The Wave Function and the Minimum Uncertainty Function of the Time Dependent Harmonic Oscillator

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ABSTRACT

The time dependent harmonic oscillator is solved explicitly for quantum mechanics by the operator method with an auxiliary condition as the classical solution. Two classical invariant quantities which determine whether or not the system is bound are derived by the classical equation of motion. We obtain the invariant operator from one classical invariant quantity. Its eigenfunction is related to the solution of Schrödinger equation of the system and its eigenvalue is related to another classical quantity. The wave function is evaluated exactly by the eigenfunction of the invariant operator but it is not the eigenfunction of the Hamiltonian of the system. The uncertainty which calculates with the wave function is not a minimum one. We will confirm that the function which holds minimum uncertainty is a eigenfunction of the Hamiltonian.

Over the past a few decades, many works using the several methods have been devoted to the study of the time dependent harmonic oscillator.[1-5]

In a previous paper[6], we also treated this system using the path integral method. Here, using the operator method we retreat the quantum mechanical time dependent harmonic oscillator and discuss the minimum uncertainty function of it. We began by classically considering of the system.

The Hamiltonian of the system is

$$H = \frac{p^2}{2M} + \frac{1}{2}M\omega(t)^2 q^2 \tag{1}$$

where q and p are canonical variables and $\omega(t)^2$ is a real positive function. From Hamilton's equation of motion, we obtain the classical equation of motion:

$$\ddot{q} + \omega(t)^2 q = 0 \tag{2}$$

PACS No. 03.65.Ge.

Although the differential Eq. (2) can not be easily solved, it can be expressed in the form

$$q = \rho(t)e^{i\gamma(t)} \tag{3}$$

where the function $\rho(t)$ and $\gamma(t)$ must be determined from Eq. (2); these are real and depend only on time. Substitution of Eq. (3) into Eq. (2) gives the real and imaginary parts of this equation as

$$\ddot{\rho} - \rho\dot{\gamma}^2 + \omega(t)^2\rho = 0 \tag{4}$$

$$\rho\ddot{\gamma} + 2\dot{\rho}\dot{\gamma} = 0 \tag{5}$$

One invariant quantity can be found from Eq. (5) in the form

$$\Omega = M\rho^2\dot{\gamma} \tag{6}$$

with an auxiliary condition given by the classical solution.

Substituting Eq. (6) into Eq. (4), it becomes

$$\ddot{\rho} - \frac{\Omega^2}{M^2\rho^3} + \omega(t)^2\rho = 0 \tag{7}$$

Another time invariant quantity can be evaluated from

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial I}{\partial p}\frac{\partial H}{\partial x} = 0 \tag{8}$$

From Eq. (1) and (8), we obtain the classical invariant quantity as

$$I = \frac{1}{2}\left[\frac{\Omega^2}{\rho^2}q^2 + (\rho p - M\dot{\rho}q)^2\right] \tag{9}$$

By invariant quantities Ω and I, we can determine whether or not the system is bound. If Ω is zero, Eq. (9) is an elliptic equation in the phase space. Thus, as q and p in the system is limited in some regions, it is a bound system. However, if Ω is zero, Eq. (9) is a line in the phase space. In this case, q is able to occupy everywhere. Thus, the motion of the system is unbound.

Now, we treat the system quantum mechanically. Displacing the canonical variables of the classical system to the quantum operator in Hamiltonian Eq. (1), we also obtain the quantum invariant quantity as the operator of the same form Eq. (9). In order to obtain eigenfunctions and eigenvalues of the invariant operator, we will reexpress it with the creation and annihilation operators. Thus, we define the time dependent canonical annihilation and creation operators a and $a^\dagger$ by the relations

$$a = \frac{1}{\sqrt{2M\hbar\Omega}}\left[\frac{\Omega}{\rho}q + i(\rho p - M\dot{\rho}q)\right] \tag{10}$$

$$a^\dagger = \frac{1}{\sqrt{2M\hbar\Omega}}\left[\frac{\Omega}{\rho}q - i(\rho p - M\dot{\rho}\dot{q})\right] \tag{11}$$

with an auxiliary condition Eqs. (4) and (5). These operators satisfy the canonical commutation rule $[a, a^\dagger] = 1$ and hold usual property of the creation and annihilation operators. The invariant operator Eq. (9) can be written by a and $a^\dagger$ as

$$I = \hbar\Omega\left(a^\dagger a + \frac{1}{2}\right) \tag{12}$$

The eigenstates of the invariant operator Eq. (12) are the same forms with the normalized eigenstates, $|n>$, of $a^\dagger a$, the eigenvalue spectrum of I is obtained by $\lambda = \Omega\hbar\left(n+\frac{1}{2}\right)$ where n is integer. There is a ground state, which we will denote by u_0, beyond which lowering ends. This must satisfy that $au_0 = 0$ In q-space, it becomes

$$\frac{\partial u_0}{\partial q} + \frac{M\dot{\gamma}}{\hbar}\left(1 - \frac{\dot{\rho}}{\rho\dot{\gamma}}\right) q = 0 \tag{13}$$

The normalized solution of Eq. (13) is

$$u_0 = \left(\frac{M\dot{\gamma}}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left[-\frac{M\dot{\gamma}}{2\hbar}\left(1 - i\frac{\dot{\rho}}{\rho\dot{\gamma}}\right) q^2\right] \tag{14}$$

We may also obtain the excited eigenfunctions

$$u_n = \frac{1}{\sqrt{n!}}(a^\dagger)^n u_0 \tag{15}$$

To find the explicit form of the excited eigenfunction we need a following formulas:

$$e^{\xi q^2/2}\frac{d^n}{dq^n}e^{-\xi q^2/2} = \left(\frac{d}{dq} - \xi q\right)^n \tag{16}$$

$$H_n(\sqrt{\xi}q) = (-\sqrt{\xi})^{-n} e^{\xi q^2}\frac{d^n}{dq^n}(e^{-\xi q^2}) \tag{17}$$

where n is an integer and H_n is a n-th order Hermite polynomial. With the help of the Eqs. (11), (16), and (17) in Eq. (15), we obtain the normalized eigenfunctions of the excited states by working out in detail

$$u_n(q,t) = \sqrt{\frac{1}{2^n n!}}\left(\frac{M\dot{\gamma}}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{M\dot{\gamma}}{2\hbar}\left(1 - i\frac{\rho}{\dot{\gamma}\rho}\right) q^2\right] H_n\left(\sqrt{\frac{M\dot{\gamma}}{\hbar}}q\right) \tag{18}$$

This is a eigenfunction of the invariant operator with an auxillary condition as Eqs. (4) and (5). However, we note that the eigenfunction of the invariant operator I, Eq. (18), is not the wave function of the system, but is related to the wave function of the system. The wave function of the system satisfies the Schrödinger equation of it.

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial q^2}\psi + \frac{M}{2}\omega(t)^2 q^2\psi \tag{19}$$

To obtain the wave function of the system, we try to apply Eq. (18) into Eq. (19). If so, we guess the wave function can make a following form

$$\psi_n(q,t) = e^{i\alpha_n}u_n(q,t) \tag{20}$$

Substituting Eq. (20) into Eq. (19) gives

$$\alpha_n = -\left(\frac{1}{2} + n\right)\gamma \tag{21}$$

From Eqs. (18), (20), and (21), we obtain the exact wave function of the system:

$$\psi_n(q,t) = \sqrt{\frac{1}{2^n n!}}\left(\frac{M\dot{\gamma}}{\pi\hbar}\right)^{1/4} e^{-i(1/2+n)\gamma}\exp\left[-\frac{M\dot{\gamma}}{2\hbar}\left(1 - i\frac{\dot{\rho}}{\rho\dot{\gamma}}\right) q^2\right] H_n\left(\sqrt{\frac{M\dot{\gamma}}{\hbar}}q\right) \tag{22}$$

with an auxiliary condition as Eqs. (4) and (5).

To find the propagator of the system, we use Mehler's formula[7]

$$\sqrt{1-z^2}\exp\left[\frac{2XYz-X^2-Y^2}{1-z^2}\right]=e^{-X^2-Y^2}\sum_{n=0}^{\infty}\frac{z^n}{2^n n!}H_n(X)H_n(Y) \tag{23}$$

with the help of Eqs. (22) and (23), the propagator of the system yields

$$K(q,t;q',t')=\sqrt{\frac{M\sqrt{\dot{\gamma}\dot{\gamma}'}}{2\pi\hbar\sin(\gamma-\gamma')}}\exp\left\{\frac{iM}{2\hbar}\left[\left(\frac{\dot{\rho}}{\rho}q^2-\frac{\dot{\rho}'}{\rho'}q'^2\right)\right.\right.$$
$$\left.\left.+\frac{1}{\sin(\gamma-\gamma')}[(\dot{\gamma}q^2+\dot{\gamma}'q'^2)\cos(\gamma-\gamma')-2\sqrt{\dot{\gamma}\dot{\gamma}'}qq']\right]\right\} \tag{24}$$

with an auxillary condition as Eqs. (4) and (5), where $\rho'=\rho(t')$ and $\gamma'=\gamma(t')$. These are the same as our previous result using path integral methods. We define the uncertainty product as

$$(\Delta q\Delta p)_{m,n}=\left[|<m|q^2|n>-<m|q|n>^2|\times|<m|p^2|n>-<m|p|n>^2|\right]^{1/2} \tag{25}$$

Using the Eq. (22) into Eq. (25), we get the uncertainty relation for various states

$$(\Delta q\Delta p)_{m,n}=\frac{\hbar}{2}\sqrt{1+\frac{\dot{\rho}^2}{\dot{\gamma}^2\rho^2}}[\sqrt{(n+2)(n+1)}\delta_{m,n+2}+(n+1)\delta_{m,n+1}+(2n+1)\delta_{m,n}] \tag{26}$$

Since the minimum uncertainty of the above products is larger than $\frac{\hbar}{2}$, the coherent state of the system is not the minimum uncertainty state.

We will find the minimum uncertainty state. With helping Eqs. (10) and (11). We may reexpress the Hamiltonian, Eq. (1), by a and $a^\dagger$

$$H=\frac{\hbar}{4\Omega}[\alpha a^2+\alpha^* a^{\dagger 2}+\beta\{a,a^\dagger\}] \tag{27}$$

where

$$\alpha=M(\dot{\rho}^2+\omega(t)^2\rho^2-\rho^2\dot{\gamma}^2)-2M\rho\dot{\rho}\dot{\gamma}i \tag{28}$$

$$\beta=M(\dot{\rho}^2+\omega(t)^2\rho^2+\rho^2\dot{\gamma}^2) \tag{29}$$

To diagonalize the Hamiltonian, Eq. (27), we introduce the new creation and annihilation operators

$$b=\mu a+\nu a^\dagger,\qquad b^\dagger=\mu^* a^\dagger+\nu^* a \tag{30}$$

where

$$|\mu|^2-|\nu|^2=1 \tag{31}$$

This holds the same properties of usual creation and annihilation operators. If the b and $b^\dagger$ hold the following relation

$$[H,b]=-kb \tag{32}$$

the Hamiltonian, Eq. (27), can be diagonalized in some space. The μ and ν satisfying the Eqs. (31) and (32) are

$$\mu=\frac{\alpha}{\sqrt{2k(\beta-k)}} \tag{33}$$

$$\nu = \frac{\beta - k}{\sqrt{2k(\beta - k}} \tag{34}$$

$$k = 2\omega(t)\Omega \tag{35}$$

In this case, Hamiltonian, Eq. (27), can be diagonalized and the eigenvalues of the Hamiltonian are $\lambda_n = \hbar\omega(t)\left(n + \frac{1}{2}\right)$, where n is integer. The normalized eigenstates of the Hamiltonian are

$$\begin{aligned}\phi_n &= \frac{1}{\sqrt{n!}}(b^{\dagger})^n u_0 \\ &= \sqrt{\frac{1}{2^n n!}}\left(\frac{M\omega(t)}{\pi\hbar}\right)^{1/4} e^{-\frac{M\omega(t)}{2\hbar}q^2} H_n\left(\sqrt{\frac{M\omega(t)}{\hbar}}q\right)\end{aligned} \tag{36}$$

This is not a solution of the Schrödinger equation, Eq. (23), but is a meaningful function in the system. Substituting Eq. (52) into Eq. (34), we yield the uncertainty product in the eigenstates space of the Hamiltonian of the system.

$$(\Delta q \Delta p)_{n,n} = \hbar\left(n + \frac{1}{2}\right) \tag{37}$$

Since the minimum uncertainty of Eq. (53) is $\frac{\hbar}{2}$, we understand that minimum uncertainty state is a eigenstate of the Hamiltonian of the system. That is, the eigen-coherent state of the Hamiltonian of the system is the squeezed state of the system.

All of our results are ones in the bound state because our system is a bound system. If $\omega(t)^2$ in Hamiltonian, Eq. (1), is negative, then its system is an unbound one. In this system, since the creation and anihilation operators do not tranform a new one, there are not any unbound statees.

ACKNOWLEDGEMENTS

This work was supported by the Center for Thermal and Statistical Physics, KOSEF under Contact No. 93-08-00-05 and by the BSRI Program, Ministry of Education, Republic of Korea, and by the U.S. National Science Foundation under Grant No. CHE-9196214.

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