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一般化されたカクタス樹における合流型転送行列とコヒーレント異常法

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1　序文

臨界現象を研究する時、問題を厳密に解くことができれば、それが最も信頼できる方法である。しかし、実際には厳密に解ける場合は少なく、何らかの近似を行って解いている。特に、Weiss[2]が平均場を導入して以来、平均場近似が盛んに用いられてきている。

最近、臨界現象への新しいアプローチとしてコヒーレント異常法 [3] が鈴木によって提唱された。これは、平均場的な近似の系統的な列から真の系の臨界点での振る舞いを調べる方法である。

ここでは、厳密に解けるモデルからなる新しい近似列を考える。この近似列のそれぞれのモデルは臨界点で平均場的に振る舞うが、近似の度合いが上がるにつれて系統的に真の系へ近づいていくようなモデルである。

2　コヒーレント異常法（CAM）

臨界現象を考えるとき、物理量 $f(x)$ は臨界点 x^* 付近で臨界指数 φ^* をもって発散することが期待される。一方、平均場近似からは指数 φ が得られるが、φ と φ^* は必ずしも等しくはない。しかし、真の系に収束するような近似列（『カノニカル近似列』）を考えると、指数 φ と φ^* の違いのため、発散の係数が指数 ψ をもつ『コヒーレント異常』と呼ばれる異常性を示すので、『コヒーレント異常関係式』$\varphi^* = \varphi + \psi$ から真の臨界指数が得られる [3]。

3　一般化されたカクタス樹

厳密に解けるモデルで系統的な近似列を構成することを考える。図1に示すケイリー樹は厳密に解ける最も簡単な例であるが、Bethe 近似 [6] と同じ結果を与える。図2の正方カクタス樹も容易に解けるが、接続度が小さいため対応する真の系より低い臨界点を得る。図3のようなより複雑なカクタス樹も考えられるが、このモデルの臨界点は図4に示すように真の値に近づかない [3] ので、カノニカル近似列を作ることはできない。

ここで考える一般化されたカクタス樹 [5] は、図5のような系である。この系は、ユニットセルが辺（3次元では面）で接していること、となりのセルと接していない部分があることの2つの特徴を持っている。

4　合流型転送行列（ＣＴＭ）法

通常の転送行列法では線型方程式の固有値から物理量を決めているのに対し、合流型転送行列法では、次のような非線型方程式を用いる [5]。

$$f(\boldsymbol{x}) = \mathrm{Tr}_{\{\boldsymbol{y}_j\}} S(\boldsymbol{x}; \boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_{z-1}) f(\boldsymbol{y}_1) f(\boldsymbol{y}_2) \cdots f(\boldsymbol{y}_{z-1}) \tag{4.2}$$

ここで、Sは図7に示すような１つのセルに対応する Boltzmann 因子である。式 (4.2) の固有関数 $f(\boldsymbol{x})$ はセルに接した部分の片側からの有効 Hamiltonian を与える。

外場がないとき、元の Hamiltonian はスピン反転に関し対称であり、Sも同様である。しかし、式 (4.2) には対称な解 $f^{(\mathrm{s})}(\boldsymbol{x})$ の他に、対称でない解 $f^{(\mathrm{b.s.})}(\boldsymbol{x})$ がある温度領域で存在するので、対称性の破れた解が現れる点で臨界点を決めることができる。また、臨界点以下では $f^{(\mathrm{b.s.})}(\boldsymbol{x})$ を用いて、有効 Hamiltonian$\mathcal{H}^{(\mathrm{b.s.})}(\boldsymbol{x})$ を

$$\{f^{(\mathrm{b.s.})}(\boldsymbol{x})\}^2 D(\boldsymbol{x}) = \exp\{(-\beta)\mathcal{H}^{(\mathrm{b.s.})}(\boldsymbol{x})\} \tag{4.6}$$

と表すことができるが、これは有効 Hamiltonian の自発的対称性の破れを定式化したことになっている [5]。

5　具体例と数値結果のＣＡＭ解析

ユニットセルが図１０で表される一般化された蜂の巣カクタス樹、及び、図１２で表される正方カクタス樹上の強磁性イジングモデルについて、臨界点と帯磁率の臨界係数を計算し、表１、２のような結果を得た。これをＣＡＭによって解析すると、臨界指数$\gamma = 1.705(17)$、$\gamma = 1.721(22)$と評価される。

6　ＣＴＭ法による相関距離の評価

一般化されたカクタス樹の相関関数 $C(n)$ を 0 世代目のスピンと n 世代目のスピンとの相関で定義する。ここで、n 世代目のスピンは図１５のようにして定める。このように定義した相関関数は、ＣＴＭ法を用いて具体的に計算すると、ある相関距離ξをもって指数関数的に減少する（図１６）ことがわかる。また、ξは臨界点で発散し、$\nu = 1$ となっている。

7　結論とまとめ

合流型転送行列法によって一般化されたカクタス樹の臨界現象を調べ、ＣＡＭを用いてγを評価した。得られた結果は厳密解と矛盾しないので、ＣＴＭ法が真の系の臨界現象を調べるために有効であることが確かめられた。また、このモデルではスケーリング関係式$\gamma = (2-\eta)\nu$は、$\gamma = 1$、$\eta = 1$、$\nu = 1$ で成り立っている。

Master Thesis: Confluent Transfer-Matrix and Coherent-Anomaly Method in the Generalized Cactus Trees

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1 Introduction

To study critical phenomena, it is the most reliable way to solve the problem exactly. Actually, Onsager's exact solution [1] of the two-dimensional Ising model is quite valuable. It is, however, difficult to solve realistic models. For example, even the two-dimensional Ising model is not yet exactly solved in a magnetic field. For this reason, various kinds of approximations have been made. Especially, since Weiss [2] introduced the mean-field approximation into the study of critical phenomena, mean-field type approximations have been performed actively.

Recently, a new approach to critical phenomena, the coherent-anomaly method (CAM) was proposed by Suzuki [3]. According to it, a systematic series of generalized mean-field approximations is extremely useful to study the true critical behavior of the corresponding realistic model. Therefore, various types of approximations have been used as systematic series for the CAM theory [4].

Now we introduce a new systematic series which is constructed by exactly solvable models, the generalized cactus trees [5]. Although each model shows a mean-field critical behavior, these models approach systematically the corresponding regular system.

In the present thesis, the CAM theory is reviewed in Section 2, and a systematic series is constructed by the generalized cactus trees in Section 3. In Section 4, we review the confluent transfer-matrix (CTM) method introduced by Suzuki [5] to solve these new models exactly. In Section 5, some examples of calculations are given, and critical exponents are estimated using the CAM theory. In Section 6, we try to evaluate the correlation length by use of the CTM method. These results are discussed in Section 7.

2 Coherent-Anomaly Method (CAM)

In considering a critical phenomenon, the relevant physical quantity $f(x)$ is expected to show such behavior as

$$f(x) \simeq \frac{C}{(x-x^*)^{\varphi^*}}, \tag{2.1}$$

near the critical point x^*, with the critical exponent φ^*. On the other hand, ordinary mean-field approximations yield such behavior as

$$f_n(x) \simeq \bar{f}_n \left(\frac{x_n}{x-x_n}\right)^{\varphi}, \tag{2.2}$$

where the exponent φ is not necessarily the same as φ^*.

Consider a series of approximations; $n = 1, 2, 3, \ldots$. If the series converges to the realistic system, or

$$\lim_{n\to\infty} x_n = x^*, \tag{2.3}$$
$$\lim_{n\to\infty} f_n(x) = f(x), \tag{2.4}$$

from the discrepancy between the exponents φ^* and φ, $\bar{f}_n$ shows anomaly [3]:

$$\bar{f}_n \to \infty \text{ as } n \to \infty, \text{ or } x_n \to x^*. \tag{2.5}$$

Suzuki [3] proposed that in general there exist some systematic series of approximations for which $\bar{f}_n$ can be written in the form

$$\bar{f}_n \simeq \frac{C'}{(x_n - x^*)^{\psi}} \text{ as } n \to \infty, \text{ or } x_n \to x^*, \tag{2.6}$$

and correspondingly, the relevant physical quantity takes the form

$$f_n(x) \simeq \frac{C'}{(x_n-x^*)^{\psi}} \left(\frac{x_n}{x-x_n}\right)^{\varphi} \text{ as } x_n \to x^*,\ x \to x^*. \tag{2.7}$$

This behavior is called the "coherent-anomaly", and the series which shows this anomaly is called a "canonical series". Therefore, the asymptotic behavior of the relevant physical quantity is given [3] by

$$f(x) \sim \frac{1}{(x-x^*)^{\varphi+\psi}}, \tag{2.8}$$

and the "coherent-anomaly relation" can be derived as follows:

$$\varphi^* = \varphi + \psi. \tag{2.9}$$

In the CAM theory, it is quite important how to construct the canonical series in order to study the true critical behavior of the corresponding regular system. Hitherto, various kinds of the systematic series of approximations have been made and yielded fruitful results [4]. In the next section, we introduce some systematic series of exactly solvable models [5], which play a role of systematic mean-field approximations in the CAM theory.

3 Generalized Cactus Trees

We now discuss how to construct systematic series of exactly solvable models. The simplest model of phase transition which can be solved exactly is the Cayley tree shown in Fig. 1. This model yields the same result as the Bethe approximation

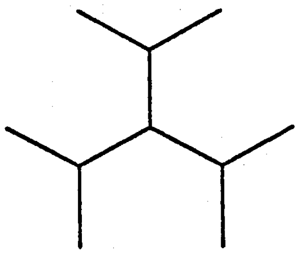

Figure 1: A Cayley tree ($z = 3$).

does [6]. The square cactus tree shown in Fig. 2 is also easily solved. The

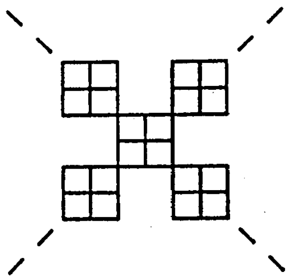

Figure 2: A square cactus tree [5].

critical point of this model is, however, lower than that of the corresponding regular system (i.e. the two-dimensional Ising model), for the connectivity of the model is very small. Then it is quite natural to extend to more complicated cactus trees as shown in Fig. 3, in which each cell connects at each boundary site. The critical points of these series do not approach the correct one, as shown [3] in Fig. 4. This is because the connectivity of such systems corresponds to the infinite dimensionality even in the limit where the cell size becomes infinite. Consequently the limiting critical point $T_c^{(\infty)}$ is different from the true one T_c^*, or such a series is not canonical for the CAM.

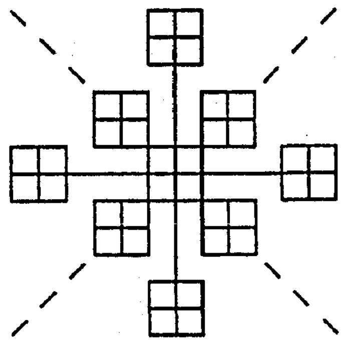

Figure 3: A cactus tree connected at each boundary site [5].

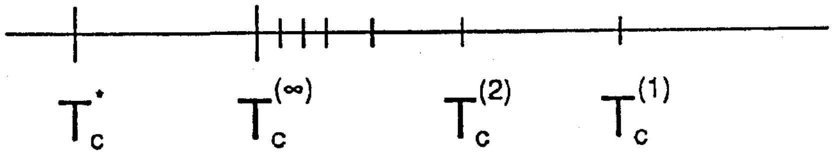

Figure 4: The series [5] of the critical points $\{T_c^{(n)}\}$ of the cactus trees connected at each boundary site as shown in Fig. 3.

Now we explain the generalized cactus trees [5] as shown in Fig. 5. The generalized cactus trees have a novel feature that they are connected not with

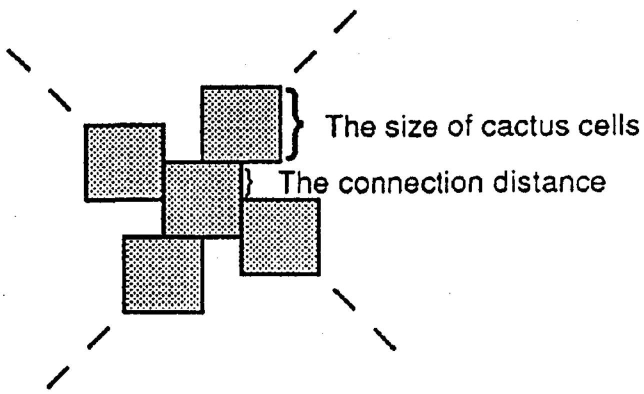

Figure 5: A generalized square cactus tree [5].

points but with lines (or surfaces). These fractal systems have the connection distance, that is, the distance between the connecting sites. Our criterion on the canonical series for the CAM is that the connection distance should be of the same order as the size of cactus cells [5].

4 Confluent Transfer-Matrix (CTM) Method

In the present section, we explain the confluent transfer-matrix (CTM) method [5] to solve such fractal systems exactly as the generalized cactus trees. The ordinary transfer-matrix method [7] has been successfully used in studying regular systems as shown in Fig. 6. In this method, various physical quantities are derived from

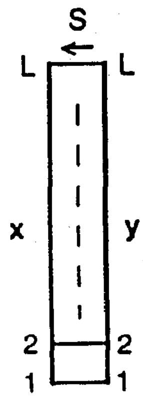

Figure 6: Transfer-matrix in the two-dimensional square lattice.

the following linear equation:

$$\lambda f(x) = \mathrm{Tr}_{y} S(x; y) f(y), \tag{4.1}$$

where x and y are defined by the set of variables $\{\sigma_1, \sigma_2, \cdots, \sigma_L\}$ on the line. $f(x)$ is the canonical probability functions on some contact region whose state specified by x. On the other hand, in the new confluent transfer-matrix method, the canonical probability function $f(x)$ is given [5] by the following nonlinear equation:

$$f(x) = \mathrm{Tr}_{\{y_j\}} S(x; y_1, y_2, \cdots, y_{z-1}) f(y_1) f(y_2) \cdots f(y_{z-1}), \tag{4.2}$$

where $S(x; y_1, y_2, \cdots, y_{z-1})$ denotes the partial Boltzmann factor due to the interaction on the relevant cell shown in Fig. 7. The kernel S in eq. (4.2) transfers

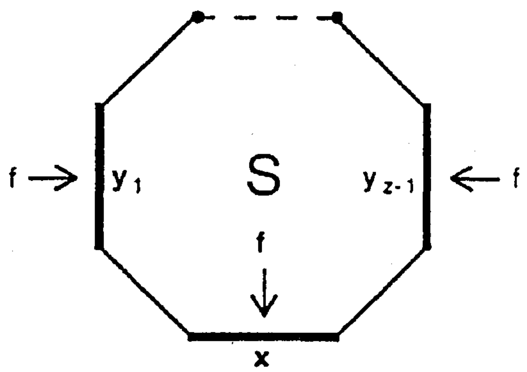

Figure 7: A general cactus cell with z connection-lines and the corresponding confluent transfer-matrix S [5].

fluctuations at the states $y_1, y_2, \cdots$ and y_{z-1} to those at x. Therefore, S is called the confluent transfer-matrix. The eigenfunction $f(\boldsymbol{x})$ gives an effective Hamiltonian induced through the half region of the system connected by $y_1, y_2, \cdots$ and y_{z-1}.

The canonical average of any physical quantity $Q(\boldsymbol{x})$ specified by the state $\boldsymbol{x}$ is given [5] by

$$\langle Q(\boldsymbol{x})\rangle = \frac{\mathrm{Tr}_{\boldsymbol{x}} Q(\boldsymbol{x}) f^2(\boldsymbol{x}) D(\boldsymbol{x})}{\mathrm{Tr}_{\boldsymbol{x}} f^2(\boldsymbol{x}) D(\boldsymbol{x})}, \tag{4.3}$$

where the square $f^2(\boldsymbol{x})$ comes from the sum of the effective fields due to the both sides at the region $\{\boldsymbol{x}\}$ and the factor $D(\boldsymbol{x})$ denotes the partial Boltzmann factor at the region $\{\boldsymbol{x}\}$. Therefore, the effective Hamiltonian $\mathcal{H}_{\mathrm{eff}}(\boldsymbol{x})$ at the region $\{\boldsymbol{x}\}$ for the total infinite system is defined [5] in

$$f^2(\boldsymbol{x}) D(\boldsymbol{x}) = \exp\{(-\beta)\mathcal{H}_{\mathrm{eff}}(\boldsymbol{x})\};\ \beta = \frac{1}{k_{\mathrm{B}}T}. \tag{4.4}$$

Provided an external field is absent, the original Hamiltonian $\mathcal{H}$ is symmetric and consequently the confluent transfer-matrix S is also symmetric with respect to the inversion transformations

$$\boldsymbol{x} \rightarrow -\boldsymbol{x},\ y_j \rightarrow -y_j. \tag{4.5}$$

Therefore, there exists always a symmetric solution $f^{(\mathrm{s})}(\boldsymbol{x})$ in eq. (4.2). The nonlinear eq. (4.2) has, however, another symmetry-breaking solution $f^{(\mathrm{b.s.})}(\boldsymbol{x})$ in some temperature region i.e. below the critical point. In the CTM method, we determine the critical point by the point where the eigenfunction $f(\boldsymbol{x})$ of eq. (4.2) bifurcates to the symmetric and symmetry-breaking solutions. Since the effective Hamiltonian is given by

$$\{f^{(\mathrm{b.s.})}(\boldsymbol{x})\}^2 D(\boldsymbol{x}) = \exp\{(-\beta)\mathcal{H}^{(\mathrm{b.s.})}(\boldsymbol{x})\} \tag{4.6}$$

below the critical point, we can formulate the spontaneous broken symmetry of an effective Hamiltonian.

5 Examples and CAM Analyses of Numerical Results

In the present section, some examples of calculations are given. We treat here the generalized honeycomb cactus trees and the generalized square cactus trees, which correspond to the two-dimensional Ising model in the limit where the cell size becomes infinite. The Hamiltonian of the Ising model is given by

$$\mathcal{H} = -J\sum_{<ij>} \sigma_i\sigma_j - \mu_{\mathrm{B}} H \sum_j \sigma_j;\ \sigma_j = \pm 1. \tag{5.1}$$

The two-dimensional Ising model is already solved exactly [1] and the critical exponents are known. Our aim is, however, to confirm the effectiveness of the CTM method by use of the well-known results.

5.1 Generalized Honeycomb Cactus Trees

The simplest model in the generalized honeycomb cactus trees is shown in Fig. 8. The unit cell of this model is a honeycomb cell shown in Fig. 9. Our general

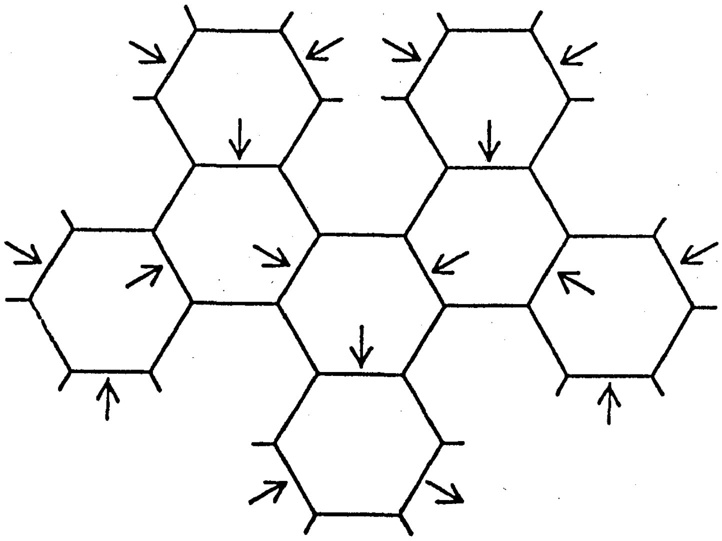

Figure 8: A generalized honeycomb cactus tree [5].

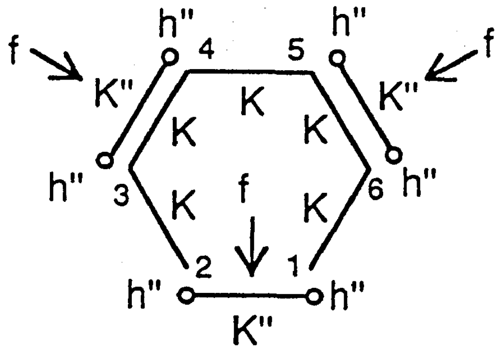

Figure 9: A honeycomb cactus cell.

eq. (4.2) is expressed [5] as

$$f(\sigma_1, \sigma_2) = \mathrm{Tr}_{3\sim 6} S(\sigma_1, \sigma_2, \cdots, \sigma_6) f(\sigma_3, \sigma_4) f(\sigma_5, \sigma_6), \tag{5.2}$$

where

$$S(\sigma_1, \sigma_2, \cdots, \sigma_6) = e^{K(\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_5+\sigma_5\sigma_6+\sigma_6\sigma_1)}; \ K = \beta J, \tag{5.3}$$

and $\mathrm{Tr}_{3\sim 6}$ denotes $\sum_{\sigma_3=\pm 1} \sum_{\sigma_4=\pm 1} \sum_{\sigma_5=\pm 1} \sum_{\sigma_6=\pm 1}$, in this model without an external field.

We assume the following form for $f(\sigma_1, \sigma_2)$

$$f(\sigma_1, \sigma_2) = a(K) e^{K''\sigma_1\sigma_2 + h''(\sigma_1+\sigma_2)}, \tag{5.4}$$

where K'' and h'' denote even- and odd-effective fields, respectively. Above the critical point, we can set $h'' = 0$. Then we obtain [5]

$$a(K) = (16)^{-1}(\cosh K)^{-3}(\cosh K')^{-2}(\cosh K''), \tag{5.5}$$

$$t^3 t'^2 = t'', \tag{5.6}$$

where

$$K' = K + K'', \tag{5.7}$$

$$t = \tanh K,\ t' = \tanh K',\ t'' = \tanh K''. \tag{5.8}$$

Below the critical point, as far as the term linear in h'' is concerned, we obtain from eq. (5.2),

$$(1 + t'')h''(\sigma_1 + \sigma_2) = t(1 + t')(1 + tt')h''(\sigma_1 + \sigma_2). \tag{5.9}$$

The critical point T_c is determined at the temperature where eq. (5.9) has a non-vanishing solution h''. Then we obtain

$$1 + t_c'' = t_c(1 + t_c')(1 + t_c t_c'), \tag{5.10}$$

where

$$t_c = \tanh K_c,\ t_c' = \tanh K_c',\ t_c'' = \tanh K_c''. \tag{5.11}$$

Consequently we obtain the critical point [5]

$$t_c = 0.52167 \tag{5.12}$$

from eqs. (5.6), (5.7) and (5.10).

The susceptibility $\hat{\chi}(T)$ is calculated as follows [5]. If we apply an external field H, then we have

$$(1 + t'')h'' = t(1 + t')(1 + tt')(h'' + h); \tag{5.13}$$

$$h = \frac{\mu_B H}{k_B T}. \tag{5.14}$$

The total susceptibility is given by

$$\begin{aligned} \hat{\chi}(T) &= \frac{J}{\mu_B H}\langle\sigma_1\rangle = \frac{K}{h}\frac{\mathrm{Tr}\sigma_1 f^2 D}{\mathrm{Tr} f^2 D} \\ &= \frac{K}{h}(1 + \tanh(2K'' + K))(2h'' + h). \end{aligned} \tag{5.15}$$

Then we obtain

$$\hat{\chi}(T) \simeq 1.33957\frac{T_c}{T-T_c}, \tag{5.16}$$

near the critical point T_c. We define the critical coefficient $\bar{\chi}(T_c)$ in

$$\hat{\chi}(T) \simeq \bar{\chi}(T_c)\frac{T_c}{T-T_c}, \tag{5.17}$$

then we have $\bar{\chi}(T_c)$ as follows:

$$\bar{\chi}(T_c) = 1.33957. \tag{5.18}$$

We also calculate the critical point T_c and the critical coefficient $\bar{\chi}(T_c)$ of some models whose unit cells are shown in Figs. 10b, 10c and 10d. Since unit

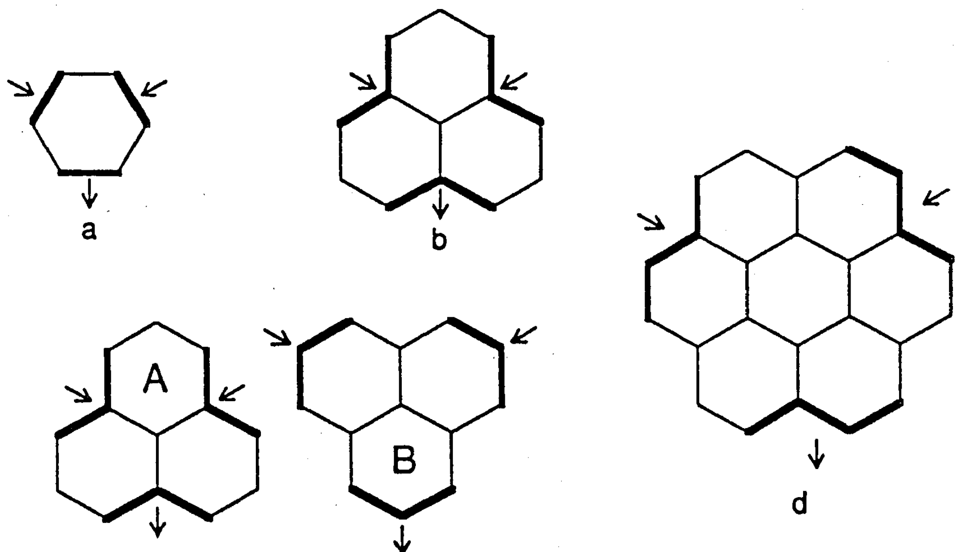

Figure 10: Unit cells of treated models.

cells shown in Fig. 10b do not connect without deformation, the model may not be canonical. Therefore, we consider also a model which has two unit cells A and B as shown in Fig. 10c.

All the results obtained by the CTM method and by the Weiss and Bethe approximations are listed in Table 1. The models whose unit cells are shown in Figs. 10c and 10d yield T_c, where $T_c < T_c^*$. We do not use these data in the CAM analyses.

In the following, we analyze the obtained data by the CAM theory, and estimate the critical exponent γ. According to the CAM theory [3], the critical coefficient $\bar{\chi}_n(T_n)$ may show the anomaly

$$\bar{\chi}_n(T_n) \simeq \frac{C'}{(T_n - T_c^*)^{\psi}}, \tag{5.19}$$

Table 1: Results of the generalized honeycomb cactus trees.

unit cell	$k_B T_c/J$	$\bar{\chi}(T_c)$
Weiss	3.00000	0.33333
Bethe	1.82048	1.00000
a	1.72822	1.33957
b	1.58131	2.33344
c	1.45546	-
d	1.40148	-

where T_n and $\bar{\chi}_n(T_n)$ are the critical point and coefficient of the n-th approximation, respectively, and T_c^* denotes the exact critical point. The total susceptibility $\hat{\chi}(T)$ is expected to show such behavior as

$$\hat{\chi}(T) \simeq \frac{C}{(T - T_c^*)^\gamma}, \tag{5.20}$$

near the critical point. Then the "coherent-anomaly relation" is given by

$$\gamma = 1 + \psi \tag{5.21}$$

from eq. (2.9).

The least-square-fitting of the data to the function (5.19) using the exact value T_c^* are plotted in Fig. 11, and we obtain the critical exponent γ as follows:

$$\gamma = 1.705(17). \tag{5.22}$$

This is consistent with the exact value:

$$\gamma = 1.75. \tag{5.23}$$

Figure 11 suggests that the model whose unit cell is shown in Fig. 10b is not canonical.

5.2 Generalized Square Cactus Trees

We also calculate the critical data in the generalized square cactus trees. The unit cells of treated models are shown in Fig. 12. We calculate the critical point T_c and the critical coefficient $\bar{\chi}(T_c)$ in the same way as shown in Section 5.1. All the results obtained by the CTM method and by the Weiss and Bethe approximations are listed in Table 2. The least-square-fitting of the data to the function (5.19)

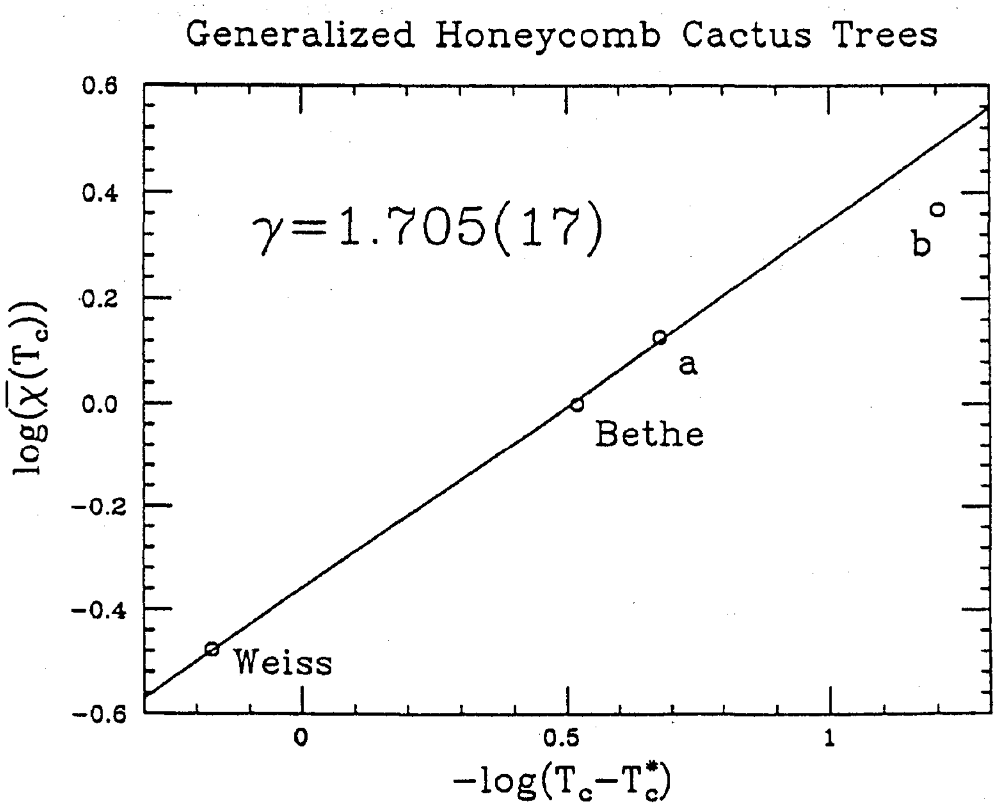

Figure 11: The data points, "Weiss", "Bethe" and "a", are fitting to the function (5.19).

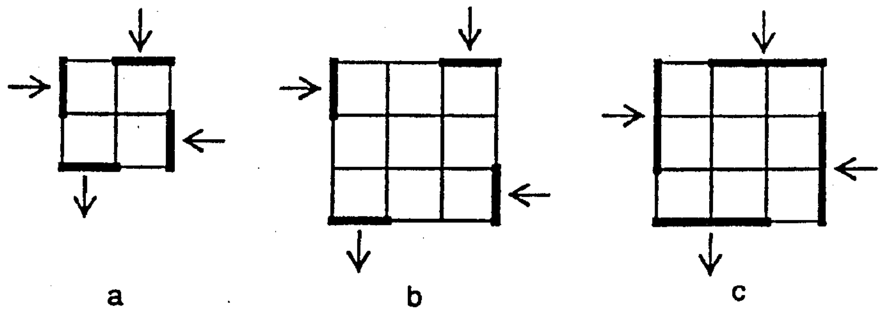

Figure 12: Unit cells of treated models.

Table 2: Results of the generalized square cactus trees.

unit cell	$k_B T_c/J$	$\bar{\chi}(T_c)$
Weiss	4.00000	0.250000
Bethe	2.88539	0.500000
a	2.62717	0.764496
b	2.13645	-
c	2.54112	0.953922

using the exact value T_c^* are plotted in Fig. 13, and we obtain the critical

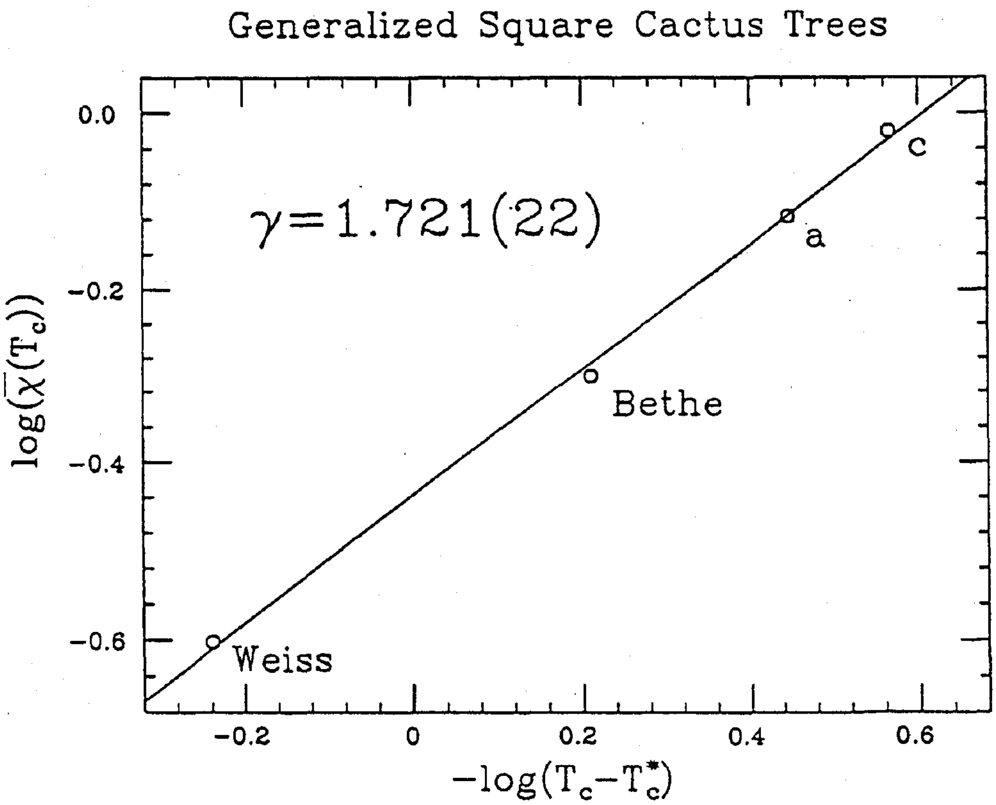

Figure 13: The data points, "Weiss", "Bethe", "a" and "c", are fitting to the function (5.19).

exponent γ as follows:

$$\gamma = 1.721(22). \tag{5.24}$$

This is also consistent with the well-known result.

6 Evaluation of the Correlation Length Based on the CTM Method

In the present section, we try to evaluate the correlation length in the generalized cactus trees by the CTM method.

6.1 Correlation Functions

In order to evaluate the correlation length, we have to define a correlation function.

First, we define two kinds of correlation functions in the Cayley tree. The point correlation function defined by

$$C^{(\mathrm{p})}(R) = \langle \sigma_0 \sigma_R \rangle \tag{6.1}$$

is calculated as

$$C^{(\mathrm{p})}(R) = t^R;\ t = \tanh\beta J, \tag{6.2}$$

where 0 and R denote 0th and R-th generation sites as shown in Fig. 14. In this

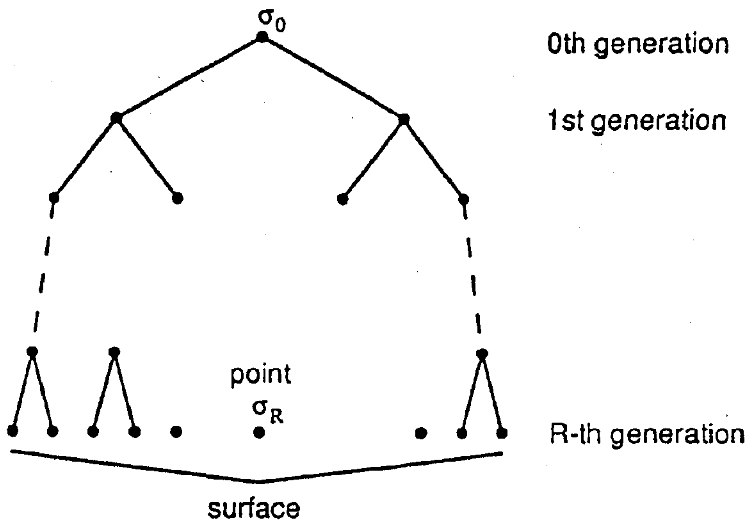

Figure 14: 0 to R-th generation spins.

definition (6.1), the correlation length ξ defined [8] in

$$C(R) \simeq \frac{e^{-\frac{R}{\xi}}}{R^{d-2+\eta}} \tag{6.3}$$

does not diverge at the critical point.

The surface correlation function defined by

$$C^{(s)}(R) = \sum_{i \in R-\mathrm{th}} \langle \sigma_0 \sigma_i \rangle \tag{6.4}$$

is calculated as

$$C^{(s)}(R) = \{(z-1)t\}^R, \tag{6.5}$$

where z denotes the coordination number. In the definition (6.4), the correlation length ξ is given by

$$\xi = \frac{1}{\log \frac{1}{(z-1)t}}, \tag{6.6}$$

and diverges at the critical point

$$t_c = \frac{1}{z-1}. \tag{6.7}$$

Therefore, we had better to define, in general, a correlation function of the generalized cactus trees in the same way as the surface correlation function of the Cayley tree.

6.2 Correlation Functions of the Generalized Cactus Trees

In order to define the surface correlation function of the generalized cactus trees, we first consider the surface of the generalized cactus trees.

We define the surface of the generalized cactus trees as shown in Fig. 15. In

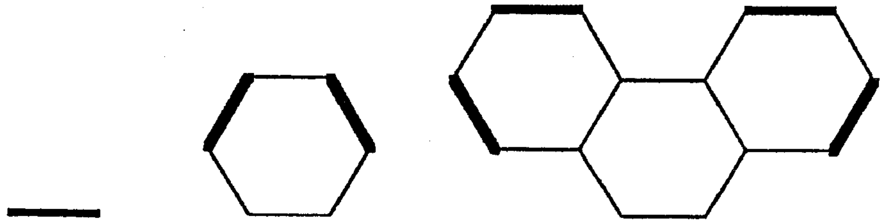

Figure 15: The surface of the n-th generation of a generalized cactus tree.

this definition, the number of sites of the surface of the n-th generation a_n is given by

$$a_n \sim (z-1)^n \tag{6.8}$$

as that of the Cayley tree. Then we define the surface correlation function of the generalized cactus trees by the correlation between the spins of the 0th generation and the n-th generation in the whole infinite system, or in

$$C(n) = \sum_{i \in 0\text{th}} \sum_{j \in n-\text{th}} \langle \sigma_i \sigma_j \rangle. \tag{6.9}$$

By use of the CTM method, the correlation functions (6.9) of the generalized cactus trees are given exactly.

For example, we consider the generalized cactus tree shown in Fig. 8. The correlation function $C(n)$ takes the form

$$C(n) \sim e^{-\frac{n}{\xi}} \tag{6.10}$$

as shown in Fig. 16, and the correlation length ξ is given by

$$\xi = -\frac{1}{\log\left(\dfrac{t(1+t')(1+tt')}{1+t^3t'^2}\right)}, \tag{6.11}$$

where t and t' are defined by eqs. (5.7) and (5.8). The correlation length ξ diverges at the critical point. Then we estimate the critical exponent ν defined [8] in

$$\xi \sim (T-T_c)^{-\nu} \tag{6.12}$$

as

$$\nu = 1. \tag{6.13}$$

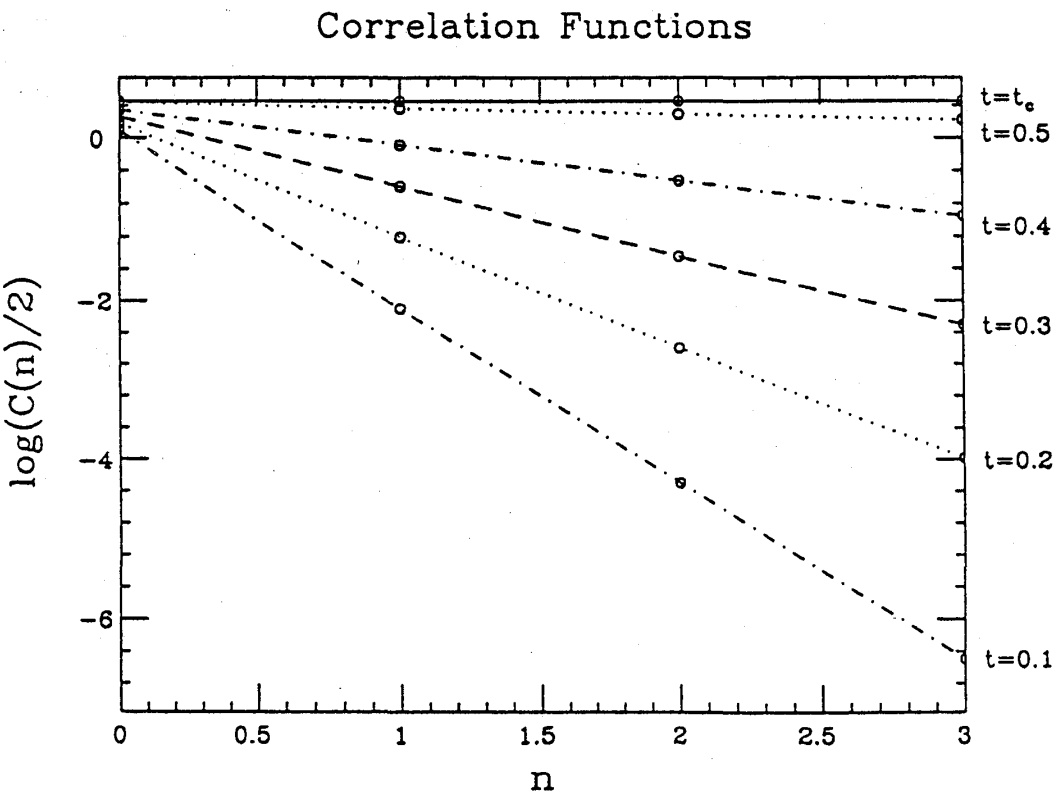

Figure 16: Correlation functions in some temperatures, where $t = \tanh \beta J$.

The surface correlation function is expected [8] to show such behavior as

$$C^{(s)}(R) \sim \frac{e^{-\frac{R}{\xi}}}{R^{d-2+\eta}} R^{d-1} = \frac{e^{-\frac{R}{\xi}}}{R^{\eta-1}}. \tag{6.14}$$

In this model, $C(n)$ is given by eq. (6.10). Then we have

$$\eta = 1. \tag{6.15}$$

We obtain also

$$\gamma = 1 \tag{6.16}$$

from eq. (5.16). Therefore, the scaling relation [8]

$$\gamma = (2 - \eta)\nu \tag{6.17}$$

is satisfied.

7 Discussions and Summary

We have studied some generalized cactus trees as the canonical series for the CAM using Suzuki's confluent transfer-matrix method in order to solve these models exactly. We have obtained $\gamma = 1.705(17)$ in the generalized honeycomb cactus trees and $\gamma = 1.721(22)$ in the generalized square cactus trees using the CAM theory. These results are consistent with the well-known results. Therefore, we have confirmed the effectiveness of the CTM method to study the true critical behavior of the corresponding regular system. We have also calculated the correlation length ξ by use of the CTM method. The scaling relation,

$$\gamma = (2 - \eta)\nu,$$

is satisfied with $\gamma = 1$, $\eta = 1$ and $\nu = 1$. However, we had better to confirm this relation in the limit where γ approaches the exact value, using the CAM analyses of η and ν.

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