BRAID GROUP AND ANYONS ON AN ANNULUS AND A TORUS

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1. Introduction

In quantum mechanics, phase factors of wave functions present a delicate problem. Typical examples are the Aharonov-Bohm effect and Berry phase. In the path integral formulation, the statistics phase factors characteristic of a many-particle system originate from the non-trivial topology of the many-particle configuration space [1]. They form a representation of the fundamental group of the latter. In three or higher dimensions, the allowed statistics is either Fermi or Bose statistics. In two dimensions, however, fractional statistics is allowed to exist . The realization of fractional statistics is first recognized in the FQH effect [2]. The discovery of the high temperature superconductivity brings a renewed interest in strongly correlated electron systems. Recently, the possibility of realizing anyons in the high-temperature superconductivity is argued by Laughlin [3]. Although it is not clear yet whether anyons are relevant to real high-T_C materials, the behavior of an anyon system is an interesting problem to investigate (at least theoretically).

2. Braid group and anyons on the annulus (cylinder)

The representation of the fundamental group of the N-body configuration space $\pi_1(C_N)$ determines the allowed statistics. Thus let us discuss $\pi_1(C_N)$ on an annulus first. For a two-dimensional system, $\pi_1(C_N)$ is the braid group B_N. If the system is a plane, it is generated by the local exchange operators σ_i, $(i=1,\cdots,N-1)$ where σ_i represents an interchange of i-th and i+1-th particle counter-clockwise without other particles enclosed in the exchange loop. For an annulus, there is another type of generators ρ_j, which represents moving the j-th particle around the hole once in the positive direction. Further the following important relation holds

$$\rho_{j+1} = \sigma_j \rho_j \sigma_j.$$

When we consider one-dimensional representation which corresponds to the usual one-component wavefunction, we get the representation $\sigma_i = e^{i\theta}$ for σ_i. and we can interpret θ as the anyon statistics. Thus we get $\rho_j = \rho_1 \exp[\, i\ 2\theta\ (j-1)\,]$ for each ρ_j. To fix the braid group representation, we need to fix the phase of ρ_1 [4]. Physically the effects of a central flux Φ threading through the hole is included by ρ_1 so we get

$$\rho_j(\Phi) = \exp[i\ \{2\theta\ (j\text{-}1)+2\pi\Phi\}].$$

We have to pay a special attention to it when we construct an anyon on the annulus which is absent in a two-dimensional plane geometries.

3. Symmetry and restriction for a cylinder system

Unlike the annular case, the two edges of a cylinder are symmetric; one should be able to identify the phase for the generator at the two edges. Using the representation for ρ_j, this condition is expressed as

$$\rho_1(\Phi) = \rho_N^{-1}(-\Phi)\ .$$

Then one obtains the constraint for the cylindrical system as

$$\exp[i\ 2\theta(N-1)] = 1$$

where N is the number of anyons. With N given, the allowed values of θ are restricted to be

$$\theta = \frac{m\pi}{N-1}\ ,\ (m = 0,\cdots, 2N-3)\ .$$

If θ/π is fixed to be fractional, then the total number N of the anyons can not be arbitrary.

This restriction also appears in the sphere case [5]. Further we get some relations for the ground state energy $E(\theta,\Phi)$ by a similar discussion. There is a symmetry as

$$E(\theta,\Phi) = E(\theta,-\Phi)$$

for a cylinder with general Φ. However, $E(\theta,\Phi) \neq E(\theta,-\Phi)$ for an annulus.

4. Numerical calculation and mean-field calculation

There are several studies of anyon on the lattice [4, 6]. Here the braid group representation is realized by string rules. We perform numerical calculations on a finite lattice[6]. In Fig.1(a), we show ground state energies from an exact calculation and a MF calculation from boson. We also perform a MF calculation from fermion in Fig.1(b). They confirm that the MF results is good unless level crossing occurs. After the level crossing, however, the MF approximation is not so good. In the MF calculation from boson, the region where the MF is good is wider than that of the MF calculation from fermion, because there are fewer level crossings near the hard-core boson point.

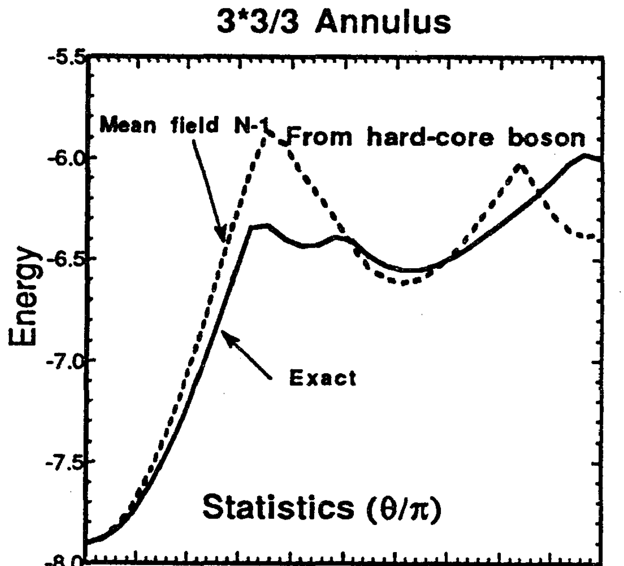

Fig.1(a)

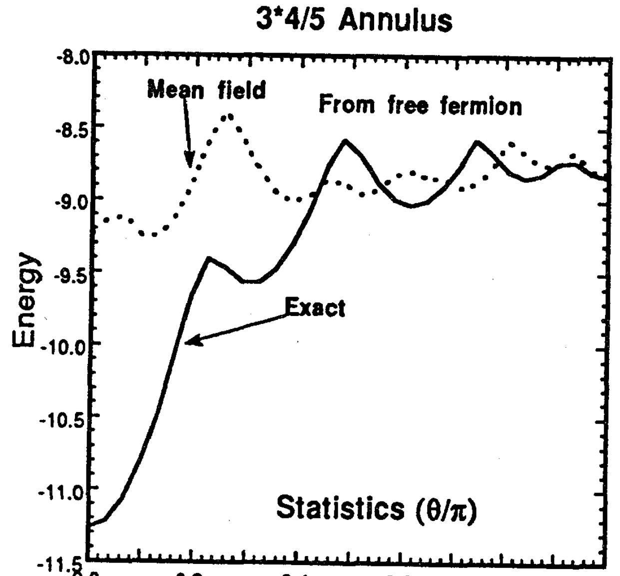

Fig.1(b)

5. Anyon on a torus

Wen, Dagotto and Fradkin [7] proposed how to put anyons on a torus. Einarrson[8] gives a mathematical structure of the braid group on the torus. We explicitly construct anyons on a toroidal lattice by starting from the braid group. The resultant rules to define anyons are basically the same (with some corrections) as those of Wen et al. We have investigate a symmetry and the Aharonov-Bohm effect of the system. The existence of non-contractible loops on the torus leads to the appearance of additional generators ρ_j and τ_j in the braid group corresponding to moving an anyon along the non-contractible loops. This in turn requires more parameters to fix the braid group representation, which can be identified as the central fluxes Φ_x and Φ_y threading through the two holes of the torus. We get the representations for ρ_j and τ_j including the effect of Φ_x and Φ_y which are given by

$$\tau_j = \exp[-i\{2\theta(j-1)+2\pi\Phi_x\}]\begin{bmatrix} 0 & 1 & & O \\ \vdots & 0 & \ddots & \\ 0 & & \ddots & 1 \\ 1 & 0 & \cdots & 0 \end{bmatrix},\ \rho_j = \exp[i\{2\theta(j-1)+2\pi\Phi_y\}]\begin{bmatrix} c^{1+\eta} & 0 & & \\ 0 & c^{2+\eta} & & O \\ & & \ddots & 0 \\ & O & 0 & c^{q+\eta} \end{bmatrix}$$

where $\theta = \pi$ (p/q), $c = \exp(i\,2\theta)$ and η is an arbitrary integer. Further the number of the particles N must satisfy $\exp[i\,2\theta N] = 1$. The representation has inevitably higher dimensions than one. Thus the wavefunction of the anyons has multi-components. To change η by 1 is a gauge transformation and it is equivalent to increase Φ_y by θ/π. This means that in addition to the usual unit period for Φ_x and Φ_y, there is a smaller period given by θ/π. If $\theta = \pi$ (p/q) with p and q mutually prime, then the smallest period is actually 1/q rather than 1. So we are led to expect that the energy spectrum of anyons on the torus exhibits a period of 1/q in Φ_x and Φ_y. For example, an eigen energy E_n must satisfy $E_n(\Phi_x,\Phi_y) = E_n(\Phi_x + 1/q,\Phi_y) = E_n(\Phi_x,\Phi_y + 1/q)$.

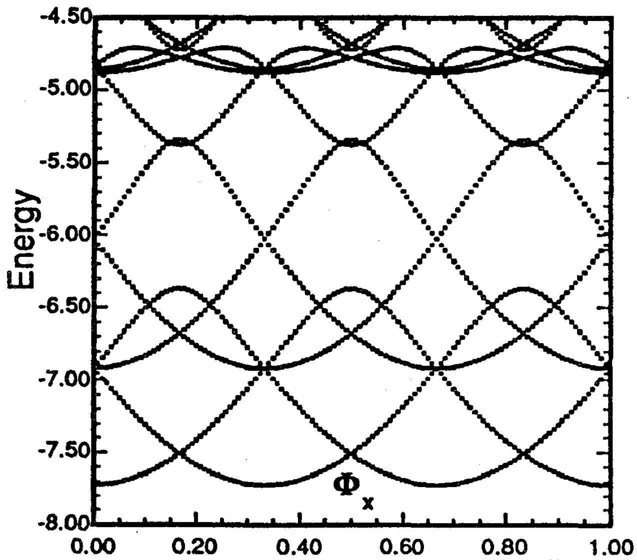

Fig.2

Ground state energy

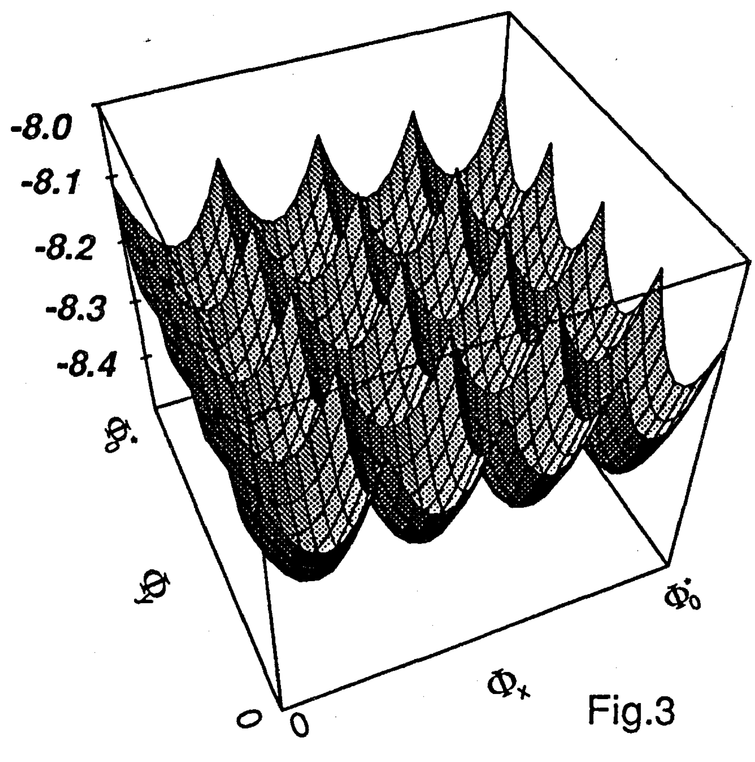

Fig.3

Since in general a given energy level does *not* necessarily have this period (it may have a larger period), we naturally expect the emergence of spectral flow and level crossing. In particular, energy levels may be rearranged as Φ_x or Φ_y changes by 1/q. In Fig.2, we show the spectrum of 3×3 with 3 anyons ($\theta = \pi/3$) as a function of Φ_x . In Fig. 3, we show the ground state energy as a function of Φ_x and Φ_y for 3×3 with 4 anyons ($\theta = \pi/4$) system.

6. Fractional Quantum Hall effect

Finally let us consider the relation of our results for the torus and the FQH effect. When the Landau level filling is $\nu = 1/q$, excitations of the system are quasiparticles with charge $e^* = e/q$ and statistics $\theta = \pi/q$. There are two types of gauge invariance in this system. The period $\Phi_0{}^* = hc/e^*$ implies the gauge invariance of the fractionally charged quasiparticle. We note that in this case the period $\Phi_0{}^*/q = \Phi_0$ coincides with that for gauge invariance of the constituent particles (electrons). Using the argument that there are two kinds of gauge invariances, we can discuss the FQH effect in the way of Laughlin [9] and Halperin [10] for the integral case. An argument for the FQH effect by the gauge invariance for electrons was discussed by Tao and Wu [11] and by Thouless [12]. We treat it in the anyon's point of view. Let us assume that the fermi energy lies in a gap. The FQH system has the property that when we increase the central flux by $\Delta\Phi = \Phi^*$, the system returns to its original state. In the process, exactly n *electrons* are transported across the potential difference V_x. (Notice that the transported particles are not *anyons* but *electrons* because the anyons are only quasiparticles in the FQH effect.) Thus the change in energy is given by $\Delta E = neV_x$. We use a formula for the current by Byers and Yang [13] $I_y = c\,(\Delta E/\Delta\Phi)$. This implies that Hall conductance is fractional as $\sigma_{xy} = (n/q)\,(e^2/h)$.

7. References

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