Abstract

Let $G$ be the classical group and let $M^k(G)$ be the based moduli space of $G$-instantons on $S^4$ with instanton number $k$. It is known that $M^k(G)$ yields real and symplectic Bott periodicity, however an explicit geometric description of the homotopy equivalence has not been known. We consider certain orbit spaces in $M^k(G)$ and show that the restriction of the inclusion of $M^k(G)$ into the moduli space of connections, which, in turn, is explicitly described by the commutator map of $G$. We prove this restriction satisfies a triple loop space version of the generating variety argument of Bott [6], and it also gives real and symplectic Bott periodicity. This also gives a new proof of real and symplectic Bott periodicity.

1 Introduction

Let $G$ be a compact connected simple Lie group. Then there is an isomorphism $\pi_3(G) \cong \pi_3(BG) \cong \mathbb{Z}$. We will fix an isomorphism $\pi_3(G) \cong \mathbb{Z}$. Then principal $G$-bundles over $S^4$ are classified by $\mathbb{Z} = \pi_3(G)$, and denote by $P_k$ the principal $G$-bundle over $S^4$ corresponding to $k \in \mathbb{Z}$. Let $C_k(G)$ be the based moduli space of connections on $P_k$. Then we have a natural homotopy equivalence

$$C_k(G) \simeq \Omega^3_k G$$

where $\Omega^3_k G$ stands for the path component of $\Omega^3 G$ corresponding to $k \in \mathbb{Z} = \pi_3(G)$. We will identify $C_k(G)$ with $\Omega^3_k G$ by this homotopy equivalence. Let $\mathcal{M}_k(G)$ be the based moduli space of instantons on $P_k$. Then we have a map

$$\theta_k : \mathcal{M}_k(G) \to \Omega^3_0 G$$

defined by the composite of the inclusion $\mathcal{M}_k(G) \to \Omega^3_k G \simeq C_k(G)$ and the homotopy equivalence $\Omega^3_k G \simeq \Omega_0 G$, the shift by $-k \in \mathbb{Z} = \pi_3(G)$.

The topology of the map $\theta_k$ was first studied by Atiyah and Jones [3], and, later, it was proved by Boyer, Hurtubise, Mann and Milgram [9], Kirwan [15] and Tian [19] that the map $\theta_k$ is a homotopy equivalence in a range, which is known as the Atiyah-Jones theorem. As a consequence of this result, Tian [19] showed that the colimit of the map $\theta_k$ yields real and symplectic
Bott periodicity. However, an explicit geometric description of the homotopy equivalence is not known. While Bott periodicity was given by a map explicitly defined by the commutator maps of the classical groups \([7]\). In \([10]\), it is shown that the map \(\theta_k\) has some relation with the commutator map of \(G\) when \(k = 1\). Recall that Bott \([6]\) also used the commutator maps to study the topology of loop spaces of Lie groups. Exploiting the above result of \([10]\) in connection with the classical result of Bott \([6]\), Kamiyama \([13]\) studied a triple loop space analogue of generating varieties of Bott \([6]\).

We will give a mild generalization of the above result of \([10]\) for arbitrary \(k\). Using this, we prove triple loop space version of the generating variety argument \([6]\) in a sense somewhat different from \([13]\), and also prove Bott periodicity. This yields a new proof of real and symplectic Bott periodicity. We will give applications of this result to the homotopy types of \(M_k(G)\).

2 Subgroups of classical groups isomorphic with \(SU(2)\)

Let \(G\) be a compact, connected, simple Lie group with a fixed isomorphism \(\pi_3(G) \cong \mathbb{Z}\). Note that \(G\) acts on \(M_k(G)\) via the action of the basepoint free gauge group of \(P_k\) on \(M_k(G)\). As is shown in \([10]\), there is an orbit of this action for \(k = 1\) such that the restriction of \(\theta_1 : M_1(G) \to \Omega^3_0 G\) is presented by the commutator map of \(G\). By putting additional assumption, we can prove this for arbitrary \(k\) by essentially the same way in \([10]\) as follows.

**Lemma 2.1.** Suppose that there exists a subgroup \(H\) of \(G\) isomorphic to \(SU(2) \cong S^3\) such that the inclusion \(\iota : H \hookrightarrow G\) represents \(k \in \mathbb{Z} = \pi_3(G)\). Then there exists \(\omega \in M_k(G)\) satisfying:

1. The orbit space \(G \cdot \omega\) is homeomorphic with \(G / C(H)\), where \(C(H)\) stands for the centralizer of \(H\).

2. Let \(\Gamma\) denote the composite:

\[
G / C(H) \approx G / \omega \hookrightarrow M_k(G) \xrightarrow{\theta_k} \Omega^3_0 G
\]

Then we have

\[
\Gamma(gC(H)) \approx g\iota(h)g^{-1}\iota(h)^{-1}
\]

for \(g \in G, h \in H\).

**Proof.** Let \(\alpha\) be an asymptotically flat connection on \(P_k\). We regard \(S^4\) as \(\mathbb{R}^4 \cup \{\infty\}\). Recall from \([3]\) that the homotopy equivalence \(\mathcal{C}_k(G) \approx \Omega^3_0 G\) takes \(\alpha \in M_k(G)\) into its 'pure gauge' \(\hat{\alpha} : S^3 \to G\) at \(\infty \in S^4\) normalized as \(\hat{\alpha}(\ast) = e\), where \(\ast\) and \(e\) are the basepoint of \(S^3\) and unity of \(G\), respectively. (See \([3]\).) The action of the basepoint free gauge group of \(P_k\) is locally
the conjugation by $G$. Then the map $\theta_k$ is $G$-equivariant under the action of $G$ on $\Omega^2_0G$ given by $g \cdot \lambda(x) = g\lambda(x)g^{-1}$ for $g \in G$, $\lambda \in \Omega^2_0G$, $x \in S^3$.

Let $P$ be a principal SU(2)-bundle over $S^4$ represented by $1 \in \mathbb{Z} \cong \pi_3(\text{SU}(2))$. In [2], an asymptotically flat instanton $\varpi$ whose pure gauge represents $1 \in \mathbb{Z} \cong \pi_3(\text{SU}(3))$. Then the proof is completed by putting $\omega$ to be the push forward of $\varpi$ by the inclusion $i : H \cong \text{SU}(2) \rightarrow G$.

The original form of Bott periodicity [7] is given by such a map $\Gamma$ in Lemma 2.1 where $\text{SU}(2) \cong S^3$ is replaced with $\text{U}(1) \cong S^1$. On the other hand, there is known a deep relation between $\mathcal{M}_k(G)$ and Bott periodicity as in [15], [18], [19]. Then we expect the map $\Gamma$ in Lemma 2.1 may yield real and symplectic Bott periodicity which has period 4. Also we expect $G/C(H)$ and $\Gamma$ in Lemma 2.1 may yield a 3-fold loop analogue of a generating variety for a loop space of a Lie group, which is already studied by Kamiyama [13] in a slightly different sense, that is, algebras over the Kudo-Araki operations. Then we introduce a family of subgroups of the classical groups which are isomorphic with SU(2) by which we can prove the above argument.

Hereafter, we put $(\mathbf{G}, \mathbf{H}, d) = (\text{Sp}, \text{O}, 1), (\text{SU}, \text{U}, 2), (\text{SO}, \text{Sp}, 4)$. We will define a family of subgroups $S_{k,l}(G)$ of $G(dk+l)$ indexed by positive integers $k$ and non-negative integers $l$. Since the Lie group $G(dk+l)$ must be simple, we will assume $dk+l > 4$ when $G = \text{SO}$.

Let $c : \text{O}(n) \rightarrow \text{U}(n)$, $q : \text{U}(n) \rightarrow \text{Sp}(n)$, $c' : \text{Sp}(n) \rightarrow \text{SU}(2n)$, and $r : \text{U}(n) \rightarrow \text{O}(2n)$ be the canonical inclusions. In order to make things clear, we write the maps $c'$ and $r$ explicitly as follows. Let $M_n(\mathbb{K})$ be the set of all square matrices of order $n$ over a field $\mathbb{K}$. For $A = (a_{ij}), B = (b_{ij}) \in M_n(\mathbb{C})$ such that $A + Bj \in \text{Sp}(n)$, we put

$$c'(A + Bj) = (c'(a_{ij} + b_{ij}j))$$

where $c'(a + jb) = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$ for $a, b \in \mathbb{C}$. We also put, for $C = (c_{ij}), D = (d_{ij}) \in M_n(\mathbb{R})$ such that $C + D\sqrt{-1} \in \text{U}(n)$,

$$r(C + D\sqrt{-1}) = (r(c_{ij} + d_{ij}\sqrt{-1}))$$

where $r(c + d\sqrt{-1}) = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$ for $c, d \in \mathbb{R}$. We denote the matrix $\begin{pmatrix} A & O \\ O & B \end{pmatrix}$ by $A \oplus B$. We consider the following family of subgroups of the classical groups isomorphic with $\text{SU}(2) \cong S^3$:

$$S_{k,l}(\text{Sp}) = \{ \alpha E_k \oplus E_l \in \text{Sp}(k+l) \mid \alpha \in \text{Sp}(1) \}$$
$$S_{k,l}(\text{SU}) = \{ A \oplus E_l \in \text{SU}(2k+l) \mid A \in c'(S_{k,0}(\text{Sp})) \}$$
$$S_{k,l}(\text{SO}) = \{ B \oplus E_l \in \text{SO}(4k+l) \mid B \in r(c'(S_{k,0}(\text{Sp})) \}$$

where $E_n$ is the identity matrix of order $n$. We easily see

$$c'(S_{k,l}(\text{Sp})) = S_{k,2l}(\text{SU})$$
$$r(S_{k,l}(\text{SU})) = S_{k,2l}(\text{SO})$$
We fix an isomorphism $\pi_3(G(dk + l)) \cong \mathbb{Z}$ such that the inclusion $S_{k,l} \rightarrow G(dk + l)$ represents $k \in \mathbb{Z}$.

Let $C_{k,l}(G)$ denote the centralizer of $S_{k,l}(G)$ in $G(dk + l)$. Then we have

$$C_{k,l}(Sp) = qc(O(k)) \oplus Sp(l).$$

We also denote by $C_{k,l}(U)$ the centralizer of $S_{k,l}(SU)$ in $U(dk + l)$. Then we have

$$C_{k,l}(U) = \{A \oplus B \in U(2k + l) \mid A = (a_{ij}E_2) \in U(2k), B \in U(l)\}.$$

In order to describe the centralizer $C_{k,l}(SO)$, we give another description of $S_{k,l}(SO)$. Define the action of $Sp(1) \times Sp(1)$ on $H$ by

$$x \cdot (p, q) = p^{-1}xq$$

for $(p, q) \in Sp(1) \times Sp(1)$ and $x \in H$. It is well known that this action yields the universal covering homomorphism $\rho: Sp(1) \times Sp(1) \cong Spin(4) \rightarrow SO(4)$. Then it easily follows that

$$S_{k,l}(SO) = \{A \oplus \cdots \oplus A \oplus E_l \mid A \in \rho(1 \times Sp(1)) \subset SO(4)\}.$$

We denote the extension $H \rightarrow M_4(\mathbb{R})$ of $\rho_{Sp(1)\times1}$ ambiguously by the same $\rho$. Then one can easily verify

$$\rho(x + yi + zj + wk) = \begin{pmatrix} x & y & z & w \\ -y & x & w & -z \\ -z & -w & x & y \\ -w & z & -y & x \end{pmatrix}$$

for $x, y, z, w \in \mathbb{R}$. The map $\rho: H \rightarrow M_4(\mathbb{R})$ induces a map $\bar{\rho}: M_n(H) \rightarrow M_{4n}(\mathbb{R})$ by $\bar{\rho}(a_{ij}) = (\rho(a_{ij}))$ for $(a_{ij}) \in M_n(H)$. Now we obtain

$$C_{k,l}(SO) = \{\bar{\rho}(A) \oplus B \in SO(4k + l) \mid A \in Sp(k), B \in SO(l)\}. \quad (2.1)$$

Summarizing the above observation on $C_{k,l}(G)$, we get:

**Proposition 2.1.** There are isomorphisms

$$C_{k,l}(Sp) \cong O(k) \times Sp(l)$$

$$C_{k,l}(U) \cong U(k) \times U(l)$$

$$C_{k,l}(SO) \cong Sp(k) \times SO(l)$$

satisfying a commutative diagram:

$$
\begin{array}{ccc}
C_{k,l}(Sp) & \xrightarrow{e'} & C_{k,2l}(U) & \xrightarrow{r} & C_{k,4l}(SO) \\
\cong & & \cong & & \cong \\
O(k) \times Sp(l) & \xrightarrow{e \times e'} & U(k) \times U(2l) & \xrightarrow{q \times r} & Sp(k) \times SO(4l)
\end{array}
$$
We now define a space and a map corresponding to the orbit space and the map $\Gamma$ in Lemma 2.1 with respect to $S_{k,l}(G)$. We define a space $X_{k,l}(G)$ by

$$X_{k,l}(G) = G(dk + l)/C_{k,l}(G)$$

and a map $\Gamma_{k,l} : S_{k,l}(G) \land X_{k,l}(G) \rightarrow G(dk + l)$ by

$$\Gamma_{k,l}(s, gC_{k,l}(G)) = gs^{-1}g^{-1}$$

for $s \in S_{k,l}(G), g \in G(dk + l)$. We will identify $S_{k,l}(G)$ with $S^3$ if there is no confusion. It is obvious that the inclusions $G(dk + l) \rightarrow G(dk + (l + 1))$ and $G(dk + l) \rightarrow G(dk + (l + 1))$ induce the commutative diagram:

$$
\begin{array}{c}
S^3 \land X_{k+1,l}(G) \\
\uparrow_{\Gamma_{k+1,l}} \\
G(dk + 1 + l)
\end{array}
\begin{array}{c}
\uparrow_{\Gamma_{k,l}} \\
S^3 \land X_{k,l}(G) \\
\uparrow_{\Gamma_{k,l}} \\
\downarrow_{\Gamma_{k,l+1}} \\
G(dk + l)
\end{array}
\begin{array}{c}
\uparrow_{\Gamma_{k,l+1}} \\
S^3 \land X_{k,l+1}(G) \\
\downarrow_{\Gamma_{k,l+1}} \\
G(dk + (l + 1))
\end{array}

(2.2)

By the above observation on $C_{k,l}(SU)$ and $C_{k,l}(U)$, we see that there is a diffeomorphism:

$$X_{k,l}(SU) \cong U(2k + l)/C_{k,l}(U)$$

(2.3)

Note that $c' : Sp(k+l) \rightarrow SU(2k+2l)$ and $r : SU(k+l) \rightarrow SO(2k+2l)$ are homomorphisms which restrict to surjections $S_{k,l}(Sp) \rightarrow S_{k,2l}(SU)$ and $S_{k,l}(SU) \rightarrow S_{k,2l}(SO)$, respectively. Then they induce maps $c' : X_{k,l}(Sp) \rightarrow X_{k,2l}(SU)$ and $r : X_{k,l}(SU) \rightarrow X_{k,2l}(SO)$ satisfying a commutative diagram:

$$
\begin{array}{c}
S^3 \land X_{k,l}(Sp) \\
\uparrow_{\Gamma_{k,l}} \\
Sp(k + l)
\end{array}
\begin{array}{c}
\uparrow_{\Gamma_{k,2l}} \\
S^3 \land X_{k,2l}(SU) \\
\uparrow_{\Gamma_{k,2l}} \\
\downarrow_{\Gamma_{k,2l}} \\
SU(2k + 2l)
\end{array}
\begin{array}{c}
\uparrow_{\Gamma_{k,2l}} \\
S^3 \land X_{k,4l}(SO) \\
\downarrow_{\Gamma_{k,4l}} \\
SO(4k + 4l)
\end{array}

(2.4)

We observe a relation between $X_{1,l}(G)$ and a projective space. It follows from Proposition 2.1 that $X_{1,l}(Sp) = \mathbb{RP}^{d+3}$ and also that $X_{1,l}(SU)$ is the total space of the unit tangent bundle of $\mathbb{CP}^{l+1}$. Note that the map $\rho : \mathbb{H} \rightarrow M_{4}(\mathbb{R})$ above induces a homomorphism $\rho : Sp(n) \rightarrow SO(4n)$. Then there is a map $\mathbb{HP}^{[\frac{d}{2}]} \rightarrow X_{1,l}(SO)$ which is natural with respect to the maps $\mathbb{HP}^{[\frac{d}{2}]} \rightarrow \mathbb{HP}^{[\frac{d}{2}]}$ and $X_{1,l}(SO) \rightarrow X_{1,l+1}(SO)$. We regard $\mathbb{HP}^{[\frac{d}{2}]}$ to be a subspace of $X_{1,l}(SO)$ by this map. Put $\Gamma'_{1,l}$ to be the restriction of $\Gamma_{1,l} : S^3 \land X_{1,l}(SO) \rightarrow SO(4+l)$ onto $\mathbb{HP}^{[\frac{d}{2}]} \subset X_{1,l}(SO)$. Then we have an obvious commutative diagram:

$$
\begin{array}{c}
S^3 \land \mathbb{HP}^{[\frac{d}{2}]} \\
\uparrow_{\Gamma'_{1,l}} \\
SO(4 + l)
\end{array}
\begin{array}{c}
\uparrow_{\Gamma'_{l+1}} \\
S^3 \land \mathbb{HP}^{[\frac{d}{2}]} \\
\uparrow_{\Gamma'_{l+1}} \\
\downarrow_{\Gamma'_{l+1}} \\
SO(5 + l)
\end{array}

(2.5)
We next consider the map $\Gamma_{k,l}$ when $l$ tends to $\infty$. Put $X_{k,\infty}(G) = \text{colim}_l X_{k,l}(G)$. Then, by (2.2), we have a map

$$\text{colim}_l \Gamma_{k,l} : S^3 \wedge X_{k,\infty}(G) \to G(\infty).$$

which we denote by

Now for $G = \text{Sp}, \text{SO}$, there is a principal bundle

$$H(k) \to G(dk + l)/G(l) \to X_{k,l}(G)$$

by Proposition 2.1 where $G(dk + l)/G(l)$ is $(4l + 2)$-connected and $(l - 1)$-connected according as $G = \text{Sp}, \text{SO}$. By Proposition 2.1 and (2.3), we also have a principal bundle

$$U(k) \to U(2k + l)/U(l) \to X_{k,l}(\text{SU})$$

in which $U(2k + l)/U(l)$ is $2l$-connected. Then it follows that there is a homotopy equivalence

$$X_{k,\infty}(G) \simeq BH(k)$$

and thus we obtain a map

$$\Gamma_{k,\infty} : S^3 \wedge BH(k) \to G(\infty).$$

Moreover, by Proposition 2.1 and (2.4), we get:

**Proposition 2.2.** There is a homotopy commutative diagram:

$$
\begin{array}{ccc}
S^3 \wedge BO(k) & \xrightarrow{1/\ell} & S^3 \wedge BU(k) \\
\Gamma_{k,\infty} & & \Gamma_{k,\infty} \\
\text{Sp}(\infty) & \xrightarrow{\ell} & \text{SU}(\infty) \\
& & \xrightarrow{r} \text{SO}(\infty)
\end{array}
$$

Note that, by (2.5), we also have a map $\Gamma_{1,\infty} : S^3 \wedge H^P \to SO(\infty)$ which coincides with the map $\Gamma_{1,\infty} : S^3 \wedge BSp(1) \to SO(\infty)$.

We see from (2.2) that $\Gamma_{k,\infty}$ satisfies a homotopy commutative diagram

$$
\begin{array}{ccc}
S^3 \wedge BH(k) & \longrightarrow & S^3 \wedge BH(k + 1) \\
\Gamma_{k,\infty} & & \Gamma_{k+1,\infty} \\
G(\infty) & \longrightarrow & G(\infty)
\end{array}
$$

(2.6)

where the top horizontal arrow is induced from the inclusion $H(k) \to H(k + 1)$. Then we get a map

$$\Gamma_{\infty,\infty} = \text{colim}_k \Gamma_{k,\infty} : S^3 \wedge BH(\infty) \to G(\infty).$$

Let $\mu : G(n) \times G(n) \to G(2n)$ be an inclusion such as by $\mu(A, B) = A \oplus B$ for $A, B \in G(n)$. Then $\mu$ induces a map $X_{k,l}(G) \times X_{k,l}(G) \to X_{2k,2l}(G)$, denoted by the same symbol $\mu$, which
yields the standard H-space structure on $BH(\infty) \simeq \mathcal{X}_{\infty,\infty}(G)$. Moreover, the map $\mu$ satisfies a commutative diagram

$$
\begin{array}{ccc}
S^3 \wedge (\mathcal{X}_{k,l}(G) \times \mathcal{X}_{k,l}(G)) & \xrightarrow{1\wedge \mu} & S^3 \wedge \mathcal{X}_{2k,2l}(G) \\
\Delta & & \\
(S^3 \wedge \mathcal{X}_{k,l}(G)) \times (S^3 \wedge \mathcal{X}_{k,l}(G)) & \xrightarrow{\Gamma_{k,l}} & \Gamma_{2k,2l}
\end{array}
$$

where $\Delta$ is defined by $\Delta(s, x, y) = (s, x, s, y)$ for $s \in S^3, x, y \in \mathcal{X}_{k,l}(G)$. Let $ad : [\Sigma X, Y] \cong [X, \Omega Y]$ denote the adjoint congruence. Then we have established:

**Lemma 2.2.** The map $ad^3 \Gamma_{\infty,\infty} : BH(\infty) \to \Omega_3^3 G(\infty)$ is an H-map.

We will show that the image of $ad^3 \Gamma_{1,l}$ in homology generates the Pontrjagin ring of $\Omega_3^3 G(2k+2l)$ in a range, which is an analogue of the generating variety for a loop space of a Lie group, and that the map $ad^3 \Gamma_{\infty,\infty}$ yields Bott periodicity.

### 3 Cohomology calculation for $\Gamma_{1,l}$

In this section, we give a cohomology calculation for the map $\Gamma_{1,l}$ and $\Gamma_{1,l}'$. We first consider the case $G = SO$. In this case, we calculate $\Gamma_{1,l}'$ in cohomology instead of $\Gamma_{1,l}$ since the cohomology of $\mathcal{X}_{1,l}(SO)$ is complicated as is seen in [14].

**Proposition 3.1.** For $l \geq 4$, the map $(\Gamma_{1,l})^* : H^*(SO(4 + l); \mathbb{Z}/2) \to H^*(S^3 \wedge \mathbb{H}P^2; \mathbb{Z}/2)$ is surjective.

**Proof.** Recall first that the mod 2 cohomology of $SO(4 + l)$ is given as

$$
H^*(SO(4 + l); \mathbb{Z}/2) = \mathbb{Z}/2[x_1, x_3, \ldots] \text{ for } * \leq 3 + l,
$$

where $x_i$ is the suspension of the Stiefel-Whitney class $w_{i+1}$. Let $u_3$ be a generator of $H^3(S^3; \mathbb{Z}/2)$. Then, by definition, the inclusion $i : S^3 = S_{1,l}(SO) \to SO(4 + l)$ induces the map in cohomology such as $i^*(x_3) = u_3$.

Let us consider the case $l = 12$. Let $PSO(n)$ denote the $n$-dimensional projective orthogonal group, that is, $SO(n)$ divided by its center. It is well known that

$$
H^*(PSO(16); \mathbb{Z}/2) = \mathbb{Z}/2[v, \bar{x}_1, \bar{x}_3, \bar{x}_5, \bar{x}_7] \text{ for } * \leq 7
$$
where $|v| = 1$ and $\pi^*(\bar{x}_i) = x_i$ for the projection $\pi : \text{SO}(16) \to \text{PSO}(16)$. Moreover, we see from [4] that the Hopf algebra structure of $H^*(\text{PSO}(16); \mathbb{Z}/2)$ is given as

$$\bar{\phi}^*(v) = 0, \quad \bar{\phi}^*(\bar{x}_i) = \sum_{j=1}^{i} a_{ij} \bar{x}_j \otimes v^{i-j}$$

for $i = 1, 3, 5, 7$ in which $a_{53} = 0, a_{73} = 1$, where $\bar{\phi}$ stands for the reduced comultiplication. Let $\gamma : \text{PSO}(16) \wedge \text{PSO}(16) \to \text{PSO}(16)$ be the reduced commutator map and let $\tilde{\gamma} : \text{SO}(16) \wedge \text{PO}(16) \to \text{SO}(16)$ be a lift of $\gamma$. Then by a straightforward calculation, we have

$$\tilde{\gamma}^*(x_7) = u_3 \otimes v^4.$$ 

On the other hand, since the center of $\text{SO}(16)$ is included in $C_{1,12}(\text{SO})$, we have the projection $\text{PSO}(16) \to \mathcal{K}_{1,12}(\text{SO})$ satisfying a commutative diagram

\[
\begin{array}{ccc}
\text{SO}(16) & \longrightarrow & \text{SO}(16)/\text{SO}(12) \overset{\text{re}}{\longrightarrow} \text{Sp}(4)/\text{Sp}(3) \\
\pi & \downarrow & \downarrow \\
\text{PSO}(16) & \longrightarrow & \mathcal{K}_{1,12}(\text{SO}) \leftarrow \mathbb{H}P^3 \\
& \downarrow & \downarrow \\
B(\mathbb{Z}/2) & \longrightarrow & B\text{Sp}(1) \longrightarrow \text{BSp}(1) \\
\end{array}
\]

where $\mathbb{Z}/2$ is the center of $\text{Sp}(1)$. Then we see that a generator $x$ of $H^4(\mathcal{K}_{1,12}(\text{SO}); \mathbb{Z}/2)$ satisfies

$$\pi^*(x) = v^4, \quad (\text{re})^*(x) = q,$$

where $q$ is a generator of $H^4(\mathbb{H}P^n; \mathbb{Z}/2)$. Now we have a commutative diagram:

\[
\begin{array}{ccc}
S^3 \wedge \text{PSO}(16) & \longrightarrow & S^3 \wedge \mathcal{K}_{1,12}(\text{SO}) \\
\downarrow & & \downarrow \\
\text{SO}(16) & \longrightarrow & \text{SO}(16)
\end{array}
\]

Then we obtain

$$(\Gamma'_{1,12})^*(x_7) = u_3 \otimes q.$$ 

By (2.5), we have established

$$(\Gamma'_{1,12})^*(x_7) = u_3 \otimes q. \quad (3.1)$$

By the Wu formula, we have

$$\text{Sq}^i_{\text{SO}(4 + l)} x_{4i-1} = (i-1)x_{4i+3}, \quad \text{Sq}^8_{\text{SO}(4 + l)} x_{4i-1} = \left(\frac{i-1}{2}\right)x_{4i+7}$$

in $H^*(\text{SO}(4 + l); \mathbb{Z}/2)$ for $* < 4 + l$. Then, applying this to (3.1), the proof is completed.
Proposition 3.2. For $i > 0$, the map $\Gamma_{1,i}^* : H^{4i+3}(\text{Sp}(1+l); \mathbb{Z}/2) \to H^{4i+3}(S^3 \wedge \mathbb{R}P^{4i+3}; \mathbb{Z}/2)$ is surjective.

Proof. Let $w$ and $g$ be generators of $H^1(\mathbb{R}P^\infty; \mathbb{Z}/2)$ and $H^i(\mathbb{H}P^\infty; \mathbb{Z}/2)$, respectively. Then the map $\text{qc} : \mathbb{R}P^\infty \to \mathbb{H}P^\infty$ induces $(\text{qc})^*(g) = w^i$ in cohomology. Recall that the mod 2 cohomology of $\text{Sp}(n)$ is given as

$$H^*(\text{Sp}(n); \mathbb{Z}/2) = \Lambda(y_3, y_7, \ldots, y_{4n-1})$$

where $y_{4i-1}$ is the suspension of the modulo 2 reduction of the symplectic Pontrjagin class $q_i$. Then we have $(\text{rc})^*(x_{4i-1}) = y_{4i-1}$ here we use the same notation for the mod 2 cohomology of $\text{SO}(\infty)$ as in the proof of Proposition 3.1. Then, for $l = \infty$, the proposition follows from Proposition 3.1 and (2.2). Thus the proof is completed by (2.2). \hfill \Box

Let $X(\langle n \rangle)$ denote the $n$-connective cover of a path-connected space $X$. Then, in general, any map $f : S^3 \wedge A \to X$ with $A$ path-connected lifts to $X(3)$ which we denote by $\tilde{f}$.

Proposition 3.3. Any lift $\tilde{\Gamma}_{1,\infty} : S^3 \wedge \mathbb{C}P^\infty \to (\text{SU}(\infty))(\langle 3 \rangle)$ of $\Gamma_{1,\infty} : S^3 \wedge \mathbb{C}P^\infty \to \text{SU}(\infty)$ induces an isomorphism $\tilde{\Gamma}_{1,\infty}^* : H^5((\text{SU}(\infty))(\langle 3 \rangle); \mathbb{Z}) \cong H^5(S^3 \wedge \mathbb{C}P^\infty; \mathbb{Z}).$

Proof. We will denote the modulo $p$ reduction in cohomology by $\rho_p$ for a prime $p$.

The integral cohomology of $\text{SU}(n)$ is

$$H^*(\text{SU}(n); \mathbb{Z}) = \Lambda(e_3, e_5, \ldots, e_{2n-1}),$$

where $e_{2i-1}$ is the suspension of the Chern class $c_i$. Then, by considering the Serre spectral sequence of a fibre sequence $\mathbb{C}P^\infty \to (\text{SU}(\infty))(\langle 3 \rangle) \xrightarrow{q} \text{SU}(\infty)$, we see that $H^5((\text{SU}(\infty))(\langle 3 \rangle); \mathbb{Z}) \cong \mathbb{Z}$ is generated by $\epsilon$ such that

$$q^*(\epsilon) = 2\epsilon.$$ \hfill (3.2)

Let $\text{PSU}(n)$ be the $n$-dimensional projective unitary group, that is, $\text{SU}(n)$ divided by its center. Let $p$ be an odd prime. In [4], it is shown that

$$H^*(\text{PSU}(p^r); \mathbb{Z}/p) = \mathbb{Z}/p[v]/(v^{p^r}) \otimes \Lambda(\tilde{e}_1, \tilde{e}_3, \ldots, \tilde{e}_{2p^r-1})$$

where $|v| = 2$ and $\tilde{\pi}^*(\tilde{e}_i) = \rho_p(e_i)$ for the projection $\tilde{\pi} : \text{SU}(p^r) \to \text{PSU}(p^r)$. Moreover, for the reduced comultiplication $\tilde{\phi}$, we have

$$\tilde{\phi}(\tilde{e}_3) = a_1\tilde{e}_3 \otimes v + a_2\tilde{e}_1 \otimes v^2$$

for $a_1, a_2 \in (\mathbb{Z}/p)^\times$. Let $c$ and $u_3$ be generators of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$ and $H^3(S^3; \mathbb{Z})$ respectively. Then, as in the proof of Proposition 3.1, we see that

$$\Gamma_{1,\infty}^*(\rho_p(e_3)) = a\rho_p(u_3 \otimes c)$$
for $a \in (\mathbb{Z}/p)^\times$. Note that the above equation holds for any odd prime $p$. Then we have obtained, in the integral cohomology, that

$$\Gamma^*_{1,\infty}(e_3) = \pm 2^b u_3 \otimes c$$

for some non-negative integer $b$, and thus by (3.2),

$$\tilde{\Gamma}^*_{1,\infty}(\epsilon) = \pm 2^{b-1} u_3 \otimes c$$

which implies that $b$ is positive. Since $H^5(S^3 \wedge \mathbb{R}P^2; \mathbb{Z}) \cong \mathbb{Z}/2$, it follows from Lemma 3.2 below and (2.2) that $\tilde{\Gamma}_{1,\infty}(\rho_2(\epsilon)) \neq 0$ in the mod 2 cohomology, which yields $b = 1$. Thus the proof is done.

**Lemma 3.1.** Let $\theta : \mathbb{R}P^2 \rightarrow SO(6)$ and $\iota : S^3 = S_{1,2}(SO) \rightarrow SO(6)$ be the inclusions. Then the Samelson product $\langle \iota, \theta \rangle$ is essential.

**Proof.** By the adjointness of Whitehead products and Samelson products, we show that the Whitehead product of $\text{ad}^{-1} \iota : S^4 \rightarrow BSO(6)$ and $\text{ad}^{-1} \theta : \Sigma \mathbb{R}P^2 \rightarrow BSO(6)$ is essential. Suppose now that $[\text{ad}^{-1} \iota, \text{ad}^{-1} \theta] = 0$. Then there exists a map $\kappa : S^4 \times \Sigma \mathbb{R}P^2 \rightarrow BSO(6)$ satisfying the homotopy commutative diagram:

$$\begin{array}{ccc}
S^4 & \cup & \Sigma \mathbb{R}P^2 \text{ad}^{-1} \iota \vee \text{ad}^{-1} \theta \rightarrow BSO(6) \\
\downarrow & & \downarrow \kappa \\
S^4 \times \Sigma \mathbb{R}P^2 & \rightarrow & BSO(6)
\end{array}$$

Let $w$ and $u_4$ be generators of $H^1(\mathbb{R}P^2; \mathbb{Z}/2)$ and $H^4(S^4; \mathbb{Z}/2)$, respectively. Then, by definition, we have $\kappa^*(w_3) = 1 \otimes \Sigma w^2$ and $\kappa^*(w_4) = u_4 \otimes 1$, where $w_i$ is the Stiefel-Whitney class. On the other hand, it follows from the Wu formula that $\text{Sq}^3 w_4 = w_3 w_4$. Thus we obtain

$$0 = \text{Sq}^3(u_4 \otimes 1) = \text{Sq}^3 \kappa^*(w_4) = \kappa^*(\text{Sq}^3 w_4) = \kappa^*(w_3 w_4) = u_4 \otimes \Sigma w^2 \neq 0$$

which is a contradiction. Therefore we have established the Whitehead product $[\text{ad}^{-1} \iota, \text{ad}^{-1} \theta]$ is essential.

Recall that there is an isomorphism $SU(4) \cong Spin(6)$. Since the center of $SU(4) \cong Spin(6)$ is included in $C_{1,2}(SU)$, there is a projection $\pi : SO(6) \rightarrow X_{1,2}(SU)$.

**Lemma 3.2.** Let $\theta : \mathbb{R}P^2 \rightarrow SO(6)$ be the inclusion and let $\lambda : S^3 \wedge \mathbb{R}P^2 \rightarrow (SU(4)) \langle 3 \rangle$ be the composite:

$$S^3 \wedge \mathbb{R}P^2 \xrightarrow{1 \wedge \theta} S^3 \wedge SO(6) \xrightarrow{1 \wedge \pi} S^3 \wedge X_{1,2}(SU) \xrightarrow{\tilde{\Gamma}_{1,2}} (SU(4)) \langle 3 \rangle$$

Then $\lambda^*(\epsilon) \neq 0$, where $\epsilon$ is a generator of $H^5((SU(4)) \langle 3 \rangle; \mathbb{Z}) \cong \mathbb{Z}$ as above.
Proof. Since $S^3 \wedge \mathbb{R}P^2$ is 3-connected, the projection $(\text{SO}(6)) \langle 3 \rangle \to \text{SO}(6)$ induces an injection $[S^3 \wedge \mathbb{R}P^2, (\text{SO}(6)) \langle 3 \rangle] \to [S^3 \wedge \mathbb{R}P^2, \text{SO}(6)]$ of pointed homotopy set. By Lemma 3.1, we know that the Samelson product $\langle i, \theta \rangle$ is essential, and then so is its lift $S^3 \wedge \mathbb{R}P^2 \to (\text{SO}(6)) \langle 3 \rangle$.

Let $\tilde{\gamma} : S^3 \wedge \text{SO}(6) \to (\text{SO}(6)) \langle 3 \rangle$ be a lift of the restriction of the reduced commutator of SO(6) to $S^3 \wedge \text{SO}(6) = S_{1,2}(\text{SO}) \wedge \text{SO}(6)$. Then we have a homotopy commutative diagram:

$$
\begin{array}{ccc}
S^3 \wedge \text{SO}(6) & \xrightarrow{1 \wedge \pi} & S^3 \wedge X_{1,2}(\text{SU}) \\
\tilde{\gamma} & \Downarrow & \tilde{\Gamma}_{1,2} \\
(\text{SO}(6)) \langle 3 \rangle & \xrightarrow{} & (\text{SU}(4)) \langle 3 \rangle
\end{array}
$$

Thus we have established that $\lambda$ is essential. Now since $S^3 \wedge \mathbb{R}P^2$ is of dimension 5 and $(\text{SU}(4)) \langle 3 \rangle$ is 4-connected, it follows from the J.H.C. Whitehead theorem that $\lambda^*(\epsilon) \neq 0$. 

\section{Generating variety for $\Omega^n_0 \mathbb{G}(n)$}

The aim of this section is to prove that it holds for $\Omega^n_0 \mathbb{G}(d + l)$ by the map $\Gamma_{1,l}$ and $\Gamma'_{1,l}$ in the stable range of $\Omega^n_0 \mathbb{G}(d + l)$, the generating variety argument which is analogous to single loop spaces of Lie groups in [6]. The proofs are done by a similar calculation in [16].

\textbf{Theorem 4.1.} For $* \leq l$, the Pontrjagin ring $H_*(\Omega^n_0 \text{SO}(4 + l); \mathbb{Z}/2)$ is a polynomial ring generated by the image of $(\text{ad}^3 \Gamma'_{1,l})_* : H_*(\mathbb{H}P^{l+1}; \mathbb{Z}/2) \to H_*(\Omega^n_0 \text{SO}(4 + l); \mathbb{Z}/2)$.

\textbf{Proof.} We first prove the case $l = \infty$. We will use the same notation for the mod 2 cohomology of SO($\infty$) as in the proof of Proposition 3.1. Then, in particular, we have

$$
\text{Sq}^{2i-2}x_{2i-1} = x_{4i-3}, \text{ Sq}^{4i-3}x_{4i-1} = 0.
$$

Let $q$ and $u_n$ be generators of $H^i(\mathbb{H}P^{\infty}; \mathbb{Z}/2)$ and $H^n(S^n; \mathbb{Z}/2)$ as above, respectively. Then it follows from Proposition 3.1 that

$$(\Gamma'_{1,\infty})^*(x_{4i-1}) = u_3 \otimes q^{i-1}.$$

Since $\pi_1(\text{SO}(\infty)) \cong \mathbb{Z}/2$, we have

$$H^*(\text{SO}(\infty))(1); \mathbb{Z}/2) = \mathbb{Z}/2[\pi^*(x_3), \pi^*(x_5), \pi^*(x_7), \ldots],$$

where $\pi : (\text{SO}(\infty))(1) \to \text{SO}(\infty)$ denotes the projection. Then, by the Borel transgression theorem, we have

$$H^*(\Omega_0 \text{SO}(\infty); \mathbb{Z}/2) = \Delta(y_2, y_4, y_6, \ldots), \ (\text{ad} \Gamma'_{1,\infty})^*(y_{4i-2}) = u_2 \otimes q^{i-1}$$
where $y_i$ is the suspension of $x_{i+1}$ and $\Delta(a_1, a_2, \ldots)$ stands for the simple system of generators $\{a_1, a_2, \ldots\}$. It is rewritten as

$$H^*(\Omega_0\text{SO}(\infty); \mathbb{Z}/2) = \mathbb{Z}/2[y_2, y_6, y_{10}, \ldots].$$

Then it follows from the Borel transgression theorem that

$$H^*(\Omega^2\text{SO}(\infty); \mathbb{Z}/2) = \Delta(z_1, z_5, z_9, \ldots), \quad (\text{ad}^2\Gamma'_1, \infty)^*(z_{4i-3}) = u_1 \otimes q^{i-1}$$

and

$$z^2_{4i-3} = \text{Sq}^{4i-3}z_{4i-3} = 0,$$

where $z_i$ is the suspension of $y_{i+1}$. Namely, we have

$$H^*(\Omega^2\text{SO}(\infty); \mathbb{Z}/2) = \Lambda(z_1, z_5, z_9, \ldots).$$

Now we take the dual Hopf algebra of $H^*(\Omega^2\text{SO}(\infty); \mathbb{Z}/2)$ to get

$$H_*(\Omega^2\text{SO}(\infty); \mathbb{Z}/2) = \Lambda(z^*_1, z^*_5, z^*_9, \ldots), \quad (\text{ad}^2\Gamma'_1, \infty)_*(u^*_1 \otimes (q^{i-1})^*) = z^*_{4i-3},$$

where $x^*$ means the Kronecker dual of $x$. Since $\pi_3(\text{SO}(\infty)) \cong \mathbb{Z}$, we have

$$H_*(((\Omega^2\text{SO}(\infty)) (1); \mathbb{Z}/2) = \Lambda(s_5, s_9, s_{13}, \ldots),$$

where $s_l$ is defined by $\pi'_* (s_l) = z^*_l$ for the projection $\pi' : (\Omega^2\text{SO}(\infty))(1) \to \Omega^2\text{SO}(\infty)$. Then, by the Borel transgression theorem, we have, for $* < l$,

$$H_*(((\Omega^2\text{SO}(\infty)); \mathbb{Z}/2) = \mathbb{Z}/2[t_4, t_8, t_{12}, \ldots], \quad (\text{ad}^3\Gamma'_1, \infty)_*((q^{i-1})^*) = t_{4i-4}$$

in which $s_{i+1}$ is the transgression image of $t_i$, and therefore the proof is completed.

Note that the inclusion $\text{SO}(4 + l) \to \text{SO}(\infty)$ is a $(4 + l)$-equivalence. Then the inclusion $\Omega^2\text{SO}(4 + l) \to \Omega^2\text{SO}(\infty)$ is a $(1 + l)$-equivalence, and thus the theorem follows from (2.5).

In proving the generating variety argument for $\Omega^3\text{Sp}(1 + l)$, we will use:

**Lemma 4.1** (S. Araki and T. Kudo [1]). *Let $X$ be a simply connected homotopy associative $H$-space. If $H_*(X; \mathbb{Z}/2) = \mathbb{Z}/2[x_1, x_2, \ldots]$ and each $x_i$ is transgressive, then we have

$$H_*(\Omega X; \mathbb{Z}/2) = \mathbb{Z}/2[y^0_1, y^1_1, \ldots, y^0_2, y^1_2, \ldots]$$

where $y^l_k$ is the transgression image of $x^l_k$.*
**Theorem 4.2.** For \( * \leq 4l + 2 \), the Pontrjagin ring \( H_*(\Omega^3_0 \text{Sp}(1 + l); \mathbb{Z}/2) \) is a polynomial ring generated by the image of \((\text{ad}^3 \Gamma_{1,l})_* : H_*(\mathbb{R}P^{4l+3}; \mathbb{Z}/2) \rightarrow H_*(\Omega^3_0 \text{Sp}(1 + l); \mathbb{Z}/2)\).

**Proof.** We first prove the case \( l = \infty \). We will use the same notation for the mod 2 cohomology of \( \text{Sp}(1 + l) \) as in the proof of Proposition 3.2. Let \( u_n \) and \( w \) be generators of \( H^n(S^n; \mathbb{Z}/2) \) and \( H^1(\mathbb{R}P^{\infty}; \mathbb{Z}/2) \), respectively, as well as above. Then it follows from Proposition 3.2 that

\[
\Gamma_{1,\infty}^*(y_{4l-1}) = u_3 \otimes w^{4l-4}.
\]

Now we take the dual Hopf algebra of \( H^*(\text{Sp}(\infty); \mathbb{Z}/2) \) so that

\[
H_*(\text{Sp}(\infty); \mathbb{Z}/2) = \Lambda(y_3, y_5, \ldots), \quad (\Gamma_{1,\infty})_*(u_3 \otimes (w^{4l-4})^\sharp) = y_{4l-1}^\sharp.
\]

Then, by the Borel transgression theorem, we get

\[
H_*(\Omega\text{Sp}(\infty); \mathbb{Z}/2) = \mathbb{Z}/2[z_2, z_6, \ldots], \quad (\text{ad} \Gamma_{1,\infty})_*(u_2^\sharp \otimes (w^{4l-4})^\sharp) = z_{4l-2}
\]

in which \( z_i \) is the transgression image of \( y_{i+1} \).

By Lemma 5.1 in the next section, Theorem 4.1 implies the map

\[
(\text{ad}^3 \Gamma_{\infty,\infty})_* : H_*(B\text{Sp}(\infty); \mathbb{Z}/2) \rightarrow H_*(\Omega^3_0 \text{SO}(\infty); \mathbb{Z}/2)
\]

is an isomorphism. Then since \( B\text{Sp}(\infty) \) and \( \Omega^3_0 \text{SO}(\infty) \) are of finite type, we deduce that the map \((\text{ad}^3 \Gamma_{\infty,\infty})_{(2)} : B\text{Sp}(\infty)_{(2)} \simeq \Omega^3_0 \text{SO}(\infty)_{(2)}\) is a homotopy equivalence, where \(-_{(2)}\) means the 2-localization in the sense of Bousfield and Kan [8]. In particular, we can consider the action of the Kudo-Araki operation \( Q^4i \) on \( q_i^\sharp \in H_*(B\text{Sp}(\infty); \mathbb{Z}/2) \), where \( q_i \) is the mod 2 reduction of the symplectic Pontrjagin class. (See [11].) Recall that in \( H^*(B\text{Sp}(\infty); \mathbb{Z}/2) \), we have

\[
q_i^\sharp = (q_i^\sharp)^i.
\]

Then, in particular,

\[
Q^4iz_i^\sharp = (q_i^\sharp)^2 = (q_1^\sharp)^{2i} = q_{2i}.
\]

Since \( Q^4i \) commutes with the transgression, we obtain

\[
Q^4iz_{4l-2} = z_{8l-2}.
\]

Then it follows from the Nishida relation \( \text{Sq}^2 Q^s = (\frac{s-2}{2}) Q^{s-2} + Q^{s-1} \text{Sq}_1^i \)

\[
\text{Sq}^2 z_{8l-2} = z_{4l-2}^2,
\]

where \( \text{Sq}^k \) denotes the dual of \( \text{Sq}^k \). (See [11].) Since \( (\text{ad} \Gamma_{1,\infty})_*(u_2^\sharp \otimes (w^{4l-4})^\sharp) = z_{4l-2} \) and \( \text{Sq}^2_i (w^{4i})^\sharp = (w^{4i-2})^\sharp \), we have established

\[
(\text{ad} \Gamma_{1,\infty})_*(u_2^\sharp \otimes (w^{2i-2})^\sharp) = z_{2i}.
\]
Applying Lemma 4.1, we get

\[ H_\ast(\Omega^2 \text{Sp}(\infty); \mathbb{Z}/2) = \mathbb{Z}/2[s_1, s_3, \ldots], \quad (\text{ad}^2 \Gamma_{1,\infty})_\ast(u_i^2 \otimes (w^{2i-2})^t) = s_{2i-1} \]

where \( s_{2m(4n-2)-1} \) is the transgression image of \( z_{4n-2}^{2m} \). Note that we can consider the operation \( Q^i \) and \( Q^j \) on \( z_i \). By the Nishida relation \( \text{Sq}^i Q^j = (s-1)Q^{s-1} \), we see that, for \( m \geq 1 \),

\[ \text{Sq}^i z_{4n-2}^{2m} = \text{Sq}^i Q^{2m-1}(4n-2)z_{4n-2}^{2m-1} = Q^{2m-1}(4n-2)-1z_{4n-2}^{2m-1} \]

and then

\[ \text{Sq}^i s_{2m(4n-2)-1} = s_{2m-1(4n-2)-1}. \]

Thus we can deduce that

\[ (\text{ad}^2 \Gamma_{1,\infty})_\ast(u_i^2 \otimes (w^{i-1})^t) = s_i, \]

where we put \( s_{2i} = s_i^2 \).

Since \( \pi_3(\text{Sp}(\infty)) \cong \mathbb{Z} \), we have

\[ H_\ast((\Omega^2 \text{Sp}(\infty))(1); \mathbb{Z}/2) = \mathbb{Z}/2[\bar{s}_2, \bar{s}_3, \bar{s}_5, \ldots] \]

in which \( \bar{s}_i \) is defined by \( \pi_\ast(\bar{s}_i) = s_i \) for the projection \( \pi : (\Omega^2 \text{Sp}(\infty))(1) \to \Omega^2 \text{Sp}(\infty) \). Then, by Lemma 4.1, we obtain

\[ H_\ast(\Omega^3_0 \text{Sp}(\infty); \mathbb{Z}/2) = \mathbb{Z}/2[t_1, t_2, t_3, \ldots], \quad (\text{ad}^3 \Gamma_{1,\infty})_\ast((w^{i-1})^t) = t_{i-1} \]

and thus the proof is done.

Since the inclusion \( \text{Sp}(1+l) \to \text{Sp}(\infty) \) is an \( (4l+6) \)-equivalence, the inclusion \( \Omega^3_0 \text{Sp}(1+l) \to \Omega^3_0 \text{Sp}(\infty) \) is an \( (4l+3) \)-equivalence. Therefore the proof is completed by (2.2).

We next consider the case \( G = \text{SU} \). Only in this case, we will use a result related with Bott periodicity which is an easy consequence of [20].

**Lemma 4.2.** Let \( a \) be a generator of \( H^2(\Omega^3_0 \text{SU}(\infty); \mathbb{Z}) \cong \mathbb{Z} \). Then the integral homology of \( \Omega^3_0 \text{SU}(\infty) \) is

\[ H_\ast(\Omega^3_0 \text{SU}(\infty); \mathbb{Z}) = \mathbb{Z}[b_2, b_4, \ldots], \quad b_{2i} = (a^i)^2. \]

**Theorem 4.3.** For \( * \leq 2l \), the Pontrjagin ring \( H_\ast(\Omega^3_0 \text{SU}(\infty); \mathbb{Z}) \) is a polynomial ring generated by the image of \( (\text{ad}^3 \Gamma_{1,\ell})_\ast : H_\ast(X_{1,\ell}(\text{SU}); \mathbb{Z}) \to H_\ast(\Omega^3_0 \text{SU}(2+l); \mathbb{Z}) \).

**Proof.** The case \( l = \infty \) follows from Proposition 3.3 and Lemma 4.2. One can easily verify that the inclusion \( \Omega^3_0 \text{SU}(2+l) \to \Omega^3_0 \text{SU}(\infty) \) and the natural map \( X_{1,l}(\text{SU}) \to X_{1,\infty}(\text{SU}) \) are \( (2l+4) \)-equivalences. Thus the theorem follows from (2.2).
5 Bott periodicity

In this section, we prove that the map \( \text{ad}^3 \Gamma_\infty,\infty : BH(\infty) \to \Omega^3_0 G(\infty) \) is a homotopy equivalence. Notice here that we have not used any result concerning real and symplectic Bott periodicity. We have only used the result of Toda [20] to get the ring structure of \( \Omega^3_0 SU(\infty) \) in the last section. Then our result provides a new proof for real and symplectic Bott periodicity.

We start with an easy algebraic lemma. Let \( V \) be a graded free module over a PID. As usual, we will call \( V \) of finite type if, in each dimension, \( V \) is finitely generated. We will denote the free commutative graded algebra generated by \( V \) by \( \Lambda V \). Then we can easily see:

**Lemma 5.1** (Kono and Tokunaga [17]). Let \( V \) and \( W \) be of finite type graded free modules over a PID \( R \) such that \( V \cong W \), and let \( U \) be a graded module over \( R \). Given a graded algebra map \( f : \Lambda V \to \Lambda W \) and a graded module map \( g : U \to \Lambda V \). If the image of \( f \circ g : U \to \Lambda W \) generates \( \Lambda W \), then \( f \) is an isomorphism.

Now we prove our main theorem.

**Theorem 5.1.** The map \( \text{ad}^3 \Gamma_\infty,\infty : BH(\infty) \to \Omega^3_0 G(\infty) \) is a homotopy equivalence.

**Proof.** We first prove the case \( G = SU \). By Lemma 2.2, Theorem 4.3 and Lemma 5.1 together with the homotopy commutative diagram (2.6), we see that the map \( \text{ad}^3 \Gamma_\infty,\infty : BU(\infty) \to \Omega^3_0 SU(\infty) \) induces an isomorphism in the integral homology. Then, by the J.H.C. Whitehead theorem, we obtain that \( \text{ad}^3 \Gamma_\infty,\infty \) is a homotopy equivalence. Thus, in particular, from \( \pi_*(BU(2)) \) is for \( * \leq 4 \), we deduce:

\[
\pi_*(BU(\infty)) \cong \begin{cases} 
\mathbb{Z} & * = 2, 4, \ldots \\
0 & * = 1, 3, \ldots 
\end{cases} \tag{5.1}
\]

Note here that we do not need to use Bott periodicity of \( BU(\infty) \).

We next consider the case \( G = SO \). We may assume \( \Gamma'_{1,\infty} = \Gamma_{1,\infty} \) as noted above. Then it follows from Lemma 2.2, Theorem 4.1, Lemma 5.1 and (2.6) that the map \( \text{ad}^3 \Gamma_\infty,\infty : BSp(\infty) \to \Omega^3_0 SO(\infty) \) induces an isomorphism in the mod 2 homology. On the other hand, we have \( qc' = 1 : BSp(\infty) \to BSp(\infty) \) and \( rc = 2 : BSO(\infty) \to BSO(\infty) \). Then it follows from (5.1) that the homotopy groups of \( BSp(\infty) \) and \( \Omega^3_0 SO(\infty) \) are odd torsion free. Then, by considering the rational cohomology of \( BSp(\infty) \) and \( BSO(\infty) \), we obtain

\[
\pi_*(BSp(\infty)) \otimes \mathbb{Z}[\frac{1}{2}] \cong \begin{cases} 
\mathbb{Z}[\frac{1}{2}] & * = 4, 8, \ldots \\
0 & * \neq 4, 8, \ldots 
\end{cases}
\]

and

\[
\pi_*(\Omega^3_0 SO(\infty)) \otimes \mathbb{Z}[\frac{1}{2}] \cong \begin{cases} 
\mathbb{Z}[\frac{1}{2}] & * = 4, 8, \ldots \\
0 & * \neq 4, 8, \ldots 
\end{cases}
\]
This implies that the maps $c'_b : \pi_*(BSp(\infty)) \otimes \mathbb{Z}[\frac{1}{2}] \to \pi_*(BU(\infty)) \otimes \mathbb{Z}[\frac{1}{2}]$ and $c'_a : \pi_*(\Omega^3_0SO(\infty)) \otimes \mathbb{Z}[\frac{1}{2}] \to \pi_*(\Omega^3_0SU(\infty)) \otimes \mathbb{Z}[\frac{1}{2}]$ are split monomorphisms. Thus since $\text{ad}^3\Gamma_{\infty,\infty} : BU(\infty) \to \Omega^3_0SU(\infty)$ is a homotopy equivalence as above, the map $\left(\text{ad}^3\Gamma_{\infty,\infty}\right)_* : \pi_*(BSp(\infty)) \otimes \mathbb{Z}[\frac{1}{2}] \to \pi_*(\Omega^3_0SO(\infty)) \otimes \mathbb{Z}[\frac{1}{2}]$ is an isomorphism by Proposition 2.2. On the other hand, we can apply Lemma 5.1 to the map $\text{ad}^3\Gamma_{\infty,\infty} : BSp(\infty) \to \Omega^3_0SO(\infty)$ in the mod 2 homology by Lemma 2.2 and Theorem 4.1. Then we obtain the map $\text{ad}^3\Gamma_{\infty,\infty} : BSp(\infty) \to \Omega^3_0SO(\infty)$ induces an isomorphism in the mod 2 homology. Summarizing, we have established that this map is a homology equivalence and therefore by a generalized J.H.C. Whitehead theorem [12], the proof is completed.

The case $G = Sp$ is quite similar to the case $G = SO$. □

**Corollary 5.1.** Let $d_{k,l} = \min\{2k+1, 2l+1\}, \min\{k, 4l+3\}, \min\{4k+3, l\}$ according as $G = SU, Sp, SO$. Then the map $\text{ad}^3\Gamma_{k,l} : X_{k,l}(G) \to \Omega^3_0G(dk+l)$ is a $d_{k,l}$-equivalence.

**Proof.** Let $a_k = 2k+1, k, 4k+3$ according as $G = SU, Sp, SO$. Then it is easy to see that the projection $BH(k) \to BH(\infty)$ is an $a_k$-equivalence. By definition, there is a principal bundles

$$H(k) \to G(dk+l)/G(l) \to X_{k,l}(G)$$

for $G = Sp, SO$ and

$$U(k) \to U(2k+l)/U(l) \to X_{k,l}(SU).$$

Let $b_{k,l} = 2l+1, 4l+3, l$ according as $G = SU, Sp, SO$. Then it follows from the above principal bundles that the composite of the inclusion $X_{k,l}(G) \to X_{k,\infty}(G)$ and the homotopy equivalence $X_{k,\infty}(G) \cong BH(k)$ is a $b_{k,l}$-equivalence. Let $c_{k,l} = 4k + 2l - 3, 4k + 4l - 1, 4k + l - 4$ according as $G = SU, Sp, SO$. Then the inclusion $\Omega^3_0G(dk+l) \to \Omega^3_0G(\infty)$ is a $c_{k,l}$-equivalence. Now let us consider a homotopy commutative diagram:

$$\begin{array}{ccc}
X_{k,l}(G) & \xrightarrow{\text{ad}^3\Gamma_{k,l}} & BH(k) \\
\downarrow \text{ad}^3\Gamma_{k,\infty} & & \downarrow \text{ad}^3\Gamma_{\infty,\infty} \\
\Omega^3_0G(dk+l) & \xrightarrow{\text{ad}^3\Gamma_{k,l}} & \Omega^3_0G(\infty)
\end{array}$$

Then it follows from Theorem 5.1 that the map $\text{ad}^3\Gamma_{k,l} : X_{k,l}(G) \to \Omega^3_0G(dk+l)$ is a $\min\{a_k, b_{k,l}, c_{k,l}\}$-equivalence. Thus the proof is completed. □

## 6 Applications to instanton moduli spaces

In this section, we give applications of the results obtained so far to the homotopy types of instanton moduli spaces $\mathcal{M}_k(G)$.  

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Recall from Lemma 2.1 that the map $\Gamma_{k,l} : S^3 \wedge \mathcal{X}_{k,l}(G) \to G(dk+l)$ was constructed from the moduli space of $G(dk+l)$-instantons on $S^4$. In particular, $\mathcal{X}_{k,l}(G)$ is a subspace of $\mathcal{M}_k(G(dk+l))$. We denote the inclusion $\mathcal{X}_{k,l}(G) \to \mathcal{M}_k(G(dk+l))$ by $i_{k,l}$. Then, by definition, we have

$$\text{ad}^3 \Gamma_{k,l} = j_{k,l} \circ i_{k,l} \quad (6.1)$$

where $j_{k,l} : \mathcal{M}_k(G(dk+l)) \to \Omega^3 \Omega G(dk+l)$ is the inclusion. We also have a commutative diagram:

$$\begin{array}{ccc}
\mathcal{X}_{k,l}(G) & \longrightarrow & \mathcal{X}_{k,l+1}(G) \\
i_{k,l} & & \downarrow \quad i_{k,l+1} \\
\mathcal{M}_k(G(dk+l)) & \longrightarrow & \mathcal{M}_k(G(dk+l+1))
\end{array}$$

Here the horizontal arrows are induced from the inclusion $G(dk+l) \to G(dk+l+1)$. Then we have a map

$$\text{colimi}_{k,l} \mathcal{X}_{k,\infty}(G) \to \text{colimi} \mathcal{M}_k(G(dk+l))$$

which we denote by $i_{k,\infty} : \mathcal{X}_{k,\infty}(G) \to \mathcal{M}_k(G(\infty))$.

**Proposition 6.1.** The map $i_{k,\infty}$ is a homotopy equivalence.

**Proof.** We first prove the case $G = SU, SO$. Recall from [18] that there is a homotopy equivalence $\mathcal{M}_k(G(\infty)) \simeq BH(k)$. On the other hand, we know that $\mathcal{X}_{k,\infty}(G) \simeq BH(k)$. Then we have $H^*(\mathcal{X}_{k,\infty}(G); \mathbb{Z}) \cong H^*(\mathcal{M}_k(G(\infty)); \mathbb{Z})$ as abstract rings. Note that $H^*(\mathcal{X}_{k,\infty}(G); \mathbb{Z})$ is a polynomial ring. By Corollary 5.1, we see that $H^*(\mathcal{X}_{k,\infty}(G); \mathbb{Z})$ is generated by $\text{Im}(\text{ad}^3 \Gamma_{k,\infty})^*$. Therefore, by Lemma 5.1 and (6.1), the proof is completed.

We next prove the case $G = Sp$. By Corollary 5.1, the map $\left(\text{ad}^3 \Gamma_{k,\infty}\right)^* : H^*(\Omega^3 \Omega Sp(\infty); \mathbb{Z}/2) \to H^*(\mathcal{X}_{k,\infty}(G); \mathbb{Z}/2)$ is an isomorphism for $* \leq k$. Then, in particular, it follows from (6.1) that the map $(i_{k,\infty})_* : \pi_1(\mathcal{X}_{k,\infty}(G)) \to \pi_1(\mathcal{M}_k(G(\infty)))$ is an isomorphism, where both $\pi_1(\mathcal{X}_{k,\infty}(G))$ and $\pi_1(\mathcal{M}_k(G(\infty)))$ are isomorphic to $\mathbb{Z}/2$. Since both $\mathcal{X}_{k,\infty}(1)$ and $\mathcal{M}_k(G(\infty))(1)$ have the homotopy type of $BSO(k)$, we can see the map $(i_{k,\infty})(1) : \mathcal{X}_{k,\infty}(1) \to \mathcal{M}_k(G(\infty))(1)$ induces an isomorphisms in the cohomology with the coefficients $\mathbb{Z}/2$ and $\mathbb{Z}[\frac{1}{2}]$ quite analogously to the above case. Then the map $(i_{k,\infty})(1)$ is a homotopy equivalence, and hence a homotopy equivalence. Therefore the map $i_{k,\infty}$ is a homotopy equivalence.

We estimate a range that the map $i_{k,l} : \mathcal{X}_{k,l}(SU) \to \mathcal{M}_k(SU(dk+l))$ is a homotopy equivalence.

**Theorem 6.1.** The map $i_{k,l} : \mathcal{X}_{k,l}(G) \to \mathcal{M}_k(G(dk+l))$ is a $\min\{2k+1, 2l+1\}$-equivalence.
Proof. In [15], it is shown that the map $\mathcal{M}_k(G(dk + l)) \rightarrow \mathcal{M}_k(G(\infty))$ induced from the inclusion $G(dk + l) \rightarrow G(dk + l + 1)$ is a $(2k + 1)$-equivalence. On the other hand, the map $X_{k,l}(G) \rightarrow X_{k,\infty}(G)$ induced from the inclusion $G(dk + l) \rightarrow G(dk + l + 1)$ is a $(2l + 1)$-equivalence as is seen in the proof of Corollary 5.1. Then the theorem follows from (2.2) and Proposition 6.1.

References


