TIME-DEPENDENT TFD

Y. Yamanaka
Waseda Univ. Senior High School

1 Introduction

This note reports a new formulation of time-dependent nonequilibrium thermo field dynamics (TFD), developed in the last few years. TFD is a real-time operator formalism of thermal field theory and well-established in equilibrium case [1, 2]. An regular attempt to extend TFD to nonequilibrium situations started with the paper [3], but it was not an easy task to complete it. After many attempts in this direction (see for example [4]), our recent collaboration which took place at University of Alberta led to a new formulation which is rather simple in principle [5, 6, 7, 8, 9]. This note is a brief summary of parts of these recent references.

Our formulation of TFD is designed to deal with time-dependent nonequilibrium thermal phenomena for an *isolated* system of *quantum field*, so it will find interesting applications to the evolution of the Universe, high energy particle reactions (quark-gluon plasma) and so on other than to many problems in solid state physics, and particularly to time-dependent phase transitions.

Before presenting our theory we point out the following difference between quantum field theory (QFT in short, quantum theory with *infinite degrees of freedom*) and quantum mechanics with finite degrees of freedom: Problems in quantum mechanics are solved by simply integrating the Heisenberg (equivalently Schrödinger) equation under given initial conditions. On the other hand, one obtains various physical results from a single Heisenberg equation in QFT, due to the existence of *inequivalent representations*. In quantum mechanics such a problem do not arise because of von Neumann's theorem. Thus QFT additionally requires us to make a choice of relevant state vector space, usually done by a self-consistent renormalization procedure. The choice of a state vector space corresponds to a choice of a quasi particle picture. This situation is phrased in [4] as QFT is a theory of dual language while quantum and classical mechanics are ones of single language. In the conventional approaches in the density matrix formalism of statistical physics and Green's function method of the path ordering method the procedure to choose state vector spaces is forgotten or is not yet formulated properly. Our study using TFD aims at formulating the choice of state vector space (equivalently of thermal vacuum or of quasi particle picture) in thermal situation.

2 Basic Structure of TFD

The basic structure of equilibrium and nonequilibrium TFD is common, and we just list it below.

(I) Every degree of freedom is doubled in a way that to every (nontilde) operator A is associated its tilde partner $\tilde{A}$ according to the so called tilde conjugation rules:

$$(AB)^{\sim} = \tilde{A}\tilde{B} \tag{1}$$

$$(c_1A + c_2B)^{\sim} = c_1^*\tilde{A} + c_2^*\tilde{B} \quad (c_i : \text{c-numbers}) \tag{2}$$

$$(A^\dagger)^{\sim} = \tilde{A}^\dagger \tag{3}$$

$$(\tilde{A})^{\sim} = \sigma A \tag{4}$$

$$|0\rangle^{\sim} = |0\rangle \tag{5}$$

$$\langle 0|^{\sim} = \langle 0| \tag{6}$$

where $\sigma = +1(-1)$ for bosonic (fermionic) operators and $|0\rangle$ is the thermal vacuum. (II) The thermal average of (dynamical) observable A is given by

$$\langle A\rangle = \langle 0|A|0\rangle. \tag{7}$$

(III) The total Hamiltonian governing both of nontilde and tilde operators, denoted by $\hat{H}$, is

$$\hat{H} = H - \tilde{H}. \tag{8}$$

We better distinguish eigenvalues or expectation values of $\hat{H}$ and H and call them the hat-energy ($\hat{E}$) and the dynamical energy (E), respectively.

The structure of $\hat{H}$ in (8) immediately implies that the hat-energy is unbounded from below contrary to the usual QFT without thermal degree of freedom. This nature of $\hat{H}$ is the origin of thermal degree of freedom (also of dissipation) [5, 6], as shown more explicitly next.

Let us find possible candidates for stationary thermal vacua which are the eigenvectors of $\hat{H}$ and are invariant under the tilde conjugation, then one can prove

$$\hat{H}|0(\theta)\rangle = 0 \tag{9}$$

Because of (8) there are uncountable number of such thermal vacua, classified by a continuous multiple components parameter θ. It is very important to note the fact that the stationary thermal vacua, each of which gives rise to respective inequivalent representation, form an *uncountable* set $\{|0(\theta)\rangle\}$ degenerate in hat-energy. This fact enables us to deal with thermal degree of freedom in such a manner that the thermal vacua with different θ correspond to different thermal situations. (In equilibrium case θ is identified as the temperature.)

It was shown that the existence of uncountable thermal vacua is closely related to the first law of thermodynamics; heat energy appears when one compares the thermal average of dynamical energies (H) for different thermal vacua. Details of this subject is found in Ref. [5] and is skipped here.

Now it is clear that the main theme in TFD is to formulate a definite way of finding a particular thermal vacuum among uncountable ones for each thermal situation, which is given in the rest of this note.

3 Perturbation Scheme

As is well-known in equilibrium TFD, the degeneracy in thermal vacua induces the degree of the thermal Bogoliubov transformation, each thermal vacua being specified by each thermal Bogoliubov matrix or vice versa. We attempt below to describe time-dependent thermal situation simply by making thermal Bogoliubov matrix time-dependent [10, 8, 9], keeping thermal vacuum time-independent.

In the unperturbed representation, this implies that bosonic oscillator operators $a_k^\mu(t)$ is written as

$$a_k(t)^\mu = B_k^{-1}(t)^{\mu\nu}\xi_k(t)^\nu \tag{10}$$

$$\xi_k(t)^\mu = \xi_k^\mu \exp[-i\int_{t_0}^t ds\, \omega_k(s)] \tag{11}$$

and time-independent ξ-operators define *time-independent vacuum*,

$$\xi_k|0\rangle = 0, \qquad \langle 0|\xi_k^\dagger = 0. \tag{12}$$

Here thermal doublet notation is used

$$a^\mu = \begin{bmatrix} a \\ \tilde{a}^\dagger \end{bmatrix}^\mu \tag{13}$$

$$\bar{a}^\mu = [a^\dagger \quad -\tilde{a}]^\mu. \tag{14}$$

For the field operator of Schrödinger type we have

$$\psi(x)^\mu = \int \frac{d^3k}{(2\pi)^{(3/2)}} e^{i\vec{k}\cdot\vec{x}} a_k(t)^\mu. \tag{15}$$

We take the following particular form for $B_k(t)$ (so-called in the $\alpha = 1$ representation and linear in $n_k(t)$),

$$B_k(t) = \begin{bmatrix} 1 + n_k(t) & -n_k(t) \\ -1 & 1 \end{bmatrix}, \tag{16}$$

$n_k(t)$ being the number density,

$$n_k(t)\delta(\vec{k} - \vec{l}) = \langle 0|a_k^\dagger(t) a_l(t)|0\rangle. \tag{17}$$

The main reason for the choice in (16) is that the time-ordered Feynman method is usable as in the usual QFT [7, 11, 8].

From (11) follow the equations of motion,

$$[\frac{d}{dt} + i\omega_k(t) - iP_k(t)]a_k(t) = 0, \tag{18}$$

$$[\frac{d}{dt} + i\omega_k(t)]\xi_k(t)^\mu = 0, \tag{19}$$

where

$$P_k(t) \equiv iB_k^{-1}(t)\frac{d}{dt}B_k(t) = i\dot{n}_k(t)T_0 \tag{20}$$

with

$$T_0 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}. \tag{21}$$

We now see the *two* different unperturbative Hamiltonians: one for $\xi_k(t)^\mu$,

$$\hat{H}_0(t) = \int d^3k\, \omega_k(t)\, [a_k^\dagger(t)a_k(t) - \tilde{a}_k^\dagger(t)\tilde{a}_k(t)], \tag{22}$$

and the other for $a_k(t)^\mu$,

$$\hat{H}_Q(t) = \hat{H}_0(t) - \hat{Q}(t) \tag{23}$$

with

$$\hat{Q}(t) = \int d^3k\, \bar{a}_k(t)P_k(t)a_k(t). \tag{24}$$

In the interaction picture $\hat{H}_Q(t)$ is taken to be the unperturbed Hamiltonian, and therefore the perturbed Hamiltonian becomes because of (8)

$$\hat{H}_I(t) = \hat{H}_{int}(t) + \hat{Q}(t), \tag{25}$$

where $\hat{H}_{int}$ consists of nonlinear terms and usual renormalization counter terms. The unperturbed 2×2-matrix propagator for Schrödinger type field $\psi(x)^\mu$ to be used for internal lines in Feynman diagram is, in $(t, \vec{k})$-space,

$$\Delta_k(t,t')^{\mu\nu} = \left[B_k^{-1}(t)\begin{bmatrix} -i\theta(t-t') & 0 \\ 0 & i\theta(t'-t) \end{bmatrix} B_k(t')\right]^{\mu\nu} e^{-i\int_{t'}^{t} ds\,\omega_k(s)}. \tag{26}$$

Note the structure of this propagator that the diagonal propagator is sandwiched between the Bogoliubov matrices.

It can be shown that the full propagator corrected by interactions in $(t,\vec{k})$-space is put into the following form, similar to that of the unperturbed one above,

$$G_k(t,t')^{\mu\nu} = B^{-1}[N_k^*(t',t)]^{\mu\mu'} \begin{bmatrix} -i\theta(t-t')g(t,t':\vec{k}) & 0 \\ 0 & i\theta(t'-t)g^*(t',t:\vec{k}) \end{bmatrix}^{\mu'\nu'} B[N_k(t,t')]^{\nu'\nu} \quad (27)$$

where

$$\begin{aligned} N_k(t,t') &= n_{H,k}(t') + \nu_k(t,t') && (28)\\ n_{H,k}(t)\delta(\vec{k}-\vec{l}) &\equiv \langle 0|\bar{a}_{kH}(t)^1 a_{lH}(t)^1|0\rangle && (29)\\ \nu_k(t,t) &= 0. && (30) \end{aligned}$$

This expression leads us to a simple interpretation that the quasi particle experiences time-dependent thermal effects that the particle number fluctuates around the average observable number $n_{H,k}(t)$ with the fluctuation ν_k.

We can write down a similar form in equilibrium case [8] as in (27), using the spectral representation in $(k_0,\vec{k})$-space [1]. Thus even in equilibrium situations the quasi particle sees the thermal fluctuation through ν.

4 Self-Consistent Renormalization on Self-energy and Entropy Law

In this section we first calculate the self-energy in the approximation explained soon, and then impose a self-consistent renormalization condition. This step of renormalization condition is intended to pick up a unique thermal vacuum self-consistently when interaction effects are taken account of. As will be seen, our renormalization condition indeed derives the kinetic equation, to which the parameter $n_k(t)$ in $B_k(t)$ is subject [8, 9].

Let us calculate the self-energy diagram of two vertices without vertex corrections, taking time-independent unperturbed ω_k. For definiteness we consider a model interaction of $(\psi^\dagger\psi)^2$, thus the corresponding self-energy diagram has three internal lines connecting two vertices. The result [8] is

$$\Sigma(t,t',\vec{k})^{\mu\nu} = \int[dq]\left[B^{-1}[N(t)]\begin{bmatrix}1 & 0\\ 0 & s(t)\end{bmatrix}V(t-t')\begin{bmatrix}s(t') & 0\\ 0 & 1\end{bmatrix}B[N(t')]\right]^{\mu\nu} - \delta(t-t')P_k(t)^{\mu\nu} \quad (31)$$

with

$$\begin{aligned} [dq] &= \prod_{i=1}^{3}\left(\frac{d^3q_i}{(2\pi)^3}\right)(2\pi)^3\delta(\vec{k}-\vec{q}_1-\vec{q}_2+\vec{q}_3) && (32)\\ s(t) &= C\frac{n_{q_1}(t)n_{q_2}(t)(1+n_{q_3}(t))}{N(t)} && (33)\\ N(t) &= \frac{n_{q_1}n_{q_2}(1+n_{q_3})}{(1+n_{q_1})(1+n_{q_2})n_{q_3} - n_{q_1}n_{q_2}(1+n_{q_3})} && (34)\\ W &= \omega_{q_1}+\omega_{q_2}-\omega_{q_3} && (35)\\ V(t-t')^{\mu\nu} &= \begin{bmatrix} -i\theta(t-t') & 0\\ 0 & i\theta(t'-t)\end{bmatrix}^{\mu\nu}\exp[-iW(t-t')], && (36) \end{aligned}$$

where C is a positive number. The last term (see (21)) in (31) is a contribution from $\hat{Q}$ in $\hat{H}_I$.

The question here is how one can separate an on-shell part from the above total self-energy (31), for a renormalization condition is to be imposed on the on-shell part. In this respect, we propose our guiding principle that *a definition of an on-shell part and subsequently a self-consistent renormalization condition should be formulated in terms of not a-operators but of ξ-ones*, because the thermal vacuum is specified by ξ.

Then we note the fact that the diagonal matrix $V(t-t')^{\mu\nu}$ represents the wave propagation of ξ-operators and has the same Fourier transform as the self-energy in usual QFT,

$$V(k_0)^{\mu\nu} \equiv \int d(t-t')\, e^{ik_0(t-t')} V(t-t')^{\mu\nu} = \left[\frac{1}{k_0 - W + i\epsilon\tau_3}\right]^{\mu\nu}, \tag{37}$$

τ_3 being the Pauli matrix. We therefore put k_0 on shell, $k_0 = \omega_k$, in this $V(k_0)^{\mu\nu}$ to get

$$V(t-t')^{\mu\nu}_{on-shell} = V(k_0 = \omega_k)^{\mu\nu}\delta(t-t') \tag{38}$$

and define the on-shell part of (31) by replacing $V(t-t')$ with $V(t-t')_{on-shell}$. The real and imaginary parts of this on-shell self-energy are

$$\Re\Sigma(t,t',\vec{k})^{\mu\nu}_{on-shell} = \delta(t-t')\int [dq]\, s(t)\mathcal{P}\frac{1}{\omega_k - W} B[N(t)]\delta^{\mu\nu} \tag{39}$$

$$\Im\Sigma(t,t',\vec{k})^{\mu\nu}_{on-shell} = -i\delta(t-t')\left[\int [dq]\, s(t)\pi\delta(\omega_k - W)A[N(t)] - \dot{n}_k(t)T_0\right]^{\mu\nu}, \tag{40}$$

respectively, where $A \equiv B^{-1}\tau_3 B$. The real part gives the time-dependent energy shift $\delta\omega_k(t)$. The imaginary part contributes to the Hamiltonian for ξ-operators denoted by $\delta\hat{H}$, adding to $\hat{H}_0$ (not $\hat{H}_Q$). We now require that the Hamiltonian for ξ-operators, $\hat{H}_0(t) + \delta\hat{H}(t)$ should *be diagonal at any t in terms of ξ^μ_k and $\bar{\xi}^\mu_k$*. Thus this is called the *self-consistent renormalization condition.*

The result of this self-consistent renormalization condition is summarized as follows: The condition is satisfied by the following single relation:

$$\dot{n}_k(t) = -2\kappa_k(t)n_k(t) + 2\pi C\int [dq]\,\delta(\omega_k - W)n_{q_1}(t)n_{q_2}(t)[1 + n_{q_3}(t)], \tag{41}$$

with the time-dependent dissipative coefficient

$$\kappa_k(t) \equiv \int [dq]\, s(t)\pi\delta(\omega_k - W). \tag{42}$$

It is remarkable that the single equation is sufficient to fulfill the requirement on four components (2×2-matrix).

Equation (41) is the kinetic equation for the Bogoliubov number density $n_k(t)$ in the two vertex approximation without vertex correction. In this approximation *the kinetic equation has the structure of the Boltzmann equation*, as is expected.

From this kinetic (Boltzmann) equation we can prove [8] that the entropy increases in time: Consider the entropy density given by

$$\mathcal{S}(t) = \int d^3k\,\{[1 + n_k(t)]\ln[1 + n_k(t)] - n_k(t)\ln n_k(t)\}, \tag{43}$$

and use the fact that $n_k(t)$ is a solution of (41), then it turns out that

$$\dot{\mathcal{S}}(t) = \frac{(2\pi)^7 C}{4}\int \prod_{i=0}^{3}\left(\frac{d^3q_i}{(2\pi)^3}\right)\delta^4(q_0 - q_1 - q_2 + q_3)(X - Y)\ln\left(\frac{X}{Y}\right) \geq 0 \tag{44}$$

where

$$
\begin{aligned}
X &= n_{q_0} n_{q_3} (1+n_{q_1})(1+n_{q_2}) && (45)\\
Y &= (1+n_{q_0})(1+n_{q_3}) n_{q_1} n_{q_2}, && (46)
\end{aligned}
$$

because of $(X-Y)\ln(X/Y) \geq 0$ for any positive X and Y. This is *how the second law of thermodynamics follows from TFD.*

5 Future Study

Finally we just list the two subjects in future study along the line presented in this paper.

First of all extension of the analysis in this section to the self-energy of higher order (with vertex correction) is of vital importance, it will show how interaction effects modify the Boltzmann equation and possibly the definition of entropy.

Secondly, the present spatially homogeneous time-dependent formulation should be extended to time-space dependent one. Practically almost all the nonequilibrium in nature take place in spatially inhomogeneous situation. The study in this direction is at preliminary stage [12] and is being developed. So far introduction of momentum mixing thermal Bogoliubov transformation [13] is useful. The formulation is to be applied to heat conduction.

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