Chaotic Force in Brownian Motion

Kokushikan Univ.

Toshihiro Shimizu

A Brownian particle, which interacts with molecules of the surrounding medium, receives the force with irregular magnitude at irregular intervals from the surrounding molecules. The irregular nature of the force was characterized as the random noise in the original Langevin equation. If we observe it from a purely mechanical point of view, however, we may find that the magnitude of the force and the time interval between successive collisions are deterministic, although they look stochastic at first sight. This situation can be elucidated also from the derivation of a Langevin-type equation from the Hamiltonian equation in generalized Brownian motion theory.

In this article we discuss a Langevin-type equation with a deterministic random force, which is called chaotic force The main interest is to investigate (i) what kind of stationary state exists, (ii) how the system relaxes to the statinary state, (iii) what kind of the fluctuation-dissipation theorem exists and (iv) how relaxation processes depend on a bifurcation parameter, if the chaotic force is dependent on a bifurcation parameter.

1. Model

To take into account of this deterministic nature of the random force we employed a chaotic sequence of iterates of some maps as a model of a deterministic random force [1].

Let us denote the position of the Brownian particle by $x(t)$ and let us assume that the velocity of the Browninan particle changes from $y_n/\sqrt{\tau}$ to $y_{n+1}/\sqrt{\tau}$ at time $t=n\tau$ due to a collision and the velocity remains constant between successive collsions, if the damping is neglected.

Then we have a Langevin-type equation

$$x(t) = -\gamma x(t) + f(t), \qquad (1)$$

where γ is a damping constant. The force $f(t)$ is assumed to be constant in the time interval τ

$$f(t) = \frac{1}{\sqrt{\tau}} y_{n+1} \qquad \text{for } n\tau \leqq t < (n+1)\tau , \quad (2)$$

where y_{n+1} is the (n+1)th iterate of a map $F(y)$

$$y_{n+1} = F(y_n). \qquad (3)$$

Here $F(y)$ may depend on a bifurcation parameter.

In (2) we have chosen $1/\sqrt{\tau}$ as the magnitude of the force to get a finite diffusion constant in the limit of small time interval, as will be shown later.

If the initial values $x(0) = x_0$ and y_0 are given, the solution of (1) is uniquely determined,

$$x(t, x_0, y_0) = e^{-\gamma(t-n\tau)} x_n(x_0, y_0) + \frac{1- e^{-\gamma(t-n\tau)}}{\gamma\sqrt{\tau}} y_{n+1}$$

$$\text{for } n\tau < t \leqq (n+1)\tau , \qquad (4)$$

where $x_n(x_0, y_0) = x(n\tau, x_0, y_0)$. If we observe the system stroboscopically at time interval τ, the position at time $t = (n+1)\tau$ is

$$x_{n+1}(x_0, y_0) = a\, x_n(x_0, y_0) + b\sqrt{\tau}\, y_{n+1}(y_0)$$

$$= a^{n+1} x_0 + b\sqrt{\tau} \sum_{m=0}^{n} a^m\, y_{n+1-m}(y_0), \qquad (5)$$

where

$$a = e^{-\gamma\tau} , \quad b = (1-a)/\gamma\tau . \qquad (6)$$

2. Stationary distribution

Let us assume that the initial position x_0 is definite, but the surroundng molecules have different initial values so that the initial distribution of y_0 is given by $\rho_0(y_0)$.
The initial distribution developes according to the Frobenius-Perron equation. The distribution $\rho_{n+1}(y)$ of y_{n+1} at $t = (n+1)\tau$ is given by

$$\rho_{n+1}(y) = \int dy' \, \delta(y-F(y'))\rho_n(y'). \tag{7}$$

If the mapping function $F(y)$ has an invariant density $\rho_*(y)$, the arbitrary initial distribution tends to the invariant density $\rho_*(y)$. Since we are interested in the long time behaviour, we assume that the initial distribution of y_0 is given by $\rho_*(y_0)$. According to (5), the average position of the particle over $\rho_*(y_0)$ at time $t = (n+1)\tau$ is given by

$$\langle x_{n+1}(x_0,y_0)\rangle = a^{n+1} x_0 + b\sqrt{\tau} \sum_{m=0}^{n} a^m \langle y_0\rangle. \tag{8}$$

Here we have used the stationary property $\langle y_n(y_0)\rangle = \langle y_0\rangle$ and x_0 was assumed to be statistically independent of y_0. Let us denote the deviation of x_n and y_n from the average value (8) by $x_n(y_0) = x_n(x_0,y_0) - \langle x_n(x_0,y_0)\rangle$ and $y_n = y_n - \langle y_0\rangle$, respectively. It should be noted here that x_n depends on y_0 only.

For later convenience we introduce the generating function

$$\psi_{n+1}(x,y) = \int dy_0 \, \rho_*(y_0) \, \delta(x-x_{n+1}(y_0)) \delta(y-y_{n+2}(y_0)), \tag{9}$$

which satisfies the recurrence relation

$$\psi_{n+1}(x,y) = \iint dx \, dy \, \delta(x-(ax+b\sqrt{\tau}y))\psi_n(x,y). \tag{10}$$

Then the distribution function for the position is defined by

$$W_{n+1}(x) = \int dy \, \psi_{n+1}(x,y) \tag{11}$$

with the initial distribution $W_0(x) = \delta(x)$. Noting the property of the invariant density, we get

$$\int dx \, \psi_{n+1}(x,y) = \rho_*(y). \tag{12}$$

However, (10) and (12) do not mean $\psi_{n+1}(x,y)$ can be factorized as $W_{n+1}(x)\rho_*(y)$ and rather it has a correlation term $g_{n+1}(x,y)$:

$$\psi_{n+1}(x,y) = W_{n+1}(x)\rho_*(y) + g_{n+1}(x,y). \tag{13}$$

As will be shown later, the distribution $W_n(x)$ and the generating function $\psi_n(x,y)$ tend to sationary ones

$$\lim_{n \to \infty} W_n(x) = W_*(x), \tag{14}$$

$$\lim_{n \to \infty} \psi_n(x,y) = \psi_*(x,y).$$

In the following limiting cases the explicit form of $W_*(x)$ can be easily calculated.

Case(i): $\gamma \tau \gg 1$

If τ is much longer than the decay time $\tau_d = 1/\gamma$, a Brownian particle, which starts with x_0 and y_0, can follow the change of the force $f(t)$ very quickly and the deviation x in (5) reaches $y_{n+1}/\gamma\sqrt{\tau}$ very rapidly during each time interval,

$$x_{n+1}(y_0) = y_{n+1}/\gamma\sqrt{\tau}, \tag{15}$$

because $a \ll 1$. Substituting this into (11), we get

$$W_{n+1}(x) = \gamma\sqrt{\tau} \rho_*(\gamma\sqrt{\tau} x + \langle y_0 \rangle). \tag{16}$$

In this case the distribution function $W_n(x)$ converges to the stationary distribution (16) very rapidly, whose shape has the same form as that of the invariant density $\rho_*(y)$.

Let us next consider the small τ limit. It should be noted here that τ is very small but not zero. We must treat this case very carefully.

Case (ii): small τ limit and $\gamma \tau \ll 1$

If $\tau \ll \tau_d$, the position x_{n+1} depends on those at previous times, because the position decays very slowly. If we expand a and b in τ, and retain the lowest order term in (10), we can derive a Fokker-Planck type equation

$$\frac{W_{n+1}(x) - W_n(x)}{\tau} = \frac{\partial}{\partial x}(\gamma x W_n(x)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\langle y_0^2 \rangle W_n(x))$$

$$+ \frac{\partial^2}{\partial x^2}[\sum_{m=1}^{n} \langle y_m(y_0)y_0 \rangle W_{n-m}(x)]. \tag{17}$$

If the correlation $\langle y_n(y_0)y_0 \rangle$ decays more rapidly than $W_{n-m}(x)$, we can introduce the Markovian approximation

$$\sum_{m=1}^{n} <y_m(y_0)y_0> W_{n-m}(\mathbf{x}) = \sum_{m=1}^{n} <y_m(y_0)y_0> W_n(\mathbf{x})$$

$$\cong \sum_{m=1}^{\infty} <y_m(y_0)y_0> W_n(\mathbf{x}) . \qquad (18)$$

In this case (17) can be represented as a Fokker-Planck equation with the diffusion constant D

$$D = \frac{1}{2}<y_0^2> + \sum_{m=1}^{\infty} <y_m(y_0)y_0> . \qquad (19)$$

Then the stationary distribution $W_*(x)$ has a Gaussian form

$$W_*(\mathbf{x}) = \sqrt{\frac{\gamma}{2\pi D}} \quad \exp\left(- \frac{\gamma x^2}{2D} \right) . \qquad (20)$$

In this case the shape of $W_*(x)$ does not depend on the detail of the invariant density.

By taking the logistic map as an example of $F(y)$, we investigated the bifurcation parameter dependence of relaxation processes.

3. Correlation in the stationary state

Let us define the correlation functions in the stationary state $\Psi_*(x,y)$ by

$$\ll x\, x_n(x,y) \gg = \iint dxdy\, x\, x_n(x,y)\psi_*(x,y),$$

$$\ll y\, y_n(y) \gg = \iint dxdy\, y\, y_n(y)\psi_*(x,y). \qquad (21)$$

The Fourier transformations are defined by

$$\Phi_x(\omega) = \tau \sum_{n=-\infty}^{\infty} e^{-in\omega\tau} \ll x\, x_n(x,y) \gg ,$$

$$\Phi_f(\omega) = \tau \sum_{n=-\infty}^{\infty} e^{-in\omega\tau} \ll \frac{y}{\sqrt{\tau}} \frac{y_n(y)}{\sqrt{\tau}} \gg . \qquad (22)$$

If $<y\, y_n(y)>$ is δ-correlated, we get

$$\ll x^2 \gg = 2\, b^2 \tau D/(1-a^2),$$

$$\Phi_x(\omega) = 2(b\tau)^2 D/|1-\exp((-\gamma+i\omega)\tau)|^2, \qquad (23)$$

$$\Phi_f(\omega) = 2D.$$

In the small τ limit (23) reduces to

$$\ll x^2 \gg = D/\gamma ,$$

$$\Phi_x(\omega) = 2D/(\gamma^2 + \omega^2), \qquad (24)$$

$$\Phi_f(\omega) = 2D.$$

These relations coincide exactly with the results of the usual Brownian motion theory, which are obtained by assuming that f(t) is a white-Gaussian process.

4. Change from the shape of the invariant density to the Gaussian form

As will be shown later, the recurrence relation (5) has a fractal structure. If τ is decreased or a is increased from zero to one, the fractal structure will be enlaged. If $\gamma \tau \gg 1$, the lowest order term of (5) is given by

$$x_{n+1} = b\sqrt{\tau} \; y_{n+1}$$
$$x_n = b\sqrt{\tau} \; y_n. \tag{25}$$

Therefore the recurrence relation (x_{n+1}, x_n) has the same form as that of (y_{n+1}, y_n).

If τ is slightly decreased, we must take into account of the first order term in a, and we get

$$x_{n+1} = b\sqrt{\tau} \; (y_{n+1} + a \; y_n),$$
$$x_n = b\sqrt{\tau} \; (y_n + a \; y_{n-1}). \tag{26}$$

It was shown that the recurrence relation (26) exibits doubling of (25), if F(y) is a non-invertible function with two-to-one correspondence. In a similar way we get for the recurrence relation up to the second order in a,

$$x_{n+1} = b\sqrt{\tau} \; (y_{n+1} + a \; y_n + a^2 y_{n-1}),$$
$$x_n = b\sqrt{\tau} \; (y_n + a \; y_{n-1} + a^2 y_{n-2}). \tag{27}$$

The recurrence relation (27) exhibits doubling of (26). This means that the recurrence relation (5) has the fractal structure. The fractal structure is not visible for small a. This fractal structure is similar to the Henon map. If τ is decreased, however, the fractal structure is enlarged automatically. If τ approaches zero, the recurrence relation tends to a diagonal strip with very small width, which does not depend on the detail of the map F(y). The senario of this change has universality [3].

5. Chaotic time interval

So far the time interval was assumed to be constant. This

assumption is, in general, not appropriate. The time interval also changes chaotically as the magnitude of the force. The force f(t) is generalized to

$$f(t) = \frac{1}{\sqrt{T}} f_{n+1} \quad \text{for } t_n \leq t < t_{n+1} , \tag{28}$$

where f_{n+1} is a function of $\{y_n\}$ and a time series $\{t_n\}$ is defined by

$$t_n = T \sum_{m=0}^{n-1} \tau_m \quad (0 \leq \tau_m \leq 1). \tag{29}$$

Here T is the magnitude of the time interval and $\{\tau_n\}$ is a chaotic series, which is generated by another mapping function

$$\tau_{n+1} = G(\tau_n). \tag{30}$$

The effect of chaotic time interval $\{\tau_n\}$ and chaotic magnitude $\{y_n\}$ on the stationary distribution of x was invetigated in the follwing cases:(i) y_n is chaotic and τ_n is constant, (ii) y_n is regular and τ_n is chaotic, (iii) y_n and τ_n are chaotic and (iv) y_n is strongly correlated with τ_n . The theoretical results were shown to be in a good agreement with numerical ones [2].

References

[1] T. Shimizu, Physica A 164 (1990) 123.

[2] T. Shimizu, Physica A 195 (1993) 113.

[3] T. Shimizu, Physica A 196 (1993) 42.