Quantum Calogero-Moser Model and its Algebraic Structures

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The quantum Calogero-Moser model is a one-dimensional many-body system with pairwise inverse square interactions. The Hamiltonian in the quantum theory is

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2}g \sum_{\substack{j,k=1 \\ j\neq k}}^{N} \frac{1}{(x_j - x_k)^2}, \tag{1}$$

$$p_j = -i\frac{\partial}{\partial x_j}, \tag{2}$$

where N and g mean the number of particles and the coupling constant, respectively. Note that we set the Planck constant $\hbar$ and the mass m to be unity.

Let us introduce the quantum inverse scattering method. Let L and M be operator-valued $N \times N$ matrices. We choose them such that the Lax equation

$$\begin{aligned} \frac{d}{dt}L_{jk} &\equiv i[H, L_{jk}] \\ &= i[L, M]_{jk} = i\sum_{l}(L_{jl}M_{lk} - M_{jl}L_{lk}), \end{aligned} \tag{3}$$

is equivalent to the Heisenberg equation of motion given by the Hamiltonian H. We refer to the L and M operators as a Lax pair.

For the quantum Calogero-Moser model, the Lax pair is found to be [1,2]

$$L_{jk} = p_j\delta_{jk} + ia(1-\delta_{jk})\frac{1}{x_j - x_k}, \tag{4}$$

$$M_{jk} = -a(1-\delta_{jk})\frac{1}{(x_j - x_k)^2} + a\delta_{jk}\sum_{l\neq j}\frac{1}{(x_j - x_l)^2}, \tag{5}$$

where a is a constant related to the coupling constant g by

$$g = a^2 - a. \tag{6}$$

With $\hbar$ written explicitly, the above relation reads $g = a^2 - \hbar a$, which becomes $g = a^2$ in the classical limit.

From the knowledge of the Lax equation (3) and the Lax pair, (4) and (5), conserved operators for the quantum Calogero-Moser model are constructed in the following way [1,2,3, 4,5,6]. The Lax equation yields

$$[H, (L^n)_{jk}] = [L^n, M]_{jk}, \qquad n = 1, 2, \cdots. \tag{7}$$

Since the M operator satisfies

$$\sum_{j=1}^{N} M_{jk} = \sum_{j=1}^{N} M_{kj} = 0, \tag{8}$$

we readily see that

$$[H, \sum_{j,k=1}^{N} (L^n)_{jk}] = \sum_{j,k=1}^{N} [L^n, M]_{jk} = 0. \tag{9}$$

Namely, a set of conserved operators $\{I_n\}$ is given by

$$I_n = \frac{1}{n} \sum_{j,k=1}^{N} (L^n)_{jk}, \tag{10}$$

where the coefficient $1/n$ in (10) is included just for convenience. This formula suggests that $\mathrm{Tr} L^n = \sum_j (L^n)_{jj}$, which is conserved in the classical case, is not necessarily conserved operator in the quantum theory. A condition (8) on the M operator is the key to our derivation of the formula (10). We call it "sum-to-zero" condition.

The explicit forms of the first four conserved operators are

$$\begin{aligned}
I_1 &= \sum_{j,k} L_{jk} = \sum_j p_j, \quad I_2 = \frac{1}{2}\sum_{j,k} (L^2)_{jk} \equiv H, \\
I_3 &= \frac{1}{3}\sum_{j,k} (L^3)_{jk} \\
&= \frac{1}{3}\sum_j p_j^3 + \frac{1}{3} g \sum_{j,k}{}' \left\{ p_j \frac{1}{x_{jk}^2} + \frac{1}{x_{jk}} p_k \frac{1}{x_{jk}} + \frac{1}{x_{jk}^2} p_j \right\}, \\
I_4 &= \frac{1}{4}\sum_{j,k} (L^4)_{jk} \\
&= \frac{1}{4}\sum_j p_j^4 + \frac{1}{4} g \sum_{j,k}{}' \left\{ p_j^2 \frac{1}{x_{jk}^2} + p_j \frac{1}{x_{jk}} p_k \frac{1}{x_{jk}} + p_j \frac{1}{x_{jk}^2} p_j + \frac{1}{x_{jk}} p_k^2 \frac{1}{x_{jk}} \right. \\
&\quad \left. + \frac{1}{x_{jk}} p_k \frac{1}{x_{jk}} p_j + \frac{1}{x_{jk}^2} p_j^2 \right\} + \frac{1}{2} g^2 \sum_{j,k,l}{}' \frac{1}{x_{jk}^2} \frac{1}{x_{kl}^2} + \frac{1}{4}(g^2 - g) \sum_{j,k}{}' \frac{1}{x_{jk}^4}.
\end{aligned} \tag{11}$$

Here and sometimes hereafter, we use $x_{jk} = x_j - x_k$ etc, and $\sum'$ means that all the indices which appear in the summand do not coincide. We can see that $\{I_n\}$ for $n = 1, 2, \cdots, N$ are functionally independent. Comparing with the classical case, we observe the coefficient of the last term in I_4, which is $1/4 \cdot (g^2 - \hbar^2 g)$ with the Planck constant explicitly written, contains a quantum correction.

Moreover, we introduce another series of operators, which we call boost operators [7], made from the Lax operator:

$$J_n = \frac{1}{2i} \left[\sum_j x_j^2, I_{n+1} \right] = \frac{1}{2i(n+1)} \left[\sum_j x_j^2, \sum_{l,m} \left(L^{n+1} \right)_{lm} \right]. \tag{12}$$

The first of them is $J_0 = \sum_j x_j$ which satisfies a commutation relation:

$$[J_0, I_n] = i(n-1)I_{n-1}. \tag{13}$$

Motivated by the existence of the conserved operators $\{I_n\}$, we assume that I_n for $n = 1, 2, \cdots$ gives the Lax equation with $M^{(n)}$ operator satisfying the sum-to-zero condition:

$$[I_n, L] = [L, M^{(n)}], \tag{14}$$

$$\sum_j M_{jk}^{(n)} = \sum_j M_{kj}^{(n)} = 0. \tag{15}$$

The Lax equation (3) is nothing but (14) for $n = 2$. The condition (15) suggests that $M_{jk}^{(n)}$ has a form;

$$M_{jk}^{(n)} = (1-\delta_{jk})N_{jk}^{(n)} - \delta_{jk}\sum_{l\neq j} N_{jl}^{(n)}. \tag{16}$$

Using the conserved operators (11), we obtain explicit expressions for the first four of $N_{jk}^{(n)}, j \neq k$:

$$\begin{aligned}
N_{jk}^{(1)} &= 0, \qquad N_{jk}^{(2)} = -\frac{a}{x_{jk}^2}, \\
N_{jk}^{(3)} &= -\frac{a}{x_{jk}^2}(p_j + p_k) - ia^2 \sum{}' \frac{1}{x_{jl}x_{lk}x_{jk}}, \\
N_{jk}^{(4)} &= -\frac{a}{x_{jk}^2}(p_j^2 + p_j p_k + p_k^2) - ia\frac{1}{x_{jk}^3}(p_j - p_k) \\
&\quad -ia^2 \sum_l{}' \frac{1}{x_{jl}x_{lk}x_{jk}}(p_j + p_l + p_k) - (a^3 - 3a)\frac{1}{x_{jk}^4} \\
&\quad -a^2 \sum_l{}' \{\frac{1}{x_{jl}x_{lk}x_{jk}^2} - \frac{1}{x_{jl}^2 x_{lk}^2}\} - a^3 \sum_l{}' \{\frac{1}{x_{jl}^2 x_{jk}^2} + \frac{1}{x_{lk}^2 x_{jk}^2}\} \\
&\quad +\frac{1}{2}a^3 \sum_{l,m}{}' \frac{1}{x_{jl}x_{lk}x_{jm}x_{mk}}.
\end{aligned} \tag{17}$$

Using (17), we can show that the following relation is satisfied up to $n = 4$:

$$[I_n, X] = [X, M^{(n)}] - i(L^{n-1}), \tag{18}$$

$$X_{jk} = x_j \delta_{jk}. \tag{19}$$

Shortly later, we shall prove the existence of $M^{(n)}$ for $n \geq 5$. We refer to (18) as "additional" relation. Here and hereafter, we often use matrix notation as is seen in (18). There should be no confusion about it.

From the above knowledge, we get a useful formula:

$$[J_2, I_n] = i(n+1)I_{n+1}. \tag{20}$$

Note that the J_2 operator raises the conserved operators.

Now, we are ready to prove the Lax equation (14) with the sum-to-zero condition (15) and the additional relation (18).

First, we prove the Lax equation (14) and (15) by mathematical induction. Let (14) and (15) be satisfied up to n. Because of the relation (20), we have

$$[[J_2, I_n], L] = i(n+1)[I_{n+1}, L]. \tag{21}$$

Using the Jacobi's identity repeatedly, we can reduce (21) to

$$i(n+1)[I_{n+1}, L] = -\left[L, \left[M^{(n)}, J_2\right]\right] - \frac{1}{2i}\left[L, \left[O, M^{(n)} + I_n\right]\right], \tag{22}$$

where an operator O is defined by

$$O \equiv \left[M^{(3)}, \sum_j x_j^2\right]. \tag{23}$$

Note that the operator-valued matrix O does satisfy the sum-to-zero condition.

In order that (22) is the Lax equation for I_{n+1}, $M^{(n+1)}$ is identified as

$$M^{(n+1)} = \frac{i}{n+1}\left\{[M^{(n)}, J_2] + \frac{1}{2i}[O, M^{(n)} + I_n]\right\}. \tag{24}$$

Since $M^{(n)}$ and O satisfy the sum-to-zero condition, the operator $M^{(n+1)}$ satisfies the sum-to-zero condition as well. Equation (24) is a recursion formula for the $M^{(n)}$ operator.

Next, we prove the additional relation (18) again by mathematical induction. Let the equation (18) be satisfied up to n. Using (20) and applying the Jacobi's identity, we can show the following realation:

$$\begin{aligned}[I_{n+1}, X] &= \frac{1}{i(n+1)}[[J_2, I_n], X] \\ &= \frac{i}{n+1}\left\{\left[X, \left[M^{(n)}, J_2\right]\right] + \frac{1}{2i}\left[X, \left[O, I_n + M^{(n)}\right]\right]\right\} - iL^n \\ &= \left[X, M^{(n+1)}\right] - iL^n. \end{aligned} \tag{25}$$

This is (18) for $n+1$. Then, the proof is completed.

We are in position to calculate commutation relations among $\{I_n\}$ and $\{J_n\}$. We first show that commutation relations among $\{I_n\}$ vanish, which means that the conserved operators are in involution. Since (14) holds for all n, we have

$$\begin{aligned}[I_n, I_m] &= \frac{1}{m}\sum_{j,k}[I_n, (L^m)_{jk}] \\ &= \frac{1}{m}\sum_{j,k}\left[L^m, M^{(n)}\right]_{jk}. \end{aligned} \tag{26}$$

Applying the sum-to-zero condition (15) to (26), we get

$$[I_n, I_m] = 0. \tag{27}$$

For the N-particle system, this shows that N independent conserved operators $\{I_n\}$ are involutive and thus the integrability of the quantum Calogero-Moser model is proved.

Commutation relations between I_m and J_n are calculated as follows. From the definition of J_n operator, we have

$$
\begin{aligned}
[J_n, I_m] &= \frac{1}{2i}\left[\left[\sum_j x_j^2, I_{n+1}\right], I_m\right] \\
&= \frac{1}{2i}\left[\left[\sum_j x_j^2, \frac{1}{n+1}\sum_{k,l}(L^{n+1})_{kl}\right], I_m\right] \\
&= \frac{1}{n+1}\sum_{k,l}\left[\left\{XL^n + LXL^{n-1} + \cdots + L^nX\right\}_{kl}, I_m\right].
\end{aligned} \tag{28}
$$

Applying the additional relation (18), we get

$$
\begin{aligned}
[J_n, I_m] &= \frac{1}{n+1}\sum_{r=0}^{n}\sum_{k,l}\left\{\left[M^{(m)}, L^r\right]XL^{n-r} + L^r\left[M^{(m)}, X\right]L^{n-r}\right. \\
&\quad \left. + iL^rL^{m-1}L^{n-r} + L^rX\left[M^{(m)}, L^{n-r}\right]\right\}_{kl} \\
&= \sum_{k,l}\left\{iL^{n+m-1} + \frac{1}{n+1}\left[M^{(m)}, XL^n + LXL^{n-1} + \cdots + L^nX\right]\right\}_{kl} \\
&= i(n+m-1)I_{n+m-1}.
\end{aligned} \tag{29}
$$

Finally, we calculate commutation relations among $\{J_n\}$. This is a straightforward, but rather laborious task. By definition and using the Jacobi's identity, we have

$$
\begin{aligned}
[J_n, J_m] &= \left[\left[\frac{1}{2i}\sum_j x_j^2, I_{n+1}\right], J_m\right] \\
&= -\frac{1}{2i}\left\{\left[[I_{n+1}, J_m], \sum_j x_j^2\right] + \left[\left[J_m, \sum_j x_j^2\right], I_{n+1}\right]\right\}.
\end{aligned} \tag{30}
$$

By using (29), the first term in the r.h.s. of (30) is calculated as

$$
\begin{aligned}
-\frac{1}{2i}[[I_{n+1}, J_m], \sum_j x_j^2] &= -\frac{1}{2i}[-i(n+m)I_{n+m}, \sum_j x_j^2] \\
&= -i(n+m)J_{n+m-1}.
\end{aligned} \tag{31}
$$

In addition, we can prove that

$$
\begin{aligned}
\left[J_m, \sum_j x_j^2\right] &= \frac{1}{2}\sum_{k,l}\left[\{XL^m + L^mX\}_{kl}, \sum_j x_j^2\right] \\
&= -i\sum_{k,l}\sum_{r=1}^{m}\left\{XL^{m-r}XL^{r-1} + L^{r-1}XL^{m-r}X\right\}_{kl}.
\end{aligned} \tag{32}
$$

Substituting (32) into the second term in the r.h.s. of (30) is calculated as

$$
-\frac{1}{2i}\left[\left[J_m, \sum_j x_j^2\right], I_{n+1}\right]
$$

$$= \frac{1}{2}\sum_{j,l}\sum_{r=1}^{m}\left[\{XL^{m-r}XL^{r-1}+L^{r-1}XL^{m-r}X\}_{jl}, I_{n+1}\right]$$
$$= \frac{1}{2}\sum_{j,l}\sum_{r=1}^{m}\{iL^{n+m-r}XL^{r-1}+iXL^{n+m-1}+iL^{n+m-1}X+iL^{r-1}XL^{n+m-r}\}_{jl}$$
$$= \frac{i}{2}\sum_{j,l}2m\{XL^{n+m-1}+L^{n+m-1}X\}_{jl} = 2imJ_{n+m-1}. \tag{33}$$

Thus combining (31) and (33), we get

$$[J_n, J_m] = i(m-n)J_{n+m-1}. \tag{34}$$

We have proved the commutation relations, (27), (29) and (34) among $\{I_n\}$ and $\{J_n\}$). It is interesting to notice that the relations constitute a closed algebra.

After slight modifications of the definitions,

$$A_n \equiv nI_n, \quad B_n \equiv iJ_{n+1}, \tag{35}$$

the commutation relations among them become

$$[A_n, A_m] = 0, \quad [A_m, B_n] = mA_{m+n}, \quad [B_m, B_n] = (m-n)B_{n+m}. \tag{36}$$

These relations are the U(1)-current algebra, which has been extensively studied related to the conformal field theory and anyon physics.

In conclusion, we have proved that the quantum Calogero-Moser model is integrable and is a realization of the U(1)-current algebra.

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