Unstable Periodic Orbits and Chaos

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Investigation of ensemble of unstable periodic orbits in dissipative chaos has long been an important subject in constructing a statistical theory on chaos, because a measure zero set of this ensemble reconstruct a chaotic attractor, provided that a single chaotic orbit wanders from one unstable periodic orbit to another. This wandering phenomenon is meaningful, if each unstable periodic orbit has a stable manifold along which a chaotic orbit becomes close.

We make an attmpt to construct semi-empirical statistical theories for statistically stationally quantities such as long time averages in 3D ODE systems and in 1D difference systems.

In the first part, we describe a theory in 3D ODE systems based on an assumption such that

$$\mathbf{X}(t) = \mathbf{X}_0(\alpha) + x_1\mathbf{u}_1(\alpha) + x_2\mathbf{u}_2(\alpha), \tag{1}$$

where $\mathbf{X}_0, \mathbf{u}_1$ and $\mathbf{u}_2$ represent a unstable periodic trajectory and corresponding Floquet engenvectors which are all 2π- periodic with respect to a phase denoted by α. Equation (1) represents a transformation from the old variables (X_1, X_2, X_2) to the new variables the phase α and two amplitudes (x_1, x_2). This transformation to a rotating coordinate enables us to rewrite a statistically stationally quantity by employing an invariant density defined on a Poincaré cross section. The derived formula must be verified by applying this formula to a concrete example, because the formula has some umbiguities such as a difficulty in determining the invariant density and a difficulty of finding a region of the phase space which can be described by a phase of an unstable limit cycle. The second difficulty mentioned above is closely related to a phase singularity phenomenon which indicates the fact that some set in the phase space can not be described by the phase of the limit cycle, as is known in 2D ODE systems, so that investigation of this phenomenon of the phase singularity in a disspative chaos will be reported elsewhere separately.

We adopt a famous Lorenz system as a representative 3D ODE system. We adopt a fixed point approximation which implies that the deviation vectors are always negligible. The results show that the long time averages of $< Z >$ and $< Z^2 >$ are in good agreement with the directly calculated values. See the details of calculation of statistical weights imposed on the imbedded unstable limit cycles in Ref.1.

In the second part of our theory, we present a semi-empirical formula for evaluating invariant desities in 1D difference system in a standpoint similar to the case of 3D ODE systems.

In this case, the situation becomes considerably simplified. We find that a formula for an invariant density is exactly an invariant solution of a corresponding Frobenius-Perron equation provided that an artificially constructed map which is called a modified map satisfies a correspoding Frobenius-Perron equation.

We find that, in the case of fixed point approximation, long time averages may be well approximated by this formula by considering several maps such as a tent map, an r-adic transformation, and a logistic map. In the case of step-wise approximation, we find that a Markovian property is realized for a certain class of unimordal maps which are not of Markovian type. In fact we find that discretized probabilities satisfy a Markov chain. This step-wise approximation enables us to evaluate an invariant density which may possesses several discontinuous points, by applying Frobenius-Perron operator several times. This step-wise approximation can be well-employed for a tent map, an r-adic transformation, and a logistic map.

We make several comments here. In 3D ODE systems. the derived formula has some umbiguities which must be clarified. For one point, we take up the problem of finding the well-defined Poincaré cross section. In other words, each imbedded unstable limit cycle has a different phase-zero plane, because the corresponding Floquet eigenvectors are different from each other in general. For another point, the problem is to find a phase singularity set, as was mentioned.

We have derived the formulas for 3D ODE system and 1D difference system. However, the derived formulas are mutially independent. Therefore the formula for 3D ODE system is not possible to be derived from the one for 1D difference system, e.g., by introducing a small time increment such as an Euler difference scheme, and vice versa. The reason is as follows; As in the case of a logistic map, chaotic behavior may be found if the time increment becomes finite in the Euler difference scheme, whereas a temporally continuous system exhibit a non-chaotic behavior. Therefore existence of chaotic behavior must be treated seperately in 3D ODE system and in 1D difference system. Instead, the derived formulas have been constructed in the same sense such that a single chaotic orbit or trajectory wanders in the measure zero set of unstable periodic orbits or trajectories. Therefore the formulas exhibit a considerably similar structure in both 3D ODE system and 1D difference system. And essential correspondence between the formulas in 3D ODE system and 1D difference system must be be clarified when we are confronted with the case such that both the temporally continuous system and the temporally discretized system which is derived by making the time increment finite exhibit chaotic behaviors.

References

[1] Shinji Koga, Prog. Theor. Phys. **76**, 335, (1986)

[2] Shinji Koga, Prog. Theor. Phys. **80**, 216, (1988)