Thermodynamic Theory of Hydrodynamic Fluctuations

東工大・理　　北原　和夫

When we extend the thermodynamics to a non-equilibrium system, we introduce the momentum of a convective system as an independent variable in the expression of entropy. We also introduce the total energy which is the sum of the kinetic energy and the internal energy. Thus the entropy is a function of conserved variables.

We denote the velocity, the mass density of the species α, the internal energy density and the total energy density by $\mathbf{v}$, ρ_α, e and ε, respectively. Then, we have

$$ds = \frac{1}{T}d\varepsilon - (\frac{\mathbf{v}}{T}) \cdot d(\rho\mathbf{v}) - \sum_{\alpha=1}^{c} \frac{1}{T}(\mu_\alpha - \frac{v^2}{2})d\rho_\alpha$$

Since the differential of the entropy is written in terms of differentials of density variables of conserved quantities x_i, namely, $ds = \sum_i F_i dx_i$,we may regard the coefficients F_i as thermodynamic forces. Therefore, the thermodynamic forces for the transport of conserved quantities are the gradients of these forces.

We will write $\alpha_i = (\rho, \mathbf{j}, \varepsilon)$, where, ρ is the mass density, $\mathbf{j} = \rho\mathbf{v}$ is the momentum density. The probability functional of the realization of the spatial profiles of these conserved quantities at time t is denoted by$P(\{\alpha\}, t)$. In the equilibrium state, we should have

$$P_{eq}(\{\alpha\}) \propto \exp(\frac{S}{k_B})$$

Here S is the entropy of the whole system. The probability functional for a non-equilibrium state obeys the Fokker-Planck equation,

$$\frac{\partial}{\partial t}P(\{\alpha\}, t) = \int d^3\mathbf{r} \sum_i \frac{\delta}{\delta\alpha_i(\mathbf{r})}[F_i(\mathbf{r})P(\{\alpha\}, t)]$$

$$+ \int d^3\mathbf{r} \sum_{ij} \frac{\delta}{\delta\alpha_i(\mathbf{r})} \int d^3\mathbf{r}' D_{ij}(\mathbf{r}, \mathbf{r}')[-\frac{\delta S}{\delta\alpha_j(\mathbf{r}')} + \frac{\delta}{\delta\alpha_j(\mathbf{r}')}]P(\{\alpha\}, t)$$

We can construct such a Fokker-Planck equation in such a way that the probability functional tends to the equilibrium probability functional in the long time limit if there is no external constraint. The conditions for that are

$$\int d^3\mathbf{r} \sum_j [\frac{\delta F_j(\mathbf{r})}{\delta\alpha_j(\mathbf{r})} + F_j(\mathbf{r})\frac{\delta S}{\delta\alpha_j(\mathbf{r})}] = 0$$

and the positive definiteness of $D_{ij}(\mathbf{r},\mathbf{r}')$. Here $F_j(\mathbf{r})$ are the reversible parts of phenomenological equations,

$$F_j(\mathbf{r}) = \begin{bmatrix} -\nabla\cdot\mathbf{j} \\ -\nabla(\rho\mathbf{v}\mathbf{v}) - \nabla P \\ -\nabla\{\mathbf{v}(\varepsilon + P)\} \end{bmatrix}$$

When the fluctuations are small, the equations for the first moments become

$$\frac{\partial}{\partial t}\alpha_i(\mathbf{r}) = F_i(\mathbf{r}) - \sum_j \int d^3\mathbf{r}' D_{ij}(\mathbf{r},\mathbf{r}') \frac{\delta S}{\delta\alpha_j(\mathbf{r}')}$$

We can choose $D_{ij}(\mathbf{r},\mathbf{r}')$ in such a way that the resulting equations for the first moments have correct dissipative terms.

$D_{ij}(\mathbf{r},\mathbf{r}')$ are related to random forces in the fluctuating hydrodynamic equations. From these arguments, we can show that Landau-Lifshitz formulation is still valid in a non-linear regime and appropriate for the case of finite shear flow.