8. A section of the ground state energy of the spin glass in the phase space

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§1. Introduction

In literatures of spin glasses figures of the energy as a function of the phase space, as shown in Fig.1, are often listed qualitatively. In this note a quantitative figure of similar nature will be shown. The Ising spin glass for the ±J model on the Bethe lattice (pair approximation in the cluster variation method (CVM)) at T=0 is considered.

§2. Discrete distribution

The free energy F in the pair approximation is shown to be given by[1)]

$$F=(1-z)F_1+(z/2)F_2 \qquad (1)$$

where F_1 and F_2 are one- and two-body free energies and z is the coordination number.

First we consider the case where the distribution function g(h) of the single bond effective field h is expressed by a superposition of 2n+1 delta functions:

$$g(h)=\Sigma_{\ell=-n}^{n}\mu_{\ell}\delta(h-\ell/n) \tag{2}$$

The exchange energy J is taken to be a unity. The one- and two-body effective fields, $H^{(1)}$ and $H^{(2)}$, are given by convolutions of z and z-1 single bond effective fields.

$$G^{(1)}(H^{(1)})=\Sigma_{\ell=-nz}^{nz}\ b_{\ell}\delta(H^{(1)}-\ell/n) \tag{3}$$

$$G^{(2)}(H^{(2)})=\Sigma_{\ell=-(z-1)n}^{(z-1)n}\ a_{\ell}\delta(H^{(2)}-\ell/n) \tag{4}$$

where

b_{ℓ} is a coefficient of ξ^{ℓ} in $(\Sigma_{\ell=-n}^{n}\mu\ \xi^{\ell})^{z}$ $(-nz<\ell<nz)$

a_{ℓ} is a coefficient of ξ^{ℓ} in $(\Sigma_{\ell=-n}^{n}\mu_{\ell}\xi^{\ell})^{z-1}$

$$(-n(z-1)<\ell<n(z-1)) \tag{5}$$

The one and two-body free energies F_1 and F_2 are given[2)] in terms of b_{ℓ} and a_{ℓ}:

$$-F_1=(1/n)\Sigma_{\ell=-zn}^{zn}\ b_{\ell}\max(\ell,-\ell) \tag{6}$$

$$-F_2=(1/n)\Sigma_{\ell=-(z-1)n}^{(z-1)n}\ \Sigma_{m=-(z-1)n}^{(z-1)n}\ a_{\ell}a_{m}$$

$$\times\ \max(-n+\ell+m,\ -n-\ell-m,\ n-\ell+m,\ n+\ell-m) \tag{7}$$

The distribution function g(h) is determined by the integral equation. At T=0 it reads

$$g(h)=\int\delta(h-\mathrm{sgn}(H^{(2)})\min(|H^{(2)}|))\Pi_{K=1}^{z-1}\ g(h_K)dh_K \tag{8}$$

$$H^{(2)}=\Sigma_{K=1}^{z-1}\ h_K$$

The integral equation (7) leads a system of algebraic equtions for unknown coefficients ~~functions~~ μ_ℓ.

$$\mu_\ell = \Sigma_{\ell_1 \ell_2 \ldots \ell_{z-1}} \mu_{\ell_1} \mu_{\ell_2} \cdots \mu_{\ell_{z-1}} \tag{9}$$

$$\Sigma \ell_i = \ell, \qquad \ell_i = -n+1,\ -n+2,\ \ldots\ n-1$$

$$\Sigma \mu_\ell = 1$$

We consider the symmetric (spin glss, $\mu_\ell = -\mu_{-\ell}$) case for z=3. The algebraic equations (8) are reduced to

$$f_0(\mu_0) = 0 \tag{10}$$

$$\mu_\ell = f_\ell(\mu_0) \qquad \ell = 1,\ 2,\ \ldots\ ,\ n \tag{11}$$

where f_0 and f_ℓ are polynomials of of μ_0 of 2^n and 2^{n-1}th degree.

The solution of (10) which gives the solution of the integral equation are obtained by Katsura et al[3] for n=1, 2, 3, and 4. The values of the energies in these points except the paramagnetic state are almost the same (agree with three digits)

We calculated F in terms of μ_0 for $0<\mu_0<1$ by using (10), (11), (5), (6), and (1) successively. Figures 2 and 3 show the negative of the free energies vs μ_0 for n=2 and 3, respectively. The energy surface for n=2 in the μ_0-μ_1 plane are shown in Fig.4, (similar to ref 6). The points for the solutions of integral equations are shown by close circles in the figures. The maxima (minima) in Figs. 2 and 3 do not necessarily give the maxima (minima) of the energy surface. The reason is that Fig.2 (or 3) is a section of the energy surface on a curve given by (11) in μ_1, μ_2..., μ_n space. These figures are similar to Fig.1 but with numerical axis.

§3. Continuous distribution

The integral equation (7) has solutions expressed by superpositions of 2n+1 delta functions. As $n\to\infty$, it tends to a continous distribution with three delta functions[4]. The solution of the integral equation in that case is shown to be

$$g(h)=a\delta(h)+(b/2)[\delta(h-1)+\delta(h+1)]+c_0-c_2(3h^2-1)/4 \quad (12)$$

The coefficients a, b, c_0, c_2 are solution of a system of algebraic equations and they are reduced to

$$f_0(a)=0,\ b=f_1(a),\ c_0=f_2(a),\ c_2=f_3(a) \quad (13)$$

where f_0 is a polynomial of of a of 8th degree, and f_1, f_2, f_3 of 7th degree.

The energy was calculated as a function of a, b, c_0, and c_2 and expressed as a function of a, $F=f^*(a)$, polynomial of a of 28th degree. The stationary points of f^* are calculated and are shown in Table 1. In table 1 the value of a with * are the solution of the integral equation, (7). These points are neither maxima nor minima of $f^*(a)$, again since this is a section in a, b, c_0, c_2 space. Among them a=1/3 and 0.10683 are spin glass states, the former is the state of discrete distribution and the latter the continuous distribution. The energies by Morita[5] and by Wong et al[6] are 1.276 and 1.2749, respectively.

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References

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Fig. 1 Schematic diagram of the free energy in the phase space appearing in literatures.

Fig. 2 Negative of the energy, $-E(\mu_0)$, as a function of μ_0. $z=3$, $n=2$.

Fig. 3 Negative of the energy, $-E(\mu_0)$, as a function of μ_0. $z=3$, $n=3$.

Fig.4 Negative of the energy, $-E(\mu_0, \mu_1)=mF$. $z=3$, $n=2$.

Table 1 Stationary poins of $F=f^{*}(a)$. The value of a with * are solutions of integral equation.

a	f(a)
0.03815	366.972
*0.04171	-77584.08
0.06734	-41476862
0.10681	1.31011
*0.10683	1.27367
0.20446	-1649870784
0.33059	130.862
*0.33333	1.27778
*0.42219	-2342966
0.45170	-0.68646
0.45452	284.973
*0.45809	-7168.45
0.61111	-4759339956
*0.72049	-5145002
0.73281	387.355
0.84812	-6858432078
0.95890	2769.61
*0.97162	-1096078
*1.00000	1.5

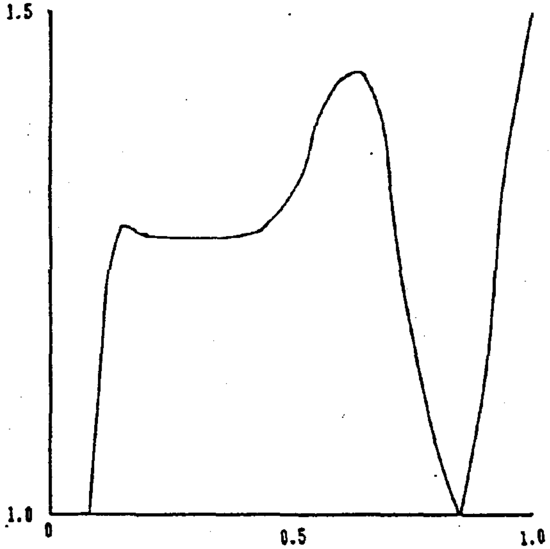

Fig.2

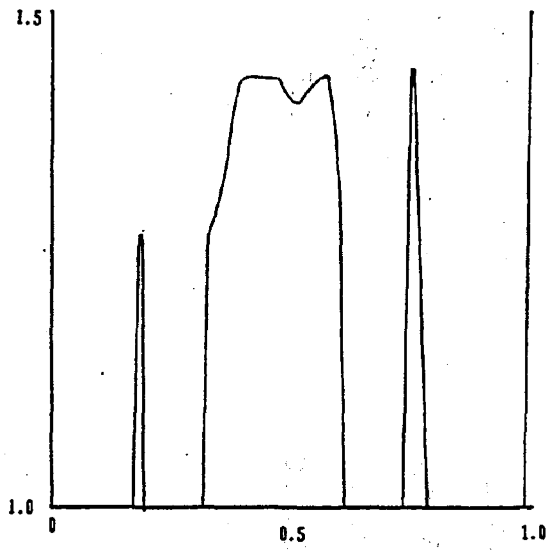

Fig.3

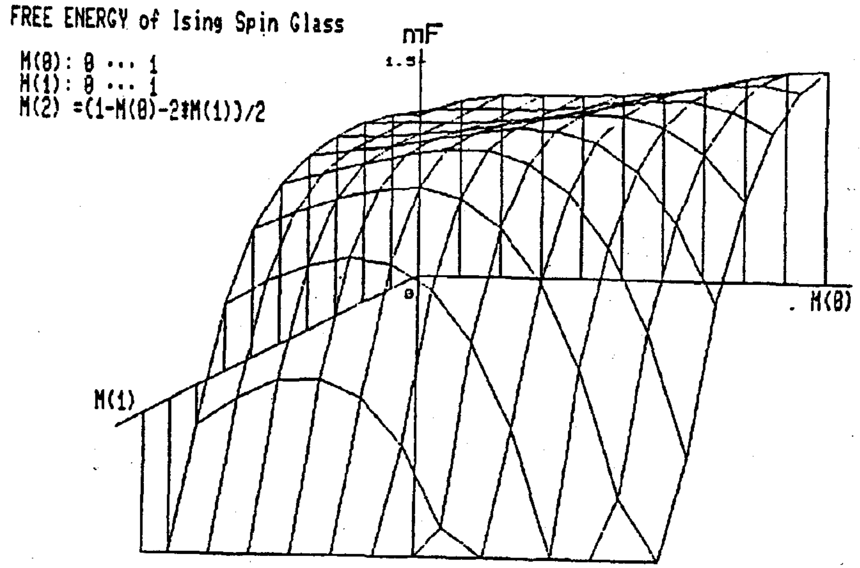

Fig.4

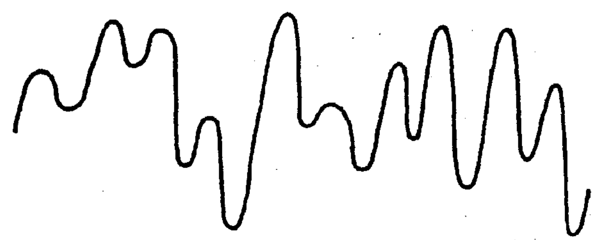

Fig.1