CHAOS AND CRITICAL PHENOMENA IN HAMILTONIAN DYNAMICS

— A New Interpretation of $1/f^{\nu}$ noises in Quartz Crystal —

Yoji Aizawa and Kenji Tanaka

Department of Applied Physics, Waseda University
Okubo 3–4–1, Shinjuku 161, Tokyo, Japan

Stagnant Layer Theory, $1/f^{\nu}$ Fluctuations, Nearly Integrable Systems, Quartz Oscillator

Recent developments of the stagnant layer theory are reviewed, and are applied to the universality of 1/f noises in quartz-oscillator experiments. Geometric and kinetic complexity of nearly integrable systems is discussed in relation to critical phenomena in Hamiltonian dynamics.

§ 1 Introduction

Some universal aspects of Hamiltonian dynamics have been elucidated by use of the stagnant layer theory and the renormalization technique. Here we will quickly explain the basic idea of the stagnant layer theory and some universal results, which can be used in a new interpretation of $1/f$ noises in quartz crystals.

Let us consider the following nearly integrable Hamiltonian systems,

$$H(\mathrm{p},\mathrm{q}) = H_0(\mathrm{p}) + \varepsilon H_1(\mathrm{p},\mathrm{q}) \tag{1}$$

where (p,q) are action-angle variables of 2n-dimension, and εH_1 is a small perturbation. When ε is small enough, almost all integrable tori (so-called KAM tori) survive invariantly in the off-resonant region,

$$|(\mathrm{k}\cdot\frac{\partial H_0}{\partial \mathrm{p}})| > C\,|\mathrm{k}|^{-a}, \tag{2}$$

but on the other hand the tori in resonance regions break up and there appear instability zones where chaos and higher order small tori (so-called Poincaré-Birkhoff tori) are generated.

Chaos in the instability zone has some characteristic features which can be understood in terms of critical phenomena in the transition regime between chaos and torus [1]. The characteristic time of the diffusion in phase space T was estimated by Nekhoroshev [2],

$$T \simeq \frac{1}{\varepsilon} exp[\varepsilon^{-b}] \tag{3}$$

where ε is the smallness parameter in Eq.(1). Furthermore, the distribution of the stagnant time T obeys a universal law [1],

$$P(T) \simeq \frac{1}{T\log T} \tag{4}$$

and the power spectral density for appropriate dynamical variables $S(f)$ satisfies,

$$S(f) \simeq 1/f^{\nu} \tag{5}$$

According to these scaling laws the distribution of Lyapunov exponent λ also obeys $P(\lambda) \simeq 1/\lambda^{\delta}$. The stagnant layer theory predicts that the values of these indices are universal constants ($\delta = 1, \nu = 2$). Some numerical evidence are shown in Figs.1 and 2. [1]

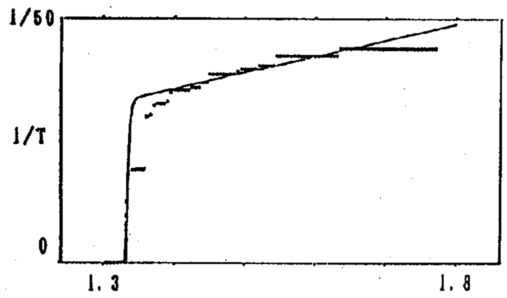


Fig.1 Nekhoroshev bound for lattice system [1]

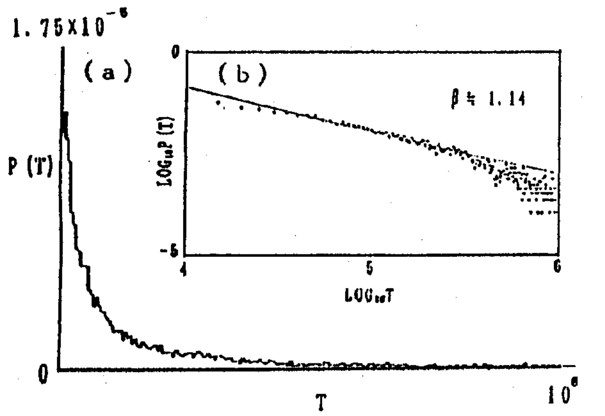


Fig.2 Distribution of the stagnant time T (in Fig.1)

§ 2 Critical Phenomena of Resonant Tori

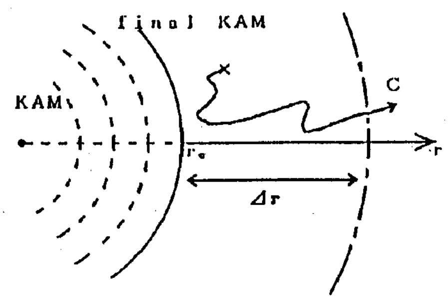
KAM tori distributed in chaotic sea also have a universal nature. Let us consider the inside of the outermost KAM torus ($r \leq r_C$) as is shown in Fig.3-(a). The rotation number R, which is a characteristics of torus, reveals an inherent singularity at the transition point $r = r_C$,

$$|\frac{\partial R}{\partial r}| \to \infty \quad (r \to r_C) \tag{1}$$

Here r stands for the distance in phase space. The following scaling was used in Fig.3-(b),

$$|R - R_C| \propto |r - r_C|^{\alpha} \quad (\alpha \simeq 0.13) \tag{2}$$

where R_C is the rotation number of the outermost KAM torus. The index α is also surmised to be a universal constant, but this is still open [1].



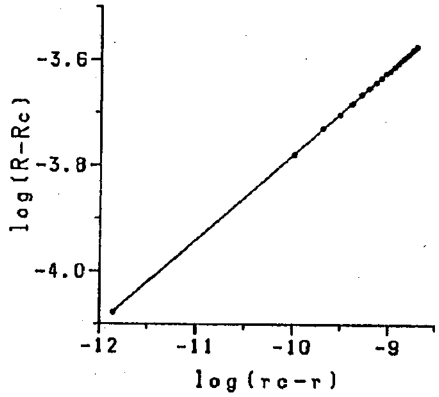


Fig.3 Phase transition between torus and chaos

The same kind of critical phenomena have been discovered near the squeezed point and the collapsation point of twin Poincare-Birkhoff tori [3]. Each critical phenomenon mentioned here is a new type of phase transition in dynamical systems.

§ 3 Model for Quartz Oscillators

Let us consider the following nearly integrable system as a basic model for quartz crystal lattice,

$$H(\mathrm{p},\mathrm{q}) = \frac{1}{2m}\sum \mathrm{p}_i^2 + \frac{k}{2}\sum(\mathrm{q}_i - \mathrm{q}_{i+1})^2 + \frac{\mu}{4}\sum(\mathrm{q}_i - \mathrm{q}_{i+1})^4 \tag{3}$$

The unperturbed system is the harmonic chain just like a Debye model, and the last term is nonlinear perturbations. By an appropriate canonical transformation the equation of motion for an eigen mode (I_i, θ_i) can be written ,

$$\begin{aligned} \dot{I}_i &= -\varepsilon \frac{\partial \tilde{H}_1}{\partial \theta_i} \\ \dot{\theta}_i &= \omega_i + \varepsilon \frac{\partial \tilde{H}_1}{\partial I_i} \end{aligned} \tag{4}$$

where ω_i is the proper angular frequency.

The $1/f$ noises in quartz-experiments are interrelated to the chaotic behavior of a certain resonant mode (I_r, θ_r), e.g., the phase noise [4] is the fluctuation of the angular frequency $\dot{\theta}_r - \omega_r$ and the phonon-number fluctuations [5] is obtained from I_r. Figure 4 shows the time course for each fluctuation observed in a discrete lattice vibration model [1]. The remarkable point is that both time courses in Fig.4 reveal a synchronized intermittent turbulence. Indeed, the PSD functions for both time courses become $1/f^{\nu}$ ($\nu \simeq 2$) as were predicted by the stagnant layer theory.

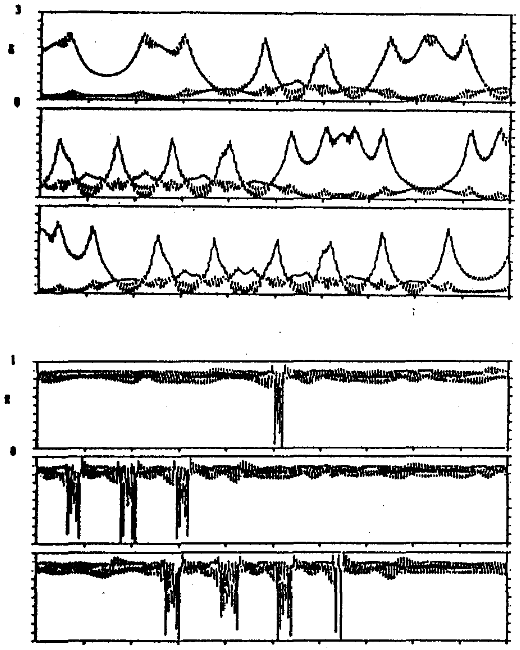


Fig.4 Intermittent time courses for I_i and $\dot{\theta}_i$

In quartz-oscillator experiments, however, the spectral indices are much smaller than 2,i.e., $1 < \nu < 1.5$. The difference between theoretical and experimental results are crucial. To overcome these difficulties we have extended our model to the more realistic ones. The first step of our approach is to introduce dissipative interactions into Eq.(4),because the eigen mode in quartz systems is always accompanied by the free-energy dissipation which comes not only from the dielectric loss but also from the cavity loss. Taking account of these dissipation effects, the equation of motion for the resonant mode should be changed,

$$\begin{aligned} \dot{I}_r &= -\frac{\partial \tilde{H}_1}{\partial \theta_r} + \gamma g(I, \theta) \\ \dot{\theta}_r &= \omega_r + \frac{\partial \tilde{H}_1}{\partial I_r} + \gamma h(I, \theta) \end{aligned} \tag{5}$$

The damping effect plays a very important role in chaotic motions even if the value of γ is small enough ($\gamma \simeq 10^{-15}$), namely the time courses of θ_r and I_r become quite different from those in Fig.4 though the mean energy of each eigen mode is almost invariant. In other words, the statistical properties of these fluctuations are sensitively controlled by the damping coefficient γ. Figure 5(a,b,c) show the PSD functions for the phase noises (a), the phonon-number fluctuation (b) and the fluctuation of Rayleigh's dissipation functions (c). The last one (Fig.5-c) is corresponding to the dielectric loss in real experiments [6].

One of the most striking points is that the spectral indices obtained from Fig.5 coincide well with experimental results at least qualitatively ($1 < \nu < 1.5$) [4, 5, 6]. From these numerical studies, we can say that the dissipativre mechanism plays an essential role to generate $1/f$ fluctuations in quartz-oscillator systems, though the rigorous dynamical theory has not yet been established beyond the stagnant layer theory.

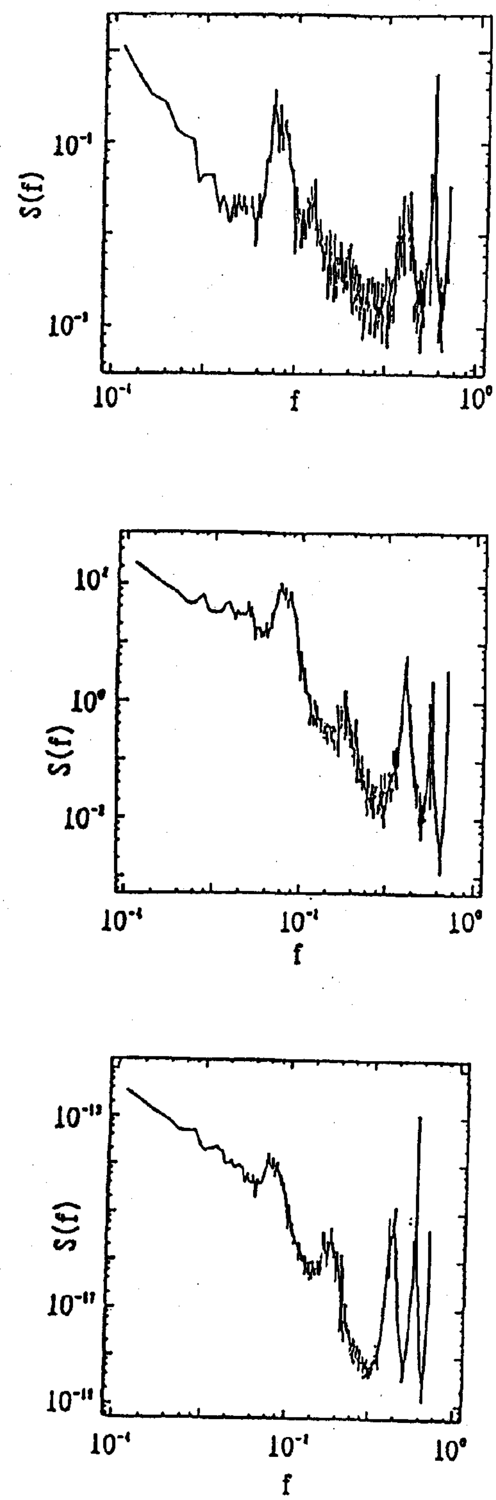


Fig.5 PSD functions for $\dot{\theta}_r, I_r$ and S

§ 4 Conclusions - Universality of nearly integrable systems -

From the view point of dynamical system theory, nearly integrable systems have two essential features; one is non-stationarity and another is multi-ergodicity. The results of § 3 is that these two features guarantee the universality of the $1/f$ fluctuations in quartz crystals. Here the non-stationarity means the divergence of the recurrence time in dynamics (i.e., $\nu \geq 1$), and the multi-ergodicity can be identified by the non-uniqueness of the long time average of dynamical variables. The important point is that the simple ergodicity based on the Lebesgue measure in phase space can never be satisfied in the nearly integrable systems.

The statistical mechanical problems, such as transport phenomena and irreversibility, are now studied from the viewpoints of non-stationarity and multi-ergodicity.

References

[1] Y.Aizawa et. al. Prog. Theor. Phys. 81 (1989), 249; Prog. Theor. Phys. Suppl. No.98 (1989), 36

[2] N.N.Nekhoroshev, Russ. Math. Surveys 32 (1977), 1

[3] T.Hino and Y.Aizawa, in preparation

[4] V.F.Kroupa, Frequency Stability, (IEEE, 1983)

[5] T. Musha, B. Gábor and M. Shoji, Phys. Rev. Lett., 64 (1990), 2394

[6] T. Musha, A. Nakajima and H. Akabane, Jpn. J. Appl. Phys. Pt.2, 27 (1988), L311