A random walk with random obstacles and a tagged particle of an infinite hard core particle system in $\mathbf{R}^d$

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Let $\mathfrak{M}$ be the set of all countable subsets η of $\mathbf{R}^d$ satisfying $N_K(\eta) < \infty$ for any compact subset K , where $N_A(\eta)$ is the number of points of η in $A \subset \mathbf{R}^d$ $(d \geq 2)$. We regard $\eta \in \mathfrak{M}$ as a non-negative integer valued Radon measure on $\mathbf{R}^d$: $\eta(\cdot) = \sum_{x\in\eta} \delta_x(\cdot)$ and accordingly equip $\mathfrak{M}$ with the vague topology, where δ_x denotes the δ-measure at x. We define σ-fields $\mathcal{B}(\mathfrak{M})$ and $\mathcal{B}_K(\mathfrak{M})$ by

$$\mathcal{B}(\mathfrak{M}) = \sigma(N_A; A \in \mathcal{B}(\mathbf{R}^d)),$$

and

$$\mathcal{B}_K(\mathfrak{M}) = \sigma(N_A; A \in \mathcal{B}(\mathbf{R}^d), A \subset K).$$

The σ-field $\mathcal{B}(\mathfrak{M})$ coincides with the topological Borel field of $\mathfrak{M}$.

For any $\eta \in \mathfrak{M}$ we define a measurable kernel $q_\eta(x, dy)$ on $\mathbf{R}^d \times \mathcal{B}(\mathbf{R}^d)$ by

$$q_\eta(x, dy) = p(|x - y|)\chi(x|\eta)\chi(y|\eta)dy,$$

where $p(\cdot)$ is a non-negative function on $[0, \infty)$ satisfying

(p.1) $$\int_{\mathbf{R}^d} dx p(|x|) = 1,$$

(p.2) $$\int_{\mathbf{R}^d} dx |x|^2 p(|x|) < \infty,$$

(p.3) $\{\alpha \in [0, \infty) : p(\alpha) > 0\} = [0, h),$ for some $h \in (0, \infty]$,

(p.4) $\text{ess.inf}\{p(\alpha) : \alpha \in [0, c)\} > 0$ for any $c \in (0, h)$,

and for any $\eta \in \mathfrak{M}$ and $x \in \mathbf{R}^d$

$$\chi(x|\eta) = \exp\{-\sum_{y\in\eta} \Psi(|x - y|)\}.$$

Here Ψ is a given measurable function on $[0, \infty)$ which is bounded from below and satisfies

(Ψ.1) $\Psi(\alpha) = \infty$ if and only if $\alpha \in [0, r)$,

(Ψ.2) $\Psi(\alpha) = 0$ if $\alpha \in [r_0, \infty)$,

for some positive constants r and r_0 with $r \leq r_0$.

Let $\mathbf{C}_\infty(\mathbf{R}^d)$ be the space of continuous functions φ on $\mathbf{R}^d$ such that $\varphi(x) \to 0$ as $|x| \to \infty$. We denote by $(\Omega, \mathcal{F}, P_\eta, x(t))$ the right continuous Markov process starting from 0 with generator

$$L_\eta \varphi(x) = \int_{\mathbf{R}^d} q_\eta(x, dy)\{\varphi(y) - \varphi(x)\}, \quad \varphi \in \mathbf{C}_\infty(\mathbf{R}^d).$$

For any probability measure ν on $\mathfrak{M}$ we write $P_\nu = \int_{\mathfrak{M}} \nu(d\eta) P_\eta$. We call the process $(\Omega, \mathcal{F}, P_\nu, x(t))$ a random walk with random obstacles.

Denote the r-neighborhood of $A \subset \mathbf{R}^d$ by $U_r(A)$ and abbreviate $U_r(\{x\})$ to $U_r(x)$. For $x \in \mathbf{R}^d$ and $\eta \in \mathfrak{M}$ put

$$C(x,\eta) = \begin{cases} A_{x,\eta} \backslash \overline{U_r(\eta)}, & x \notin \overline{U_r(\eta)}, \\ \emptyset, & x \in \overline{U_r(\eta)}, \end{cases}$$

where $A_{x,\eta}$ is the connected component of $U_{\frac{h}{2}}(U_r(\eta)^c)$ containing x. We call the set $C(x, \eta)$ the cluster containing x for η. Define a measurable subset $\mathfrak{M}^*$ of $\mathfrak{M}$ by

$$\mathfrak{M}^* = \{\eta \in \mathfrak{M} : |C(0,\eta)| = \infty\}.$$

For a probability measure on $\mathfrak{M}$ satisfying $\mu(\mathfrak{M}^*) > 0$, we define

$$\mu^*(d\eta) = \frac{\mathbb{I}_{\mathfrak{M}^*}(\eta)}{\mu(\mathfrak{M}^*)} \mu(d\eta),$$

where $\mathbb{I}_A$ stands for the indicator function for a set A.

We study the asymptotic behavior of $(x(t), P_{\mu^*})$ in the case where μ is a Gibbs state. We introduce terminologies for Gibbs states. Let Φ be a real valued measurable function on $[0, \infty)$ which is bounded from below and satisfies the following condition (Φ.1) called *regularity condition*:

$$(\Phi.1) \qquad \int_{\mathbf{R}^d} dx |\exp(-\Phi(|x|)) - 1| < \infty.$$

Next we assume either one of the following conditions (Φ.2) and (Φ.2'):

$$(\Phi.2) \qquad \Phi(\cdot) \geq 0,$$

($\Phi.2'$) (i) There exists a positive number r' such that

$$\Phi(\alpha) = \infty, \quad \text{if and only if } \alpha \in [0, r'),$$

(ii) There exists a non-negative number c_0 such that

$$\sum_{i=1}^{m} \Phi(|x_i|) \geq -c_0$$

$$\text{for all } m \text{ and } x_1, x_2, \cdots, x_m \in \mathbf{R}^d \text{ with } |x_i - x_j| \geq r' \text{ for } i \neq j.$$

Φ is regarded as a pair potential which is rotation invariant and translation invariant. For $x_1, x_2, \cdots, x_n \in \mathbf{R}^d$ and $\eta \in \mathfrak{M}$ we associate a potential energy

$$U(x_1, x_2, \cdots, x_n|\eta) = \sum_{1 \leq i < j \leq n} \Phi(|x_i - x_j|) + \sum_{i=1}^{n} \sum_{y \in \eta} \Phi(|x_i - y|).$$

For any compact subset $K \subset \mathbf{R}^d$, we denote by $\mathfrak{M}(K)$ and $\mathfrak{M}(K, n)$ the set of all finite subsets of K and the set of all subsets of K having n points, respectively. An alternative description of $\mathfrak{M}(K, n)$ is given by

$$\mathfrak{M}(K, n) = \begin{cases} \{\emptyset\}, & \text{if } n = 0, \\ (K^n)'/\mathbf{S}_n, & \text{if } n \geq 1, \end{cases} \tag{1}$$

where $(K^n)' = \{(x_1, x_2, \cdots, x_n) \in K^n : x_i \neq x_j \text{if } i \neq j\}$ and $\mathbf{S}_n$ is the symmetric group of degree n. By means of the factorization (1) we introduce a measure $\lambda_{K,z}$ on $\mathfrak{M}(K) = \bigcup_{n=0}^{\infty} \mathfrak{M}(K, n)$ (direct sum) such that

$$\lambda_{K,z}(\emptyset) = 1,$$

and

$$\lambda_{K,z}(A) = \frac{z^n}{n!} \int_{\tilde{A}} dx_1 dx_2 \cdots dx_n,$$

for a Borel set A in $\mathfrak{M}(K, n), n \geq 1$, where $z \geq 0$ and $\tilde{A}$ is a preimage of A by the factor mapping in the factorization (1.3).

Now, we are going to define a Gibbs state.

Definition 1.1. A probability measure μ on $\mathfrak{M}$ is called a Gibbs state with respect to the activity $z \geq 0$ and the potential Φ, if for any compact subset K of $\mathbf{R}^d$,

$$\mu(\,\cdot\,|\mathcal{B}_{K^c}(\mathfrak{M}))(\eta) = \mu_{K,\eta,z}(\,\cdot\,), \qquad \mu\text{-a.s. } \eta,$$

where $\mu_{K,\eta,z}$ is the probability measure on $\mathfrak{M}(K)$ defined by

$$\mu_{K,\eta,z}(d\mathbf{x}) = \frac{1}{Z_{K,\eta,z}} \exp\{-U(\mathbf{x}|\eta \cap K^c)\} \lambda_{K,z}(d\mathbf{x}),$$

$$Z_{K,\eta,z} = \int_{\mathfrak{M}(K)} \lambda_{K,z}(d\mathbf{x}) \exp\{-U(\mathbf{x}|\eta \cap K^c)\}.$$

Denote by $\mathcal{G}(z, \Phi)$ the set of all Gibbs states with respect to the activity $z \geq 0$ and the potential Φ, and by $\mathcal{G}_{\odot}(z, \Phi)$ the set of all elements of $\mathcal{G}(z, \Phi)$ which are translation invariant.

Remark 1.1. (i) The set $\mathcal{G}_{\odot}(z,\Phi)$ is convex and any element of $\mathcal{G}_{\odot}(z,\Phi)$ is represented by the extremal points of $\mathcal{G}_{\odot}(z,\Phi)$, which are characterized by their ergodicity under translation (see [1]). We denote the set of all extremal points of $\mathcal{G}_{\odot}(z,\Phi)$ by $\mathrm{ex}\mathcal{G}_{\odot}(z,\Phi)$. If $\sharp\mathcal{G}(z,\Phi)=1$ and $\mu\in\mathcal{G}(z,\Phi)$, then μ is rotation invariant, translation invariant and ergodic under translation.

(ii) There exists a positive constant $z_1>0$ such that if $z\in(0,z_1)$ and $\mu\in\mathcal{G}(z,\Phi)$, then $\mu(\mathfrak{M}^*)>0$. In particular, $z_1=\infty$ in case $h=\infty$.

Now, we shall state our first main result.

Theorem 1.1. *There exists $z_2\in(0,z_1]$ such that if $z\in(0,z_2)$ and $\mu\in\mathrm{ex}\mathcal{G}(z,\Phi)$, then the process $\varepsilon x(\frac{t}{\varepsilon^2})$ on $(\Omega,\mathcal{F},P_{\mu^*})$ converges to $D^*B(t)$ as $\varepsilon\to 0$ in distribution with respect to J_1-topology on Skorohod's function space $\mathbf{D}[0,\infty)$, where $B(t)$ is a d-dimensional Brownian motion and D^* is a positive definite $d\times d$-matrix. In particular, $z_2=\infty$ in case $h=\infty$.*

In the previous paper [2] we studied a system of infinitely many hard balls with the same diameter r moving discontinuously in $\mathbf{R}^d$. We denote the configuration space of hard balls by $\mathfrak{X}$:

$$\mathfrak{X}=\{\xi=\{x_i\}:|x_i-x_j|\geq r, i\neq j\},$$

the position of a ball being represented by its center. The space $\mathfrak{X}$ is a compact subset of $\mathfrak{M}$ with the vague topology.

Let $\mathbf{C}(\mathfrak{X})$ be the space of all real valued continuous functions on $\mathfrak{X}$ and $\mathbf{C}_0(\mathfrak{X})$ be the set of functions of $\mathbf{C}(\mathfrak{X})$ each of which depends only on the configurations in some compact set K. The system is described by the $\mathfrak{X}$-valued Markov process ξ_t whose generator is the smallest closed extension of the operator $\bar{\mathbf{K}}$ on $\mathbf{C}_0(\mathfrak{X})$ given by

$$\mathbf{K}f(\xi)=\sum_{x\in\xi}\int_{\mathbf{R}^d}dy\{f(\xi^{x,y})-f(\xi)\}p(|x-y|)\chi(y|\xi\backslash\{x\}),\quad f\in\mathbf{C}_0(\mathfrak{X}),$$

where

$$\xi^{x,y}=\begin{cases}(\xi\backslash\{x\})\cup\{y\} & \text{if } x\in\xi,\ y\notin\xi,\\ \xi & \text{otherwise.}\end{cases}$$

We study the behavior of a tagged particle in the process. In order to follow the motion of the tagged particle it is convenient to regard the process ξ_t as a Markov process $(y(t),\zeta_t)$ on the locally compact space $\mathbf{R}^d\times\mathfrak{X}_0$, where

$$\mathfrak{X}_0=\{\zeta\in\mathfrak{X}:\zeta\cap U_r(0)=\emptyset\}.$$

$y(t)$ is the position of the tagged particle and ζ_t is the entire configuration seen from the tagged particle. We can see that ζ_t is a Markov process whose generator $\bar{\mathcal{K}}$ is the smallest closed extension of the operator on $\mathbf{C}_0(\mathfrak{X}_0)$ given by

$$\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2,$$

$$\mathcal{K}_1 f(\zeta) = \int_{\mathbf{R}^d} du\{f(\tau_{-u}\zeta) - f(\zeta)\}p(|u|)\chi(u|\zeta),$$

$$\mathcal{K}_2 f(\zeta) = \sum_{x\in\zeta}\int_{U_r(0)^c} dy\{f(\zeta^{x,y}) - f(\zeta)\}p(|x-y|)\chi(y|\zeta\backslash\{x\}), \quad f \in \mathbf{C}_0(\mathfrak{X}_0),$$

where $\mathbf{C}(\mathfrak{X}_0)$ and $\mathbf{C}_0(\mathfrak{X}_0)$ are defined by the same way as $\mathbf{C}(\mathfrak{X})$ and $\mathbf{C}_0(\mathfrak{X})$, respectively. We denote by S_t the semigroup associated with generator $\bar{\mathcal{K}}$ and by $(\Omega, \mathcal{F}, P^0_\nu, \zeta_t)$ the associated process with initial distribution ν.

For any $\mu \in \mathcal{G}(z, \Psi)$ we define

$$\mu_0(d\eta) = \frac{\chi(0|\eta)}{c_3}\mu(d\eta),$$

where $c_3 = \int_{\mathfrak{M}} \chi(0|\eta)\mu(d\eta)$. In [2] we proved that there exists $z_3 \in (0, \infty)$ such that if $z \in (0, z_3)$ and if $\sharp\mathcal{G}(z,\Psi) = 1$ and $\mu \in \mathcal{G}(z,\Psi)$, then $(P^0_{\mu_0}, \zeta_t)$ is an ergodic reversible Markov process.

The process $y(t)$ is driven by the process ζ_t in the following way. Let $A \in \mathcal{B}(\mathbf{R}^d)$ and let $\Xi(A)$ be the measurable subset of $\mathfrak{X}_0 \times \mathfrak{X}_0$ defined by

$$\Xi(A) = \{(\eta,\zeta) \in (\mathfrak{X}_0 \times \mathfrak{X}_0)\backslash\Delta : \zeta = \tau_{-u}\eta \text{ for some } u \in A\},$$

where

$$\Delta = \{(\zeta,\zeta) : \zeta \in \mathfrak{X}_0\} \cup (\{\zeta \in \mathfrak{X}_0 : \zeta = \tau_{-u}\zeta \text{ for some } u \in \mathbf{R}^d\backslash\{0\}\}^2).$$

Define a σ-finite random measure N by

$$N((0,t] \times A) = \sum_{s\in(0,t]} \mathbb{1}_{\Xi(A)}(\eta_{s-}, \eta_s), \qquad t > 0.$$

Then,

$$y(t) = y(0) + \int_0^t \int_{\mathbf{R}^d} N(dsdu)u.$$

Our second main result is the following theorem.

Theorem 1.2. *If $z \in (0, z_2 \wedge z_3)$ and if $\sharp\mathcal{G}(z,\Psi) = 1$ and $\mu \in \mathcal{G}(z,\Psi)$, then the process $\varepsilon y(\frac{t}{\varepsilon^2})$ on $(\Omega, \mathcal{F}, P^0_{\mu_0})$ converges to $\sigma_0 B(t)$ as $\epsilon \to 0$ in distribution with respect to J_1-topology on Skorohod's function space $\mathbf{D}[0,\infty)$, where σ_0 is a positive constant.*

References

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2. Tanemura, H., *Ergodicity for an infinite particle system in $\mathbf{R}^d$ of jump type with hard core interaction*, J.Math.Soc.Japan **41** (1989), 681–697.