Quantum Chaos*

Motion of energy levels of isolated systems and quasi-energy spectra of periodically kicked systems

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The name *Quantum Chaos* is rather confusing since quantum mechanics is basically expressed by a set of linear equations while chaos is regarded as typically non-linear phenomena. Under these circumstances one naturally wonder how does quantum chaos exist? At the moment the answer to this question could be either positive or negative. If the answer is YES, he need to give a universal and quantitative definition. On the other hand if the answer is NO, he should be able to describe how does (classical) chaos appear in such a linear dynamical systems at the classical limit. In this talk I shall take the second position. Since there exists no canonical definition yet, by the quantum chaos I mean "quantum properties of a system whose classical motion is chaotic" whenever it will appear in the following discussions.

Among various routes to approach quantum chaos we concentrate on the energy spectrum assuming that nature of any quantum system can be characterized essentially by it. We shall then consider i) fluctuation property of level distribution for isolated systems, and ii) nature of quasi-energy spectrum for a system under external periodical perturbation. Through these studies key problems I would like to solve are; i) how the fluctuation property of energy levels can reflect in chaos for isolated systems?,and ii) is there transition in the quasi-energy spectra from pure point to absolute continuous for periodically perturbed systems?. The meaning of these questions will be clarified as the discussion proceeds.

The talk consists of two parts, *i.e.* on the level fluctuation property of isolated systems and the spectrum nature of the quasi-energy of periodically perturbed systems. Brief introductions with some numerical results are attached to each subjects, followed by theoretical formulations for the motion of spectra. At the end we present discussions concerning what we have clarified and what we could not show. The plan of the talk is as follows;

I. Level statistics of isolated systems
- I-1. Introduction: Relation between chaos and level statistics in the billiard problem.
- I-2. Motion of Levels: Dynamical equation of levels through a coupling strength as the time, and relation to the Calogero-Moser equation.
- I-3. Statistical Theory of Levels: Calculation of the joint distribution function and the transition between the Poisson to the Gauss distribution.

II. Quasi-energy Spectra
- II-1. Introduction: Classical and quantum mapping of the periodically kicked rotator.
- II-2. Motion of the Quasi-energy: Dynamical equation of quasi-energy with coupling strength as the time.
- II-3. Resonance and Energy Diffusion: Continuous spectra.

III. Discussions
- III-1. Is there relation between chaos and level fluctuation?
- III-2. Is there transition from pure point to absolute continuous in quasi-energy spectra?

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** 1-1 Oho, Tsukuba, Ibaraki

I. Level Statistics of an Isolated System

I-1 Introduction *The billiard problem*

We study the classical motion of a free particle moving in a convex billiard table of the shape

$$r(\theta) = r_0[1 + a\cos(2\theta)], (0 \leq a \leq 0.2).$$

Historically the billiard system is studied through the Birkhoff mapping; plot $(\ell, \sin\psi)$, where ℓ is the length along periphery and ψ is the reflection angle, every time the ball hits the wall. This is the area preserving mapping similar to the Poincare mapping. In order to measure chaos quantitatively we calculate the Lyapunov exponents defined formally by

$$L(x) = lim_{t\to\infty, d(0)\to 0} ln\{d(t)/d(0)\}t^{-1},$$

where $d(t)$ is the distance of two neighboring trajectories in the phase space at time t, and x represents initial point in the phase space. While the Lyapunov exponent measures the local orbital instability the K-S entropy $\langle L(x)\rangle$ represents the global instability by averaging the Lyapunov exponent over the phase space x.

Quantized spectra of the billiard system can be obtained by solving the Laplace equation with the hard wall boundary condition;

$$0 = \sum_n c_n J_n[kr(\theta)]e^{in\theta},$$

where $J_n(\theta)$ is the Bessel function. In practice this equation is solved by expanding in the Fourier series, and impose the determinant to be zero. The energy of the system is given by $E_i = k_i^2$. Numerical studies are made by calculating i) the cumulative level number

$$N(E) = \sum_i \theta(E - E_i),$$

and the average level density

$$\rho(E) = \frac{d\bar{N}(E)}{dE},$$

for the average properties of the system. Here $\bar{N}(E)$ is a smoothed function of $N(E)$, which can be obtained either by smoothing data or from the semi-classical formula

$$\bar{N}(E) = \int dqdp\theta(E - H(q,p)).$$

By making use of this function we construct a new level sequence $x_i = \bar{N}(E_i)$, so that $\rho(x) = 1$. After extracting the average property by changing the variable from E_i to x_i, the following two quantities are customary chosen to represent the fluctuation properties: ii) The nearest neighbor level spacing distribution

$$P_0(s)ds \sim \sum_i \{\theta(s + ds - s_i) - \theta(s - s_i)\}$$

for the short range correlation property, where $s_i = x_i - x_{i-1}$, and iii) the mean square deviation

$$\Sigma^2(L) = \langle \frac{1}{L}\int_a^{a+L} \{N(x) - x\}^2 dx\rangle_a$$

for the long range property.

The random matrix theory fits surprisingly well for level fluctuation properties in the chaotic limit: the Gaussian orthogonal ensemble(GOE) gives for the nearest-neighbor spacing

$$P_0(s) \sim (\frac{\pi s}{2})\exp(-\frac{\pi s^2}{4})$$

known as the Wigner distribution almost exactly, and for the mean-square deviation

$$\Sigma^2(L) \sim \frac{1}{\pi^2} lnL,$$

which is a slight modification of the Dyson-Mehta Δ_3 statistics.

I-2 Motion of Levels *and a relation to the Calogero-Moser equation*

We consider a class of isolated systems with the Hamiltonian

$$H(t) = H(0) + tH',$$

where the interaction strength t runs on the real axis. The motion of eigenvalues of this system as t varies can be studied by the equations;

$$\frac{d\mathbf{X}}{dt} = \mathbf{P}, \frac{d\mathbf{P}}{dt} = 0,$$

where $\mathbf{X}$ and $\mathbf{P}$ are Hermite matrices corresponding to $H(t)$ and H' represented by the complete set of eigen-functions of $H(0)$. Changing the basis set by the transformation $\mathcal{O}$:

$$\mathbf{X} = \mathcal{O}^{-1}\mathbf{E}\mathcal{O}, \mathbf{P} = \mathcal{O}^{-1}\mathbf{V}\mathcal{O},$$

so that $\mathbf{E}$ is diagonal, equations of the motion now become

$$\frac{d\mathbf{E}}{dt} + i[\mathbf{M}, \mathbf{E}] = \mathbf{V}, \tag{1}$$

$$\frac{d\mathbf{V}}{dt} + i[\mathbf{M}, \mathbf{V}] = 0, \tag{2}$$

where $\mathbf{M} = i(d\mathcal{O}/dt)\mathcal{O}^{-1}$. Introducing a Hermitian matrix

$$\mathbf{G} = i[\mathbf{E}, \mathbf{V}] \tag{3}$$

the equation of motion for $\mathbf{G}$ is given by

$$\frac{d\mathbf{G}}{dt} + i[\mathbf{M}, \mathbf{G}] = 0. \tag{4}$$

Off-diagonal parts of eq.(1),which do not involve time derivatives, define the matrix $\mathbf{M}$:

$$M_{mn} = i\frac{V_{mn}}{(E_m - E_n)}, (m \neq n)$$

except diagonal components. In order to have finite matrix elements for $M_{mn}(m \neq n)$ eigen-values of the unperturbed Hamiltonian should not be degenerate. Diagonal parts of eqs.(1) and (2):

$$\frac{dE_n}{dt} = V_{nn}, \tag{1d}$$

$$\frac{dV_{nn}}{dt} = i\sum_l (V_{nl}M_{ln} - M_{nl}V_{ln}), \tag{2d}$$

and eq.(4),

$$\frac{dG_{mn}}{dt} = i\sum_l (G_{ml}M_{ln} - M_{ml}G_{ln}),$$

together with eq.(3) constitute a closed set of equations.

I now wish to show that the undetermined elements M_{nn} will not cause any ambiguities. By writing

$$G_{mn} = g_{mn}e^{-i(a_m - a_n)}, \tag{5}$$

we can always choose a_n to cancel M_{nn}. For example if we impose $da_n/dt = M_{nn}$, eq.(4) becomes

$$\frac{dg_{mn}}{dt} = i \sum_{l(\neq m,n)} (g_{ml}M_{ln} - M_{ml}g_{ln}).$$

(note. $g_{nn} = 0$ from eq.(3)) In terms of g_{mn} eqs.(2d) and (4) are written as

$$\frac{dV_{nn}}{dt} = 2\sum_{l(\neq n)} \frac{g_{nl}g_{ln}}{(E_n - E_l)^3}, \tag{6}$$

$$\frac{dg_{mn}}{dt} = i \sum_{l(\neq m,n)} g_{ml}g_{ln}\{\frac{1}{(E_n - E_l)^2} - \frac{1}{(E_m - E_l)^2}\}. \tag{7}$$

The set of equations (1d),(6) and (7) is called the generalized Calogero-Moser (GCM) equation. The original Calogero-Moser(CM) equation can be obtained by making an ansatz

$$G_{mn} = g_m^* g_n - |g_n|^2\delta_{mn}$$

in eq.(4), which reads

$$g_n^{-1}\frac{dg_n}{dt} = i\{M_{nn} + \sum_{l(\neq n)} \frac{|g_l|2}{(E_n - E_l)^2}\}.$$

This equation fixes the phase of g_n, and we obtain a set of solutions with $|g_m| = const.$. Thus, we have

$$\frac{dE_n}{dt} = V_{nn},$$

$$\frac{dV_{nn}}{dt} = 2\sum_{m(\neq n)} \frac{|g_m|^2|g_n|^2}{(E_n - E_m)^3},$$

where the CM eq. is a special case when $|g_m|^2 = g$. The CM eq. can also be derived from the Hamilton equation with the Hamiltonian

$$H_{CM} = \frac{1}{2}\sum_n V_{nn}^2 + \frac{g^2}{2}\sum_{m\neq n} \frac{1}{(E_m - E_n)^2},$$

while the first two of the GCM equations is obtained from

$$H_{GCM} = \frac{1}{2}\sum_n V_{nn}^2 + \frac{1}{2}\sum_{mn}\frac{|g_{mn}|^2}{(E_m - E_n)^2}.$$

These Hamiltonian can be interpreted as the sum of the translational energy of a particle in N-dimensional space with unit mass and the rotational energy with the moment of inertia $(E_m - E_n)^2$ for rotation along the axis perpendicular to the (m,n) plane. Other possible interpretation of this Hamiltonian is the one dimensional N-body Hamiltonian with centrifugal potentials interacting between m-th and n-th particles with the relative angular momentum g_{mn}.

I-3 Statistical Theory of Levels

The system we are dealing can be regarded as an N-particle system with inverse square repulsion and relative angular momenta g_{mn}, instead of the N^2 free particle system where we have started. Let us consider the statistical properties of this system. In order to introduce the statistical ensemble we have to be able to construct statistically independent sub-systems. We choose the member of the ensemble to be a sequence of N levels. Once we define the sub-system our next task is to find additive constants of the motion. From eqs.(2) and (4) , which are written in the Lax form, there are in general infinite number of constants of the motion when $[\mathbf{V},\mathbf{G}] \neq 0$. It is consistent with the complete integrability of our starting equations. It is commonly believed that the statistical treatment is only possible for those systems which is not completely integrable. In our case we construct sub-systems confining N-particles in a potential well in order to have constant density as we have change energy sequence from $\{E_i\}$ to $\{x_i\}$ in the last section. This corresponds to change the equation from $d\mathbf{P}/dt = 0$ to $d\mathbf{P}/dt = \mathbf{F}(\mathbf{X})$ where $\mathbf{F}(\mathbf{X})$ is obtained from an appropriate potential $U(\mathbf{X})$. In this case constants of the motion are greatly reduced to

$$\frac{1}{2}Tr(\mathbf{V}^2) + U(\mathbf{X}), \text{and } Tr(\mathbf{G}^n).$$

The Lax equation for $\mathbf{G}$ is a consequence of the completeness of the basis function or in other word $U(N)$ invariance. Then by constructing sub-systems its invariance should also be broken. How bad is the braking will depend on the size of N as well as the initial conditions. Here, we assume that only $Tr(\mathbf{G}^2)$ survives as the good conserved quantity.

The equilibrium distribution is then given by

$$dw \sim e^{-\beta h - \gamma q} d\Gamma$$

with

$$h = \frac{1}{2}Tr(\mathbf{V}^2) + U(\mathbf{X}), \text{and } q = \frac{1}{2}Tr(\mathbf{G}^2).$$

The phase volume $d\Gamma$ is defined by

$$d\Gamma = \mathcal{F}(\mathbf{X},\mathbf{P})\frac{[d\mathbf{X}][d\mathbf{P}]}{(2\pi\hbar)^{\mathcal{N}}}$$

as the ordinary classical statistical mechanics with an appropriate constraint $\mathcal{F}$ on the type of matrices $\mathbf{X}$ and $\mathbf{P}$. $\mathcal{N}$ corresponds to one-half of the dimension of phase space. By changing the variables to $\{E_n, \theta_i, V_{nn}, g_{mn}\}$ the probability distribution becomes

$$\exp[-\frac{\beta}{2}\{\sum_n V_{nn}^2 + \sum_{m\neq n}\frac{g_{mn}^2}{(E_m - E_n)^2}\} - \beta\sum_n U(E_n) - \frac{\gamma}{2}\sum_{m\neq n} g_{mn}^2]$$

with

$$d\Gamma \sim J(\{\theta_i\})\prod_n dE_n \prod_i d\theta_i \prod_n dV_{nn} \prod_{m>n} dg_{mn}$$

for a time reversal invariant system. Here J is the Jacobian depending only on rotation angles to diagonalize $\mathbf{X}$, while the familiar factor

$$\prod_{m>n} |E_m - E_n|$$

is canceled by the Jacobian of transformation from $\{V_{mn}\}$ to $\{g_{mn}\}$. Since we are interested on the level distribution, we carry out integrations over unobserved variables $\{\theta_i, V_{nn}, g_{mn}\}$. Then we get the joint distribution function as

$$P(\{E_n\}) \sim \prod_{m>n} [\frac{(E_m - E_n)^2}{1 + \frac{\gamma}{\beta}(E_m - E_n)^2}]^{\frac{\alpha}{2}} \prod_n e^{-\beta U(E_n)},$$

where α should be 1 for real symmetric Hamiltonian and 2 for Hermitian (complex) Hamiltonian.

Let us consider the distribution function in two limits: i) high temperature low angular momentum limit $(\gamma/\beta \to \infty)$

$$P(\{E_n\}) \to \prod_n e^{-\beta U(E_n)}.$$

In this case distribution of E_n is mutually independent and we have the Poisson distribution;

$$P_0(s) \sim \rho e^{-\rho s},$$

$$\Sigma^2(L) \sim L.$$

ii) low temperature high angular momentum limit$(\gamma/\beta \to 0, N \to \infty$, and $N\gamma/\beta \to const.)$

$$P(\{E_n\}) \sim \prod_{m>n} |E_m - E_n|^\alpha \prod_n \exp[-\frac{a}{2}(E_n - E_c)^2 - \beta U(E_n)],$$

with $a = N\alpha\gamma/\beta$ and E_c is the center of gravity of all levels. If we set $U(E_n)$ to be zero, this distribution coincides with the well-known GOE($\alpha = 1$) or GUE($\alpha = 2$) .

II Quasi-Energy Spectra

II-1. Introduction *Periodically Kicked Rotator*

In order to study the dynamical chaos of Hamiltonian systems the *Standard Map*

$$I_{j+1} = I_j + K \sin\theta_j$$

$$\theta_{j+1} = \theta_j + I_{j+1}$$

has been one of the standard tool. This mapping can be obtained from the Hamilton equation of the Hamiltonian

$$H = \frac{1}{2}P^2 + \{\sum_j \delta(t - jT)\}\lambda\cos\theta$$

i.e. the periodically kicked rotator with period T, where

$$I_j = TP(jT^-),\ \theta_j = \theta(jT^-),\ K = \lambda T^2$$

at time jT^- just before the j-th kick. When $T \sim 0$, this system is the pendulum. There exist three types of classical motion,namely 1) *regular motion* at $K < K_c$: Motion stays on a torus or bounded by tori, and thus orbits are periodic or quasi-periodic. 2)*Chaotic motion* at $K > K_c$: Disappearance of the KAM tori causes stochastic motion in phase space. 3)*Accelerator mode* at $K > K_a$: Each kick adds up coherently at specific initial conditions.

As an observable of chaos, we consider the energy diffusion: the time variation of energy defined by

$$E_j = \langle I_j^2 \rangle$$

where $\langle ... \rangle$ represents averaging over the initial ensemble. Iterating the standard map we obtain

$$I_{j+1} = I_0 + K \sum_{i=0}^{j} sin\theta_i,$$

and by taking the ensemble average we have

$$\langle I_{j+1}^2 \rangle \sim K^2 \sum_{i,k=0}^{j} \langle \sin\theta_i \sin\ \theta_k \rangle.$$

Corresponding to each type of motions for the regular motion 1) the energy E_j varies periodically or quasi-periodically . As for the chaotic motion 2) θ_i behaves like random white noise, and thus I_j is the random walk in I-direction. In this case E_j corresponds to the mean square fluctuation which is known to increase proportional to j,

$$E_j \sim \frac{1}{2} K^2 j.$$

Finally for the accelerator mode 3) the sum in I_j accumulates coherently, and I_j increases proportional to j, or E_j increases proportional to j^2. From these results we can summarize that " appearance of the *absorptive energy diffusion* is a good signal of chaos".

We now quantize the kicked rotator. The wave function at time jT^- just before the j-th kick is written as

$$\psi_j(\theta) = \psi(\theta, jT^-) = \sum_n A_n(j) e^{in\theta}.$$

The quantum standard map

$$A_m(j+1) = \sum_n U_{m,n} A_n(j)$$

is generated by the one period evolution operator

$$U_{m,n} = i^{-m+n} J_{m-n}(q) \exp^{-\frac{i}{2}\tau m^2},$$

where $J_m(q)$ is the Bessel function and where $\tau = \hbar T, q = \lambda T/\hbar$. The classical parameter K is written as the product of τ and q ($K = \tau q$). Corresponding to the classical case we calculate the energy in quantum motion,

$$E_j = T^2\langle\psi_j|P^2|\psi_j\rangle = \tau^2\sum_n n^2|A_n(j)|^2.$$

Characteristic features of quantum motion observed in the numerical calculations are 1) *limited diffusion*: No absorptive energy diffusion is seen for most of the parameter values τ and q. The wave function in momentum space is localized exponentially:

$$|A_n(j)| \sim e^{-|n|/L}, (j \to \infty),$$

where the localization length L behaves as q^α. In this case E_j is bounded asymptotically by

$$E_j \to \tau^2 q^{2\alpha},$$

for large q, i.e. for a fixed K asymptotic value of E_j increases as $\hbar^{-2\alpha+2}$ for $\alpha > 1$ as $\hbar$ vanishes. Another important quantum phenomena is 2) the *resonance* which appears when $\tau = 2\pi M/N$ for an even integer N, or $\tau = 4\pi M/N$ for an odd integer N. In this case E_j increases asymptotically proportional to j^2 regardless of the value of q.

II-2 Motion of Quasi-energies

By exactly the same manner as the energy eigen-value we consider the motion of the quasi-energy defined by the eigen-value problem

$$U(t)\phi = e^{-iE(t)}\phi,$$

where the unitary operator is given by the one period evolution operator of a kicked system:

$$U(t) = e^{-ith'}e^{-ih_0}$$

with interaction strength t running on the real axis. The motion of quasi-energies as t varies can be studied by the equations;

$$i\frac{d\mathbf{X}}{dt}\mathbf{X}^{-1} = \mathbf{P},\ \frac{d\mathbf{P}}{dt} = 0,$$

where $\mathbf{X}$ and $\mathbf{P}$ are matrices corresponding to $U(t)$ and h' represented by a complete set of eigen-functions of h_0 independent of t. Changing the basis set by the transformation $\mathcal{O}$:

$$\mathbf{X} = \mathcal{O}^{-1}\mathbf{D}\mathcal{O}, \mathbf{P} = \mathcal{O}^{-1}\mathbf{V}\mathcal{O},$$

so that $\mathbf{D}$ is diagonal, equations of the motion becomes

$$\frac{d\mathbf{D}}{dt} + i[\mathbf{M},\mathbf{D}] = -i\mathbf{V}\mathbf{D}, \tag{11}$$

$$\frac{d\mathbf{V}}{dt} + i[\mathbf{M},\mathbf{V}] = 0, \tag{12}$$

where $\mathbf{M} = i(d\mathcal{O}/dt)\mathcal{O}^{-1}$ is Hermitian. Introducing a matrix $\mathbf{G}$ by

$$\mathbf{DG} = -[\mathbf{D}, \mathbf{V}] \tag{13}$$

the equation for $\mathbf{G}$ is given by

$$\frac{d\mathbf{G}}{dt} + i[\mathbf{M}, \mathbf{G}] = 0. \tag{14}$$

Off-diagonal parts of eq.(11) define the matrix $\mathbf{M}$:

$$M_{mn} = iV_{mn}\frac{D_n}{D_m - D_n}, (m \neq n)$$

except diagonal elements. Diagonal parts of eqs.(11) and (12):

$$\frac{dE_n}{dt} = V_{nn}, (D_n = e^{-iE_n}) \tag{11d}$$

$$\frac{dV_{nn}}{dt} = i\sum_l (V_{nl}M_{ln} - M_{nl}V_{ln}), \tag{12d}$$

and eq.(4),

$$\frac{dG_{mn}}{dt} = i\sum_l (G_{ml}M_{ln} - M_{ml}G_{ln}),$$

together with eq.(13) constitute a closed set of equations.

Similar to the previous case the undetermined elements M_{nn} will not cause any ambiguities. By writing

$$G_{mn} = g_{mn}\exp\{-i(a_m - a_n)\} \tag{15}$$

we can always choose a_n to cancel M_{nn}. For example if we choose $da_n/dt = M_{nn}$, eq.(14) becomes

$$\frac{dg_{mn}}{dt} = i\sum_{l(\neq m,n)} (g_{ml}M_{ln} - M_{ml}g_{ln})$$

noting $g_{nn} = 0$ from eq.(13). In terms of g_{mn} eqs(12d) and (14) become

$$\frac{dV_{nn}}{dt} = \frac{1}{4}\sum_{l(\neq n)} g_{nl}g_{ln}\frac{\cos\{\frac{1}{2}(E_n - E_l)\}}{\sin^3\{\frac{1}{2}(E_n - E_l)\}}, \tag{16}$$

$$\frac{dg_{mn}}{dt} = \frac{i}{4}\sum_{l(\neq m,n)} g_{ml}g_{ln}\left[\frac{1}{\sin^2\{\frac{1}{2}(E_n - E_l)\}} - \frac{1}{\sin^2\{\frac{1}{2}(E_m - E_l)\}}\right]. \tag{17}$$

Similar to the Calogero-Moser(CM) equation we can obtain a special class of solutions by making an ansatz

$$G_{mn} = g_m^* g_n - |g_n|^2\delta_{mn}$$

in eq.(14), which reads

$$g_n^{-1}\frac{dg_n}{dt} = \frac{i}{4}\left[M_{nn} + \sum_{l(\neq n)}\frac{|g_l|^2}{sin^2\{\frac{1}{2}(E_n - E_l)\}}\right].$$

This fixes the phase of g_n, and we have a set of solutions with $|g_m| = const.$. Thus we have

$$\frac{dE_n}{dt} = V_{nn},$$

$$\frac{dV_{nn}}{dt} = \frac{1}{4}\sum_{m(\neq n)} |g_m|^2|g_n|^2 \frac{\cos\{\frac{1}{2}(E_n - E_l)\}}{\sin^3\{\frac{1}{2}(E_n - E_l)\}}.$$

These equations are obtained from the Hamilton equation with the Hamiltonian

$$H_{OP} = \frac{1}{2}\sum_n V_{nn}^2 + \frac{1}{2}\sum_{m\neq n} \frac{|g_m|^2|g_n|^2}{\sin^2\{\frac{1}{2}(E_n - E_l)\}}$$

while the equations (11d) and (16) can be obtained from

$$H_{GOP} = \frac{1}{2}\sum_n V_{nn}^2 + \frac{1}{2}\sum_{m\neq n} \frac{|g_{mn}|^2}{\sin^2\{\frac{1}{2}(E_n - E_l)\}}.$$

The equation for quasi-energies is more complicated than that of energy eigenvalues because of the periodicity of interactions. In the previous case the interaction between the m-th level and the n-th level reduces as $(E_m - E_n)^{-2}$, while in the present case $\sin^{-2}\{\frac{1}{2}(E_n - E_l)\}$ can be arbitrary large regardless to the difference of two quasi-energies. In such a case the range of interaction in g_{mn} can be very long and it will not be possible to divide the system into independent sub-systems for constructing the statistical ensemble. Nevertheless, when we give a set of initial values,for example

$$E_n = \frac{1}{2}\tau n^2,\ V_{nn} = 0, \text{and } g_{mn} = -\frac{1}{2}\{1 - e^{i(E_m - E_n)}\}(\delta_{m,n+1} + \delta_{m,n-1})$$

for the kicked rotator, the equation of motion can be solved in principle for most of the parameter values of τ away from resonance. There we expect the discreteness of quasi-energy spectra is preserved.

II-3.Resonance

Period -N resonance appears when $\tau = 2\pi M/N$ for even N, or $\tau = 4\pi M/N$ for odd N. In such cases we have

$$U_{m+N,n+N} = U_{m,n}.$$

Quantum Mapping for the resonance case can be written as

$$\mathbf{A}_m(j+1) = \sum_n \mathbf{U}_{m-n}\mathbf{A}_n(j)$$

with

$$\mathbf{A}_n(j) = \{A_{s+nN}(j)\}, \mathbf{U}_{m-n} = \{U_{s+(m-n)N,t}\}, (s,t = 1,...,N).$$

It should be noticed that this mapping has translational invariance, and in this case we cannot expect localization. Quasi-energy spectrum of this equation is obtained from the eigen-value equation

$$e^{-i\epsilon}\mathbf{b}(a) = \mathbf{U}(a)\mathbf{b}(a),$$

where we have defined

$$\mathbf{U}(a) = \sum_n e^{-ina}\mathbf{U}_n,$$

with a continuous parameter $a = [0, 2\pi]$. Since quasi-energy ϵ now depends on the continuous parameter a, the spectrum is absolute continuous forming the energy bands.

Let me show that this system exhibits energy diffusion proportional to j^2: When the initial condition is chosen to be

$$\mathbf{A}_n(0) = \sum_\mu \int c_\mu(a)e^{ina}\mathbf{b}_\mu(a)\frac{da}{2\pi},$$

where $\mathbf{b}_\mu(a)$ is the eigen-function with the quasi-energy $\epsilon_\mu(a)$, the state at time j is given by

$$\mathbf{A}_n(j) = \sum_\mu \int c_\mu(a)e^{ina}e^{-i\epsilon_\mu(a)j}\mathbf{b}_\mu(a)\frac{da}{2\pi}.$$

Then the energy expectation value in the asymptotic limit is given by

$$E_j \sim \tau^2 N^2 j^2 \sum_\mu \int \{\frac{d\epsilon_\mu(a)}{da}\}^2 |c_\mu(a)|^2 \frac{da}{2\pi} + \mathcal{O}(j).$$

This expression indicates that the diffusion occurs only when the quasi-energy has continuous spectrum.

III.Discussion

III-I Relations between the Level Fluctuation and Chaos

Let me summarize the present status of the theory of level statistics: i) The GOE can explain surprisingly well the existing data. ii) The GOE is not a unique ensemble, but rather an ensemble at the "chaotic" limit. iii) With two quantities, TrH'^2 and $Tr[H,H']^2$, the statistics of level fluctuations can be well describe in both integrable and chaotic limits; i.e. the level ststistics makes a smooth transition from the Poisson to the Gauss distribution. Now the following two problems seem to me essential.

Problem(1): Once we have a Hamiltonian H, we can calculate energy eigen-values, and thus we can take level statistics. However, in order to predict the statistical properties from our theory we need to know both H and H'.

Resolution: If we can assume the ergodicity, the statistical nature of the Hamiltonian $H = H_0 + tH'$ will be dominated by that of $t \neq 0$. In this case the theorem says that the statistical average along t can be replaced by the ensemble average of a certain time, say $t = 1$, larger than the relaxation time.

Problem(2): If the two invariants are the order parameters of chaos for the quantized system, do their classical correspondences,

$$\int dqdpH'(q,p)^2$$

and

$$\int dqdp\{H(q,p), H'(q,p)\}^2$$

are related to the classical order parameter such as the K-S entropy?

Resolution: I do not have any answer for this problem. I would rather like to ask you the following question. "If we know a Hamiltonian $H = H_0 + H'$, can we estimate the K-S entropy without calculating the Lyapunov exponent by the direct integration of equations of the motion?" If it were possible, it is natural to expect that the K-S entropy is a function of above two quantities.

III-2. Is there transition from pure point to absolute continuous in quasi-energy spectra?

When the system has only pure point spectrum, the motion is always periodic or quasi-periodic, and no energy diffusion is expected. In order to have the time asymmetric phenomena such as diffusion it is essential to have continuous spectrum. There is a theorem due to Casati et.al. that for certain class of irrational values α of the parameter $\tau = 4\pi\alpha$ there exist continuous quasi-energy spectrum.

Problem: Where can we find the absolutely continuous spectra except at the resonance?

Answer(not complete): When α belong to a class of numbers which are well approximated by the continued fraction such as a Liouville number, we construct a series approximating α by rational numbers;

$$|\alpha - M_i/N_i| < cN_i^{-\sigma}.$$

The problem we like to answer is whether the energy for $\alpha = M/N$

$$E_j = \tau^2 \sum_{n,s} (s + nN)^2 |A_{s+nN}(j)|^2$$

can still increase as N_i increases to approximate an irrational number by the limit of rationals. Let us write

$$E_j = \tau^2 \{e_2(\alpha) j^2 + e_1(\alpha) j + e_0(\alpha)\}.$$

Here, $e_2(\alpha)$ has been given before. In order to know the existence of energy diffusion at generic values of α we need to estimate the behavior at $N \to \infty$. It is not a simple task at all, and we do not have answer yet. But, there is a special case where we have the large N limit: in the semi-classical limit ($\tau = \frac{4\pi}{N}, q = \frac{KN}{4\pi}, N \to \infty$) the energy dissipate as $E_j = \frac{1}{2}K^2 j$. In this case we should have

$$(\frac{4\pi}{N})^2 e_2(\frac{1}{N}) \sim 0, \text{ and } (\frac{4\pi}{N})^2 e_1(\frac{1}{N}) \sim \frac{1}{2}K^2,$$

i.e. $e_1(\frac{1}{N})$ is asymptotically proportional to q^2. Numerical investigation of the behavior of $e_2(\frac{M}{N})$ and $e_1(\frac{M}{N})$ will help to guess the values at generic α.

References

These references are only lists of the minimum requirement. Reader will find more references in the article listed below.

[1] Yukawa T. and Ishikawa T.; Prog.Theor. Phys. Suppl.**98**(1989)157. More articles related to Chaos and Quantum chaos will be found in this supplement.

[2] Olshansky M.A. and Prelomov A.M.; Phys.Rep. **71**(1981)313.

[3] Gutzwiller M.C.; *Chaos in Classical and Quantum Mechanics* (Springer-Verlag,1990).