Generating varieties, Bott periodicity and instantons

Daisuke Kishimoto

January 22, 2010

Abstract

Let G be the classical group and let $\mathcal{M}_k(G)$ be the based moduli space of G-instantons on S^4 with instanton number k. It is known that $\mathcal{M}_k(G)$ yields real and symplectic Bott periodicity, however an explicit geometric description of the homotopy equivalence has not been known. We consider certain orbit spaces in $\mathcal{M}_k(G)$ and show that the restriction of the inclusion of $\mathcal{M}_k(G)$ into the moduli space of connections, which, in turn, is explicitly described by the commutator map of G. We prove this restriction satisfies a triple loop space version of the generating variety argument of Bott [6], and it also gives real and symplectic Bott periodicity. This also gives a new proof of real and symplectic Bott periodicity.

1 Introduction

Let G be a compact connected simple Lie group. Then there is an isomorphism $\pi_3(G) \cong \pi_4(BG) \cong \mathbb{Z}$. We will fix an isomorphism $\pi_3(G) \cong \mathbb{Z}$. Then principal G-bundles over S^4 are classified by $\mathbb{Z} = \pi_3(G)$, and denote by P_k the principal G-bundle over S^4 corresponding to $k \in \mathbb{Z}$. Let $\mathcal{C}_k(G)$ be the based moduli space of connections on P_k. Then we have a natural homotopy equivalence

$$\mathcal{C}_k(G) \simeq \Omega^3_k G$$

where $\Omega^3_k G$ stands for the path component of $\Omega^3 G$ corresponding to $k \in \mathbb{Z} = \pi_3(G)$. We will identify $\mathcal{C}_k(G)$ with $\Omega^3_k G$ by this homotopy equivalence. Let $\mathcal{M}_k(G)$ be the based moduli space of instantons on P_k. Then we have a map

$$\theta_k : \mathcal{M}_k(G) \to \Omega^3_0 G$$

defined by the composite of the inclusion $\mathcal{M}_k(G) \to \Omega^3_k(G) \simeq \mathcal{C}_k(G)$ and the homotopy equivalence $\Omega^3_k G \simeq \Omega_0 G$, the shift by $-k \in \mathbb{Z} = \pi_3(G)$.

The topology of the map θ_k was first studied by Atiyah and Jones [3], and, later, it was proved by Boyer, Hurtubise, Mann and Milgram [9], Kirwan [15] and Tian [19] that the map θ_k is a homotopy equivalence in a range, which is known as the Atiyah-Jones theorem. As a consequence of this result, Tian [19] showed that the colimit of the map θ_k yields real and symplectic

Bott periodicity. However, an explicit geometric description of the homotopy equivalence is not known While Bott periodicity was given by a map explicitly defined by the commutator maps of the classical groups [7]. In [10], it is shown that the map θ_k has some relation with the commutator map of G when $k = 1$. Recall that Bott [6] also used the commutator maps to study the topology of loop spaces of Lie groups. Exploiting the above result of [10] in connection with the classical result of Bott [6], Kamiyama [13] studied a triple loop space analogue of generating varieties of Bott [6].

We will give a mild generalization of the above result of [10] for arbitrary k. Using this, we prove triple loop space version of the generating variety argument [6] in a sense somewhat different from [13], and also prove Bott periodicity. This yields a new proof of real and symplectic Bott periodicity. We will give applications of this result to the homotopy types of $\mathcal{M}_k(G)$.

2 Subgroups of classical groups isomorphic with $\mathrm{SU}(2)$

Let G be a compact, connected, simple Lie group with a fixed isomorphism $\pi_3(G) \cong \mathbb{Z}$. Note that G acts on $\mathcal{M}_k(G)$ via the action of the basepoint free gauge group of P_k on $\mathcal{M}_k(G)$. As is shown in [10], there is an orbit of this action for $k = 1$ such that the restriction of $\theta_1 : \mathcal{M}_1(G) \to \Omega_0^3 G$ is presented by the commutator map of G. By putting additional assumption, we can prove this for arbitrary k by essentially the same way in [10] as follows.

Lemma 2.1. *Suppose that there exists a subgroup H of G isomorphic to $\mathrm{SU}(2) \approx S^3$ such that the inclusion $\iota : H \hookrightarrow G$ represents $k \in \mathbb{Z} = \pi_3(G)$. Then there exists $\omega \in \mathcal{M}_k(G)$ satisfying:*

1. *The orbit space $G{\cdot}\omega$ is homeomorphic with $G/C(H)$, where $C(H)$ stands for the centralizer of H.*
2. *Let Γ denote the composite:*

$$G/C(H) \approx G \cdot \omega \hookrightarrow \mathcal{M}_k(G) \xrightarrow{\theta_k} \Omega_0^3 G$$

Then we have

$$\Gamma(gC(H)) \simeq g\iota(h)g^{-1}\iota(h)^{-1}$$

for $g \in G, h \in H$.

Proof. Let α be an asymptotically flat connection on P_k. We regard S^4 as $\mathbb{R}^4 \cup \{\infty\}$. Recall from [3] that the homotopy equivalence $\mathcal{C}_k(G) \xrightarrow{\simeq} \Omega_0^3 G$ takes $\alpha \in \mathcal{M}_k(G)$ into its 'pure gauge' $\hat{\alpha} : S^3 \to G$ at $\infty \in S^4$ normalized as $\hat{\alpha}(*) = e$, where $*$ and e are the basepoint of S^3 and unity of G, respectively. (See [3].) The action of the basepoint free gauge group of P_k is locally the conjugation by G. Then the map θ_k is G-equivariant under the action of G on $\Omega_0^3 G$ given by $g \cdot \lambda(x) = g\lambda(x)g^{-1}$ for $g \in G, \lambda \in \Omega_0^3 G, x \in S^3$.

Let P be a principal SU(2)-bundle over S^4 represented by $1 \in \mathbb{Z} \cong \pi_3(\mathrm{SU}(2))$. In [2], an asymptotically flat instanton ϖ whose pure gauge represents $1 \in \mathbb{Z} \cong \pi_3(\mathrm{SU}(3))$. Then the proof is completed by putting ω to be the push forward of ϖ by the inclusion $\iota : H \cong \mathrm{SU}(2) \to G$. □

The original form of Bott periodicity [7] is given by such a map Γ in Lemma 2.1 where $\mathrm{SU}(2) \approx S^3$ is replaced with $\mathrm{U}(1) \approx S^1$. On the other hand, there is known a deep relation between $\mathcal{M}_k(G)$ and Bott periodicity as in [15], [18], [19]. Then we expect the map Γ in Lemma 2.1 may yield real and symplectic Bott periodicity which has period 4. Also we expect $G/C(H)$ and Γ in Lemma 2.1 may yield a 3-fold loop analogue of a generating variety for a loop space of a Lie group, which is already studied by Kamiyama [13] in a slightly different sense, that is, algebras over the Kudo-Araki operations. Then we introduce a family of subgroups of the classical groups which are isomorphic with SU(2) by which we can prove the above argument.

Hereafter, we put $(\mathbf{G}, \mathbf{H}, d) = (\mathrm{Sp}, \mathrm{O}, 1), (\mathrm{SU}, \mathrm{U}, 2), (\mathrm{SO}, \mathrm{Sp}, 4)$. We will define a family of subgroups $S_{k,l}(\mathbf{G})$ of $\mathbf{G}(dk+l)$ indexed by positive integers k and non-negative integers l. Since the Lie group $\mathbf{G}(dk + l)$ must be simple, we will assume $dk + l > 4$ when $\mathbf{G} = \mathrm{SO}$.

Let $\mathbf{c} : \mathrm{O}(n) \to \mathrm{U}(n)$, $\mathbf{q} : \mathrm{U}(n) \to \mathrm{Sp}(n)$, $\mathbf{c}' : \mathrm{Sp}(n) \to \mathrm{SU}(2n)$, and $\mathbf{r} : \mathrm{U}(n) \to \mathrm{O}(2n)$ be the canonical inclusions. In order to make things clear, we write the maps $\mathbf{c}'$ and $\mathbf{r}$ explicitly as follows. Let $\mathrm{M_n}(\mathbb{K})$ be the set of all square matrices of order n over a field $\mathbb{K}$. For $A = (a_{ij}), B = (b_{ij}) \in \mathrm{M}_n(\mathbb{C})$ such that $A + B\mathbf{j} \in \mathrm{Sp}(n)$, we put

$$\mathbf{c}'(A + B\mathbf{j}) = (\mathbf{c}'(a_{ij} + b_{ij}\mathbf{j}))$$

where $\mathbf{c}'(a + \mathbf{j}b) = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$ for $a, b \in \mathbb{C}$. We also put, for $C = (c_{ij}), D = (d_{ij}) \in \mathrm{M}_n(\mathbb{R})$ such that $C + D\sqrt{-1} \in \mathrm{U}(n)$,

$$\mathbf{r}(C + D\sqrt{-1}) = (\mathbf{r}(c_{ij} + d_{ij}\sqrt{-1}))$$

where $\mathbf{r}(c + d\sqrt{-1}) = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$ for $c, d \in \mathbb{R}$. We denote the matrix $\begin{pmatrix} A & O \\ O & B \end{pmatrix}$ by $A \oplus B$. We consider the following family of subgroups of the classical groups isomorphic with $\mathrm{SU}(2) \approx S^3$:

$$\begin{aligned} S_{k,l}(\mathrm{Sp}) &= \{\alpha E_k \oplus E_l \in \mathrm{Sp}(k + l) \mid \alpha \in \mathrm{Sp}(1)\} \\ S_{k,l}(\mathrm{SU}) &= \{A \oplus E_l \in \mathrm{SU}(2k + l) \mid A \in \mathbf{c}'(S_{k,0}(\mathrm{Sp}))\} \\ S_{k,l}(\mathrm{SO}) &= \{B \oplus E_l \in \mathrm{SO}(4k + l) \mid B \in \mathbf{rc}'(S_{k,0}(\mathrm{Sp}))\} \end{aligned}$$

where E_n is the identity matrix of order n. We easily see

$$\mathbf{c}'(S_{k,l}(\mathrm{Sp})) = S_{k,2l}(\mathrm{SU}), \ \mathbf{r}(S_{k,l}(\mathrm{SU})) = S_{k,2l}(\mathrm{SO}).$$

We fix an isomorphism $\pi_3(\mathbf{G}(dk+l)) \cong \mathbb{Z}$ such that the inclusion $S_{k,l} \to \mathbf{G}(dk+l)$ represents $k \in \mathbb{Z}$.

Let $C_{k,l}(\mathbf{G})$ denote the centralizer of $S_{k,l}(\mathbf{G})$ in $\mathbf{G}(dk+l)$. Then we have

$$C_{k,l}(\mathrm{Sp}) = \mathbf{qc}(\mathrm{O}(k)) \oplus \mathrm{Sp}(l).$$

We also denote by $C_{k,l}(\mathrm{U})$ the centralizer of $S_{k,l}(\mathrm{SU})$ in $\mathrm{U}(dk+l)$. Then we have

$$C_{k,l}(\mathrm{U}) = \{A \oplus B \in \mathrm{U}(2k+l) \mid A = (a_{ij}E_2) \in \mathrm{U}(2k), B \in \mathrm{U}(l)\}.$$

In order to describe the centralizer $C_{k,l}(\mathrm{SO})$, we give another description of $S_{k,l}(\mathrm{SO})$. Define the action of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ on $\mathbb{H}$ by

$$x \cdot (p,q) = p^{-1}xq$$

for $(p,q) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ and $x \in \mathbb{H}$. It is well known that this action yields the universal covering homomorphism $\rho : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \cong \mathrm{Spin}(4) \to \mathrm{SO}(4)$. Then it easily follows that

$$S_{k,l}(\mathrm{SO}) = \{\underbrace{A \oplus \cdots \oplus A}_{k} \oplus E_l \mid A \in \rho(1 \times \mathrm{Sp}(1)) \subset \mathrm{SO}(4)\}.$$

We denote the extension $\mathbb{H} \to \mathrm{M}_4(\mathbb{R})$ of $\rho\,|_{\mathrm{Sp}(1)\times 1}$ ambiguously by the same ρ. Then one can easily verify

$$\rho(x + y\mathbf{i} + z\mathbf{j} + w\mathbf{k}) = \begin{pmatrix} x & y & z & w \\ -y & x & w & -z \\ -z & -w & x & y \\ -w & z & -y & x \end{pmatrix}$$

for $x, y, z, w \in \mathbb{R}$. The map $\rho : \mathbb{H} \to \mathrm{M}_4(\mathbb{R})$ induces a map $\bar{\rho} : \mathrm{M}_n(\mathbb{H}) \to \mathrm{M}_{4n}(\mathbb{R})$ by $\bar{\rho}(a_{ij}) = (\rho(a_{ij}))$ for $(a_{ij}) \in \mathrm{M}_n(\mathbb{H})$. Now we obtain

$$C_{k,l}(\mathrm{SO}) = \{\bar{\rho}(A) \oplus B \in \mathrm{SO}(4k+l) \mid A \in \mathrm{Sp}(k), B \in \mathrm{SO}(l)\}. \tag{2.1}$$

Summarizing the above observation on $\mathbb{C}_{k,l}(\mathbf{G})$, we get:

Proposition 2.1. *There are isomorphisms*

$$\begin{aligned} C_{k,l}(\mathrm{Sp}) &\cong \mathrm{O}(k) \times \mathrm{Sp}(l) \\ C_{k,l}(\mathrm{U}) &\cong \mathrm{U}(k) \times \mathrm{U}(l) \\ C_{k,l}(\mathrm{SO}) &\cong \mathrm{Sp}(k) \times \mathrm{SO}(l) \end{aligned}$$

satisfying a commutative diagram:

$$\begin{array}{ccccc} C_{k,l}(\mathrm{Sp}) & \xrightarrow{\mathbf{c}'} & C_{k,2l}(\mathrm{U}) & \xrightarrow{\mathbf{r}} & C_{k,4l}(\mathrm{SO}) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \mathrm{O}(k) \times \mathrm{Sp}(l) & \xrightarrow{\mathbf{c}\times\mathbf{c}'} & \mathrm{U}(k) \times \mathrm{U}(2l) & \xrightarrow{\mathbf{q}\times\mathbf{r}} & \mathrm{Sp}(k) \times \mathrm{SO}(4l) \end{array}$$

We now define a space and a map corresponding to the orbit space and the map Γ in Lemma 2.1 with respect to $S_{k,l}(\mathbf{G})$. We define a space $\mathcal{X}_{k,l}(\mathbf{G})$ by

$$\mathcal{X}_{k,l}(\mathbf{G}) = \mathbf{G}(dk+l)/C_{k,l}(\mathbf{G})$$

and a map $\Gamma_{k,l} : S_{k,l}(\mathbf{G}) \wedge \mathcal{X}_{k,l}(\mathbf{G}) \to \mathbf{G}(dk+l)$ by

$$\Gamma_{k,l}(s, gC_{k,l}(\mathbf{G})) = gsg^{-1}s^{-1}$$

for $s \in S_{k,l}(\mathbf{G}), g \in \mathbf{G}(dk+l)$. We will identify $S_{k,l}(\mathbf{G})$ with S^3 if there is no confusion. It is obvious that the inclusions $\mathbf{G}(dk+l) \to \mathbf{G}(dk+(l+1))$ and $\mathbf{G}(dk+l) \to \mathbf{G}(d(k+1)+l)$ induce the commutative diagram:

$$\begin{array}{ccccc} S^3 \wedge \mathcal{X}_{k+1,l}(\mathbf{G}) & \longleftarrow & S^3 \wedge \mathcal{X}_{k,l}(\mathbf{G}) & \longrightarrow & S^3 \wedge \mathcal{X}_{k,l+1}(\mathbf{G}) \\ \big\downarrow{\scriptstyle \Gamma_{k+1,l}} & & \big\downarrow{\scriptstyle \Gamma_{k,l}} & & \big\downarrow{\scriptstyle \Gamma_{k,l+1}} \\ \mathbf{G}(d(k+1)+l) & \longleftarrow & \mathbf{G}(dk+l) & \longrightarrow & \mathbf{G}(dk+(l+1)) \end{array} \tag{2.2}$$

By the above observation on $C_{k,l}(\mathrm{SU})$ and $C_{k,l}(\mathrm{U})$, we see that there is a diffeomorphism:

$$\mathcal{X}_{k,l}(\mathrm{SU}) \cong \mathrm{U}(2k+l)/C_{k,l}(\mathrm{U}) \tag{2.3}$$

Note that $\mathbf{c}' : \mathrm{Sp}(k+l) \to \mathrm{SU}(2k+2l)$ and $\mathbf{r} : \mathrm{SU}(k+l) \to \mathrm{SO}(2k+2l)$ are homomorphisms which restrict to surjections $S_{k,l}(\mathrm{Sp}) \to S_{k,2l}(\mathrm{SU})$ and $S_{k,l}(\mathrm{SU}) \to S_{k,2l}(\mathrm{SO})$, respectively. Then they induce maps $\mathbf{c}' : \mathcal{X}_{k,l}(\mathrm{Sp}) \to \mathcal{X}_{k,2l}(\mathrm{SU})$ and $\mathbf{r} : \mathcal{X}_{k,l}(\mathrm{SU}) \to \mathcal{X}_{k,2l}(\mathrm{SO})$ satisfying a commutative diagram:

$$\begin{array}{ccccc} S^3 \wedge \mathcal{X}_{k,l}(\mathrm{Sp}) & \xrightarrow{1\wedge \mathbf{c}'} & S^3 \wedge \mathcal{X}_{k,2l}(\mathrm{SU}) & \xrightarrow{1\wedge \mathbf{r}} & S^3 \wedge \mathcal{X}_{k,4l}(\mathrm{SO}) \\ \big\downarrow{\scriptstyle \Gamma_{k,l}} & & \big\downarrow{\scriptstyle \Gamma_{k,2l}} & & \big\downarrow{\scriptstyle \Gamma_{k,2l}} \\ \mathrm{Sp}(k+l) & \xrightarrow{\mathbf{c}'} & \mathrm{SU}(2k+2l) & \xrightarrow{\mathbf{r}} & \mathrm{SO}(4k+4l) \end{array} \tag{2.4}$$

We observe a relation between $\mathcal{X}_{1,l}(\mathbf{G})$ and a projective space. It follows from Proposition 2.1 that $\mathcal{X}_{1,l}(\mathrm{Sp}) = \mathbb{R}P^{4l+3}$ and also that $\mathcal{X}_{1,l}(\mathrm{SU})$ is the total space of the unit tangent bundle of $\mathbb{C}P^{l+1}$. Note that the map $\rho : \mathbb{H} \to \mathrm{M}_4(\mathbb{R})$ above induces a homomorphism $\rho : \mathrm{Sp}(n) \to \mathrm{SO}(4n)$. Then there is a map $\mathbb{H}P^{[\frac{l}{4}]} \to \mathcal{X}_{1,l}(\mathrm{SO})$ which is natural with respect to the maps $\mathbb{H}P^{[\frac{l}{4}]} \to \mathbb{H}P^{[\frac{l+1}{4}]}$ and $\mathcal{X}_{1,l}(\mathrm{SO}) \to \mathcal{X}_{1,l+1}(\mathrm{SO})$. We regard $\mathbb{H}P^{[\frac{l}{4}]}$ to be a subspace of $\mathcal{X}_{1,l}(\mathrm{SO})$ by this map. Put $\Gamma'_{1,l}$ to be the restriction of $\Gamma_{1,l} : S^3 \wedge \mathcal{X}_{4,l}(\mathrm{SO}) \to \mathrm{SO}(4+l)$ onto $\mathbb{H}P^{[\frac{l}{4}]} \subset \mathcal{X}_{4,l}(\mathrm{SO})$. Then we have an obvious commutative diagram:

$$\begin{array}{ccc} S^3 \wedge \mathbb{H}P^{[\frac{l}{4}]} & \longrightarrow & S^3 \wedge \mathbb{H}P^{[\frac{l+1}{4}]} \\ \big\downarrow{\scriptstyle \Gamma'_{1,l}} & & \big\downarrow{\scriptstyle \Gamma'_{1,l+1}} \\ \mathrm{SO}(4+l) & \longrightarrow & \mathrm{SO}(5+l) \end{array} \tag{2.5}$$

We next consider the map $\Gamma_{k,l}$ when l tends to ∞. Put $\mathcal{X}_{k,\infty}(\mathbf{G}) = \operatorname{colim}_{l} \mathcal{X}_{k,l}(\mathbf{G})$. Then, by (2.2), we have a map

$$\operatorname{colim}_{l} \Gamma_{k,l} : S^3 \wedge \mathcal{X}_{k,\infty}(\mathbf{G}) \to \mathbf{G}(\infty).$$

which we denote by Now for $\mathbf{G} = \mathrm{Sp}, \mathrm{SO}$, there is a principal bundle

$$\mathbf{H}(k) \to \mathbf{G}(dk+l)/\mathbf{G}(l) \to \mathcal{X}_{k,l}(\mathbf{G})$$

by Proposition 2.1 where $\mathbf{G}(dk+l)/\mathbf{G}(l)$ is $(4l+2)$-connected and $(l-1)$-connected according as $\mathbf{G} = \mathrm{Sp}, \mathrm{SO}$. By Proposition 2.1 and (2.3), we also have a principal bundle

$$\mathrm{U}(k) \to \mathrm{U}(2k+l)/\mathrm{U}(l) \to \mathcal{X}_{k,l}(\mathrm{SU})$$

in which $\mathrm{U}(2k+l)/\mathrm{U}(l)$ is $2l$-connected. Then it follows that there is a homotopy equivalence

$$\mathcal{X}_{k,\infty}(\mathbf{G}) \simeq B\mathbf{H}(k)$$

and thus we obtain a map

$$\Gamma_{k,\infty} : S^3 \wedge B\mathbf{H}(k) \to \mathbf{G}(\infty).$$

Moreover, by Proposition 2.1 and (2.4), we get:

Proposition 2.2. *There is a homotopy commutative diagram:*

$$\begin{array}{ccccc} S^3 \wedge B\mathrm{O}(k) & \xrightarrow{1\wedge \mathbf{c}} & S^3 \wedge B\mathrm{U}(k) & \xrightarrow{1\wedge \mathbf{q}} & S^3 \wedge B\mathrm{Sp}(k) \\ \downarrow{\scriptstyle \Gamma_{k,\infty}} & & \downarrow{\scriptstyle \Gamma_{k,\infty}} & & \downarrow{\scriptstyle \Gamma_{k,\infty}} \\ \mathrm{Sp}(\infty) & \xrightarrow{\mathbf{c}'} & \mathrm{SU}(\infty) & \xrightarrow{\mathbf{r}} & \mathrm{SO}(\infty) \end{array}$$

Note that, by (2.5), we also have a map $\Gamma'_{1,\infty} : S^3 \wedge \mathbb{H}P^\infty \to \mathrm{SO}(\infty)$ which coincides with the map $\Gamma_{1,\infty} : S^3 \wedge B\mathrm{Sp}(1) \to \mathrm{SO}(\infty)$.

We see from (2.2) that $\Gamma_{k,\infty}$ satisfies a homotopy commutative diagram

$$\begin{array}{ccc} S^3 \wedge B\mathbf{H}(k) & \longrightarrow & S^3 \wedge B\mathbf{H}(k+1) \\ \downarrow{\scriptstyle \Gamma_{k,\infty}} & & \downarrow{\scriptstyle \Gamma_{k+1,\infty}} \\ \mathbf{G}(\infty) & = & \mathbf{G}(\infty) \end{array} \tag{2.6}$$

where the top horizontal arrow is induced from the inclusion $\mathbf{H}(k) \to \mathbf{H}(k+1)$. Then we get a map

$$\Gamma_{\infty,\infty} = \operatorname{colim}_{k} \Gamma_{k,\infty} : S^3 \wedge B\mathbf{H}(\infty) \to \mathbf{G}(\infty).$$

Let $\mu : \mathbf{G}(n) \times \mathbf{G}(n) \to \mathbf{G}(2n)$ be an inclusion such as by $\mu(A,B) = A \oplus B$ for $A, B \in \mathbf{G}(n)$. Then μ induces a map $\mathcal{X}_{k,l}(\mathbf{G}) \times \mathcal{X}_{k,l}(\mathbf{G}) \to \mathcal{X}_{2k,2l}(\mathbf{G})$, denoted by the same symbol μ, which yields the standard H-space structure on $B\mathbf{H}(\infty) \simeq \mathcal{X}_{\infty,\infty}(\mathbf{G})$. Moreover, the map μ satisfies a commutative diagram

$$\begin{array}{ccc} S^3 \wedge (\mathcal{X}_{k,l}(\mathbf{G}) \times \mathcal{X}_{k,l}(\mathbf{G})) & \xrightarrow{1\wedge\mu} & S^3 \wedge \mathcal{X}_{2k,2l}(\mathbf{G}) \\ \Big\downarrow{\scriptstyle\Delta} & & \\ (S^3 \wedge \mathcal{X}_{k,l}(\mathbf{G})) \times (S^3 \wedge \mathcal{X}_{k,l}(\mathbf{G})) & & \Big\downarrow{\scriptstyle\Gamma_{2k,2l}} \\ \Big\downarrow{\scriptstyle\Gamma_{k,l}\times\Gamma_{k,l}} & & \\ \mathbf{G}(dk+l) \times \mathbf{G}(dk+l) & \xrightarrow{\mu} & \mathbf{G}(2dk+2l) \end{array}$$

where Δ is defined by $\Delta(s,x,y) = (s,x,s,y)$ for $s \in S^3, x,y \in \mathcal{X}_{k,l}(\mathbf{G})$. Let $\mathrm{ad} : [\Sigma X, Y] \cong [X, \Omega Y]$ denote the adjoint congruence. Then we have established:

Lemma 2.2. *The map* $\mathrm{ad}^3\Gamma_{\infty,\infty} : B\mathbf{H}(\infty) \to \Omega_0^3\mathbf{G}(\infty)$ *is an H-map.*

We will show that the image of $\mathrm{ad}^3\Gamma_{1,l}$ in homology generates the Pontrjagin ring of $\Omega_0^3\mathbf{G}(dk+l)$ in a range, which is an analogue of the generating variety for a loop space of a Lie group, and that the map $\mathrm{ad}^3\Gamma_{\infty,\infty}$ yields Bott periodicity.

3 Cohomology calculation for $\Gamma_{1,l}$

In this section, we give a cohomology calculation for the map $\Gamma_{1,l}$ and $\Gamma'_{1,l}$. We first consider the case $\mathbf{G} = \mathrm{SO}$. In this case, we calculate $\Gamma'_{1,l}$ in cohomology instead of $\Gamma_{1,l}$ since the cohomology of $\mathcal{X}_{1,l}(\mathrm{SO})$ is complicated as is seen in [14].

Proposition 3.1. *For $l \geq 4$, the map* $(\Gamma'_{1,l})^* : H^*(\mathrm{SO}(4+l);\mathbb{Z}/2) \to H^*(S^3 \wedge \mathbb{H}P^{[\frac{l}{4}]};\mathbb{Z}/2)$ *is surjective.*

Proof. Recall first that the mod 2 cohomology of $\mathrm{SO}(4+l)$ is given as

$$H^*(\mathrm{SO}(4+l);\mathbb{Z}/2) = \mathbb{Z}/2[x_1, x_3, \ldots] \text{ for } * \leq 3+l,$$

where x_i is the suspension of the Stiefel-Whitney class w_{i+1}. Let u_3 be a generator of $H^3(S^3;\mathbb{Z}/2)$. Then, by definition, the inclusion $\iota : S^3 = S_{1,l}(\mathrm{SO}) \to \mathrm{SO}(4+l)$ induces the map in cohomology such as $\iota^*(x_3) = u_3$.

Let us consider the case $l = 12$. Let $P\mathrm{SO}(n)$ denote the n-dimensional projective orthogonal group, that is, $\mathrm{SO}(n)$ divided by its center. It is well known that

$$H^*(P\mathrm{SO}(16);\mathbb{Z}/2) = \mathbb{Z}/2[v, \bar{x}_1, \bar{x}_3, \bar{x}_5, \bar{x}_7] \text{ for } * \leq 7$$

where $|v| = 1$ and $\pi^*(\bar{x}_i) = x_i$ for the projection $\pi : \mathrm{SO}(16) \to P\mathrm{SO}(16)$. Moreover, we see from [4] that the Hopf algebra structure of $H^*(P\mathrm{SO}(16); \mathbb{Z}/2)$ is given as

$$\bar{\phi}^*(v) = 0,\ \bar{\phi}^*(\bar{x}_i) = \sum_{j=1}^{i} a_{ij}\bar{x}_j \otimes v^{i-j}$$

for $i = 1, 3, 5, 7$ in which $a_{53} = 0, a_{73} = 1$, where $\bar{\phi}$ stands for the reduced comultiplication. Let $\gamma : P\mathrm{SO}(16) \wedge P\mathrm{SO}(16) \to P\mathrm{SO}(16)$ be the reduced commutator map and let $\tilde{\gamma} : \mathrm{SO}(16) \wedge PO(16) \to \mathrm{SO}(16)$ be a lift of γ. Then by a straightforward calculation, we have

$$\tilde{\gamma}^*(x_7) = u_3 \otimes v^4.$$

On the other hand, since the center of SO(16) is included in $C_{1,12}(\mathrm{SO})$, we have the projection $P\mathrm{SO}(16) \to \mathcal{X}_{1,12}(\mathrm{SO})$ satisfying a commutative diagram

$$\begin{array}{ccccc}
\mathrm{SO}(16) & \longrightarrow & \mathrm{SO}(16)/\mathrm{SO}(12) & \xleftarrow{\mathbf{rc}'} & \mathrm{Sp}(4)/\mathrm{Sp}(3) \\
{\scriptstyle\pi}\downarrow & & \downarrow & & \downarrow \\
P\mathrm{SO}(16) & \longrightarrow & \mathcal{X}_{1,12}(\mathrm{SO}) & \longleftarrow & \mathbb{H}P^3 \\
\downarrow & & \downarrow & & \downarrow \\
B(\mathbb{Z}/2) & \longrightarrow & B\mathrm{Sp}(1) & = & B\mathrm{Sp}(1)
\end{array}$$

where $\mathbb{Z}/2$ is the center of Sp(1). Then we see that a generator x of $H^4(\mathcal{X}_{1,12}(\mathrm{SO}); \mathbb{Z}/2)$ satisfies

$$\pi^*(x) = v^4,\ (\mathbf{rc}')^*(x) = q,$$

where q is a generator of $H^4(\mathbb{H}P^n; \mathbb{Z}/2)$. Now we have a commutative diagram:

$$\begin{array}{ccccc}
S^3 \wedge P\mathrm{SO}(16) & \longrightarrow & S^3 \wedge \mathcal{X}_{1,12}(\mathrm{SO}) & \longleftarrow & S^3 \wedge \mathbb{H}P^3 \\
\downarrow{\scriptstyle\tilde{\gamma}} & & \downarrow{\scriptstyle\Gamma_{1,12}} & & \downarrow{\scriptstyle\Gamma'_{1,12}} \\
\mathrm{SO}(16) & = & \mathrm{SO}(16) & = & \mathrm{SO}(16)
\end{array}$$

Then we obtain

$$(\Gamma'_{1,12})^*(x_7) = u_3 \otimes q.$$

By (2.5), we have established

$$(\Gamma'_{1,l})^*(x_7) = u_3 \otimes q. \tag{3.1}$$

By the Wu formula, we have

$$\mathrm{Sq}^4 x_{4i-1} = (i-1)x_{4i+3},\ \mathrm{Sq}^8 x_{4i-1} = \binom{i-1}{2} x_{4i+7}$$

in $H^*(\mathrm{SO}(4+l); \mathbb{Z}/2)$ for $* < 4 + l$. Then, applying this to (3.1), the proof is completed. □

Proposition 3.2. *For $i > 0$, the map* $\Gamma_{1,l}^* : H^{4i+3}(\mathrm{Sp}(1+l); \mathbb{Z}/2) \to H^{4i+3}(S^3 \wedge \mathbb{R}P^{4l+3}; \mathbb{Z}/2)$ *is surjective.*

Proof. Let w and q be generators of $H^1(\mathbb{R}P^\infty; \mathbb{Z}/2)$ and $H^4(\mathbb{H}P^\infty; \mathbb{Z}/2)$, respectively. Then the map $\mathbf{qc} : \mathbb{R}P^\infty \to \mathbb{H}P^\infty$ induces $(\mathbf{qc})^*(q) = w^4$ in cohomology. Recall that the mod 2 cohomology of $\mathrm{Sp}(n)$ is given as

$$H^*(\mathrm{Sp}(n); \mathbb{Z}/2) = \Lambda(y_3, y_7, \ldots, y_{4n-1})$$

where y_{4i-1} is the suspension of the modulo 2 reduction of the symplectic Pontrjagin class q_i. Then we have $(\mathbf{rc}')^*(x_{4i-1}) = y_{4i-1}$ here we use the same notation for the mod 2 cohomology of $\mathrm{SO}(\infty)$ as in the proof of Proposition 3.1. Then, for $l = \infty$, the proposition follows from Proposition 3.1 and (2.2). Thus the proof is completed by (2.2). □

Let $X\langle n \rangle$ denote the n-connective cover of a path-connected space X. Then, in general, any map $f : S^3 \wedge A \to X$ with A path-connected lifts to $X\langle 3 \rangle$ which we denote by $\tilde{f}$.

Proposition 3.3. *Any lift* $\tilde{\Gamma}_{1,\infty} : S^3 \wedge \mathbb{C}P^\infty \to (\mathrm{SU}(\infty))\langle 3 \rangle$ *of* $\Gamma_{1,\infty} : S^3 \wedge \mathbb{C}P^\infty \to \mathrm{SU}(\infty)$ *induces an isomorphism* $\tilde{\Gamma}_{1,\infty}^* : H^5((\mathrm{SU}(\infty))\langle 3 \rangle; \mathbb{Z}) \xrightarrow{\cong} H^5(S^3 \wedge \mathbb{C}P^\infty; \mathbb{Z})$.

Proof. We will denote the modulo p reduction in cohomology by ρ_p for a prime p.

The integral cohomology of $\mathrm{SU}(n)$ is

$$H^*(\mathrm{SU}(n); \mathbb{Z}) = \Lambda(e_3, e_5, \ldots, e_{2n-1}),$$

where e_{2i-1} is the suspension of the Chern class c_i. Then, by considering the Serre spectral sequence of a fibre sequence $\mathbb{C}P^\infty \to (\mathrm{SU}(\infty))\langle 3 \rangle \xrightarrow{q} \mathrm{SU}(\infty)$, we see that $H^5((\mathrm{SU}(\infty))\langle 3 \rangle; \mathbb{Z}) \cong \mathbb{Z}$ is generated by ϵ such that

$$q^*(e_5) = 2\epsilon. \tag{3.2}$$

Let $P\mathrm{SU}(n)$ be the n-dimensional projective unitary group, that is, $\mathrm{SU}(n)$ divided by its center. Let p be an odd prime. In [4], it is shown that

$$H^*(P\mathrm{SU}(p^r); \mathbb{Z}/p) = \mathbb{Z}/p[v]/(v^{p^r}) \otimes \Lambda(\bar{e}_1, \bar{e}_3, \ldots, \bar{e}_{2p^r-1})$$

where $|v| = 2$ and $\bar{\pi}^*(\bar{e}_i) = \rho_p(e_i)$ for the projection $\bar{\pi} : \mathrm{SU}(p^r) \to P\mathrm{SU}(p^r)$. Moreover, for the reduced comultiplication $\bar{\phi}$, we have

$$\bar{\phi}(\bar{e}_5) = a_1\bar{e}_3 \otimes v + a_2\bar{e}_1 \otimes v^2$$

for $a_1, a_2 \in (\mathbb{Z}/p)^\times$. Let c and u_3 be generators of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$ and $H^3(S^3; \mathbb{Z})$ respectively. Then, as in the proof of Proposition 3.1, we see that

$$\Gamma_{1,\infty}^*(\rho_p(e_5)) = a\rho_p(u_3 \otimes c)$$

for $a \in (\mathbb{Z}/p)^\times$. Note that the above equation holds for any odd prime p. Then we have obtained, in the integral cohomology, that

$$\Gamma_{1,\infty}^*(e_5) = \pm 2^b u_3 \otimes c$$

for some non-negative integer b, and thus by (3.2),

$$\tilde{\Gamma}_{1,\infty}^*(\epsilon) = \pm 2^{b-1} u_3 \otimes c$$

which implies that b is positive. Since $H^5(S^3 \wedge \mathbb{R}P^2; \mathbb{Z}) \cong \mathbb{Z}/2$, it follows from Lemma 3.2 below and (2.2) that $\tilde{\Gamma}_{1,\infty}(\rho_2(\epsilon)) \neq 0$ in the mod 2 cohomology, which yields $b = 1$. Thus the proof is done. □

Lemma 3.1. *Let $\theta : \mathbb{R}P^2 \to \mathrm{SO}(6)$ and $\iota : S^3 = S_{1,2}(\mathrm{SO}) \to \mathrm{SO}(6)$ be the inclusions. Then the Samelson product $\langle \iota, \theta \rangle$ is essential.*

Proof. By the adjointness of Whitehead products and Samelson products, we show that the Whitehead product of $\mathrm{ad}^{-1}\iota : S^4 \to B\mathrm{SO}(6)$ and $\mathrm{ad}^{-1}\theta : \Sigma\mathbb{R}P^2 \to B\mathrm{SO}(6)$ is essential. Suppose now that $[\mathrm{ad}^{-1}\iota, \mathrm{ad}^{-1}\theta] = 0$. Then there exists a map $\kappa : S^4 \times \Sigma\mathbb{R}P^2 \to B\mathrm{SO}(6)$ satisfying the homotopy commutative diagram:

$$\begin{array}{ccc} S^4 \vee \Sigma\mathbb{R}P^2 & \xrightarrow{\mathrm{ad}^{-1}\iota \vee \mathrm{ad}^{-1}\theta} & B\mathrm{SO}(6) \\ \downarrow & & \| \\ S^4 \times \Sigma\mathbb{R}P^2 & \xrightarrow{\kappa} & B\mathrm{SO}(6) \end{array}$$

Let w and u_4 be generators of $H^1(\mathbb{R}P^2; \mathbb{Z}/2)$ and $H^4(S^4; \mathbb{Z}/2)$, respectively. Then, by definition, we have $\kappa^*(w_3) = 1 \otimes \Sigma w^2$ and $\kappa^*(w_4) = u_4 \otimes 1$, where w_i is the Stiefel-Whitney class. On the other hand, it follows from the Wu formula that $\mathrm{Sq}^3 w_4 = w_3 w_4$. Thus we obtain

$$0 = \mathrm{Sq}^3(u_4 \otimes 1) = \mathrm{Sq}^3 \kappa^*(w_4) = \kappa^*(\mathrm{Sq}^3 w_4) = \kappa^*(w_3 w_4) = u_4 \otimes \Sigma w^2 \neq 0$$

which is a contradiction. Therefore we have established the Whitehead product $[\mathrm{ad}^{-1}\iota, \mathrm{ad}^{-1}\theta]$ is essential. □

Recall that there is an isomorphism $\mathrm{SU}(4) \cong \mathrm{Spin}(6)$. Since the center of $\mathrm{SU}(4) \cong \mathrm{Spin}(6)$ is included in $C_{1,2}(\mathrm{SU})$, there is a projection $\pi : \mathrm{SO}(6) \to \mathcal{X}_{1,2}(\mathrm{SU})$.

Lemma 3.2. *Let $\theta : \mathbb{R}P^2 \to \mathrm{SO}(6)$ be the inclusion and let $\lambda : S^3 \wedge \mathbb{R}P^2 \to (\mathrm{SU}(4))\langle 3 \rangle$ be the composite:*

$$S^3 \wedge \mathbb{R}P^2 \xrightarrow{1\wedge\theta} S^3 \wedge \mathrm{SO}(6) \xrightarrow{1\wedge\pi} S^3 \wedge \mathcal{X}_{1,2}(\mathrm{SU}) \xrightarrow{\tilde{\Gamma}_{1,2}} (\mathrm{SU}(4))\langle 3 \rangle$$

Then $\lambda^*(\epsilon) \neq 0$, where ϵ is a generator of $H^5((\mathrm{SU}(4))\langle 3 \rangle; \mathbb{Z}) \cong \mathbb{Z}$ as above.

Proof. Since $S^3 \wedge \mathbb{R}P^2$ is 3-connected, the projection $(\mathrm{SO}(6))\langle 3\rangle \to \mathrm{SO}(6)$ induces an injection $[S^3 \wedge \mathbb{R}P^2, (\mathrm{SO}(6))\langle 3\rangle] \to [S^3 \wedge \mathbb{R}P^2, \mathrm{SO}(6)]$ of pointed homotopy set. By Lemma 3.1, we know that the Samelson product $\langle \iota, \theta\rangle$ is essential, and then so is its lift $S^3 \wedge \mathbb{R}P^2 \to (\mathrm{SO}(6))\langle 3\rangle$.

Let $\tilde{\gamma} : S^3 \wedge \mathrm{SO}(6) \to (\mathrm{SO}(6))\langle 3\rangle$ be a lift of the restriction of the reduced commutator of $\mathrm{SO}(6)$ to $S^3 \wedge \mathrm{SO}(6) = S_{1,2}(\mathrm{SO}) \wedge \mathrm{SO}(6)$. Then we have a homotopy commutative diagram:

$$\begin{array}{ccc} S^3 \wedge \mathrm{SO}(6) & \xrightarrow{1\wedge\pi} & S^3 \wedge \mathcal{X}_{1,2}(\mathrm{SU}) \\ \tilde{\gamma}\downarrow & & \downarrow\tilde{\Gamma}_{1,2} \\ (\mathrm{SO}(6))\langle 3\rangle & = & (\mathrm{SU}(4))\langle 3\rangle \end{array}$$

Thus we have established that λ is essential. Now since $S^3 \wedge \mathbb{R}P^2$ is of dimension 5 and $(\mathrm{SU}(4))\langle 3\rangle$ is 4-connected, it follows from the J.H.C. Whitehead theorem that $\lambda^*(\epsilon) \neq 0$. □

4 Generating variety for $\Omega_0^3\mathbf{G}(n)$

The aim of this section is to prove that it holds for $\Omega_0^3\mathbf{G}(d+l)$ by the map $\Gamma_{1,l}$ and $\Gamma'_{1,l}$ in the stable range of $\Omega_0^3\mathbf{G}(d+l)$, the generating variety argument which is analogous to single loop spaces of Lie groups in [6]. The proofs are done by a similar calculation in [16].

Theorem 4.1. *For $* \leq l$, the Pontrjagin ring $H_*(\Omega_0^3\mathrm{SO}(4+l);\mathbb{Z}/2)$ is a polynomial ring generated by the image of* $(\mathrm{ad}^3\Gamma'_{1,l})_* : H_*(\mathbb{H}P^{[\frac{l}{4}]};\mathbb{Z}/2) \to H_*(\Omega_0^3\mathrm{SO}(4+l);\mathbb{Z}/2)$.

Proof. We first prove the case $l = \infty$. We will use the same notation for the mod 2 cohomology of $\mathrm{SO}(\infty)$ as in the proof of Proposition 3.1. Then, in particular, we have

$$\mathrm{Sq}^{2i-2}x_{2i-1} = x_{4i-3},\ \mathrm{Sq}^{4i-3}x_{4i-1} = 0.$$

Let q and u_n be generators of $H^4(\mathbb{H}P^\infty;\mathbb{Z}/2)$ and $H^n(S^n;\mathbb{Z}/2)$ as above, respectively . Then it follows from Proposition 3.1 that

$$(\Gamma'_{1,\infty})^*(x_{4i-1}) = u_3 \otimes q^{i-1}.$$

Since $\pi_1(\mathrm{SO}(\infty)) \cong \mathbb{Z}/2$, we have

$$H^*((\mathrm{SO}(\infty))\langle 1\rangle;\mathbb{Z}/2) = \mathbb{Z}/2[\pi^*(x_3), \pi^*(x_5), \pi^*(x_7), \ldots],$$

where $\pi : (\mathrm{SO}(\infty))\langle 1\rangle \to \mathrm{SO}(\infty)$ denotes the projection. Then, by the Borel transgression theorem, we have

$$H^*(\Omega_0\mathrm{SO}(\infty);\mathbb{Z}/2) = \Delta(y_2, y_4, y_6, \ldots),\ (\mathrm{ad}\Gamma'_{1,\infty})^*(y_{4i-2}) = u_2 \otimes q^{i-1}$$

and

$$y_{2i-2}^2 = \mathrm{Sq}^{2i-2} y_{2i-2} = y_{4i-4},\ \mathrm{Sq}^{4i-3} y_{4i-2} = 0,$$

where y_i is the suspension of x_{i+1} and $\Delta(a_1, a_2, \ldots)$ stands for the simple system of generators $\{a_1, a_2, \ldots\}$. It is rewritten as

$$H^*(\Omega_0 \mathrm{SO}(\infty); \mathbb{Z}/2) = \mathbb{Z}/2[y_2, y_6, y_{10}, \ldots].$$

Then it follows from the Borel transgression theorem that

$$H^*(\Omega_0^2 \mathrm{SO}(\infty); \mathbb{Z}/2) = \Delta(z_1, z_5, z_9, \ldots),\ (\mathrm{ad}^2 \Gamma'_{1,\infty})^*(z_{4i-3}) = u_1 \otimes q^{i-1}$$

and

$$z_{4i-3}^2 = \mathrm{Sq}^{4i-3} z_{4i-3} = 0,$$

where z_i is the suspension of y_{i+1}. Namely, we have

$$H^*(\Omega_0^2 \mathrm{SO}(\infty); \mathbb{Z}/2) = \Lambda(z_1, z_5, z_9, \ldots).$$

Now we take the dual Hopf algebra of $H^*(\Omega_0^2 \mathrm{SO}(\infty); \mathbb{Z}/2)$ to get

$$H_*(\Omega_0^2 \mathrm{SO}(\infty); \mathbb{Z}/2) = \Lambda(z_1^\sharp, z_5^\sharp, z_9^\sharp, \ldots),\ (\mathrm{ad}^2 \Gamma'_{1,\infty})_*(u_1^\sharp \otimes (q^{i-1})^\sharp) = z_{4i-3}^\sharp,$$

where $x^\sharp$ means the Kronecker dual of x. Since $\pi_3(\mathrm{SO}(\infty)) \cong \mathbb{Z}$, we have

$$H_*((\Omega_0^2 \mathrm{SO}(\infty))\langle 1 \rangle; \mathbb{Z}/2) = \Lambda(s_5, s_9, s_{13}, \ldots),$$

where s_i is defined by $\pi'_*(s_i) = z_i^\sharp$ for the projection $\pi' : (\Omega_0^2 \mathrm{SO}(\infty))\langle 1 \rangle \to \Omega_0^2 \mathrm{SO}(\infty)$. Then, by the Borel transgression theorem, we have, for $* < l$,

$$H_*(\Omega_0^3 \mathrm{SO}(\infty); \mathbb{Z}/2) = \mathbb{Z}/2[t_4, t_8, t_{12}, \ldots],\ (\mathrm{ad}^3 \Gamma'_{1,\infty})_*((q^{i-1})^\sharp) = t_{4i-4}$$

in which s_{i+1} is the transgression image of t_i, and therefore the proof is completed.

Note that the inclusion $\mathrm{SO}(4+l) \to \mathrm{SO}(\infty)$ is a $(4+l)$-equivalence. Then the inclusion $\Omega_0^3 \mathrm{SO}(4+l) \to \Omega_0^3 \mathrm{SO}(\infty)$ is a $(1+l)$-equivalence, and thus the theorem follows from (2.5). □

In proving the generating variety argument for $\Omega_0^3 \mathrm{Sp}(1+l)$, we will use:

Lemma 4.1 (S. Araki and T. Kudo [1]). *Let X be a simply connected homotopy associative H-space. If $H_*(X; \mathbb{Z}/2) = \mathbb{Z}/2[x_1, x_2, \ldots]$ and each x_i is transgressive, then we have*

$$H_*(\Omega X; \mathbb{Z}/2) = \mathbb{Z}/2[y_1^0, y_1^1, \ldots, y_2^0, y_2^1, \ldots]$$

where y_k^l is the transgression image of $x_k^{2^l}$.

Theorem 4.2. *For $* \leq 4l+2$, the Pontrjagin ring $H_*(\Omega_0^3\mathrm{Sp}(1+l);\mathbb{Z}/2)$ is a polynomial ring generated by the image of $(\mathrm{ad}^3\Gamma_{1,l})_* : H_*(\mathbb{R}P^{4l+3};\mathbb{Z}/2) \to H_*(\Omega_0^3\mathrm{Sp}(1+l);\mathbb{Z}/2)$.*

Proof. We first prove the case $l=\infty$. We will use the same notation for the mod 2 cohomology of $\mathrm{Sp}(1+l)$ as in the proof of Proposition 3.2. Let u_n and w be generators of $H^n(S^n;\mathbb{Z}/2)$ and $H^1(\mathbb{R}P^\infty;\mathbb{Z}/2)$, respectively, as well as above. Then it follows from Proposition 3.2 that

$$\Gamma_{1,\infty}^*(y_{4i-1}) = u_3 \otimes w^{4i-4}.$$

Now we take the dual Hopf algebra of $H^*(\mathrm{Sp}(\infty);\mathbb{Z}/2)$ so that

$$H_*(\mathrm{Sp}(\infty);\mathbb{Z}/2) = \Lambda(y_3^\sharp, y_7^\sharp, \ldots),\ (\Gamma_{1,\infty})_*(u_3^\sharp \otimes (w^{4i-4})^\sharp) = y_{4i-1}^\sharp.$$

Then, by the Borel transgression theorem, we get

$$H_*(\Omega\mathrm{Sp}(\infty);\mathbb{Z}/2) = \mathbb{Z}/2[z_2, z_6, \ldots],\ (\mathrm{ad}\Gamma_{1,\infty})_*(u_2^\sharp \otimes (w^{4i-4})^\sharp) = z_{4i-2}$$

in which z_i is the transgression image of y_{i+1}.

By Lemma 5.1 in the next section, Theorem 4.1 implies the map

$$(\mathrm{ad}^3\Gamma_{\infty,\infty})_* : H_*(B\mathrm{Sp}(\infty);\mathbb{Z}/2) \to H_*(\Omega_0^3\mathrm{SO}(\infty);\mathbb{Z}/2)$$

is an isomorphism. Then since $B\mathrm{Sp}(\infty)$ and $\Omega_0^3\mathrm{SO}(\infty)$ are of finite type, we deduce that the map $(\mathrm{ad}^3\Gamma_{\infty,\infty})_{(2)} : B\mathrm{Sp}(\infty)_{(2)} \simeq \Omega_0^3\mathrm{SO}(\infty)_{(2)}$ is a homotopy equivalence, where $-_{(2)}$ means the 2-localization in the sense of Bousfield and Kan [8]. In particular, we can consider the action of the Kudo-Araki operation Q^{4i} on $q_i^\sharp \in H_*(B\mathrm{Sp}(\infty);\mathbb{Z}/2)$, where q_i is the mod 2 reduction of the symplectic Pontrjagin class. (See [11].) Recall that in $H^*(B\mathrm{Sp}(\infty);\mathbb{Z}/2)$, we have

$$q_i^\sharp = (q_1^\sharp)^i.$$

Then, in particular,

$$Q^{4i}q_i^\sharp = (q_i^\sharp)^2 = (q_1^\sharp)^{2i} = q_{2i}.$$

Since Q^{4i} commutes with the transgression, we obtain

$$Q^{4i}z_{4i-2} = z_{8i-2}.$$

Then it follows from the Nishida relation $\mathrm{Sq}_*^2 Q^s = \binom{s-2}{2}Q^{s-2} + Q^{s-1}\mathrm{Sq}_*^1$ that

$$\mathrm{Sq}_*^2 z_{8i-2} = z_{4i-2}^2,$$

where Sq_*^k denotes the dual of Sq^k. (See [11].) Since $(\mathrm{ad}\Gamma_{1,\infty})_*(u_2^\sharp \otimes (w^{4i-4})^\sharp) = z_{4i-2}$ and $\mathrm{Sq}_*^2(w^{4i})^\sharp = (w^{4i-2})^\sharp$, we have established

$$(\mathrm{ad}\Gamma_{1,\infty})_*(u_2^\sharp \otimes (w^{2i-2})^\sharp) = z_{2i}.$$

Applying Lemma 4.1, we get

$$H_*(\Omega^2\mathrm{Sp}(\infty);\mathbb{Z}/2)=\mathbb{Z}/2[s_1,s_3,\ldots],\ (\mathrm{ad}^2\Gamma_{1,\infty})_*(u_1^\sharp\otimes(w^{2i-2})^\sharp)=s_{2i-1}$$

where $s_{2^m(4n-2)-1}$ is the transgression image of $z_{4n-2}^{2^m}$. Note that we can consider the operation Q^{i-1} and Q^i on z_i. By the Nishida relation $\mathrm{Sq}^1_*Q^s=(s-1)Q^{s-1}$, we see that, for $m\geq 1$,

$$\mathrm{Sq}^1_*z_{4n-2}^{2^m}=\mathrm{Sq}^1_*Q^{2^{m-1}(4n-2)}z_{4n-2}^{2^{m-1}}=Q^{2^{m-1}(4n-2)-1}z_{4n-2}^{2^{m-1}}$$

and then

$$\mathrm{Sq}^1_*s_{2^m(4n-2)-1}=s^2_{2^{m-1}(4n-2)-1}.$$

Thus we can deduce that

$$(\mathrm{ad}^2\Gamma_{1,\infty})_*(u_1^\sharp\otimes(w^{i-1})^\sharp)=s_i,$$

where we put $s_{2i}=s_i^2$.

Since $\pi_3(\mathrm{Sp}(\infty))\cong\mathbb{Z}$, we have

$$H_*((\Omega^2\mathrm{Sp}(\infty))\langle 1\rangle;\mathbb{Z}/2)=\mathbb{Z}/2[\bar{s}_2,\bar{s}_3,\bar{s}_5,\ldots]$$

in which $\bar{s}_i$ is defined by $\pi_*(\bar{s}_i)=s_i$ for the projection $\pi:(\Omega^2\mathrm{Sp}(\infty))\langle 1\rangle\to\Omega^2\mathrm{Sp}(\infty)$. Then, by Lemma 4.1, we obtain

$$H_*(\Omega^3_0\mathrm{Sp}(\infty);\mathbb{Z}/2)=\mathbb{Z}/2[t_1,t_2,t_3,\ldots],\ (\mathrm{ad}^3\Gamma_{1,\infty})_*((w^{i-1})^\sharp)=t_{i-1}$$

and thus the proof is done.

Since the inclusion $\mathrm{Sp}(1+l)\to\mathrm{Sp}(\infty)$ is an $(4l+6)$-equivalence, the inclusion $\Omega^3_0\mathrm{Sp}(1+l)\to\Omega^3_0\mathrm{Sp}(\infty)$ is an $(4l+3)$-equivalence. Therefore the proof is completed by (2.2). □

We next consider the case $\mathbf{G}=\mathrm{SU}$. Only in this case, we will use a result related with Bott periodicity which is an easy consequence of [20].

Lemma 4.2. *Let a be a generator of $H^2(\Omega^3_0\mathrm{SU}(\infty);\mathbb{Z})\cong\mathbb{Z}$. Then the integral homology of $\Omega^3_0\mathrm{SU}(\infty)$ is*

$$H_*(\Omega^3_0\mathrm{SU}(\infty);\mathbb{Z})=\mathbb{Z}[b_2,b_4,\ldots],\ b_{2i}=(a^i)^\sharp.$$

Theorem 4.3. *For $*\leq 2l$, the Pontrjagin ring $H_*(\Omega^3_0\mathrm{SU}(\infty);\mathbb{Z})$ is a polynomial ring generated by the image of $(\mathrm{ad}^3\tilde{\Gamma}_{1,l})_*:H_*(\mathcal{X}_{1,l}(\mathrm{SU});\mathbb{Z})\to H_*(\Omega^3_0\mathrm{SU}(2+l);\mathbb{Z})$.*

Proof. The case $l=\infty$ follows from Proposition 3.3 and Lemma 4.2. One can easily verify that the inclusion $\Omega^3_0\mathrm{SU}(2+l)\to\Omega^3_0\mathrm{SU}(\infty)$ and the natural map $\mathcal{X}_{1,l}(\mathrm{SU})\to\mathcal{X}_{1,\infty}(\mathrm{SU})$ are $(2l+4)$-equivalences. Thus the theorem follows from (2.2). □

5 Bott periodicity

In this section, we prove that the map $\mathrm{ad}^3\Gamma_{\infty,\infty} : B\mathbf{H}(\infty) \to \Omega_0^3\mathbf{G}(\infty)$ is a homotopy equivalence. Notice here that we have not used any result concerning real and symplectic Bott periodicity. We have only used the result of Toda [20] to get the ring structure of $\Omega_0^3\mathrm{SU}(\infty)$ in the last section. Then our result provides a new proof for real and symplectic Bott periodicity.

We start with an easy algebraic lemma. Let V be a graded free module over a PID. As usual, we will call V of finite type if, in each dimension, V is finitely generated. We will denote the free commutative graded algebra generated by V by ΛV. Then we can easily see:

Lemma 5.1 (Kono and Tokunaga [17])**.** *Let V and W be of finite type graded free modules over a PID R such that $V \cong W$, and let U be a graded module over R. Given a graded algebra map $f : \Lambda V \to \Lambda W$ and a graded module map $g : U \to \Lambda V$. If the image of $f \circ g : U \to \Lambda W$ generates ΛW, then f is an isomorphism.*

Now we prove our main theorem.

Theorem 5.1. *The map $\mathrm{ad}^3\Gamma_{\infty,\infty} : B\mathbf{H}(\infty) \to \Omega_0^3\mathbf{G}(\infty)$ is a homotopy equivalence.*

Proof. We first prove the case $\mathbf{G} = \mathrm{SU}$. By Lemma 2.2, Theorem 4.3 and Lemma 5.1 together with the homotopy commutative diagram (2.6), we see that the map $\mathrm{ad}^3\Gamma_{\infty,\infty} : B\mathrm{U}(\infty) \to \Omega_0^3\mathrm{SU}(\infty)$ induces an isomorphism in the integral homology. Then, by the J.H.C. Whitehead theorem, we obtain that $\mathrm{ad}^3\Gamma_{\infty,\infty}$ is a homotopy equivalence. Thus, in particular, from $\pi_*(B\mathrm{U}(2))$ is for $* \leq 4$, we deduce:

$$\pi_*(B\mathrm{U}(\infty)) \cong \begin{cases} \mathbb{Z} & * = 2, 4, \dots \\ 0 & * = 1, 3, \dots \end{cases} \tag{5.1}$$

Note here that we do not need to use Bott periodicity of $B\mathrm{U}(\infty)$.

We next consider the case $\mathbf{G} = \mathrm{SO}$. We may assume $\Gamma'_{1,\infty} = \Gamma_{1,\infty}$ as noted above. Then it follows from Lemma 2.2, Theorem 4.1, Lemma 5.1 and (2.6) that the map $\mathrm{ad}^3\Gamma_{\infty,\infty} : B\mathrm{Sp}(\infty) \to \Omega_0^3\mathrm{SO}(\infty)$ induces an isomorphism in the mod 2 homology. On the other hand, we have $\mathbf{qc}' = 1 : B\mathrm{Sp}(\infty) \to B\mathrm{Sp}(\infty)$ and $\mathbf{rc} = 2 : B\mathrm{SO}(\infty) \to B\mathrm{SO}(\infty)$. Then it follows from (5.1) that the homotopy groups of $B\mathrm{Sp}(\infty)$ and $\Omega_0^3\mathrm{SO}(\infty)$ are odd torsion free. Then, by considering the rational cohomology of $B\mathrm{Sp}(\infty)$ and $B\mathrm{SO}(\infty)$, we obtain

$$\pi_*(B\mathrm{Sp}(\infty)) \otimes \mathbb{Z}[\tfrac{1}{2}] \cong \begin{cases} \mathbb{Z}[\frac{1}{2}] & * = 4, 8, \dots \\ 0 & * \neq 4, 8, \dots \end{cases}$$

and

$$\pi_*(\Omega_0^3\mathrm{SO}(\infty)) \otimes \mathbb{Z}[\tfrac{1}{2}] \cong \begin{cases} \mathbb{Z}[\frac{1}{2}] & * = 4, 8, \dots \\ 0 & * \neq 4, 8, \dots. \end{cases}$$

This implies that the maps $\mathbf{c}'_* : \pi_*(B\mathrm{Sp}(\infty))\otimes\mathbb{Z}[\frac{1}{2}] \to \pi_*(B\mathrm{U}(\infty))\otimes\mathbb{Z}[\frac{1}{2}]$ and $\mathbf{c}_* : \pi_*(\Omega^3_0\mathrm{SO}(\infty))\otimes \mathbb{Z}[\frac{1}{2}] \to \pi_*(\Omega^3_0\mathrm{SU}(\infty)) \otimes \mathbb{Z}[\frac{1}{2}]$ are split monomorphisms. Thus since $\mathrm{ad}^3\Gamma_{\infty,\infty} : B\mathrm{U}(\infty) \to \Omega^3_0\mathrm{SU}(\infty)$ is a homotopy equivalence as above, the map $(\mathrm{ad}^3\Gamma_{\infty,\infty})_* : \pi_*(B\mathrm{Sp}(\infty)) \otimes \mathbb{Z}[\frac{1}{2}] \to \pi_*(\Omega^3_0\mathrm{SO}(\infty)) \otimes \mathbb{Z}[\frac{1}{2}]$ is an isomorphism by Proposition 2.2. On the other hand, we can apply Lemma 5.1 to the map $\mathrm{ad}^3\Gamma_{\infty,\infty} : B\mathrm{Sp}(\infty) \to \Omega^3_0\mathrm{SO}(\infty)$ in the mod 2 homology by Lemma 2.2 and Theorem 4.1. Then we obtain the map $\mathrm{ad}^3\Gamma_{\infty,\infty} : B\mathrm{Sp}(\infty) \to \Omega^3_0\mathrm{SO}(\infty)$ induces an isomorphism in the mod 2 homology. Summarizing, we have established that this map is a homology equivalence and therefore by a generalized J.H.C. Whitehead theorem [12], the proof is completed.

The case $\mathbf{G} = \mathrm{Sp}$ is quite similar to the case $\mathbf{G} = \mathrm{SO}$. □

Corollary 5.1. *Let* $d_{k,l} = \min\{2k+1, 2l+1\}, \min\{k, 4l+3\}, \min\{4k+3, l\}$ *according as* $\mathbf{G} = \mathrm{SU}, \mathrm{Sp}, \mathrm{SO}$. *Then the map* $\mathrm{ad}^3\Gamma_{k,l} : \mathcal{X}_{k,l}(\mathbf{G}) \to \Omega^3_0\mathbf{G}(dk+l)$ *is a* $d_{k,l}$*-equivalence.*

Proof. Let $a_k = 2k+1, k, 4k+3$ according as $\mathbf{G} = \mathrm{SU}, \mathrm{Sp}, \mathrm{SO}$. Then it is easy to see that the projection $B\mathbf{H}(k) \to B\mathbf{H}(\infty)$ is an a_k-equivalence. By definition, there is a principal bundles

$$\mathbf{H}(k) \to \mathbf{G}(dk+l)/\mathbf{G}(l) \to \mathcal{X}_{k,l(\mathbf{G})}$$

for $\mathbf{G} = \mathrm{Sp}, \mathrm{SO}$ and

$$\mathrm{U}(k) \to \mathrm{U}(2k+l)/\mathrm{U}(l) \to \mathcal{X}_{k,l}(\mathrm{SU}).$$

Let $b_{k,l} = 2l+1, 4l+3, l$ according as $\mathbf{G} = \mathrm{SU}, \mathrm{Sp}, \mathrm{SO}$. Then it follows from the above principal bundles that the composite of the inclusion $\mathcal{X}_{k,l}(\mathbf{G}) \to \mathcal{X}_{k,\infty}(\mathbf{G})$ and the homotopy equivalence $\mathcal{X}_{k,\infty}(\mathbf{G}) \simeq B\mathbf{H}(k)$ is a $b_{k,l}$-equivalence. Let $c_{k,l} = 4k+2l-3, 4k+4l-1, 4k+l-4$ according as $\mathbf{G} = \mathrm{SU}, \mathrm{Sp}, \mathrm{SO}$. Then the inclusion $\Omega^3_0\mathbf{G}(dk+l) \to \Omega^3_0\mathbf{G}(\infty)$ is a $c_{k,l}$-equivalence. Now let us consider a homotopy commutative diagram:

$$\begin{array}{ccccc}
\mathcal{X}_{k,l}(\mathbf{G}) & \longrightarrow & B\mathbf{H}(k) & \longrightarrow & B\mathbf{H}(\infty) \\
\downarrow{\scriptstyle \mathrm{ad}^3\Gamma_{k,l}} & & \downarrow{\scriptstyle \mathrm{ad}^3\Gamma_{k,\infty}} & & \downarrow{\scriptstyle \mathrm{ad}^3\Gamma_{\infty,\infty}} \\
\Omega^3_0\mathbf{G}(dk+l) & \longrightarrow & \Omega^3_0\mathbf{G}(\infty) & = & \Omega^3_0\mathbf{G}(\infty)
\end{array}$$

Then it follows from Theorem 5.1 that the map $\mathrm{ad}^3\Gamma_{k,l} : \mathcal{X}_{k,l}(\mathbf{G}) \to \Omega^3_0\mathbf{G}(dk+l)$ is a $\min\{a_k, b_{k,l}, c_{k,l}\}$-equivalence. Thus the proof is completed. □

6 Applications to instanton moduli spaces

In this section, we give applications of the results obtained so far to the homotopy types of instanton moduli spaces $\mathcal{M}_k(G)$.

Recall from Lemma 2.1 that the map $\Gamma_{k,l} : S^3 \wedge \mathcal{X}_{k,l}(\mathbf{G}) \to \mathbf{G}(dk+l)$ was constructed from the moduli space of $\mathbf{G}(dk+l)$-instantons on S^4. In particular, $\mathcal{X}_{k,l}(\mathbf{G})$ is a subspace of $\mathcal{M}_k(\mathbf{G}(dk+l))$. We denote the inclusion $\mathcal{X}_{k,l}(\mathbf{G}) \to \mathcal{M}_k(\mathbf{G}(dk+l))$ by $i_{k,l}$. Then, by definition, we have

$$\mathrm{ad}^3\Gamma_{k,l} = j_{k,l} \circ i_{k,l} \tag{6.1}$$

where $j_{k,l} : \mathcal{M}_k(\mathbf{G}(dk+l)) \to \Omega_0^3\mathbf{G}(dk+l)$ is the inclusion. We also have a commutative diagram:

$$\begin{array}{ccc} \mathcal{X}_{k,l}(\mathbf{G}) & \longrightarrow & \mathcal{X}_{k,l+1}(\mathbf{G}) \\ {\scriptstyle i_{k,l}}\downarrow & & \downarrow{\scriptstyle i_{k,l+1}} \\ \mathcal{M}_k(\mathbf{G}(dk+l)) & \longrightarrow & \mathcal{M}_k(\mathbf{G}(dk+l+1)) \end{array}$$

Here the horizontal arrows are induced from the inclusion $\mathbf{G}(dk+l) \to \mathbf{G}(dk+l+1)$. Then we have a map

$$\operatorname*{colim}_l i_{k,l} : \mathcal{X}_{k,\infty}(\mathbf{G}) \to \operatorname*{colim}_l \mathcal{M}_k(\mathbf{G}(dk+l))$$

which we denote by $i_{k,\infty} : \mathcal{X}_{k,\infty}(\mathbf{G}) \to \mathcal{M}_k(\mathbf{G}(\infty))$.

Proposition 6.1. *The map $i_{k,\infty}$ is a homotopy equivalence.*

Proof. We first prove the case $\mathbf{G} = \mathrm{SU}, \mathrm{SO}$. Recall from [18] that there is a homotopy equivalence $\mathcal{M}_k(\mathbf{G}(\infty)) \simeq B\mathbf{H}(k)$. On the other hand, we know that $\mathcal{X}_{k,\infty}(\mathbf{G}) \simeq B\mathbf{H}(k)$. Then we have $H^*(\mathcal{X}_{k,\infty}(\mathbf{G});\mathbb{Z}) \cong H^*(\mathcal{M}_k(\mathbf{G}(\infty));\mathbb{Z})$ as abstract rings. Note that $H^*(\mathcal{X}_{k,\infty}(\mathbf{G});\mathbb{Z})$ is a polynomial ring. By Corollary 5.1, we see that $H^*(\mathcal{X}_{k,\infty}(\mathbf{G});\mathbb{Z})$ is generated by $\mathrm{Im}(\mathrm{ad}^3\Gamma_{k,\infty})^*$. Therefore, by Lemma 5.1 and (6.1), the proof is completed.

We next prove the case $\mathbf{G} = \mathrm{Sp}$. By Corollary 5.1, the map $(\mathrm{ad}^3\Gamma_{k,\infty})^* : H^*(\Omega_0^3\mathrm{Sp}(\infty)\ \mathbb{Z}/2) \to H^*(\mathcal{X}_{k,\infty}(\mathbf{G});\mathbb{Z}/2)$ is an isomorphism for $* \le k$. Then, in particular, it follows from (6.1) that the map $(i_{k,\infty})_* : \pi_1(\mathcal{X}_{k,\infty}(\mathbf{G})) \to \pi_1(\mathcal{M}_k(\mathbf{G}(\infty)))$ is an isomorphism, where both $\pi_1\mathcal{X}_{k,\infty}(\mathbf{G})$ and $\pi_1(\mathcal{M}_k(\mathbf{G}(\infty)))$ are isomorphic to $\mathbb{Z}/2$. Since both $\mathcal{X}_{k,\infty}\langle 1\rangle$ and $\mathcal{M}_k(\mathbf{G}(\infty))\langle 1\rangle$ have the homotopy type of $B\mathrm{SO}(k)$, we can see the map $(i_{k,\infty})\langle 1\rangle : \mathcal{X}_{k,\infty}\langle 1\rangle \to \mathcal{M}_k(\mathbf{G}(\infty))\langle 1\rangle$ induces an isomorphisms in the cohomology with the coefficients $\mathbb{Z}/2$ and $\mathbb{Z}[\frac{1}{2}]$ quite analogously to the above case. Then the map $(i_{k,\infty})\langle 1\rangle$ is a homology equivalence, and hence a homotopy equivalence. Therefore the map $i_{k,\infty}$ is a homotopy equivalence. □

We estimate a range that the map $i_{k,l} : \mathcal{X}_{k,l}(\mathrm{SU}) \to \mathcal{M}_k(\mathrm{SU}(dk+l))$ is a homotopy equivalence.

Theorem 6.1. *The map $i_{k,l} : \mathcal{X}_{k,l}(\mathbf{G}) \to \mathcal{M}_k(\mathbf{G}(dk+l))$ is a $\min\{2k+1, 2l+1\}$-equivalence.*

Proof. In [15], it is shown that the map $\mathcal{M}_k(\mathbf{G}(dk+l)) \to \mathcal{M}_k(\mathbf{G}(\infty))$ induced from the inclusion $\mathbf{G}(dk+l) \to \mathbf{G}(dk+l+1)$ is a $(2k+1)$-equivalence. On the other hand, the map $\mathcal{X}_{k,l}(\mathbf{G}) \to \mathcal{X}_{k,\infty}(\mathbf{G})$ induced from the inclusion $\mathbf{G}(dk+l) \to \mathbf{G}(dk+l+1)$ is a $(2l+1)$-equivalence as is seen in the proof of Corollary 5.1. Then the theorem follows from (2.2) and Proposition 6.1. □

References

[1] S. Araki and T. Kudo, *Topology of H_n-spaces and H-squaring operations*, Mem. Fac. Sci. Kyūsyū Univ. Ser. A **10** (1956), 85-120.

[2] M.F. Atiyah, *Geometry of Yang-Mills Fields*, Scuola Norm. Sup. Pisa (1979).

[3] M.F. Atiyah and J.D.S. Jones, *Topological aspects of Yang-Mills theory*, Commun. Math. Phys. **61** (1978), 97-118.

[4] P.F. Baum and W. Browder, *The cohomology of quotients of classical groups*, Topology **3** (1965), 305-336.

[5] R. Bott, *An application of the Morse theory to the topology of Lie groups*, Bull. Soc. Math. Fr. **84**, 251-281.

[6] R. Bott, *The space of loops on a Lie group*, Michigan Math. J. **5** (1958), 35-68.

[7] R. Bott, *The stable homotopy of the classical groups*, Ann. of Math. (2) **70** (1959), 313-337.

[8] A.K. Bousfield and D.M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, **304**, Springer-Verlag, Berlin-New York, 1972.

[9] C.P. Boyer, J.C. Hurtubise, B.M. Mann and R.J. Milgram, *The topology of instanton moduli spaces. I. The Atiyah-Jones conjecture*, Ann. of Math. (2) **137** (1993), no. 3, 561-609.

[10] C.P. Boyer, B.M. Mann and D. Waggoner, *On the homology of* SU(n) *instantons*, Trans. Amer. Math. Soc. **323** (1991), no. 2, 529-561.

[11] F.R. Cohen, T.J. Lada and J.P. May, *The homology of iterated loop spaces*, Lecture Notes in Mathematics, **533**, Springer-Verlag, Berlin-New York, 1976.

[12] E. Dror, *A generalization of the Whitehead theorem*, Symposium on Algebraic Topology (Battelle Seattle Res. Center, Seattle, Wash., 1971), pp. 13–22. Lecture Notes in Mathematics, **249**, Springer, Berlin, 1971.

[13] Y. Kamiyama, *Generating varieties for the triple loop space of classical Lie groups*, Fund. Math. **177** (2003), 269-283.

[14] Y. Kamiyama, A. Kono, M. Tezuka, *Cohomology of the moduli space of* SO(n)*-instantons with instanton number 1*, Topology Appl. **146/147** (2005), 471-487.

[15] F. Kirwan, *Geometric invariant theory and the Atiyah-Jones conjecture*, Proc. S. Lie Mem. Conf., Scand. Univ. Press (1994), 161-188.

[16] D. Kishimoto, *A topological proof of real and symplectic Bott periodicity theorem*, J. Math. Kyoto Univ. **41** (2001), no. 1, 33-41.

[17] A. Kono and K. Tokunaga, *A topological proof of Bott periodicity theorem and a characterization of* BU, J. Math. Kyoto Univ. **34** (1994), no. 4, 873-880.

[18] M. Sanders, *Classifying spaces and Dirac operators coupled to instantons*, Trans. Amer. Math. Soc. **347** (1995), 4037-4072.

[19] Y. Tian, *The Atiyah-Jones conjecture for classical groups and Bott periodicity*, J. Diff. Geom. **44** (1996), 178-199.

[20] H. Toda, *A topological proof of theorems of Bott and Borel-Hirzebruch for homotopy groups of unitary groups*, Mem. Coll. Sci. Univ. Kyoto Ser. A, vol. **32** (1959), 103-119.