A numerical study on multiple attractors in a laser

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In this report we would like to present some of the results of a numerical study on multiple attractors in a laser. Namely, a- the bifurcation diagram for a region of the chosen control-parameter, and b- the basins of attraction for two sets of parameter values.

The laser is a good example for studying multiple attractors. There exist reliable models in terms of a small set of ordinary differential equations the time scales involved are very short (10^{-3} to 10^{-7} s) and numerical results can be comparatively easily verified by experiment. In an experiment, usually the laser is modulated with a frequency ω and the control parameter is the drive amplitude δ. It has been observed that the period doubling cascade to chaos is interrupted by periodic windows with a periodicity increasing by one for each successive window. Furthermore, it has been shown that this phenomenon is related to an infinit series of so-called Primary Saddle-Node Bifurcations (PSNB) that lead to the existence of multiple orbits, and to internal and boundary crisises.

Besides the series of PSNB there turns out to exist a previousely unknown (series) of period n attractors, that bifurcate and undergo inverse bifurcation and that are assumed to come into existence shortly after the first bifurcation of a period n PSNB. The new attractor appears to be 3-fold for every branch of a period n PSNB (up to the second periodic window the existence of this attractor could be numerically verified) and may therefore also be assumed to form an infinite series.

Let us start with equations that describe a single mode laser:

$$\begin{aligned} \frac{du}{dt} &= -u\left[\delta\cos(\omega t+\phi)-v\right] \\ \frac{dv}{dt} &= -\epsilon v - u - \kappa u v + 1, \end{aligned} \tag{1}$$

where u and v denote (scaled) intensity and population inversion. δ, ϵ and κ are small positive real numbers and ϕ is the phase. δ is the strength of

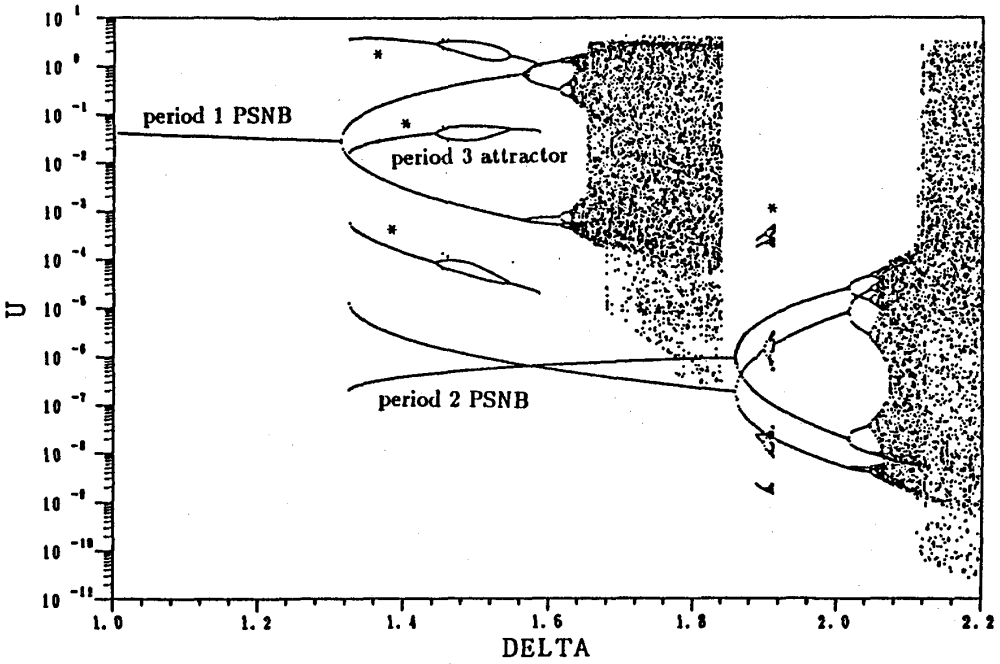

Figure 1: Bifurcation-diagram with 3 resp. 2 coexisting attractors

the intensity's damping which is modulated with a frequency ω, ϵ gives the linear loss rate for the inverse population and κ represents the loss of the intensity per photon.

Fig. 1 shows the bifurcation diagram for eqns. 1, where the attractors stemming from the PSNB and the newly discovered attractors are superimposed. The windows are thought to be caused by the collision of the unstable branch of a PSNB with a chaotic region. Since the region up to $\delta \simeq 1.65$ contains a period 1 PSNB it seems to be reasonable to call this region the 0th window.

The new attractor is marked by a '*'. Clearly a bifurcation and an inverse bifurcation can be seen. Although fig. 1 suggests that the vanishing of the period 3 attractor is caused by a collision with the bifurcating period 1 PSNB, this is actually not the case. It turns out to be an artefact of the projection. The reasons for both the creation and the vanishing of this attractor are not understood yet. From the numerical results however it can be inferred that it comes into existence shortly after the first bifurcation of a stable branch of a PSNB and that it vanishes before the chaotic region starts. It has not been possible yet to trace the abovementioned attractor for a substantial range of the parameter δ in the first and second window. This might be due to a fractal boundary (see below).

In all cases the initial conditions were chosen on the respective initial branches of the attractors. For the parameters other than δ we chose the following values: $\omega = 0.9$, $\phi = 0.079516$, $\epsilon = 0.09$ and $\kappa = 0.003$.

To determine the basins of attraction and the trajectories we transformed equations (1) where $u \rightarrow \log u$ and $v \rightarrow v$ for computational convenience.

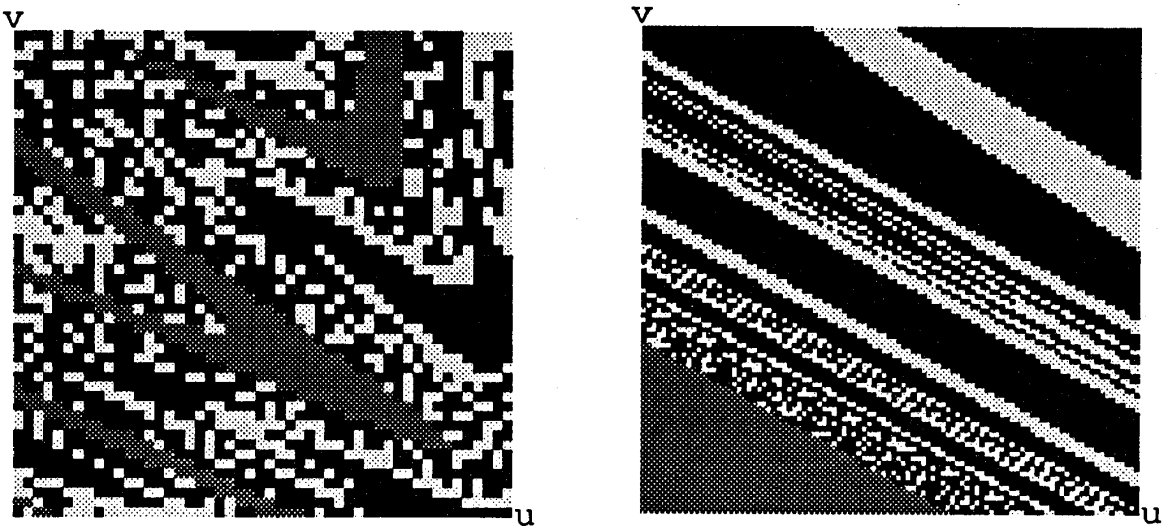

Figure 2: Basins of attraction, $\delta = 1.4$

First, for every set of parameters, with δ ranging from 0.2 to 2.0 stepsize 0.2 and for $\delta = 3.5$, we traced a 20 by 20 grid of 400 initial conditions to get a global idea of the basins of attraction (u = [-50.0, 2.25], v = [-7.0, 6.3]). This grid gives an upper bound for the maximum size of an undiscovered attractor in the the 3 dimensional δ, u, v space, implying such an attractor would be quite small and that the effects on the global dynamics might be limited. For $\delta = 1.4$ we increased the grid to 50 by 50. The result is displayed at the left hand side in fig. 2.

In the left hand side of fig. 2 clearly 3 different ω-limits can be distinguished. The grey zone corresponds to the period 2 PSNB, the black zone to the period 1 PSNB and the white zone to the period 3 attractor. It also suggests a fractal boundary between the black and the white zone. This might explain why the period 3 attractor has escaped attention until now. To investigate the nature of this boundary more closely we enlarged the grid to 100 by 100 for a blow up of the region u = [-19.5, -14.25], v = [-2.8, 0], see the right hand side of fig. 2.

To conclude, a new series of attractors coexisting with known multiple attractors was found, revealing a complicated dynamical structure that greatly influences the basins of attraction. The boundary between two of the three basins seems to be fractral. Bifurcation-diagrams, basins of attraction and trajectories were computed.