Delta x^{\mu}_{(\alpha)} = \{\gamma_V \frac{{}^{\circ}\mathbf{r}^{(\alpha)}_{\parallel} V}{c} ,\ {}^{\circ}\mathbf{r}^{(\alpha)}_{\perp} + \gamma_V {}^{\circ}\mathbf{r}^{(\alpha)}_{\parallel}\} \quad , \tag{3-35}$$

$$\dot{\bar{x}}^{\mu} = \{\gamma_V \ ,\ \gamma_V \frac{\mathbf{V}}{c}\} , \qquad \Delta\dot{x}^{\mu}_{(\alpha)} = \{\gamma_V \frac{{}^{\circ}\mathbf{v}^{(\alpha)}_{\parallel} V}{c^2} ,\ \frac{{}^{\circ}\mathbf{v}^{(\alpha)}_{\perp}}{c} + \gamma_V \frac{{}^{\circ}\dot{\mathbf{v}}^{(\alpha)}_{\parallel}}{c}\} \quad , \tag{3-36}$$

and each point (α) moves with the velocity of light, presence of the following five constraints is obvious for the variation of $\delta\bar{x}^{\mu}(s)$ and $\varepsilon^{\mu\nu}(s)$.

$$\left.\begin{aligned} &(1) \quad \dot{\bar{x}}^{\mu}\dot{\bar{x}}_{\mu} = 1 \ , \qquad (2) \quad \dot{\bar{x}}^{\mu}\Delta x^{(\alpha)}_{\mu} = 0 \ , \qquad (3) \quad \dot{\bar{x}}^{\mu}\Delta\dot{x}^{(\alpha)}_{\mu} = 0 \ , \\ &(4) \quad \Delta x^{\mu}_{(\alpha)}\Delta x^{(\beta)}_{\mu} = -{}^{\circ}R^2\cos{}^{\circ}\phi_{(\alpha\beta)} \ , \qquad (5) \quad \Delta\dot{x}^{\mu}_{(\alpha)}\Delta\dot{x}^{(\alpha)}_{\mu} = -1 \ , \end{aligned}\right\} \tag{3-37}$$

with

$$\ddot{\bar{x}}^{\mu}\dot{\bar{x}}_{\mu} = 0 \ , \quad \ddot{\bar{x}}^{\mu}\Delta x^{(\alpha)}_{\mu} = -\dot{\bar{x}}^{\mu}\Delta\dot{x}^{(\alpha)}_{\mu} = 0 \ , \quad \ddot{\bar{x}}^{\mu}\Delta\dot{x}^{(\alpha)}_{\mu} = -\dot{\bar{x}}^{\mu}\Delta\ddot{x}^{(\alpha)}_{\mu} \neq 0 \ , \tag{3-38}$$

$$\left.\begin{aligned} &\Delta x^{\mu}_{(\alpha)}\Delta\dot{x}^{(\alpha)}_{\mu} = 0 \ , \qquad \Delta\dot{x}^{\mu}_{(\alpha)}\Delta x^{(\beta)}_{\mu} + \Delta x^{\mu}_{(\alpha)}\Delta\dot{x}^{(\beta)}_{\mu} = 0 \ , \\ &\Delta x^{\mu}_{(\alpha)}\Delta\ddot{x}^{(\alpha)}_{\mu} = -\Delta\dot{x}^{\mu}_{(\alpha)}\Delta\dot{x}^{(\alpha)}_{\mu} = 1 \end{aligned}\right\} \tag{3-39}$$

Constraints (2) - (5) are special for VR, the presence of which can easily be checked in °K by using Eqs.(3-35) and (3-36). The variation of constraint (2) gives

$$0 = \delta\dot{\bar{x}}^{\mu}\Delta x^{(\alpha)}_{\mu} + \dot{\bar{x}}^{\mu}\varepsilon_{\mu\nu}\Delta x^{\nu}_{(\alpha)} = (\delta\dot{\bar{x}}^{\mu} + \varepsilon^{\nu\mu}\dot{\bar{x}}_{\nu})\Delta x^{(\alpha)}_{\mu} \ . \tag{3-40}$$

Since $\Delta x^{(\alpha)}_{\mu}$ has all the necessary freedom, Eq.(3-40) requests

$$\delta\dot{\bar{x}}^{\mu} = -\varepsilon^{\nu\mu}\dot{\bar{x}}_{\nu} = \varepsilon^{\mu\nu}\dot{\bar{x}}_{\nu} \quad , \tag{3-41}$$

indicating the presence of strong correlation[8] between the four dimensional rotational variation, $\varepsilon^{\mu\nu}$, and the Lorentz boosts, $\delta\dot{\bar{x}}^{\mu}$. Constraint (3) gives the same relation. Constraint (3) and (5) are relevant to the situation that each point (α) moves with the velocity of light, i. e.,

$$\dot{x}^{\mu}_{(\alpha)}\dot{x}^{(\alpha)}_{\mu} = \dot{\bar{x}}^{\mu}\dot{\bar{x}}_{\mu} + 2\dot{\bar{x}}^{\mu}\Delta\dot{x}^{(\alpha)}_{\mu} + \Delta\dot{x}^{\mu}_{(\alpha)}\Delta\dot{x}^{(\alpha)}_{\mu} = 0 \tag{3-42}$$

Constraint (4) indicates the mutual angle ${}^{\circ}\phi_{(\alpha\beta)}$ between (α) and (β) unchanged. In the following calculations, we are careful to place excessive constraints than necessary, because, since our variation calculation is very delicate, we may get erroneous conclusion easily, if logically complicated and confused. It is noted that, except constraints (1), (2) and (3), most of other constraints will be not used as the Lagrangian multiplier term, because our variation of $\delta\dot{\bar{x}}^{\mu}(s)$ and $\varepsilon^{\mu\nu}(s)$ will not violate these constraints.

Let us explain the mathematical implication of the self term of our Lagrangian shortly. Since

$$\{j^{\mu}_{(\alpha)}\} = {}^{\circ}\rho c[\{\dot{\bar{x}}^{\mu}\} + \{\Delta\dot{x}^{\mu}_{(\alpha)}\}] \tag{3-43}$$

$$\{q^{\mu}_{(\alpha)}\} = \frac{{}^{\circ}\rho}{(-e)}\left(\frac{({}^{\circ}U_E + {}^{\circ}U_M)}{c}\{\dot{\bar{x}}^{\mu}\} + \frac{{}^{\circ}U_M}{c}\{\Delta\dot{x}^{\mu}_{(\alpha)}\}\right) \tag{3-44}$$

we have

$$\frac{c\dot{x}^{\mu}_{(\alpha)}q^{(\alpha)}_{\mu}}{2} = \frac{1}{2}\frac{{}^{\circ}\rho}{(-e)}[{}^{\circ}mc^2\dot{\bar{x}}^{\mu}\dot{\bar{x}}_{\mu} + \frac{{}^{\circ}mc^2}{(-g)}\Delta\dot{x}^{\mu}_{(\alpha)}\Delta\dot{x}^{(\alpha)}_{\mu} + ({}^{\circ}mc^2 + \frac{{}^{\circ}mc^2}{(-g)})\dot{\bar{x}}^{\mu}\Delta\dot{x}^{(\alpha)}_{\mu}] . \tag{3-45}$$

Therefore, if integrated, we have

$$\iiint\left(-\frac{c\dot{x}^{\mu}_{(\alpha)}q^{(\alpha)}_{\mu}}{2}\right)d^{\circ}V = -\frac{1}{2}{}^{\circ}mc^2\dot{\bar{x}}^{\mu}\dot{\bar{x}}_{\mu} - \sum_{(\alpha)}\frac{1}{2}\frac{{}^{\circ}\rho d^{\circ}V}{(-e)}\frac{{}^{\circ}mc^2}{(-g)}\Delta\dot{x}^{\mu}_{(\alpha)}\Delta\dot{x}^{(\alpha)}_{\mu} . \tag{3-46}$$

It is noted that the summation $\sum_{(\alpha)}$ for the term with odd number of $\Delta x^{(\alpha)}_{\mu}$ or $\Delta\dot{x}^{(\alpha)}_{\mu}$ vanishes, since it includes $\sin{}^{\circ}\phi$ or $\cos{}^{\circ}\phi$ in odd products, where ${}^{\circ}\phi$ is an Euler's angle arround the ring. Mainly from this reason, the last cross term of Eq.(3-45) has no effect in the variation procedure, as we have indicated in the second equation of (3-46), being an interesting finding for the unnecessity of the cross term in this covariant Lagrangian. Eq.(3-46) shows clearly that our Lagrangian has the structure of Eq.(3-21) in its main term. In order to match the requirement from Eq.(3-21), we replace $\dot{\bar{x}}^{\mu}\dot{\bar{x}}_{\mu}$ and $\Delta\dot{x}^{\mu}_{(\alpha)}\Delta\dot{x}^{(\alpha)}_{\mu}$ of Eq.(3-46) with $(\dot{\bar{x}}^{\mu}\dot{\bar{x}}_{\mu} + 1)$ and $(\Delta\dot{\bar{x}}^{\mu}_{(\alpha)}\Delta\dot{\bar{x}}^{(\alpha)}_{\mu} + 1)$ respectively, from now. Then, from Eq.(3-46), our L_s for the orbital motion becomes identical to Eq.(3-23), giving the value of $-{}^\circ mc^2$, which has been well established for the Lagrangian of the orbital motion.

§4. Lorentz invariant derivation of the kinematical equations of the electron by the new frame

Since we have Eq.(3-41), $\delta\dot{\bar{x}}^\mu$'s are dependent on $\varepsilon^{\mu\nu}(s)$. Furthermore, the equation which determines the orbital motion of VR is several orders of magnitude larger than that for the precessional motion of the spin of VR. Therefore, we shall solve the least action problem first for the precessional motion, i.e., for $\varepsilon^{\mu\nu}$'s, accurately with taking into account Eq.(3-41) directly. Then, after that, by using this solution, the Lagrangian multiplier method will be adopted to obtain the equation for the orbital motion in the necessary accuracy. As shown in §5, it has turned out that the Lagrangian $\bar{L}_s$, thus obtained, is also effective for the precessional motion.

Now, in Eqs.(3-32), (3-33) and (3-34), we have six $\varepsilon^{\mu\nu}(s) = -\varepsilon^{\mu\nu}(s)$, which can be all independent at least locally[8]. From Eq.(3-41), however, $\delta\dot{\bar{x}}^\mu$ is a function of $\varepsilon^{\mu\nu}$, so that $\delta\bar{x}^\mu$ must be a functional of $\varepsilon^{\mu\nu}(s)$. Since VR starts at world point (1) and arrives at world point (2) without any change, $\varepsilon^{\mu\nu}(s)$ can not be completely independent. We have developed the mathematically formal variation as

$$\begin{aligned}
\delta I &= \int_{(1)}^{(2)} \delta L_s \frac{ds}{c} = \int_{(1)}^{(2)} \left[\frac{\partial L_s}{\partial \dot{\bar{x}}_\mu}\delta\dot{\bar{x}}_\mu + \frac{\partial L_s}{\partial \bar{x}_\mu}\delta \bar{x}_\mu + \sum_{(\alpha)}\left\{ \frac{\partial L_s}{\partial(\Delta \dot{x}^{(\alpha)}_\mu)}\delta(\Delta\dot{x}^{(\alpha)}_\mu) + \frac{\partial L_s}{\partial(\Delta x^{(\alpha)}_\mu)}\delta(\Delta x^{(\alpha)}_\mu)\right\}\right] \\
&\cdot \frac{ds}{c} = \int_{(1)}^{(2)}\left(\frac{d}{ds}\left[\left(\int^s \frac{\partial L_s}{\partial \bar{x}_\mu} ds\right)\delta\bar{x}_\mu\right] + \frac{d}{ds}\left[\sum_{(\alpha)}\left\{\frac{\partial L_s}{\partial(\Delta\dot{x}^{(\alpha)}_\mu)}\delta\Delta x^{(\alpha)}_\mu\right\}\right] + \left\{\frac{\partial L_s}{\partial\dot{\bar{x}}_\mu} - \int^s \frac{\partial L_s}{\partial \bar{x}_\mu} ds\right\}\varepsilon_{\mu\nu}\dot{\bar{x}}^\nu \right. + \\
&\left. + \sum_{(\alpha)}\left[\frac{\partial L_s}{\partial(\Delta x^{(\alpha)}_\mu)} - \frac{d}{ds}\left\{\frac{\partial L_s}{\partial(\Delta\dot{x}^{(\alpha)}_\mu)}\right\}\right]\delta(\Delta x^{(\alpha)}_\mu)\right)\frac{ds}{c} \\
&= \int_{(1)}^{(2)}\left(\left\{\frac{\partial L_s}{\partial\dot{\bar{x}}_\mu} - \int^s \frac{\partial L_s}{\partial \bar{x}_\mu} ds\right\}\dot{\bar{x}}^\nu + \sum_{(\alpha)}\left[\frac{\partial L_s}{\partial(\Delta x^{(\alpha)}_\mu)} - \frac{d}{ds}\left\{\frac{\partial L_s}{\partial(\Delta\dot{x}^{(\alpha)}_\mu)}\right\}\right]\Delta x^\nu_{(\alpha)}\right)\varepsilon_{\mu\nu}\frac{ds}{c} \, . \qquad (4\text{-}1)
\end{aligned}$$

and, as a refined L_s , we have

$$\begin{aligned}
L_s = \sum_{(\alpha)}\left(\frac{1}{2}\frac{{}^\circ\rho_{(\alpha)} d{}^\circ V_{(\alpha)}}{e}\left[{}^\circ mc^2(\dot{\bar{x}}^\mu\dot{\bar{x}}_\mu + 1) + \frac{{}^\circ mc^2}{(-g)}(\Delta\dot{x}^\mu_{(\alpha)}\Delta\dot{x}^{(\alpha)}_\mu + 1)\right]\right. \\
\left. - {}^\circ\rho_{(\alpha)} d{}^\circ V_{(\alpha)}(\dot{\bar{x}}^\mu + \Delta\dot{x}^\mu_{(\alpha)})A^{(\alpha)}_\mu\right) \, . \qquad (4\text{-}2)
\end{aligned}$$

Since

$$\frac{\partial L_s}{\partial(\Delta\dot{x}^{(\alpha)}_\mu)} = \frac{{}^\circ\rho_{(\alpha)} d{}^\circ V_{(\alpha)}}{e}\frac{{}^\circ mc^2}{(-g)}\Delta\dot{x}^\mu_{(\alpha)} - {}^\circ\rho_{(\alpha)} d{}^\circ V_{(\alpha)} A^\mu_{(\alpha)} \, , \qquad (4\text{-}3)$$

$$\frac{\partial L_s}{\partial(\Delta x_\mu^{(\alpha)})} = -{}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)} (\dot{\bar{x}}^\lambda + \Delta\dot{x}^\lambda_{(\alpha)}) A^{(\alpha)\mu}_{\lambda,} \quad , \tag{4-4}$$

$$\int^s \frac{\partial L_s}{\partial \bar{x}_\mu} ds = \int^s \left[-{}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)} (\dot{\bar{x}}^\lambda + \Delta\dot{x}^\lambda_{(\alpha)}) A^{(\alpha)\mu}_{\lambda,} \right] ds \quad , \tag{4-5}$$

$$\frac{\partial L_s}{\partial \dot{\bar{x}}_\mu} = -{}^\circ mc^2 \dot{\bar{x}}^\mu - \sum_{(\alpha)} {}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)} A^\mu_{(\alpha)} \quad , \tag{4-6}$$

we get from Eq.(4-1)

$$\frac{d}{ds}\left(\sum_{(\alpha)} \left\{ - \frac{{}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)}}{e} \frac{{}^\circ mc^2}{(-g)} \Delta\dot{x}^{[\mu}_{(\alpha)} \Delta x^{\nu]}_{(\alpha)} \right\} \right)$$

$$= \left\{ + {}^\circ mc^2 \dot{\bar{x}}^{[\mu} + \sum_{(\alpha)} {}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)} A^{[\mu}_{(\alpha)} + \int^s \left[-{}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)} (\dot{\bar{x}}^\lambda + \Delta\dot{x}^\lambda_{(\alpha)}) A^{(\alpha)\,[\mu}_{\lambda,} \right] ds \right\} \dot{\bar{x}}^{\nu]}$$

$$+ \sum_{(\alpha)} {}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)} (A^{(\alpha)\,[\mu}_{\lambda,} - A^{[\mu,}_{(\alpha)\ \lambda}) (\dot{\bar{x}}^\lambda + \Delta\dot{x}^\lambda_{(\alpha)}) \Delta x^{\nu]}_{(\alpha)} \quad . \tag{4-7}$$

Here,

$$\left. \begin{aligned} & A^{\nu,\mu}_{(\alpha)} - A^{\mu,\nu}_{(\alpha)} = F^{\mu\nu}_{(\alpha)} \ , \qquad A^{\nu,\mu}_{(\alpha)} = \frac{\partial A^\nu_{(\alpha)}}{\partial x^{(\alpha)}_\mu} \ , \\ & \Delta\dot{x}^{[\mu}_{(\alpha)} \Delta x^{\nu]}_{(\alpha)} = \Delta\dot{x}^\mu_{(\alpha)} \Delta x^\nu_{(\alpha)} - \Delta\dot{x}^\nu_{(\alpha)} \Delta x^\mu_{(\alpha)} \quad . \end{aligned} \right\} \tag{4-8}$$

It is noted that $\int^s (\partial L_s/\partial \bar{x}_\mu) ds$ has a freedom of a constant term, which will be left undetermined, for the sake of convenience.

Now, let us introduce the definitions of the angular momentum and magnetic moment of VR briefly. According to the fundamental assumption of VR[1], the original definition of the angular momentum of VR is

$$S^{\mu\nu} = \sum_{(\alpha)} {}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)} \frac{{}^\circ mc^2}{c(ge)} \Delta\dot{x}^{[\mu}_{(\alpha)} \Delta x^{\nu]}_{(\alpha)} \quad . \tag{4-9}$$

From constraints (2) and (3) of Eqs.(3-37), it will be evident that

$$S^{\mu\nu} \dot{\bar{x}}_\nu = 0 \ , \qquad \dot{S}^{\mu\nu} \dot{\bar{x}}_\nu = - S^{\mu\nu} \ddot{\bar{x}}_\nu \quad . \tag{4-10}$$

For the magnetic moment, the definition is definite by electromagnetism, as

$$M^{\mu\nu} = \sum_{(\alpha)} {}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)} \frac{\Delta\dot{x}^{[\mu}_{(\alpha)} \Delta x^{\nu]}_{(\alpha)}}{2} \quad . \tag{4-11}$$

Therefore, we get

$$\frac{M^{\mu\nu}}{S^{\mu\nu}} = \frac{ge}{2^\circ mc} = -\frac{(1 + \frac{\alpha}{2\pi})e}{^\circ mc} \quad . \tag{4-12}$$

We should note that

$$M^{12} = -M_z \quad , \qquad S^{12} = -S_z \quad . \tag{4-13}$$

and this definition is in conformity with the relation **B** = **H** + **M** and **E** = **D** - **P** [9]. Here, we have to note that there are another non-antisymmetric tensors $*M^{\mu\nu}$ and $*S^{\mu\nu}$ difined by

$$*M^{\mu\nu} = \sum_{(\alpha)} {}^\circ\rho_{(\alpha)}\, d^\circ V_{(\alpha)}\, \Delta\dot{x}^{\mu}_{(\alpha)} \Delta x^{\nu}_{(\alpha)} = \frac{ge}{2^\circ mc} * S^{\mu\nu} \quad . \tag{4-14}$$

It is easily shown by direct calculation that, when VR is at rest, $*M^{\mu\nu} = M^{\mu\nu}$, but when it is precessing, $*M^{\mu\nu} \neq M^{\mu\nu}$ in the order of $({}^\circ R^\circ \omega / c)$ $\sin^\circ\theta$, or, $10^{-(9\sim 11)}$, and this small difference becomes the principal components in a few cases, such as $c\dot{S}^{\mu\nu}$ or $M^{\lambda\nu}\ddot{x}_\lambda$. For instance,

$$M^{\lambda\nu}\ddot{x}_\lambda = \frac{1}{2} * M^{\lambda\nu}\ddot{x}_\lambda \quad , \qquad -*M^{\nu\lambda}\ddot{x}_\lambda = 0 \quad . \tag{4-15}$$

In the new frame of physics, when quantized, since the c-number expression of $M^{\mu\nu}$ and $*M^{\mu\nu}$ or $S^{\mu\nu}$ and $*S^{\mu\nu}$ should be regarded as a single operator $M^{\mu\nu}$ or $S^{\mu\nu}$, this difference creates a delicate structure for the quantization. From the procedure of the least action principle of Eq.(4-7), $[\mu\nu]$ will appear, but not $[\lambda\mu]$ nor $[\lambda\nu]$. This will be one of the essential defect of the least action principle, which has to be improved for the quantization. Of course the quantization should be made so as to describe the experimental results correctly. Fortunately, however, the quantization is almost unique as we see soon.

Now, returning to Eq.(4-7), the left side is identical to $c\dot{S}^{\mu\nu}$, and the last term gives the torque given by the external fields. The first term of the right side gives a very small correction. which will be shown soon. A possible contribution of the external vector potential **A** to the $S^{\mu\nu}$ will be discussed in §6.

Since the analysis of Eq.(4-7) requires the knowledge of the orbital motion, we derive the kinematical equation of the orbital motion next. For the orbital motion, we utilize the Lagrangian multiplier method. Then, we have

$$\bar{L}_s = L_s + \frac{{}_1\Lambda}{2}(\dot{\bar{x}}^{\mu}\dot{\bar{x}}_{\mu} - 1) + \sum_{(\alpha)} {}_2\Lambda^{(\alpha)}\dot{\bar{x}}^{\mu}\Delta x^{(\alpha)}_{\mu} + \sum_{(\alpha)} {}_3\Lambda^{(\alpha)}\dot{\bar{x}}^{\mu}\Delta\dot{x}^{(\alpha)}_{\mu} \quad . \tag{4-16}$$

There will be no doubt for ${}_1\Lambda$ term. For ${}_2\Lambda^{(\alpha)}$ and ${}_3\Lambda^{(\alpha)}$, we know from Eq.(3-40), that one of them may be sufficient, but, since the problem is so delicate, we have introduced two conditions initially. Then, we have from the initial equations of Eq.(4-1),

$$-\frac{d}{ds}\left(\frac{\partial \bar{L}_s}{\partial \dot{\bar{x}}_\mu}\right) + \frac{\partial \bar{L}_s}{\partial \bar{x}_\mu} = \frac{d}{ds}(cp^\mu) + \frac{\partial \bar{L}_s}{\partial \bar{x}_\mu} = 0 \quad . \tag{4-17}$$

Here, we have introduced the definition of orbital momentum, p^μ , according to the usual definition. Then, utilizing Eqs.(4-6) and (4-5), we have

$$\frac{d}{ds}\left[({}^\circ mc^2 - {}_1\Lambda)\dot{\bar{x}}^\mu + \sum_{(\alpha)} {}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)} A^\mu_{(\alpha)} - \sum_{(\alpha)}\left({}_2\Lambda^{(\alpha)}\Delta x^\mu_{(\alpha)} + {}_3\Lambda^{(\alpha)}\Delta\dot{x}^\mu_{(\alpha)}\right)\right]$$
$$= \sum_{(\alpha)} {}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)}(\dot{\bar{x}}^\lambda + \Delta\dot{x}^\lambda_{(\alpha)})A^{(\alpha)\mu}_{\lambda,} \quad . \tag{4-18}$$

Therefore, we get

$${}^\circ mc^2\dot{\bar{x}}^\mu + \sum_{(\alpha)} {}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)} A^\mu_{(\alpha)} - \int^s \sum_{(\alpha)} {}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)}(\dot{\bar{x}}^\lambda + \Delta\dot{x}^\lambda_{(\alpha)})A^{(\alpha)\mu}_{\lambda,}ds$$
$$= {}_1\Lambda\dot{\bar{x}}^\mu + \sum_{(\alpha)}\left({}_2\Lambda^{(\alpha)}\Delta x^\mu_{(\alpha)} + {}_3\Lambda^{(\alpha)}\Delta\dot{x}^\mu_{(\alpha)}\right) \quad . \tag{4-19}$$

Replacing Eq.(4-19) in Eq.(4-7), we get

$$\frac{d}{ds}\left(cS^{\mu\nu}\right) = \left\{\sum_{(\alpha)}\left({}_2\Lambda^{(\alpha)}\Delta x^{[\mu}_{(\alpha)} + {}_3\Lambda^{(\alpha)}\Delta\dot{x}^{[\mu}_{(\alpha)}\right)\right\}\dot{\bar{x}}^{\nu]}$$
$$+ \sum_{(\alpha)} {}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)}(A^{(\alpha)[\mu}_{\lambda,} - A^{[\mu,}_{(\alpha)\ \lambda})(\dot{\bar{x}}^\lambda + \Delta\dot{x}^\lambda_{(\alpha)})\Delta x^{\nu]}_{(\alpha)} \quad . \tag{4-20}$$

It is noted that, since the left side of Eq.(4-19) is almost zero in the currently accepted equations of the electron, the terms with ${}_1\Lambda$, ${}_2\Lambda^{(\alpha)}$ and ${}_3\Lambda^{(\alpha)}$ are expected to be very small, such as $10^{-(9\sim 11)}$, as compared with the main term, ${}^\circ mc^2\dot{\bar{x}}^\mu$. This result is anticipated, because, as explained in §3, we have set up our frame in such a direction.

Multiplying $\dot{\bar{x}}_\mu$ to Eq.(4-20), we have

$$\dot{\bar{x}}_\mu c\dot{S}^{\mu\nu} = -\ddot{\bar{x}}_\mu cS^{\mu\nu} = -\left\{\sum_{(\alpha)}\left({}_2\Lambda^{(\alpha)}\Delta x^\nu_{(\alpha)} + {}_3\Lambda^{(\alpha)}\Delta\dot{x}^\nu_{(\alpha)}\right)\right\}$$
$$+ \sum_{(\alpha)} {}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)}\dot{\bar{x}}_\mu(A^{(\alpha)\mu}_{\lambda,} - A^{\mu,}_{(\alpha)\lambda})\Delta\dot{x}^\lambda_{(\alpha)}\Delta x^\nu_{(\alpha)} \quad . \tag{4-21}$$

Therefore, we get

$$c\dot{S}^{\mu\nu} = \left\{\ddot{\bar{x}}_\lambda cS^{\lambda[\mu} + \sum_{(\alpha)} {}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)}\dot{\bar{x}}_\sigma(A^{(\alpha)\sigma}_{\lambda,} - A^{\sigma,}_{(\alpha)\lambda})\Delta\dot{x}^\lambda_{(\alpha)}\Delta x^{[\mu}_{(\alpha)}\right\}\dot{\bar{x}}^{\nu]}$$
$$+ \sum_{(\alpha)} {}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)}(A^{(\alpha)[\mu}_{\lambda,} - A^{[\mu,}_{(\alpha)\ \lambda})(\dot{\bar{x}}^\lambda + \Delta\dot{x}^\lambda_{(\alpha)})\Delta x^{\nu]}_{(\alpha)} \quad . \tag{4-22}$$

From Eq.(4-21), we have

$$\sum_{(\alpha)} ({}_2\Lambda^{(\alpha)}\Delta x^{\nu}_{(\alpha)} + {}_3\Lambda^{(\alpha)}\Delta\dot{x}^{\nu}_{(\alpha)}) = \sum_{(\alpha)} {}^{\circ}\rho_{(\alpha)} d^{\circ}V_{(\alpha)} \left\{ \frac{{}^{\circ}mc^2}{ge} \Delta\dot{x}^{[\mu}_{(\alpha)}\Delta x^{\nu]}_{(\alpha)}\ddot{\bar{x}}_{\mu} + \dot{\bar{x}}_{\mu}(A^{(\alpha)\mu}_{\lambda,} - A^{\mu}_{(\alpha)}{}_{,\lambda})\Delta\dot{x}^{\lambda}_{(\alpha)}\Delta x^{\nu}_{(\alpha)} \right\} \quad . \tag{4-23}$$

Therefore, we get

$$\left.\begin{aligned} {}_2\Lambda^{(\alpha)} &= {}^{\circ}\rho_{(\alpha)} d^{\circ}V_{(\alpha)} \left\{ \frac{{}^{\circ}mc^2}{ge}\Delta\dot{x}^{\lambda}_{(\alpha)}\ddot{\bar{x}}_{\lambda} + \dot{\bar{x}}_{\sigma}(A^{(\alpha)\sigma}_{\lambda,} - A^{\sigma}_{(\alpha)}{}_{,\lambda})\Delta\dot{x}^{\lambda}_{(\alpha)} \right\} \\ {}_3\Lambda^{(\alpha)} &= {}^{\circ}\rho_{(\alpha)} d^{\circ}V_{(\alpha)} \frac{{}^{\circ}mc^2}{ge}(-\Delta x^{\mu}_{(\alpha)})\ddot{\bar{x}}_{\mu} = 0 \quad . \end{aligned}\right\} \tag{4-24}$$

Here, we can see the subtle structure of the ${}_3\Lambda^{(\alpha)}$ term. Let us approximate $\overline{A^{(\alpha)\sigma}_{\lambda,} - A^{\sigma}_{(\alpha)}{}_{,\lambda}} = \overline{F^{\sigma}_{(\alpha)\lambda}[\bar{x}^{\rho} + \Delta x^{\rho}_{(\alpha)}]}$ by $F^{\sigma}{}_{\lambda} = F^{\sigma}{}_{\lambda}[\bar{x}^{\rho}]$. Then we have

$$\sum_{(\alpha)} {}_2\Lambda^{(\alpha)}\Delta x^{\nu}_{(\alpha)} + \sum_{(\alpha)} {}_3\Lambda^{(\alpha)}\Delta\dot{x}^{\nu}_{(\alpha)} = \ddot{\bar{x}}_{\lambda} cS^{\lambda\nu} + \dot{\bar{x}}_{\sigma}F^{\sigma}{}_{\lambda}\frac{ge}{2^{\circ}mc}{}^{*}S^{\lambda\nu} \quad . \tag{4-25}$$

Therefore, we get finally

$$\begin{aligned} c\dot{S}^{\mu\nu} &= \left\{ \ddot{\bar{x}}_{\lambda} cS^{\lambda[\mu} + \frac{ge}{2^{\circ}mc}\dot{\bar{x}}_{\sigma}F^{\sigma}{}_{\lambda}{}^{*}S^{\lambda[\mu} \right\}\dot{\bar{x}}^{\nu]} + {}^{*}M^{\lambda[\nu}F^{\mu]}{}_{\lambda} + \dot{\bar{x}}^{\lambda}F^{[\mu}{}_{\lambda,\sigma}Q^{\sigma\nu]} \\ &= \ddot{\bar{x}}_{\lambda} cS^{\lambda[\mu}\dot{\bar{x}}^{\nu]} + {}^{*}M^{\lambda[\nu}\left\{ F^{\mu]}{}_{\lambda} - \dot{\bar{x}}_{\sigma}F^{\sigma}{}_{\lambda}\dot{\bar{x}}^{\mu]} \right\} + \dot{\bar{x}}^{\lambda}F^{[\mu}{}_{\lambda,\sigma}Q^{\sigma\nu]} \quad , \end{aligned} \tag{4-26}$$

where we have expanded the torque term, i.e., the last term of Eq.(4-22), and defined the electric quadrupole moment tensor of VR, $Q^{\sigma\nu}$, as

$$Q^{\sigma\nu} = \sum_{(\alpha)} {}^{\circ}\rho_{(\alpha)} d^{\circ}V_{(\alpha)}\Delta x^{\sigma}_{(\alpha)}\Delta x^{\nu}_{(\alpha)} \quad . \tag{4-27}$$

When we put ${}^{*}S^{\lambda[\mu} = S^{\lambda[\mu}$, ${}^{*}M^{\lambda[\nu} = M^{\lambda[\nu}$, Eq.(4-26) is identical to the BMT equation[10], except the last term. The first term of the last equation of Eq.(4-26) includes the Thomas precession as a main component, but it includes also the higher order torque term with the same magnitude. In this covariant least action analysis, it is not possible to get the Thomas precession term isolately.(If it could, the expression must be quite a complicated tensor transformation[11].) In addition, as shown in the middle equation, this term is almost cancelled by another higher order torque term, to the factor of $(1 + g/2) \sim -1.5 \times 10^{-3}$. These terms are directly related to the spin-orbit coupling, in which we believe that there has been an incomplete understanding. The structure will be analyzed in detail in §5.

Let us clarify the physical meaning of the additional last term. Defining the space coordinate ($\mu = 1,2,3$) by k, ℓ and m, we get in ${}^{\circ}K$, ${}^{\circ}Q^{0\nu} = {}^{\circ}Q^{\mu 0} = 0$, ${}^{\circ}Q^{k\ell} = 0$ when $k \neq \ell$, and ${}^{\circ}\dot{\bar{x}}^{k} = 0$, so that

$$\mu = 0 \ , \quad \text{or} \quad \nu = 0 \ , \qquad {}^{\circ}(\dot{\bar{x}}^{\lambda}F^{[\mu}{}_{\lambda,\sigma}Q^{\sigma\nu]}) = 0 \tag{4-28}$$

$$\mu = 1\ ,\ \nu = 2\ ,\quad {}^\circ F^1{}_{0,\sigma}{}^\circ Q^{\sigma 2} - {}^\circ F^2{}_{0,\sigma}{}^\circ Q^{\sigma 1} = {}^\circ F^1{}_{0,2}{}^\circ Q^{22} - {}^\circ F^2{}_{0,1}{}^\circ Q^{11}$$

$$= {}^\circ Q^{22}\frac{\partial^\circ E_x}{\partial y} - {}^\circ Q^{11}\frac{\partial^\circ E_y}{\partial x} \quad . \qquad (4\text{-}29)$$

Therefore, if ${}^\circ Q^{22} = {}^\circ Q^{11}$, this indicates

$$= {}^\circ Q^{kk}(-\mathbf{\nabla} \times {}^\circ\mathbf{E})_z = {}^\circ Q^{kk}\left(\frac{1}{c}\ \frac{\partial\,{}^\circ\mathbf{H}}{\partial t}\right)_z \qquad (4\text{-}30)$$

Eq.(4-30) indicates that, when ${}^\circ\mathbf{H}$ changes, VR, or, the finite size electron must experience a circling electric field, which should introduce a torque to the spin angular momentum of VR. Although the expected magnitude is very small, further beyond the limit of the present experimental technique in the usual situation, this is a new effect predicted by the new frame, which is not present in the old frame where the size of the electron is assumed to be less than 10^{-17} m .

For the orbital motion, Eq.(4-18) becomes

$$({}^\circ mc^2 - {}_1\Lambda)\ddot{\bar{x}}^\mu - {}_1\dot{\Lambda}\dot{\bar{x}}^\mu + \sum_{(\alpha)} {}^\circ\rho_{(\alpha)}d^\circ V_{(\alpha)}(A^{\mu}_{(\alpha)}{}'{}_\lambda - A^{(\alpha)\mu}_{\lambda,})(\dot{\bar{x}}^\lambda + \Delta\dot{x}^\lambda_{(\alpha)})$$

$$- \frac{d}{ds}\left\{\ddot{\bar{x}}_\lambda cS^{\lambda\mu} + \frac{ge}{2^\circ mc}\dot{\bar{x}}_\sigma F^\sigma{}_\lambda {}^*S^{\lambda\mu}\right\} = 0 \quad . \qquad (4\text{-}31)$$

Multiplying $\dot{\bar{x}}_\mu$, we get

$$-{}_1\dot{\Lambda} + \dot{\bar{x}}_\mu F_\lambda{}^\mu{}_{,\sigma}{}^*M^{\lambda\sigma} - \dot{\bar{x}}_\mu\left\{\ddot{\bar{x}}_\lambda c\dot{S}^{\lambda\mu} + \frac{ge}{2^\circ mc}\dot{\bar{x}}_\sigma F^\sigma{}_\lambda {}^*\dot{S}^{\lambda\mu}\right\} = 0 \quad . \qquad (4\text{-}32)$$

It is easy to show $\dot{\bar{x}}_\mu\ddot{\bar{x}}_\lambda\dot{S}^{\lambda\mu} = 0$, $\dot{\bar{x}}_\mu{}^*\dot{S}^{\lambda\mu} = 0$, therefore, we get

$${}_1\dot{\Lambda} = \dot{\bar{x}}_\mu(A^{\mu}{}'{}_{\lambda\sigma} - A_{\lambda,}{}^\mu{}_\sigma){}^*M^{\lambda\sigma} = -\dot{\bar{x}}_\mu A_{\lambda,}{}^\mu{}_\sigma M^{\lambda\sigma} = -M^{\lambda\sigma}\frac{d}{ds}A_{\lambda,\sigma}$$

$$= \frac{1}{2}M^{\lambda\sigma}\frac{d}{ds}(A_{\sigma,\lambda} - A_{\lambda,\sigma}) = \frac{d}{ds}\left(\frac{1}{2}M^{\lambda\sigma}F_{\lambda\sigma}\right) - \frac{1}{2}\dot{M}^{\lambda\sigma}F_{\lambda\sigma} \quad . \qquad (4\text{-}33)$$

Since we know in ${}^\circ K$ from Eq.(4-26)

$${}^\circ\dot{M}^{\lambda\sigma}{}^\circ F_{\lambda\sigma} = \frac{d^\circ\mathbf{M}}{cd^\circ t}\cdot{}^\circ\mathbf{H} = 0 \quad , \qquad (4\text{-}34)$$

we have

$${}_1\Lambda = +\frac{1}{2}M^{\lambda\sigma}F_{\lambda\sigma} = +\mu\cdot H + d\cdot E = -\Delta mc^2 \quad . \qquad (4\text{-}35)$$

Thence, finally we get the orbital equation as

$$\frac{d}{ds}\left[({}^\circ mc^2 - \frac{1}{2}M^{\lambda\sigma}F_{\lambda\sigma})\dot{\bar{x}}^\mu + \left(\frac{2}{g} + 1\right)M^{\lambda\mu}F_{\lambda\sigma}\dot{\bar{x}}^\sigma\right] = -(-e)\dot{\bar{x}}^\lambda F_\lambda{}^\mu - M^{\lambda\sigma}F_\lambda{}^\mu{}'{}_\sigma \quad . \qquad (4\text{-}36)$$

The derived equations of Eqs.(4-26) and (4-36) are identical to the accepted equations in the usual accuracy, but, contain additional terms in the higher order accuracy. Eq.(4-36) indicates the presence of additional term, $(2/g+1)M^{\lambda\mu}F_{\lambda\sigma}\dot{\bar{x}}^{\sigma}$, which is about 10^{-3} smaller than the smallest term , i.e., $-(1/2)M^{\lambda\sigma}F_{\lambda\sigma}\dot{\bar{x}}^{\mu}$, in the usual accuracy. We may call it as "anomalous orbital momentum", which may be checked in future. Our Lagrangian for the orbital motion is $\bar{L}_s$ with ${}_1\Lambda$ and ${}_2\Lambda^{(\alpha)}$ terms. With $\bar{L}_s$, we can make $\bar{x}^{\mu}$ and $\Delta x^{\mu}_{(\alpha)}$ independent for the variation and constraints (1), (2) and (3) of Eqs.(3-37) should be placed after the variation. The derived representations of ${}_1\Lambda$ and ${}_2\Lambda^{(\alpha)}$ can be safely placed in $\bar{L}_s$, because they have no effect for the variation procedure, owing to the constraints imposed after the variation.

Now, we have solved the classical dynamics of VR in the necessary accuracy. The internal structure of the results will be further discussed in the next section with emphases on the detailed mechanism of the spin-orbit coupling and the quantization.

§5. The general Lagrangian of VR, the spin-orbit coupling and the quantization

The Lagrangian, $\bar{L}_s$, thus obtained is

$$\begin{aligned}\bar{L}_s = & -\frac{1}{2}\,{}^{\circ}mc^2(\dot{\bar{x}}^{\mu}\dot{\bar{x}}_{\mu}+1) - \sum_{(\alpha)}{}^{\circ}\rho_{(\alpha)}d^{\circ}V_{(\alpha)}(\dot{\bar{x}}^{\mu}+\Delta\dot{x}^{\mu}_{(\alpha)})A^{(\alpha)}_{\mu} \\ & - \sum_{(\alpha)}\frac{1}{2}\frac{{}^{\circ}\rho_{(\alpha)}d^{\circ}V_{(\alpha)}}{(-e)}\frac{{}^{\circ}mc^2}{(-g)}(\Delta\dot{x}^{\mu}_{(\alpha)}\Delta\dot{x}^{(\alpha)}_{\mu}+1) + \frac{1}{4}M^{\lambda\sigma}F_{\lambda\sigma}(\dot{\bar{x}}^{\mu}\dot{\bar{x}}_{\mu}-1) \\ & - \left(-\frac{2}{(-g)}+1\right)\sum_{(\alpha)}({}^{\circ}\rho_{(\alpha)}d^{\circ}V_{(\alpha)}\dot{\bar{x}}^{\sigma}F_{\lambda\sigma}\Delta\dot{x}^{\lambda}_{(\alpha)}\Delta x^{\mu}_{(\alpha)})\dot{\bar{x}}_{\mu} \,. \end{aligned} \tag{5-1}$$

We should note that the magnetic self-energy term in this expression looks just like as one Lagrangian multiplier term. This mathematical structure may be utilized in general for representing hidden variables in a relativistic analysis of unknown composite particles, such as the nuclei. The variational equations are

$$\begin{aligned}\delta\bar{L}_s = & \left[-\frac{d}{ds}\left(\frac{\partial\bar{L}_s}{\partial\dot{\bar{x}}_{\mu}}\right)+\frac{\partial\bar{L}_s}{\partial\bar{x}_{\mu}}\right]\delta\bar{x}_{\mu} + \sum_{(\alpha)}\left[-\frac{d}{ds}\left(\frac{\partial\bar{L}_s}{\partial\Delta\dot{x}^{(\alpha)}_{\mu}}\right)+\frac{\partial\bar{L}_s}{\partial\Delta x^{(\alpha)}_{\mu}}\right]\delta\Delta x^{(\alpha)}_{\mu} \\ = & \; 0 \,. \end{aligned} \tag{5-2}$$

$$cp^{\mu} = \left(-\frac{\partial\bar{L}_s}{\partial\dot{\bar{x}}_{\mu}}\right) \,, \tag{5-3}$$

$$\frac{d}{ds}\left\{-\left(\frac{\partial\bar{L}_s}{\partial\dot{\bar{x}}_{\mu}}\right)\right\} = -\frac{\partial\bar{L}_s}{\partial\bar{x}_{\mu}} \,, \tag{5-4}$$

$$\frac{d}{ds}\left[\sum_{(\alpha)}\left\{\left(-\frac{\partial\bar{L}_s}{\partial\Delta\dot{x}^{(\alpha)}_{[\mu}}\right)\Delta x^{\nu]}_{(\alpha)}\right\}\right] = \sum_{(\alpha)}\left(-\frac{\partial\bar{L}_s}{\partial\Delta\dot{x}^{(\alpha)}_{[\mu}}\right)\Delta\dot{x}^{\nu]}_{(\alpha)} - \sum_{(\alpha)}\frac{\partial\bar{L}_s}{\partial\Delta x^{(\alpha)}_{[\mu}}\Delta x^{\nu]}_{(\alpha)} \,. \tag{5-5}$$

Eq.(5-4) is identical to Eq.(4-17) and becomes Eq.(4-36) finally, Eq.(5-5) leads to Eq.(4-20) and becomes Eq.(4-26).

Since VR has an essentially relativistic special structure, usual conventional way to construct Hamiltonian, such as

$$L_t = L_s \gamma_V^{-1} \quad , \qquad p_\xi = \frac{\partial L_t}{\partial \dot{q}_\xi} \tag{5-6}$$

$$\mathcal{H} = p_\xi \dot{q}_\xi - L_t \tag{5-7}$$

can not work. The constant of the motion, however, is obtainable from Eqs.(5-3), (5-4) and (4-18), such as

$$\frac{d}{ds}[cp^0] = \sum_{(\alpha)} {}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)} (\dot{\bar{x}}^\lambda + \Delta \dot{x}^\lambda_{(\alpha)}) \frac{\partial}{c\partial t} A^{(\alpha)}_\lambda \quad . \tag{5-8}$$

Therefore, if A_λ is not time dependent, we have a constant, cp^0, having the dimension of energy,

$$cp^0 = \frac{{}^\circ mc^2 - \boldsymbol{\mu}\cdot\mathbf{H} - \mathbf{d}\cdot\mathbf{E}}{\sqrt{1-(\frac{V}{c})^2}} + (-e)\phi + (\frac{2}{g}+1)\frac{\mathbf{d}\cdot(\mathbf{E}+\frac{\mathbf{V}}{c}\times\mathbf{H})}{\sqrt{1-(\frac{V}{c})^2}} \quad , \tag{5-9}$$

where

$$\mathbf{d} = \frac{\mathbf{V}}{c} \times \mu \tag{5-10}$$

is exactly the relativistically induced electric dipole moment of VR.†
Since

$$\mathbf{cp} = \left(\frac{{}^\circ mc^2 - \boldsymbol{\mu}\cdot\mathbf{H} - \mathbf{d}\cdot\mathbf{E}}{\sqrt{1-(\frac{V}{c})^2}}\right)\frac{\mathbf{V}}{c} + \sum_{(\alpha)} {}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)} \mathbf{A}^{(\alpha)}$$

$$+ (\frac{2}{g}+1)\frac{1}{\sqrt{1-(\frac{V}{c})^2}}\left((\frac{\mathbf{V}}{c}\cdot\mathbf{E})\mathbf{d} + \mu \times (\mathbf{E} + \frac{\mathbf{V}}{c}\times\mathbf{H})\right) \quad , \tag{5-11}$$

it is not difficult to eliminate the factor $1/\sqrt{1-(V/c)^2}$ from Eq.(5-9). Writing $cp^0 = E$, the result is

$$E = \sqrt{({}^\circ mc^2 - \boldsymbol{\mu}\cdot\mathbf{H} - \mathbf{d}\cdot\mathbf{E})^2 + (\mathbf{cp} - \sum_{(\alpha)} {}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)} \mathbf{A}^{(\alpha)})^2} + (-e)\Phi$$

$$\sim \sqrt{{}^\circ m^2 c^4 + (\mathbf{cp} - \sum_{(\alpha)} {}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)} \mathbf{A}^{(\alpha)})^2} + (-e)\Phi$$

$$- \frac{(\mu\cdot\mathbf{H} + \mathbf{d}\cdot\mathbf{H})\,{}^\circ mc^2}{\sqrt{{}^\circ m^2c^4 + (\mathbf{cp} - \sum_{(\alpha)} {}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)} \mathbf{A}^{(\alpha)})^2}} \quad . \tag{5-12}$$

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† $(\mu, -\mathbf{p}) = ({}^\circ\mu_\parallel + \gamma_V {}^\circ\mu_\perp, \; -\frac{\mathbf{V}}{c}\times\gamma_V {}^\circ\mu_\perp) = (\mu, -\frac{\mathbf{V}}{c}\times\mu)$.

In Eq.(5-12), the term with $-(\mathbf{d}\cdot\mathbf{E})^{\cdot}$ represents the spin-orbit coupling. In the non-relativistic approximation, if $V/c \ll 1$, $(cp)^2 \ll {}^\circ m^2c^4$, we have $-\mathbf{d}\cdot\mathbf{E}$ for this coupling, but, in the Dirac Hamiltonian, this term has a factor 1/2. We have concluded that the absence of the factor 1/2 is due in our frame, and, here, we found one of the typical interfacial structure between the c-number equation of the new frame and the Dirac Hamiltonian. Since we believe that the physical understanding of the spin-orbit coupling in the current physics is incomplete, we shall analyze the involved structure in detail by means of the new frame, hereafter.

Neglecting the last $Q^{\sigma\nu}$ term, Eq.(4-26) can be rewritten as

$$c\dot{S}^{\mu\nu} = \ddot{x}_\lambda c S^{\lambda[\mu}\dot{x}^{\nu]} + \dot{x}_\sigma F^\sigma{}_\lambda {}^*M^{\lambda[\mu}\dot{x}^{\nu]} + {}^*M^{\lambda[\nu}F^{\mu]}{}_\lambda \quad . \tag{5-13}$$

Assuming $^*M = M$, Eq.(5-13) can be rewritten as

$$c\dot{S}^{\mu\nu} = -\frac{2}{-g}\,\dot{x}_\sigma F^\sigma{}_\lambda M^{\lambda[\mu}\dot{x}^{\nu]} + \dot{x}_\sigma F^\sigma{}_\lambda M^{\lambda[\mu}\dot{x}^{\nu]} + M^{0[\nu}F^{\mu]}{}_0 + M^{k[\nu}F^{\mu]}{}_k \quad . \tag{5-14}$$

Here, different expression is used for the first term, and, the last torque term is decomposed into time related component and purely spacial component. As we show soon, each term of the right side of Eq.(5-14) has nearly the same order of magnitude. Therefore, the first and second terms cancel mutually, reducing the magnitude to $(2/g + 1) \sim 1.5 \times 10^{-3}$, which can be neglected. In order to make clear the structure of this term, however, we analyze this term briefly. Defining the usual vectors $\mathbf{S}$ and $\boldsymbol{\mu}$, as

$$\mathbf{S} = (S_x\ , S_y\ , S_z) = (S^{32}\ , S^{13}\ , S^{21})\quad , \quad \boldsymbol{\mu} = \frac{ge}{2{}^\circ mc}\,\mathbf{S}\ , \tag{5-15}$$

we have

$$-\frac{2}{-g}\,\dot{x}_\sigma F^\sigma{}_\lambda M^{\lambda[2}\dot{x}^{1]} = -\frac{2}{-g}\left[\frac{\mathbf{V}}{c}\times(\mathbf{E}\times\boldsymbol{\mu}) + \left(\frac{\mathbf{V}}{c}\cdot\boldsymbol{\mu}\right)\gamma_V^2\left\{\left(\frac{V}{c}\right)^2\mathbf{E}_\perp + \frac{\mathbf{V}}{c}\times\mathbf{H}\right\}\right]_z$$

$$= -\frac{2}{-g}\Big[\gamma_V^2\frac{1}{2}\,\boldsymbol{\mu}\times\left(\mathbf{E}\times\frac{\mathbf{V}}{c}\right) - \gamma_V^2\frac{1}{2}\left[\left(\frac{\mathbf{V}}{c}\times\boldsymbol{\mu}\right)\times\mathbf{E}_\perp + \frac{\mathbf{V}}{c}\times(\boldsymbol{\mu}\times\mathbf{E}_\perp)\right]$$

$$-\frac{1}{2}\left[\left(\frac{\mathbf{V}}{c}\times\boldsymbol{\mu}\right)\times\mathbf{E}_\parallel + \frac{\mathbf{V}}{c}\times(\boldsymbol{\mu}\times\mathbf{E}_\parallel)\right] + \gamma_V^2\left(\frac{\mathbf{V}}{c}\cdot\boldsymbol{\mu}\right)\frac{\mathbf{V}}{c}\times\mathbf{H}\Big]_z\ . \tag{5-16}$$

Let us take a 3d electron, and assume that the electron is making a circular motion at about 0.4 Å from the nucleus whose effective charge is $+Z'e \sim 20e$. Then the velocity should be $8\times10^6\,\mathrm{ms}^{-1}$ with $V/c \sim 0.3$, so that $V/c \ll 1$ can not be assumed. In this specific situation, the third and forth terms of the last equation of Eq.(5-16) are zero. As we show soon that, although the second term has the same magnitude with the first term, it gives only a ripple motion, whose integrated effect is neglegible. Therefore, Eq.(5-16) indicates that the effect of the Thomas precession in this case is equivalent to the presence of a magnetic field, H_T , of

$$H_T = -\frac{2}{(-g)}\frac{\gamma_V^2}{2}\left(E \times \frac{V}{c}\right) = -\frac{2}{(-g)}\frac{\gamma_V}{2}H' \quad . \tag{5-17}$$

Here, $H' = \gamma_V E \times (V/c)$ is the magnetic field as seen by the electron, or, in °K. We can see that the Thomas precession just reduces the effect of H' into approximately half.

Now, in Eq.(5-14), however, this term is just cancelled by the second term, or, the higher order torque term and the remaining equation is

$$\begin{aligned} c\dot{S} &= \left(\frac{V}{c} \times \mu\right) \times E + \mu \times H \\ &= \mu \times \left(\frac{1}{2}E \times \frac{V}{c} + H\right) + \frac{1}{2}\left[\left(\frac{V}{c} \times \mu\right) \times E + \frac{V}{c} \times (\mu \times E)\right] \quad . \end{aligned} \tag{5-18}$$

As we see, again the torque can be decomposed into essentially the same components with Eq.(5-16) with different sign.

Let us make a numerical calculation, in which the nucleus is located at the origin and the electron is making a circular motion in x-y plane with the angle $\psi = \omega_{orb} t$. We have

$$\left.\begin{aligned} V &= V\,(\,-\sin\omega_{orb}t\,,\ \cos\omega_{orb}t\,,\ 0\,), \\ E &= E\,(\,\cos\omega_{orb}t\,,\ \sin\omega_{orb}t\,,\ 0\,),\quad H = 0\,, \\ \mu &= \mu\,(\,\sin\theta\cos\phi\,,\ \sin\theta\sin\phi\,,\ \cos\theta\,)\,, \end{aligned}\right\} \tag{5-19}$$

here, we have represented the spin magnetic moment, μ , in the same space by μ, θ, and ϕ . It is noted that we have to assume $d\mu/dt \neq 0$. Then Eq.(5-18) becomes

$$\begin{aligned} \frac{°mc}{e}\dot{\mu} &= \frac{\mu EV}{2c}\sin\theta\,[\,(-\sin\phi\,,\ \mathrm{con}\,\phi\,,\ 0\,) + \\ &\qquad (-\sin(2\omega_{orb}t-\phi)\,,\ \cos(2\omega_{orb}t-\phi)\,,\ 0\,)\,] \\ &= \frac{°mc}{e}\Big\{\mu\,[\sin\theta\,(-\sin\phi\,,\ \cos\phi\,,\ 0)\dot{\phi} + (\cos\theta\cos\phi\,,\ \cos\theta\sin\phi\,,\ -\sin\theta\,)\dot{\theta}] \\ &\qquad + \dot{\mu}(\sin\theta\cos\phi\,,\ \sin\theta\sin\phi\,,\ \cos\theta\,)\Big\}\,. \end{aligned} \tag{5-20}$$

Eq.(5-20) gives

$$\left.\begin{aligned} \frac{°mc}{e}\frac{d\phi}{dt} &= \frac{EV}{2c}[1 + \cos(2\omega_{orb}t - 2\phi)]\,, \\ \frac{°mc}{e}\frac{d\theta}{dt} &= -\frac{EV}{4c}\sin 2\theta\,\sin(2\omega_{orb}t - 2\phi)\,. \end{aligned}\right\} \tag{5-21}$$

In Eqs.(5-20) and (5-21), we can see the real form of the ripple torque. Again, using the aforementioned numerical figures for the 3d electron, we get

$$\omega_{orb} \sim 4 \times 10^{16}\ s^{-1} \quad , \quad \frac{eEV}{2 \cdot mc^2} = \omega_\mu \sim 2.4 \times 10^{14}\ s^{-1} \quad , \tag{5-22}$$

where from Eq.(5-21), ω_μ represents the averaged precessional frequency of the spin. Therefore, we can assume $\omega_{orb} \gg \omega_\mu$, so that from Eq.(5-21), we get

$$\phi \sim \omega_\mu t + \frac{\omega_\mu}{2\omega_{orb}} \sin(2\omega_{orb}t - 2\phi) \sim 2.4 \times 10^{14}t + 3 \times 10^{-3}\sin(2\omega_{orb}t - 2\phi), \tag{5-23}$$

$$\theta \sim \frac{\omega_\mu}{4\omega_{orb}} \sin 2\theta \cos(2\omega_{orb}t - 2\phi) \sim 1.5 \times 10^{-3}\sin 2\theta \cos(2\omega_{orb}t - 2\phi) \ . \tag{5-24}$$

Therefore, the aforementioned ripple torque gives only the ripples of 0.003 radian for ϕ and, 0.0015 radian for θ . This means that, since the orbital motion is quite rapid, the ripple torque in the spin-orbit coupling, although its magnitude is comparable with the main torque#, has a very small effect for the integrated motion of the spin.

In conclusion, we have confirmed that only the first term of the last equation of Eq.(5-18) is effective, and our equations describe the spin-orbit coupling state with the Thomas precession correctly. Therefore, the expression of Eq.(5-12) must be correct classically. Of course, since Eqs.(5-9) and (5-12) are the relativistic expression, coming from Eq.(4-36) of

$$- \gamma_V \frac{1}{2} M^{\lambda\sigma}F_{\lambda\sigma} = \gamma_V(-\boldsymbol{\mu}\cdot\mathbf{H} - \mathbf{d}\cdot\mathbf{E}) \quad , \tag{5-28}$$

there is no possibility of having the factor 1/2 there. Now, Dirac theory has predicted the factor 1/2 for this coupling of $-(\mathbf{d}\cdot\mathbf{E})$. Therefore, we have a problem of disagreement. First, we may point out that the Dirac electron should get the factor one, if the treatment is extended to the large scale orbital motion of the electron in macroscopic electromagnetic fields. Second, in quantum physics and for a stationary state, the ripple terms, such as the second term of Eq.(5-18) has to be averaged out, as the ripple term is space dependent and, if averaged over the orbit, the net effect becomes zero.(If averaged by ψ in the first equation of Eq.(5-20).) There is a possibility that the effective range of the Heisenberg uncertainty and the problem of measurement in quantum physics, being similar to the case of the Stern-Gerlach experiment[23], may be involved, which may differentiate the energy of the orbital motion of a wave packet of the electron in an electromagnetic field and that of a stationary state in an atom. In this connection, there is a possibility that we may get the factor 1/2, if we extend our treatment to include the emission and absorption of light by the intra-atomic electrons, and to quantize properly. Exact evaluation of the energy transfer by induction in this very critical situation with the Thomas precession may present another point to be studied in future. It is possible to regard H' as being induced by the motion of the positive charge of the nucleus in °K. Since °K rotates by the Thomas precession,

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We found an essential fault in known literatures, which claims that the ripple terms are small in their magnitudes already. In order to show the tricky structure of the relativistic calculation, we explain this. Using Eq.(5-18) and (4-36), we have

$$(\frac{2}{-g})\frac{d\mu}{dT} = -\mu\times H - (\frac{\mathbf{V}}{c}\times\mu)\times E \quad , \quad \frac{d}{dT}(\frac{\mathbf{V}}{c}) = -[\mathbf{E}_\perp + E_\parallel(1-\beta^2) + \frac{\mathbf{V}}{c}\times H] \quad , \quad (5\text{-}25)$$

$$T = e\tau/°mc = es/mc^2 \quad , \quad E = \mathbf{E}_\perp + \mathbf{E}_\parallel = -\frac{\mathbf{V}}{c}\times H - \{\frac{d}{dT}(\frac{\mathbf{V}}{c})\} - \frac{\beta^2}{1-\beta^2}\{\frac{d}{dT}(\frac{\mathbf{V}}{c})\}_\parallel \quad ,$$

$$\mu\cdot\mathbf{E} = -(\mu\cdot\frac{\mathbf{V}}{c}\times H) - \mu\cdot\{\frac{d}{dT}(\frac{\mathbf{V}}{c})\} - \frac{\beta^2}{1-\beta^2}\left(\mu_\parallel\cdot\{\frac{d}{dT}(\frac{\mathbf{V}}{c})\}_\parallel\right) \quad , \quad \frac{\mathbf{V}}{c}\cdot\mathbf{E} = -\frac{1}{1-\beta^2}\frac{d}{dT}\frac{\beta^2}{2} \quad .$$

We have an identity of

$$\frac{1}{2}\frac{d}{dT}[-\frac{\mathbf{V}}{c}\times(\frac{\mathbf{V}}{c}\times\mu)] = \frac{1}{2}\frac{d}{dT}[\beta^2\mu - (\mu\cdot\frac{\mathbf{V}}{c})\frac{\mathbf{V}}{c}] = \frac{1}{2}\frac{d}{dT}(\beta^2\mu_\perp) \quad ,$$

$$\frac{1}{2}[-\frac{d\beta^2}{dT}\mu_\parallel - \beta^2\frac{d\mu_\parallel}{dT} + (\frac{d\mu}{dT}\cdot\frac{\mathbf{V}}{c})\frac{\mathbf{V}}{c} + \left(\mu\cdot\frac{d}{dT}(\frac{\mathbf{V}}{c})\right)\frac{\mathbf{V}}{c} + (\mu\cdot\frac{\mathbf{V}}{c})\frac{d}{dT}(\frac{\mathbf{V}}{c})] = 0 \quad .$$

Then, we get completely analytically

$$(\frac{2}{-g})\frac{d\mu}{dt} = -\mu\times H - \frac{1}{2}\mu\times(E\times\frac{\mathbf{V}}{c})$$

$$+\frac{1}{2}[-(\mu\cdot\frac{\mathbf{V}}{c}\times H)\frac{\mathbf{V}}{c} - (\frac{\mathbf{V}}{c}\cdot\mu)\frac{\mathbf{V}}{c}\times H + \frac{1}{1-\beta^2}\frac{d\beta^2}{dT}\mu_\perp + \beta^2\{(\frac{d\mu}{dT})_\parallel - \frac{d\mu_\parallel}{dT}\}] \quad . \quad (5\text{-}26)$$

Then, one had concluded that, when $V/c \ll 1$, and $d\beta^2/dT$ is not large, the last ripple term of Eq.(5-26) is at least in the second order in $\beta = V/c$, so that it can be neglected as compared $(-1/2)\mu\times(E\times \mathbf{V}/c)$. This was a serious mistake. Assuming

$$\frac{\mathbf{V}}{c} = \frac{V}{c}\nabla z \quad , \quad \frac{d}{dT}(\frac{\mathbf{V}}{c}) = a(\sin\theta_E \ , \ 0 \ , \ \cos\theta_E) \quad , \quad \mu = \mu(\sin\theta\cos\phi \ , \ \sin\theta\sin\phi \ , \ \cos\theta),$$

$$\frac{d\mu}{dT} = \nu(\sin\theta'\cos\phi', \ \sin\theta'\sin\phi', \ \cos\theta') \quad ,$$

we show easily

$$\frac{d\mu_\parallel}{dT} = \frac{1}{\Delta T}\frac{[\mu + \frac{d\mu}{dT}\Delta T]\cdot[\frac{\mathbf{V}}{c} + \frac{d}{dT}(\frac{\mathbf{V}}{c})\Delta T]}{|\frac{\mathbf{V}}{c} + \frac{d}{dT}(\frac{\mathbf{V}}{c})\Delta T|} - \mu\cos\theta = \nu\cos\theta' + \frac{c}{V}\mu\, a\sin\theta\cos\phi\sin\theta_E \ . (5\text{-}27)$$

Therefore, $(d\mu_\parallel/dT)$ contains β^{-1}, so that $-(1/2)\beta^2(d\mu_\parallel/dT)$ is in the same order with $(-1/2)\mu\times(E\times \mathbf{V}/c)$.

which, as an average and neglecting the ripple motion, is equivalent to the precession of μ in a magnetic field of $-(1/2)\mathbf{H}'$, so that the $°K^*$, the instantaneous proper frame which, as seen in K, has no Thomas precession, should see the nucleus moving by a half speed, $\mathbf{V}/2$. Therefore, the magnetic field $°\mathbf{H}^*$ will be $\mathbf{H}'/2$. The new frame regards this old picture as a second guessing, since the $°\mathbf{H}^*$ will be still $\mathbf{H}'$. It prefers another old picture, in which the factor 1/2 comes from the essentially quantal averaging character of the quantum state, in agreement with the major character of the integrated classical solution.

Let us compare the Lagrangian density in the Dirac theory[14], $\mathcal{L}_{el}$ and our original L_s . They are

$$\mathcal{L}_{el} = -\bar{\psi}[mc^2 - \frac{ic(-e)\gamma^\mu}{(-e)}\{-\frac{\hbar\partial}{i\partial\bar{x}^\mu} - \frac{(-e)}{c}A_\mu\}]\psi \qquad (5\text{-}29)$$

$$\begin{aligned} L_s &= [-\frac{°mc^2}{2} - \frac{1}{2}\frac{°mc^2}{(-g)}] - \iiint\frac{j^\mu}{(-e)}\{\frac{(-e)}{4c}b_\mu + \frac{(-e)}{c}A_\mu\}d°V \\ &= -\frac{°mc^2}{2}(1+\frac{1}{(-g)}) - \iiint\{\frac{c\dot{x}^\mu q_\mu}{2} + \frac{j^\mu A_\mu}{c}\}d°V \quad . \qquad (e>0) \end{aligned} \qquad (5\text{-}30)$$

In Eqs.(5-29) and (5-30), since, in the field theory[14], $ic(-e)\gamma^\mu$ represents the total electric current density(orbital plus spin currents), the interaction terms are both

$$-\frac{1}{c}j^\mu A_\mu \quad . \qquad (5\text{-}31)$$

Besides this, there are many similar points. Definite difference will be in the representation of spin. Namely the structure of spin is only in γ^μ in the Dirac theory and we have a very detailed structure by $\Delta\dot{x}^\mu_{(\alpha)}$, which is located in $\dot{x}^\mu$, j^μ and q^μ in Eq.(5-30). We propose that our Lagrangian has at least one advantage with respect to its simplicity and its clearness in the internal structure with $g = -2(1+\alpha/2\pi)$.

The quantization of Eqs.(4-9), (4-26), and (4-36) should be made as follows. Since it is known that the γ-matrix representation by Dirac is the simplest covariant representation of the electron spin, we have to take the Dirac four Schrödinger wave functions, $\psi_{(n)}$ and $\bar{\psi}_{(n)}$ (n = 1, 2, 3, 4), and should follow the Dirac procedure. Then, the orbital momentum, p^μ or Eqs.(5-1), (5-2), and (5-3), should be replaced by the operator

$$p^\mu = -\frac{\hbar}{i}\frac{\partial}{\partial\bar{x}_\mu} = \frac{\hbar}{i}(-\frac{\partial}{c\partial t}, +\frac{\partial}{\partial\bar{x}}, +\frac{\partial}{\partial\bar{y}}, +\frac{\partial}{\partial\bar{z}}) \qquad (5\text{-}32)$$

and, relying on the Gordon decomposition of the γ^μ operator[16),17], we should regard the spin angular momentum $S^{\mu\nu} = (2°mc/ge)M^{\mu\nu}$ as an operator

$$S^{\mu\nu} = \frac{\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}}{4i}\hbar \quad . \tag{5-33}$$

When we neglect the higher order terms, we have

$$p^{\mu} = mc\dot{\bar{x}}^{\mu} + \frac{(-e)}{c}A^{\mu} \quad , \tag{5-34}$$

$$\dot{\bar{x}}^{\mu} = \frac{1}{mc}\{-\frac{\hbar}{i}\frac{\partial}{\partial\bar{x}_{\mu}} - \frac{(-e)}{c}A^{\mu}\} = \frac{1}{mc}P^{\mu} \quad , \tag{5-35}$$

where P is called as the mechanical momentum.

We have to make the necessary considerations on $\bar{\psi}$ also. Of course the operator $(dA/d\tau)$ must be understood as representing the operator having the identity[3])

$$< \frac{dA}{d\tau} > = \frac{d}{d\tau} < A > \quad . \tag{5-36}$$

Then, in principle, we have obtained the quantized relativistic kinematical equations for the electron with $g = -2(1+\alpha/2\pi)$.

It is to be noted that most of the calculations in §'s 4 and 5 are effective so far as the electron consists of persistent currents. The special character of the VR model is only used in the numerical coefficients of Eqs.(4-9) and (4-12), where the expression of the electromagnetic momentum of VR has to be used.

§6. Quantized magnetic flux of the electron spin

An important purpose of the new frame in physics is to present new ideas or new concepts in physics, which may stimulate new investigations. In this section, we continue to present a discussion along this line.

In °K, the VR model has a quantized flux $2hc/e$ in the absence of external electromagnetic fields. What would happen when an external magnetic field is present. Let us apply the same procedure as that employed in finding the original model. Then the first question is the amount of contribution of the externally applied vector potential to the self angular momentum of VR.

First, we should note that this contribution does not affect the kinematical equation of Eq. (4-7). Because we have

$$\frac{d}{ds}\Big(\sum_{(\alpha)} {}^{\circ}\rho_{(\alpha)}\, d^{\circ}V_{(\alpha)}\Big\{\frac{{}^{\circ}mc^2}{ge}\Delta\dot{x}^{[\mu}_{(\alpha)} + \varkappa A^{[\mu}_{(\alpha)}\Big\}\Delta x^{\nu]}_{(\alpha)}\Big)$$

$$= \frac{d}{ds}\Big(\sum {}^{\circ}\rho_{(\alpha)} d^{\circ}V_{(\alpha)}\Big\{\frac{{}^{\circ}mc^2}{ge}\Delta\dot{x}^{[\mu}_{(\alpha)}\Delta x^{\nu]}_{(\alpha)}\Big\}\Big)$$

$$+ \varkappa A^{[\mu,}{}_{\lambda}(\bar{x}^{\rho})\sum_{(\alpha)} {}^{\circ}\rho_{(\alpha)} d^{\circ}V_{(\alpha)}(\Delta\dot{x}^{\lambda}_{(\alpha)}\Delta x^{\nu]}_{(\alpha)} + \Delta x^{\lambda}_{(\alpha)}\Delta\dot{x}^{\nu]}_{(\alpha)})$$

$$= \frac{d}{ds}\Big(\sum_{(\alpha)} {}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)} \Big\{ \frac{{}^\circ mc^2}{ge} \quad \Delta\dot{x}_{(\alpha)}^{\ [\mu} \Delta x_{(\alpha)}^{\nu]} \Big\}\Big) \quad , \tag{6-1}$$

in which we have neglected the quadrupole interaction term.(see Eqs.(4-9)-(4-15)). This means that the additional torque which should be added to the right side equation of Eq.(4-7) is really neglegible.

We found that the most attractive physical structure of the system can be obtained by assuming the contribution as

$${}^\circ\rho_{(\alpha)} d^\circ V_{(\alpha)} {}^\circ\mathbf{A}_{(\alpha)}/2c \quad . \tag{6-2}$$

The basis of the justification is that the angular momentum of VR is electromagnetic and, for the internal degree of freedom, ${}^\circ\mathbf{A}_{(\alpha)}$ and ${}^\circ\mathbf{a}_{(\alpha)}$ can not be distinguished. The self factor 1/2 is already used in ${}^\circ\mathbf{a}_{(\alpha)}$ with justification and this is an extension. As shown in Eq.(2-9), since the situation is really relativistic, similarly to the Thomas factor, there is no apriori reason to adopt the factor 1, which has been effective in the orbital motion, as represented in Eq.(4-18), having justification mathematically[1] and verified experimentally. Then, the angular momentum quantization becomes

$$\left| \iiint {}^\circ\mathbf{r}_{(\alpha)} \times \frac{{}^\circ\rho_{(\alpha)}}{2c}[{}^\circ\mathbf{a}_{(\alpha)} + {}^\circ\mathbf{A}_{(\alpha)}] d^\circ V_{(\alpha)} \right| = \hbar \quad . \tag{6-3}$$

From Eq.(6-3), a calculation which uses Eq.(2-10) gives

$$\frac{e^2}{8\pi^2 c}[\ln\frac{8{}^\circ R}{{}^\circ\eta} - 2] + \frac{e}{4c}{}^\circ H_1 {}^\circ R^2 \cos\theta = \hbar \quad . \tag{6-4}$$

Next, as shown in Appendix B, in the new frame[3], the Zeeman energy $-({}^\circ\mu_2 \cdot {}^\circ\mathbf{H}_1)$ is regarded as the effective Hamiltonian of the total system, such as

$$\delta^*[-({}^\circ\mu_2 \cdot {}^\circ\mathbf{H})] = \delta{}^\circ G_1 + \delta{}^\circ U_m + \delta{}^\circ G_2 \quad . \tag{6-5}$$

Here subscripts 1 and 2 refer to the source and VR[3]. ${}^\circ G_1$ and ${}^\circ G_2$ are the self energies of the source of the magnetic field, ${}^\circ\mathbf{H}_1$, and the magnetic moment, ${}^\circ\mu_2$, respectively, and ${}^\circ U_m$ is the total magnetic field energy. $\delta{}^\circ G_1$ and $\delta{}^\circ G_2$ are introduced through the energy transfer by induction. δ^* indicates that the variation should be with respect to the mutual configuration. Details are shown in Appendix B. From Eq. (B7) of the same appendix, we expect that the self energy of VR has to experience

$$\delta{}^\circ G_2 = \delta^*{}^\circ G_2 = \delta^*[-({}^\circ\mu_2 \cdot {}^\circ\mathbf{H}_1)]) = \delta(\Delta mc^2) \quad . \tag{6-6}$$

Then, from Eqs. (2-2), (2-3) and (2-10), we get

$$°mc^2 - °\mu °H_1 \cos\theta = \frac{e^2}{4\pi^2 °R}[\ln\frac{8°R}{°\eta} - 1] \quad . \tag{6-7}$$

Eqs. (6-4) and (6-7) are enough to determine °R and °η, for the new situation, or, under the applied magnetic field, °H_1. Striking result is that

$$\frac{\delta °R}{°R} = 0 \quad , \tag{6-8}$$

$$\frac{\delta °\eta}{°\eta} = -\frac{4\pi^2 °R}{e^2}\,\delta(-°\boldsymbol{\mu}_2 \cdot °\mathbf{H}_1) = -\frac{4\pi^2 °R}{e^2}\,\delta(M^{\mu\nu}A_{\mu,\nu}) \quad , \tag{6-9}$$

so that only °η will experience a change

$$\frac{\delta °\eta}{°\eta} = -\frac{1 + \frac{\alpha}{2\pi}}{\frac{\alpha}{2\pi}} \cdot \frac{\delta(-°\boldsymbol{\mu}_2 \cdot °\mathbf{H})}{mc^2} \sim \pm 10^{-(8\sim 6)} \quad , \tag{6-10}$$

for the °H_1 of $10^{3\sim 5}$Oe.

The magnetic flux in this case has two contributions. The °Φ_a due to the vector potential °**a** is, from Eq. (2-10)

$$°\Phi_a = \iint °\mathbf{h}\cdot d\mathbf{S} = \oint °\mathbf{a}\cdot d\boldsymbol{\ell} = \frac{e}{2\pi}[\ln\frac{8°R}{°\eta} - 2] \quad , \tag{6-11}$$

so that

$$\delta °\Phi_a = \frac{e}{2\pi}[-\frac{\delta °\eta}{°\eta}] = -\pi °R^2 °H_1 \cos\theta \quad , \tag{6-12}$$

which is just cancelled by the $\delta °\Phi$ due to °H_1 of

$$\pi °R^2 °H_1 \cos\theta \quad . \tag{6-13}$$

Therefore, the magnetic flux quantization is strictly maintained and VR behaves like as a superconductor, rejecting the penetration of the external magnetic field completely, by means of the induced diamagnetic change of the current loop. The result is obvious since the flux °Φ

$$°\Phi = \oint(°\mathbf{a} + °\mathbf{A})\cdot d°\boldsymbol{\ell} \quad , \tag{6-14}$$

where $d°\ell$ is the line element of the loop, whereas Eq. (6-3) can be transformed to

$$\hbar = \left|\oint °\mathbf{r} \times \frac{°\lambda}{2c}(°\mathbf{a} + °\mathbf{A})d°\ell\right| \tag{6-15}$$

in which °λ is the line density of the charge. Since, in Eq. (6-15), the ratio of the components of (°**a** + °**A**)′, i.e., of vertical to of parallel against $d°\ell$, is very small, and only the square of the ratio has an effect, the vertical component has no contribution in Eq. (6-15). Then Eqs. (6-14) and (6-15) become identical, indicating the coexistence of angular momentum and magnetic flux quantizations.

Since we have the Meissner effect for the orbital diamagnetism and, here, we have the same for the spinning motion of the electron, we may conclude that perfect diamagnetism is a virtue of any physical system with persistent currents, so far as the system is large enough or the currents are strong enough to realize it. This is a new concept obtained by the new frame. If the system is not large enough nor the current is not strong enough, the diamagnetism may be incomplete, such as the case of the Larmor diamagnetism of atoms, ions, and molecules. The concept may have an application in the structure of nuclei or in astronomical entities.

We have tried to find out the semi-classical derivation of the anomalous magnetic moment in the forth order perturbation, i.e., $-0.328(\alpha/\pi)^2$, [13] from our scheme, but, was not successful. As has been shown already, a part of the Lagrangian, $-(j^{\mu}A_{\mu})/c$, in Eq.(5-31) is essentially equivalent to the interaction term of the electron with the external electromagnetic fields in the field theory[14]. A part of this term gives the effective Hamiltonian of Eq.(6-5). Therefore, $°G_1$, $°U_m$, and $°G_2$ have already been included in the original Lagrangian. Although $_1\Lambda$ term introduces a change in °m, it is still a zero term in the Lagrangian, and, if we introduce the action of $\delta°G_2$ in L_s by $\delta°\eta/°\eta$ of Eqs.(6-7) and (6-9), then we have to include $\delta°U_m$ further, otherwise the interaction will be counted twice. But the inclusion of $\delta°U_m$ exposes the whole scheme to an entirely new reconstruction, which is not easy. Probably, in order to get this term, the reacting electromagnetic fields must be included and quantization and renormalization procedure may be needed. Accordingly, recalling the case of the spin-orbit coupling, our approximation in the present stage seems to have its limitation in this range.

§7. The principle of factor two

Here, we propose the principle of the factor two, stating that, in the new frame, if we had a difference of a factor of two between a classical c-number equation and the corresponding quantized q-number equation, we should accept the difference as granted, and lieve the clarification of the discrepancy as the next problem to be studied in future. We have such a situation already in the cases of the self-electromagnetic momentum[1], the angular momentum as shown in Eqs.(4-14) and (4-15), the spin-orbit coupling term of Eqs.(5-12) and (5-18), and the flux quantization problem in §6. As we know, although the explanations have been found in these cases, it might be a wiser way to accept such a difference as a principle, i.e., as one of the general characters of the interface between classical and quantal physics. Of course, explanation must be found out, but, it could be quite a tedious problem in some cases.

The relation between the total spin angular momentum of VR and the eigen value of the angular momentum component of the quantized electron spin is another example of this principle. The difference will not appear in the fields where the classical experiences are plenty, such as the orbital motion of the electron, but, it may appear in a field where no classical experience is present. Electron spin is the item where no classical experience is present. Therefore, we generally expect the necessity for the adoptation of this principle for spin related phenomena. The presence of this principle, however, presents just a small caution in the quantization procedure of the new frame, i.e., for the conversion of the c-number equations into q-number equations, so that no essential change in the structure of the new frame is required. We propose that the essential statement of the new frame that the classical c-number equations transit directly to the quantal q-number equations by a suitable quantization procedure is quite effective. The only lesson that we have learned in the present study is that we must be very careful for the quantization procedure, if the c-number equations are quite in detail and deal with the physics where no classical experience has been present. We note further that the concept of classical experience may need a certain reconsideration in some cases. The size of the electron is one point of this sort. Admission of the presence of persistent currents may be another example. The arrangement of the Stern-Gerlach experiment looks classical, but, the interpretation is essentially quantal[23), being deeply involved the problem of measurement. Therefore. a possibility always exists that we have to change our classical frame slightly in the c-number representation of the new frame.

§8. Conclusion

We have refined the mathematical frame of the covariant application of the principle of least action and have succeeded in deriving the dynamical equations of the classical electron from the first principle, precisely, without introducing any adjustable parameters. The equations have been quantized easily, giving the Dirac type quantal wave equations for the electron with $g = -2(1 + \alpha/2\pi)$, without having the complicated renormalization procedure. Thus, the new frame in physics has acquired a sound quantal basis for the kinematics of the electron, supporting VR as the best classical model of the electron. Although, in a few cases, we have found certain detailed structures at the interface between the c-number classical equations and the q-number quantal equations, the differences are mostly due to the improper use or incompleteness of the classical physics itself. The difference in the factor of two has appeared several times, but, the origins of these differences have been well clarified. We conclude, therefore, that the new frame can work very well down to the spin-orbit coupling in atoms.

Since VR model is so simple and reasonable, regarding the electron as a quantized electromagnetic soliton, the success of the present study may, inversely, justify the procedure of the Dirac electron, because the two approaches gave essentially the same equations in very high accuracy. We hope that further progress of the new frame will be made in future in many other directions. It is noted that it is easy to derive the Fermi contact term from VR model[2)]. Although it may be somewhat too ambitious, the Pauli principle for the symmetry of the total wave function is also derivable from our scheme, by taking into account the magnetic interactions of the approaching two VR electrons and requiring that the total orbital wave function should mathematically be regular when the two VR electrons approach to the range of °R. VR model consists of a persistent current having the velocity of light, which has been essential for the relativistic calculation. We may suggest that these relativistic persistent currents may play a role internally in other elementary particles and nuclei.

In the last, we may say that we had a definite one step forwards towards the extension of the new frame to the quantal physics of both macroscopically large inhomogeneous systems, such as biophysical functions, composite electro-optical elements, and logical circuits for high speed computations, and possibly, basic particles, such as nuclei.

Appendix A. The classical size of the electron

Although the electron is definitely a quantal existence, in the new frame of physics, in order to get the analytical continuation between quantal and classical physics, a requirement exists to find out the best self-consistent classical representation of the electron, which can be used as the basic element in the classical frame of the new physics, especially in its Maxwell-Lorentz electromagnetism. It has turned out that our VR model satisfies this requirement. With VR, we have to accept that the classical size of the electron is in the range of the Compton wave length, which, in terms of the radius, be $g\lambda\!\!\!^{-}_{c} = g\hbar/mc \sim 10^{-2}\text{Å} = 10^{-12}\text{m}$. We know that there is an old concept, in which the electron is assumed to be a point, less than 10^{-15}m , and the enormous electrostatic self-energy associated with this point charge has been just, without reason, disregarded. This concept can not be used in the new frame, because we look for a strictly selfconsistent electromagnetism and, in this old representation, the problem of the electrostatic self-energy itself affords definite selfinconsistency. Therefore, the only choise for the new frame is VR, and, the question is whether VR can represent the classical electron adequately or not.

Let us compare the old and new concepts. For the electron, the minimum Heisenberg uncertainty in the location, Δx, and the de Broglie wave length, λ_B , in the proper frame are

$$\Delta x \sim \frac{\hbar}{mc} = \lambda\!\!\!^{-}_c \quad , \qquad \lambda_B = \frac{h}{mc} = \lambda_c \quad , \tag{A1}$$

i.e., its two Compton wave lengthes. Therefore, in the old concept, the point electron is assumed to make an iteneration in the range of the Compton wave length, being called the "Zitterbewegung", and intrinsic spin magnetic moment of the electron is ascribed to the rotational orbital motion of this Zitterbewegung, leaving the g-factor problem(Why g = 2) unsolved. In the new concept, the electron itself is a persistent current, having the radial extension of $g\lambda_c \sim 10^{-2}$ Å and the spin angular momentum, $\hbar$, as its intrinsic virtue of the model. It is noted that, since the electric charge has relativistic invariance, although the ring charge is assumed to rotate with the speed of light, c , (being identical to the velocity of the Zitterbewegung in Q.E.D.) this order comes about decisively as the minimum size, being supported also by the Q.E.D. through its Darwin term[18).

Now since Eqs.(A1) exist, the question of whether a person takes the new or old model in his brain may be a matter of taste, but, in the new frame, we have to take VR model, because only this model gives a classical electron with its all non-wave virtues, enabling to construct a consistent Maxwell-Lorentz electromagnetism, and to establish an analytical continuation between quantal and classical physics[3).

It is noted that, different from the α-particle, no classically explainable quantitative Rutherford scattering data exists for the electron. Although the quantum mechanical calculation of the scattering of the electron by an idealized electrostatic Coulomb potential gives a formulus whose leading term is identical to the Rutherford classical scattering formulus[19,20), no such idealized Coulomb potential is available in nature for the electron scattering, so that, the observed scattering data of the electron by materials are rather utilized quantally to determine the electric charge distribution of the nuclei in the target[21). The electron beams of less than 10 MeV give only the electron diffraction by the target, and the beam of very high energy, having the velocity of light, e.g., 0.5 BeV[21,22], still gives diffraction by the nuclei of the target, since its de Bloglie wave length of 2.47×10^{-15} m is yet in the range of the size of the nuclei.

It is further noted that although the high energy electron-positron or electron-electron collision experiments have given the cross sectional data of the electron, from which the size is said to be less than 10^{-17} m, the experiments are of essentially quantum mechanical, and, although the theory had assumed the point charge electron, the divergence problem was left unsolved, and, under the allowance of the superposition and Pauli's exclusion principles, the obtained cross sectional data do not necessarily be related directly to the classical electromagnetic size of the electron. Classically, the two particles in these experiments, and also two VR's, may behave like as two electromagnetic solitons, which can penetrate or overlap mutually, without introducing any particle reactions. In this case, therefore, the word "size" may be the replacement of the probability of the quantum mechanical reactions, for which, the classical frame has nothing to do. Of course, in the new frame, we are mostly interested in the physics of materials, in which the relevant energies are very low, and electrons are regarded eternal. In conclusion, we state that no classically explainable Rutherford type simple data, which can appoint for the size of the electron to be less than 10^{-2} Å, has been presented.

The charge distribution range of 10^{-2} Å, given by VR model can not only describe most of the classical properties of the electron but also describe the hyperfine field to the nucleus precisely[2]. The instability of high Z number nuclei, which is known to be partly due to the capturing of their 1s electrons, might also be explained by the fact that the mean radii of 1s orbital in these nuclei approach to the range of 10^{-2} Å.

Appendix B. Exact meaning of the Zeeman energy

Let us assume two persistent current systems C_1 and C_2 with $°j_1(\mathbf{r})$ and $°j_2(\mathbf{r})$, being at rest in the frame °K. We assume that C_1 is large and is the source of the magnetic field $°\mathbf{H}_1(\mathbf{r})$, and C_2 is a magnetic moment, $°\mu_2$. Then the total magnetic energy can be represented as

$$°U_m = \iiiiiint \frac{°j_1(r_i)\cdot°j_1(r_j)}{8\pi r_{ij} c^2}\, dV_i dV_j$$

$$+ \iiiiiint \frac{°j_1(r_i)\cdot°j_2(r_j)}{4\pi r_{ij} c^2}\, dV_i dV_j + \iiiiiint \frac{°j_2(r_i)\cdot°j_2(r_j)}{8\pi r_{ij} c^2}\, dV_i dV_j \ . \quad \text{(B1)}$$

Its variation will be

$$\delta^\circ U_m = \iiiiiint [\frac{^\circ j_1(r_i)}{c} \cdot \{\frac{\delta^\circ j_1(r_j)}{4\pi r_{ij}c} + \frac{\delta^\circ j_2(r_j)}{4\pi r_{ij}c} + \frac{^\circ j_2(r_j)}{c}\,\delta(\frac{1}{4\pi r_{ij}})\}$$

$$+ \frac{^\circ j_2(r_j)}{c} \cdot \{\frac{\delta^\circ j_2(r_i)}{4\pi r_{ij}c} + \frac{\delta^\circ j_1(r_i)}{4\pi r_{ij}c} + \frac{^\circ j_1(r_i)}{c}\,\delta(\frac{1}{4\pi r_{ij}})\}$$

$$- \frac{^\circ j_1(r_i)\cdot{}^\circ j_2(r_j)}{c^2}\,\delta(\frac{1}{4\pi r_{ij}})]\,dV_i dV_j \quad . \tag{B2}$$

Here, by fixing the point i and j relatively in C_1 or in C_2 , $(1/4\pi r_{ij})$'s for $^\circ j_1(r_i)$ and $^\circ j_1(r_j)$ or for $^\circ j_2(r_i)$ and $^\circ j_2(r_j)$ can be put as zero by assumption. This assumption does not violate the generality of the analysis. Since the vector potential is represented as

$$^\circ A(r_i) = \iiint \frac{^\circ j_1(r_j) + {}^\circ j_2(r_j)}{4\pi r_{ij}c}\,dV_j \quad , \tag{B3}$$

we get

$$\delta^\circ U_m = \iiint \frac{^\circ j_1(r_i)}{c}\,\delta^\circ \mathbf{A}(r_i)dV_i + \iiint \frac{^\circ j_2(r_j)}{c}\,\delta^\circ \mathbf{A}(\mathbf{r}_j)dV_j + \delta^*[-{}^\circ\mu_2 \cdot{}^\circ H_1] \ . \tag{B4}$$

Here, * indicates that the variation must be with respect to the mutual geometrical configuration between C_1 and C_2 . Further, by considering the actual process of the variation, we get (See Eq.(6-5).)

$$\delta^\circ U_m = \iiint {}^\circ j_1(r_i)[\int_0^{\delta t_1} \frac{1}{c}\,\frac{\partial^\circ A(r_i)}{\partial t}\,dt]dV_i + \iiint {}^\circ j_2(r_j)[\int_0^{\delta t_1} \frac{1}{c}\,\frac{\partial^\circ A(r_j)}{\partial t}\,dt]dV_j$$

$$+ \delta^*[-{}^\circ\mu_2 \cdot{}^\circ H_1]$$

$$= -\delta^\circ G_1 - \delta^\circ G_2 + \delta^*[-{}^\circ\mu_2 \cdot{}^\circ H_1] \quad . \tag{B5}$$

Hence, we get the exact meaning of the Zeeman energy

$$\delta^*[-{}^\circ\mu_2 \cdot{}^\circ H_1] = \delta^\circ G_2 + \delta^\circ U_m + \delta^\circ G_1$$

$$= \delta^*[-\iiint \frac{^\circ j_2 \cdot{}^\circ A_1}{c}\,dV] = \delta^*[-\iiint \frac{^\circ j_1 \cdot{}^\circ A_2}{c}\,dV]$$

$$= \delta^*[-\iiint {}^\circ H_1 \cdot{}^\circ H_2 dV] = \delta^*[-{}^\circ U_m] \quad . \tag{B6}$$

In VR, as shown in Eq.(2-9), since the current flows with the speed of light, c , the relative change induced is very small, we can assume that $\delta^\circ G_2$ comes from the action of $\delta^\circ A_1$, while the direction and magnitude of $^\circ\mu_2$ have been kept constant. Then, we get from Eqs.(B5) and (B6)

$$\delta^\circ G_2 = \delta^{*\circ} G_2 = \delta^*[-{}^\circ\mu_2 \cdot{}^\circ H_1] \tag{B7}$$

Accordingly, we expect

$$\delta^\circ G_l + \delta^\circ U_m = 0, \tag{B8}$$

indicating that we can neglect safely both $\delta^\circ G_l$ and $\delta^\circ U_m$, as is the case of the present Dirac frame of the quantum theory for the electron spin. It will be noted that this relation does not hold for the Meissner effect, i.e., for the huge orbital diamagnetism of many electron systems[3).

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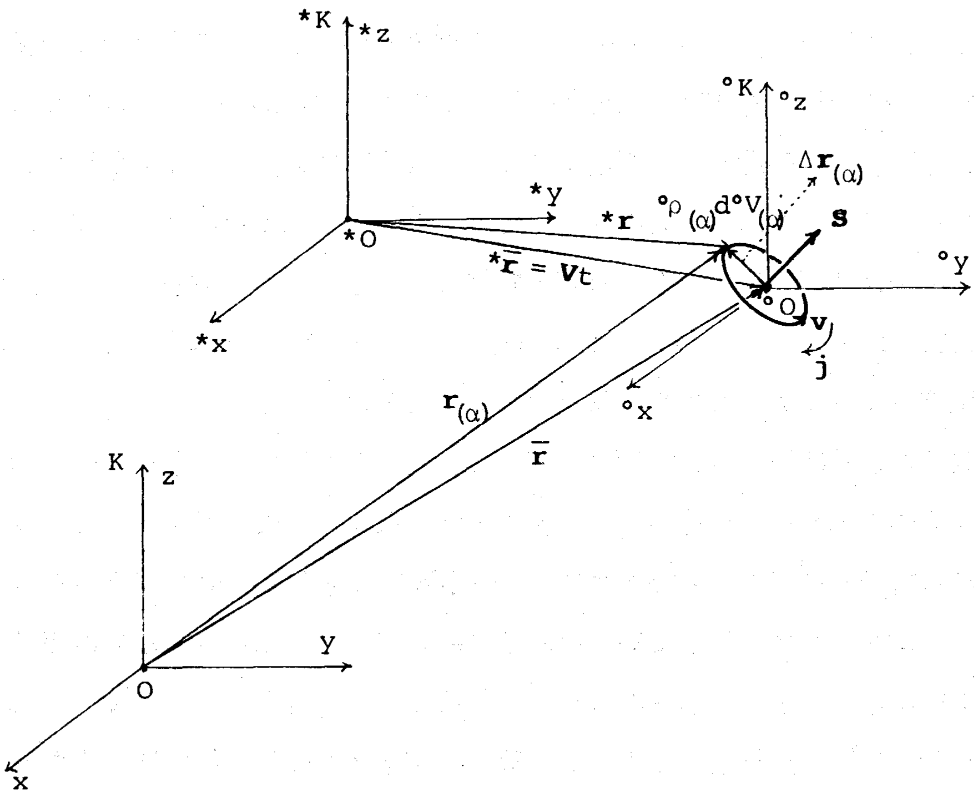

Fig.1. Relativistic geometry of the laboratory frame K, an instantaneous laboratory frame *K, and an instantaneous proper frame °K. *K and °K are related by a pure Lorentz transformation.