Multivariable Vertex Models, the Temperley-Lieb Algebra and Link Polynomials *

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Abstract

We present a family of multivariable solvable vertex models associated with representations of the Temperley-Lieb algebra. We discuss free energy and correlation lengths for the multivariable models and also for the fusion models. We also discuss connection of the multivariable vertex models to Jones polynomials and link polynomials given by Akutsu - Wadati, which are associated with representations of $U_q(sl(2,C))$.

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1 Introduction

Yang-Baxter relation: A Key to recent developments of mathematical physics

Recently various fields and branches of mathematical physics have been extensively developed. The key of this development is the Yang-Baxter relation. [1,2,3] Actually, the Yang-Baxter relation plays an important role in various interesting branches of mathematical physics such as solvable models in statistical mechanics and field theories, quantum groups, link polynomials (topological invariants for knots and links), integrable models in field theories ("Toda" field theory), rational conformal field theory (monodromy matrice), 3 dim. quantum gravity, etc..

In statistical mechanical physics, solutions of the Yang-Baxter relation give exactly solvable models We note that originally the Yang-Baxter relation was introduced as a sufficient condition for existence of commuting family of transfer matrices.

Importance of the Temperley-Lieb algebra

The Temperley-Lieb algebra plays an important role in the study of mathematical structure of intergrable systems. The algebra gives a short cut to the mathematical formulations related to exactly solvable models (the "Yang-Baxterization") [4-15]

Origin of the Temperley-Lieb algebra

The Temperley-Lieb algebra was introduced from the unification of the following two analytic approaches. [4]

(1) the Bethe ansatz method

Bethe-Hulthen, Yang-Yang, Sutherland, Lieb - Wu

(2) Transfer matrix method

Onsager, Kaufman-Onsager, Schultz-Mattis-Lieb

Presentation of the Temperley-Lieb algebra The generators $\{E_i\}$ of the Temperley-Lieb algebra satisfy the following relations.

$$\begin{aligned} E_i E_{i\pm1} E_i &= E_i, \\ E_i E_j &= E_j E_i, \quad \text{for } |i-j|>1, \\ E_i^2 &= Q^{1/2} E_i. \end{aligned} \tag{1}$$

(We call the operator E_i Temperley-Lieb operator.)

2 Multivariable vertex models

Let us introduce a family of exactly solvable vertex models that have multivariable parameters. We construct multivariable solvable vertex models for any given regular matrices. [16] We note that recently a new hierarchy of vertex models have been introduced, which also have multivariables. We call the vertex models colored vertex models. [17]

Multivariable rep. of TL algebra

(1) Let $k^{ab}(a,b=1,\cdots,N_0)$ be an arbitrary regular matrix of the size $N_0 \times N_0$. (N_0 denotes the number of state variables.) We denote the inverse matrix of k^{ab} by k_{ab}.

$$\sum_b k_{ab} k^{bc} = \delta_a^c. \tag{2}$$

(2) We introduce an operator E_i

$$E_i = \sum_{abcd} k_{cd} k^{ab} I^{(1)} \otimes \cdots \otimes I^{(i-1)} \otimes e_{ac}^{(i)} \otimes e_{bd}^{(i+1)} \otimes \cdots \otimes I^{(n)}. \tag{3}$$

$I^{(i)}$ denotes the identity matrix and e_{ab} a matrix such that $(e_{ab})_{jk} = \delta_{ja}\delta_{kb}$.

(3) The quantity $Q^{1/2}$ is

$$Q^{1/2} = \sum_{a,b} k_{ab} k^{ab}. \tag{4}$$

We define crossing parameter λ by the following relation:

$$\begin{aligned} 2\cosh(\lambda) = Q^{1/2} &= \sum_{a,b} k_{ab}k^{ab}, \text{ if } Q > 4, \\ 2\cos(\lambda) = Q^{1/2} &= \sum_{a,b} k_{ab}k^{ab}, \text{ if } Q < 4. \end{aligned} \tag{5}$$

Yang-Baxter operator

We introduce an operator $X_i(u)$, which is a unit of the diagonal-to-diagonal transfer matrix. [2]

$$X_i(u) = \sum_{abcd} X^{ab}_{cd}(u) I^{(1)} \otimes \cdots \otimes e^{(i)}_{ac} \otimes e^{(i+1)}_{bd} \otimes I^{(i+2)} \otimes \cdots \otimes I^{(n)}. \tag{6}$$

Here $X^{ab}_{cd}(u) = w(c,d,b,a;u)$.

Construction of YBO from TLO

From the Temperley-Lieb operator (3) we construct the Yang-Baxter operator $X_i(u)$. [6]

$$X_i(u) = \rho(u)(I + f(u)E_i). \tag{7}$$

Functions $\rho(u)$ and $f(u)$ are

$$f(u) = \begin{cases} \sinh u/\sinh(\lambda - u), & \text{if } Q > 4, \\ \sin u/\sin(\lambda - u), & \text{if } Q < 4, \end{cases} \tag{8}$$

$$\rho(u) = \begin{cases} \sinh(\lambda - u)/\sinh\lambda, & \text{if } Q > 4, \\ \sin(\lambda - u)/\sin\lambda, & \text{if } Q < 4. \end{cases} \tag{9}$$

3 Free energy of the multivariable model

Using the inversion relation method [2,6,18] we can calculate the free energy (partition function) of the multivariable model.

$\kappa(u)$: the partition function per site

Assumptions

(a) $\kappa(u)$ an analytic function of the spectral parameter u.

(b) the spectral parameter is in the physical domain $0 \leq u \leq \lambda$.

(This domain is invariant under the replacement $u \rightarrow \lambda - u$, and therefore invariant under the crossing symmetry.)

Expressions of the free energy

i) $Q > 4$ (massive regime) ,

$$\ln \kappa(u) = \ln \kappa(0) + 2 \sum_{n=1}^{\infty} \frac{\exp(-2n\lambda) \sinh(nu) \sinh(n(\lambda - u))}{n \cosh(n\lambda)}. \tag{10}$$

ii) $Q < 4$ (massless regime),

$$\ln \kappa(u) = \ln \kappa(0) + \int_{-\infty}^{\infty} \frac{\cosh((\pi - 2\lambda)t) \sinh ut \sinh(\lambda - u)t}{t \sinh(\pi t) \cosh \lambda t} dt. \tag{11}$$

(Here we have assumed that $0 < \lambda < \pi$.)

Key Points

(1) The free energy of the model is a function of the crossing parameter λ (or $Q^{1/2}$) in (5).

(2) There are two types of the multivariate parameters $\{k^{ab}\}$. One type is related to the crossing parameter λ, and the other type is not.

4 Fusion models

We construct fusion models of the general multivariable model. We apply the fusion method. [19,20,22,12] The fusion models can be considered as special cases of Z-invariantly generalized inhomogeneous models. [21,22] They are generalization of the fusion models associated with higher spin representations of SU(2). [19,23,24,25] Construction of fusion models from the viewpoint of the Temperley-Lieb algebra was discussed in the reference [12].

(a) The Yang-Baxter operator $X_i(u)$ becomes a projection operator when the spectral parameter u is $\pm\lambda$. [20,12]
(b) The projection operators $P_i^{[[k]]}$, which are q-analogues of the Young operators acting on the k-th order tensor product space. [26,27,12] ($[[k]]$ denotes the symmetry of the projection operator.)

Hereafter the symbol $P^{[k]}$ denotes the symmetric projection of the k-th order tensor product space.

Composite Yang-Baxter operator

$$Y_i^{[k]}(u) = P_{(i-1)k+1}^{[k]} P_{ik+1}^{[k]} \left(\prod_{j=1}^{k} \hat{X}_i^{(j)}(u) \right) P_{(i-1)k+1}^{[k]} P_{ik+1}^{[k]}, \tag{12}$$

where

$$\hat{X}_i^{(j)}(u) = \prod_{m=1}^{k} X_{ik+m-j}(u-(k-j-m+1)\lambda). \tag{13}$$

For example we consider the case $k=2$.

$$\begin{aligned} Y_i^{[2]}(u) &= P_{2i-1}^{[2]} P_{2i+1}^{[2]} X_{2i}(u-\lambda) X_{2i-1}(u) X_{2i+1}(u) \\ &\quad \times X_{2i}(u+\lambda) P_{2i-1}^{[2]} P_{2i+1}^{[2]}. \end{aligned} \tag{14}$$

We can calculate the free energy of the fusion models by inversion relation method assuming the analyticity of the free energy. We discuss the fusion model with the symmetry $[k]$ that is constructed from the multivariable vertex model with the value Q and the crossing parameter λ.
(a) The fusion model has the crossing symmetry with the same crossing parameter λ.
(b) The physical domain $0 \le u \le \lambda$.

Expression of free energy

i) $Q > 4$

$$\ln \kappa(u) = \ln \kappa(0) +$$

$$+2\sum_{n=1}^{\infty}\frac{e^{-n(k+1)\lambda}\sinh nk\lambda\sinh nu\sinh n(\lambda-u)}{n\cosh n\lambda\sinh n\lambda}. \tag{15}$$

The above expression for the free energy in terms of the crossing parameter λ is equivalent to that for the N-state vertex models, which are fusion models of the 6-vertex model. [25] It is noted that the parameter λ is given by the crossing parameter of the fusion model.

ii) $Q < 4$

$$\ln\kappa(u) = \ln\kappa(0) + $$
$$+\sum_{n=1}^{k}\int_{-\infty}^{\infty}\frac{\cosh((\pi-2\{n\lambda\})t)\sinh ut\sinh(\lambda-u)t}{t\sinh(\pi t)\cosh\lambda t}dt. \tag{16}$$

Here the symbol $\{\mu\}$ denotes the following: $\{\mu\} = \mu - n\pi$ for $n\pi < \mu < (n+1)\pi$. The expression of the free energy for the fusion model of the multivariable model is consistent with that given in the reference [25] .

5 Inversion Identity

Let us discuss correlation length for the general multivariable vertex models. We can derive inversion identity [28,29] for the multivariate vertex models, and also for the fusion models of the multivariable models. [30,16] We denote the row-to-row transfer matrix for the multivariable vertex model by $T_N(u)$. N is the size of the lattice. Then we have

$$T_N(u)T_N(\lambda+u) = C(u)^N(I_N + O(e^{-N})). \tag{17}$$

Here I_N is the identity operator. The function $C(u)$ is given by

$$C(u) = \sinh(\lambda - u). \tag{18}$$

From the inversion identity we have an equation for the ratio of the eigenvalues of the transfer matrix. When $Q > 4$ we obtain the correlation length ξ of the multivariable vertex model as

$$\xi \sim -1/\ln k_1. \tag{19}$$

Here the elliptic modulus k_1 is related to the crossing parameter λ by the relation

$$k_1 = 4\exp(-\lambda/2) \prod_{n=1}^{\infty} \left(\frac{1+\exp(-2n\lambda)}{1+\exp(-(2n-1)\lambda)} \right). \tag{20}$$

We recall that the parameter λ is given by (5). This result generalizes the calculation in the reference [10].

We can apply the discussion based on the inversion identity (17) also to the fusion models. For the fusion model with the symmetry $[k]$ constructed from the multivariable vertex models with $Q > 4$, we obtain the correlation length as $\xi \sim -1/\ln k_1$. The elliptic modulus k_1 is related to the crossing parameter λ of the fusion model by the relation (20). This result is consistent with the calculation of the Landau free-energy [31] and finite size correction [32] for the N-state vertex model. We shall discuss derivation of the inversion identity and calculation of the correlation length for the fusion models elsewhere. [33]

We discuss one-dimensional integrable systems corresponding to the multivariable vertex models. We can derive the Hamiltonian of the one-dimensional system $\hat{H}_N$ by taking the logarithmic derivative of the row-to-row transfer matrix $T_N(u)$.

$$\hat{H}_N = -\frac{d}{du}\left(\ln T_N(u)\right)_{u=0} = \sum_{i=1}^{N} H_i. \tag{21}$$

Here H_i is given by

$$H_i = -\frac{d}{du} X_i(u)_{u=0}$$

$$= \sum_{abcd} H^{ab}_{cd} I^{(1)} \otimes \cdots \otimes e^{(i)}_{ac} \otimes e^{(i+1)}_{bd} \otimes I^{(i+2)} \otimes \cdots \otimes I^{(n)}. \tag{22}$$

Here H^{ab}_{cd} is the matrix elements of the local operator H_i. For the general multivariable vertex models with $Q > 4$ we have

$$H^{ab}_{cd} = -\cosh\lambda/\sinh\lambda \cdot \delta^a_c \delta^b_d + 1/\sinh\lambda \cdot k^{ab} k_{cd}. \tag{23}$$

Thus we have shown that we have various one-dimensional integrable systems corresponding to the multivariable vertex models. The ground state energy of the one-dimensional system can be obtained by the logarithmic derivative of the free energy of the multivariable vertex model. It is noted that we can derive an infinite number of conserved quantities of the corresponding one-dimensional integrable system by taking the higher (logarithmic) derivatives of the transfer matrix of the multivariable model.

6 Connection to Link polynomials

i) From the multivariavle vertex model we have the Jones polynomial.
ii) From the fusion models of the general multivariavle vertex model we have link polynomials given by Akutsu-Wadati, which satisfy the higher order skein relations.

The link polynomials are also associated with higher representations of the quantum groups $U_q(sl(2,C))$. [34] These link polynomials can be applied to or connected to various promising future problems.

7 Future problems

The following problems seems to be closely related to representations of the Temperley-Lieb algebra.

1) Determination of the ground state of 2 dimensional antiferromagnetic Heisenberg model (or diagonalization of the Hamiltonian)

2) Spin singlet bonding state (cf. RVB state)

3) Modified spin wave theory

4) Characterization of excitation spectrum of quantum spin chains
(cf. t-J model — $gl(2|1)$ model)

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