Quantum Field Theoretical Method in Non-Equilibrium Systems — Non-Equilibrium Thermo Field Dynamics —

Physics, U Tsukuba, Toshihico ARIMITSU

1 Introduction

In this lecture, I will introduce a *canonical formalism* of non-equilibrium quantum systems, named Non-Equilibrium Thermo Field Dynamics (NETFD). This is a unified formalism which enables us to treat dissipative quantum systems (covering whole the aspects in non-equilibrium statistical mechanics listed in Table 1) by the method similar to the usual quantum mechanics and quantum field theory which accommodate the concept of the dual structure in the interpretation of nature, i.e. in terms of the *operator algebra* and the *representation space.* The representation space of NETFD (named *thermal space*) is composed of the direct product of two Hilbert spaces, the one for *non-tilde* fields and the other for *tilde* fields.* It was revealed that *dissipation* is taken into account by a *rotation* in whole the two Hilbert spaces. The terms constituted by the multiplication of tilde and non-tilde fields in the infinitesimal time-evolution generator take care of dissipative (i.e. irreversible) phenomena. This notion was discovered first when NETFD was constructed [1, 2].†

Boltzmann tried to explain the irreversibility of nature based on the microscopic and reversible Newton's mechanics. It was revealed that he had introduced a stochastic manipulation, what is called the *molecular chaos*, without knowing it in the course of the derivation of the Boltzmann equation (see [4] for a brief review of the irreversibility in statistical mechanics). Besides the technical transparency of our new method, we expect that its dual structure, as a quantum theory of dissipative fields, may provide us with a breakthrough to realize Boltzmann's original dream. The duality was not recognized in Boltzmann's days.

It is known that one can divide the fundamental aspects in non-equilibrium statistical mechanics into four categories as shown in Table 1. In category *I*, we deal with a one-particle distribution function (in the μ-phase-space within classical statistical mechanics) with the assumption of *molecular chaos* or something similar which introduces an irreversibility. In category *II*, we handle a density operator which describes the *distribution* of the *ensemble* of a system under consideration. Within the terminology of classical statistical mechanics, we treat

*In NETFD, any operator A is associated with its tilde field $\tilde{A}$ (see Tool 1 in section 2).

†This notion had not appeared in the formulation of the equilibrium thermo field dynamics (TFD) [3] which is an operator formalism of the Gibbs ensembles. This is one of the essential difference between NETFD and TFD.

Table 1: Fundamental Aspects in Non-Equilibrium Statistical Mechanics

	Founder	*Basic Equations*	*Key Words*
I	Boltzmann	Boltzmann eq. kinetic eq.	one-particle distr. func. molecular chaos
II	Gibbs	master eq. Fokker-Planck eq.	density operator ensembles
III	Einstein	Langevin eq.	random force dynamical variables
IV	Kubo	stochastic Liouville eq.	random force phase-space variables

the assembly of points in the Γ-phase-space, each point of which describes a dynamical state of an element system of the ensemble. Irreversibility is introduced by a coarse graining in Γ-space. In category *III*, we study a path of a dynamical variable which is generated by a stochastic equation with a specified random process. The correlation of random forces introduces irreversible behavior of the system. In category *IV*, we treat a distribution of the bundle of paths (flow) in the phase-space [5]-[8]. Each path is generated by an element of a set of the time sequences of a random force (stochastic process). Within the terminology of classical statistical mechanics, we chase one specific point, which represents the dynamical state of a system, in a coarse-grained Γ-space with fluctuating flows due to the stochastic process.

The framework of NETFD was constructed first [1, 2] by, so to speak, a *principle of correspondence* based upon the damping theoretical argument within the density operator formalism [9]-[11] (see Appendices A and B). It was reconstructed upon the seven axioms [12]. Then, the most general expression of the renormalized time-evolution generator in the interaction representation (the semi-free hat-Hamiltonian) was derived together with an equation for the one-particle distribution function [13, 14]. Therefore, we see that NETFD was started to be built upon the fundamental aspects *I* and *II* in Table 1. Within these aspects, the canonical formalism of dissipative quantum fields in NETFD was formulated, and the close structural resemblance between NETFD and usual quantum field theories was revealed [15, 16]. The generating functional within NETFD was derived [17]. Furthermore, the kinetic equation was derived within NETFD [21], and the relation between NETFD and the closed time-path methods [18]-[20] was shown. The extension of NETFD to the hydrodynamical region as well as the kinetic region has been started [22, 23].[‡]

Recently, the framework of NETFD has been extended [25]-[36] to take account of the aspects *III* and *IV* as well as the ones *I* and *II*. Here again NETFD allowed us to construct a unified canonical theory of quantum stochastic operators. The stochastic Liouville equations both of the Ito and of the Stratonovich types were introduced in the Schrödinger representation. Whereas, the Langevin equations both of the Ito and of the Stratonovich types were constructed as the Heisenberg equation of motion with the help of the time-evolution generator of corresponding stochastic Liouville equations. The Ito formula was derived for quantum systems.

In section 2, some fundamentals of NETFD are listed. In section 3, the general form of semi-free time-evolution generator is derived. The annihilation and creation operators are introduced by means of a time-dependent Bogoliubov transformation. The two-point function (propagator) is also derived. In section 4, the model of a damped harmonic oscillator is specified, which we will treat mainly throughout the lecture for simplicity. The Fokker-Planck equation and the Heisenberg equation of motion for coarse grained operators are explicitly handled. The irreversibility of the system is investigated in terms of the Boltzmann entropy. In section 5, the generating functional method is introduced, which gives us the relation between the method of NETFD with the one of Schwinger's closed-time path. In section 6, the general expression of the stochastic semi-free time-evolution generator is derived for a non-stationary Gaussian

[‡]Zubarev admired the method of NETFD, and he also started to use it for the investigation of these regions [24].

white quantum stochastic process. The correlation of the random force operators are also derived generally. In section 7, the stochastic Liouville equations and the Langevin equations both of Ito and Stratonovich types of the system are investigated in a unified manner. In section 8, whole the framework of NETFD is mapped to a c-number space by means of the coherent state representation within NETFD. Section 9 is devoted to discussions. Those which were not explained in this lecture are listed. The open problems and the prospect are also included. Appendices (A–H) are added in order to make the lecture note self-contained.

2 Toolbox of NETFD

NETFD is a quantum theory of dissipative fields, which enables us to construct a canonical formalism for *coarse grained* quantum fields. Here, we list the technical basics for later convenience.

Tool 1. Any operator A (the ordinary, coarse-grained and stochastic ones) is associated with its partner (tilde) operator $\tilde{A}$. The *tilde conjugation* $\sim$ is defined by:

$$\begin{aligned}
(A_1A_2)^{\sim} &= \tilde{A}_1\tilde{A}_2, &(2.1)\\
(c_1A_1 + c_2A_2)^{\sim} &= c_1^*\tilde{A}_1 + c_2^*\tilde{A}_2, &(2.2)\\
(\tilde{A})^{\sim} &= \sigma A, &(2.3)\\
(A^\dagger)^{\sim} &= \tilde{A}^\dagger, &(2.4)
\end{aligned}$$

where $\sigma = 1$ (-1) for bosonic (fermionic) operator A, and c_1 and c_2 are c-numbers.

Tool 2. The tilde and non-tilde operators at an equal time are mutually commutative, and are related with each other through the relation

$$\langle 1|A^\dagger = \langle 1|\tilde{A}. \tag{2.5}$$

Tool 3. The expectation value of an operator A is given by $\langle 1|A|0\rangle$. Observable operators consist only of non-tilde operators.

Tool 4. The thermal vacuums $\langle 1|$ and $|0\rangle$ are *tilde invariant*:

$$\langle 1|^{\sim} = \langle 1|, \quad |0\rangle^{\sim} = |0\rangle, \quad (|0_f\rangle^{\sim} = |0_f\rangle), \tag{2.6}$$

and are normalized as $\langle 1|0\rangle = 1$ $(\langle 1|0_f\rangle = 1)$. We will put the sub-script f for the quantities in stochastic (fluctuating) systems.

Tool 5. The dynamical evolution of systems is described by the *Schrödinger equation* $(\hbar = 1)$

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}|0(t)\rangle. \tag{2.7}$$

For stochastic systems, the Schrödinger equation is expressed in the forms

$$d|0_f(t)\rangle = -i\hat{\mathcal{H}}_{f,t}dt\ |0_f(t)\rangle, \tag{2.8}$$

or

$$d|0_f(t)\rangle = -i\hat{H}_{f,t}dt \circ |0_f(t)\rangle, \tag{2.9}$$

depending on the kind of stochastic multiplications. The former will be used for the Ito multiplication [37], whereas the latter the Stratonovich multiplication [38]. The symbol $\circ$ represents the Stratonovich multiplication. We usually call the Schrödinger equation as the Fokker-Planck equation for coarse grained systems, and as stochastic Liouville equation for stochastic systems. These dynamical equations are of the Schrödinger representation.

Tool 6. The *hat-Hamiltonians*, an infinitesimal time-evolution generators, $\hat{H}$, $\hat{\mathcal{H}}_{f,t}$ and $\hat{H}_{f,t}$ satisfy

$$\left(i\hat{H}\right)^{\sim} = i\hat{H}, \quad \text{the same relation for the stochastic ones.} \tag{2.10}$$

This characteristics is named *tildian*. The tildian hat-Hamiltonians are not necessarily hermitian operators.

Tool 7. The hat-Hamiltonians have zero eigenvalue for the thermal bra-vacuum:

$$\langle 1|\hat{H} = 0, \quad \text{the same relation for the stochastic ones.} \tag{2.11}$$

This is the manifestation of the conservation of probability, i.e. $\langle 1|0(t)\rangle = 1$.

3 Semi-Free Hat-Hamiltonian

The most general form of the renormalized hat-Hamiltonian $\hat{H}_t$ in the interaction representation has the form [13, 14] (see Appendix C for the derivation):[§]

$$\begin{aligned}
\hat{H}_t &= \omega(t)\left(a^\dagger a - \tilde{a}^\dagger \tilde{a}\right) \\
&\quad -i\kappa(t)\left\{[1+2n(t)]\left(a^\dagger a + \tilde{a}^\dagger \tilde{a}\right) - 2[1+n(t)]\,a\tilde{a} - 2n(t)a^\dagger \tilde{a}^\dagger\right\} \\
&\quad -i\frac{d}{dt}n(t)\bar{a}^\mu \tau^{\mu\nu} a^\nu - i2\kappa(t)n(t) \\
&= [\omega(t) - i\kappa(t)]\,\bar{a}^\mu a^\mu - i\left[\frac{d}{dt} + 2\kappa(t)\right]\bar{a}^\mu n(t)^{\mu\nu} a^\nu + \omega(t) + i\kappa(t),
\end{aligned} \tag{3.1}$$

where $\frac{d}{dt}n(t)$ is given by

$$\frac{d}{dt}n(t) = -2\kappa(t)n(t) + i\Sigma^<(t). \tag{3.2}$$

[§]Throughout this lecture note, we confine ourselves to the case of boson fields, for simplicity. The extension to the case of fermion fields are rather straightforward.

Here, we introduced the thermal doublet notation: $a^{\mu=1} = a$, $a^{\mu=2} = \tilde{a}^\dagger$ and $\bar{a}^{\mu=1} = a^\dagger$, $\bar{a}^{\mu=2} = -\tilde{a}$, and the matrices $\tau^{\mu\nu}$: $\tau^{11} = \tau^{21} = 1$, $\tau^{12} = \tau^{22} = -1$, and

$$n(t)^{\mu\nu} = \langle 1|\bar{a}(t)^\nu a(t)^\mu|0\rangle = \begin{pmatrix} n(t) & -n(t) \\ 1+n(t) & -[1+n(t)] \end{pmatrix}. \tag{3.3}$$

The *one-particle distribution function* $n(t)$ is defined by

$$n(t) = \langle 1|a^{\dagger\dagger}(t)a(t)|0\rangle, \tag{3.4}$$

and the thermal doublet notation in the interaction representation is introduced by $a(t)^{\mu=1} = a(t)$, $a(t)^{\mu=2} = \tilde{a}^{\dagger\dagger}(t)$ and $\bar{a}(t)^{\mu=1} = a^{\dagger\dagger}(t)$, $\bar{a}(t)^{\mu=2} = -\tilde{a}(t)$. The function $\Sigma^<(t)$ is given when the interaction hat-Hamiltonian is specified. The equation (3.2) is the Boltzmann equation of the system.

The operators a, $\tilde{a}^\dagger$, etc. satisfy the canonical commutation relation:

$$[a_{\mathbf{k}},\ a^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}, \qquad [\tilde{a}_{\mathbf{k}},\ \tilde{a}^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}. \tag{3.5}$$

The tilde and non-tilde operators are mutually commutative. Throughout this lecture, we do not label the operators a, $\tilde{a}^\dagger$, etc. explicitly with a subscript $\mathbf{k}$ for specifying a momentum and/or other degrees of freedom. However, remember that we are dealing with a *dissipative quantum field.*

The operators in the interaction representation are defined by

$$a(t) = \hat{S}^{-1}(t)a\hat{S}(t), \qquad \tilde{a}^{\dagger\dagger}(t) = \hat{S}^{-1}(t)\tilde{a}^\dagger\hat{S}(t), \tag{3.6}$$

where

$$\frac{d}{dt}\hat{S}(t) = -i\hat{H}_t\hat{S}(t), \qquad (i\hat{H}_t)^\sim = i\hat{H}_t, \tag{3.7}$$

with $\hat{S}(0) = 1$. The semi-free hat-Hamiltonian $\hat{H}_t$ satisfies

$$\langle 1|\hat{H}_t = 0, \tag{3.8}$$

(see Tool 7), and the semi-free operators satisfy

$$\langle 1|a^{\dagger\dagger}(t) = \langle 1|\tilde{a}(t), \qquad a(t)|0\rangle = \frac{n(t)}{1+n(t)}\tilde{a}^{\dagger\dagger}(t)|0\rangle, \tag{3.9}$$

(see Tool 2 for the former). Since the semi-free hat-Hamiltonian $\hat{H}_t$ is not necessarily hermite, we introduced the symbol $\dagger\dagger$ in order to distinguish it from the hermite conjugation $\dagger$. However, we will use $\dagger$ instead of $\dagger\dagger$, for simplicity, unless it is confusing.

The annihilation and creation operators, $\gamma(t)^{\mu=1} = \gamma(t)$, $\gamma(t)^{\mu=2} = \tilde{\gamma}^{\dagger}(t)$ and $\bar{\gamma}(t)^{\mu=1} = \gamma^{\dagger}(t)$, $\bar{\gamma}(t)^{\mu=2} = -\tilde{\gamma}(t)$, are introduced by

$$\gamma(t)^{\mu} = B(t)^{\mu\nu}a(t)^{\nu}, \qquad \bar{\gamma}(t)^{\mu} = \bar{a}(t)^{\nu}B^{-1}(t)^{\nu\mu}, \tag{3.10}$$

with the *time-dependent Bogoliubov transformation:*¶

$$B(t)^{\mu\nu} = \begin{pmatrix} 1+n(t) & -n(t) \\ -1 & 1 \end{pmatrix}. \tag{3.11}$$

The annihilation and creation operators have the properties

$$\gamma(t)|0\rangle = 0, \qquad \langle 1|\tilde{\gamma}^{\dagger}(t) = 0. \tag{3.12}$$

The two-point function $G(t,t')^{\mu\nu}$ has the form

$$\begin{aligned} G(t,t')^{\mu\nu} &= -i\langle 1|T\left[a(t)^{\mu}\bar{a}(t')^{\nu}\right]|0\rangle \\ &= \left[B^{-1}(t)\mathcal{G}(t,t')B(t')\right]^{\mu\nu}, \end{aligned} \tag{3.13}$$

where

$$\mathcal{G}(t,t')^{\mu\nu} = -i\langle 1|T\left[\gamma(t)^{\mu}\bar{\gamma}(t')^{\nu}\right]|0\rangle = \begin{pmatrix} G^{R}(t,t') & 0 \\ 0 & G^{A}(t,t') \end{pmatrix}, \tag{3.14}$$

with

$$G^{R}(t,t') = -i\theta(t-t')\exp\int_{t'}^{t} ds\left[-i\omega(s) - \kappa(s)\right], \tag{3.15}$$

$$G^{A}(t,t') = i\theta(t'-t)\exp\int_{t'}^{t} ds\left[-i\omega(s) + \kappa(s)\right]. \tag{3.16}$$

The representation space (the thermal space) of NETFD is the vector space spanned by the set of bra and ket state vectors which are generated, respectively, by cyclic operations of the annihilation operators $\gamma(t)$ and $\tilde{\gamma}(t)$ on $\langle 1|$, and of the creation operators $\gamma^{\dagger}(t)$ and $\tilde{\gamma}^{\dagger}(t)$ on $|0\rangle$.

The normal product is defined by means of the annihilation and the creation operators, i.e. $\gamma^{\dagger}(t)$, $\tilde{\gamma}^{\dagger}(t)$ stand to the left of $\gamma(t)$, $\tilde{\gamma}(t)$. The process, rewriting physical operators in terms of the annihilation and creation operators, leads to a Wick-type formula, which in turn leads to Feynman-type diagrams for multi-point functions in the renormalized interaction representation. The internal line in the Feynman-type diagrams is the unperturbed two-point function (3.13).

¶There is a minor change in the normalization of the time-dependent Bogoliubov transformation compared with the original definition given in [1, 2], [12]-[14]. This change makes the expression $\mathcal{G}(t,t')^{\mu\nu}$ simpler, and is essential in the formulation of the stochastic Liouville equation introduced below.

4 Fokker-Planck Equation —Model—

We can specify a model by writing down its Boltzmann equation. In the following through the lecture, we will use the model of a damped harmonic oscillator in order to show the heart of the formalism. The model is specified by the Boltzmann equation

$$\frac{d}{dt}n(t) = -2\kappa\left[n(t) - \bar{n}\right], \tag{4.1}$$

with

$$\bar{n} = \frac{1}{e^{\beta\omega} - 1}, \tag{4.2}$$

where β is the inverse of the temperature T of the environment, i.e. $\beta = 1/T$. The Boltzmann constant has been put to equal to 1.

Substituting the Boltzmann equation (4.1) into the semi-free hat-Hamiltonian (3.1), we have [1, 2, 12]

$$\begin{aligned}\hat{H} &= \omega\left(a^\dagger a - \tilde{a}^\dagger \tilde{a}\right) - i\kappa\left[(1+2\bar{n})\left(a^\dagger a + \tilde{a}^\dagger\tilde{a}\right) - 2(1+\bar{n})\,a\tilde{a} - 2\bar{n}a^\dagger\tilde{a}^\dagger\right] - i2\kappa\bar{n} \\ &= (\omega - i\kappa)\,\bar{a}^\mu a^\mu - i2\kappa\bar{a}^\mu\bar{n}^{\mu\nu}a^\nu + \omega + i\kappa,\end{aligned} \tag{4.3}$$

where

$$\bar{n}^{\mu\nu} = \begin{pmatrix} \bar{n} & -\bar{n} \\ 1+\bar{n} & -(1+\bar{n}) \end{pmatrix}. \tag{4.4}$$

The Hamiltonian (4.3) is the same expression as the one derived by means of the principle of correspondence when NETFD was constructed first based upon the projection operator formalism of the damping theory [1, 2] (see Appendix B).

The Fokker-Planck equation of the model is given by

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}|0(t)\rangle, \tag{4.5}$$

with (4.3), which is solved as

$$|0(t)\rangle = \exp\left[\left[n(t) - n(0)\right]\gamma^{\natural}\tilde{\gamma}^{\natural}\right]|0\rangle. \tag{4.6}$$

The ket-thermal vacuum, $|0\rangle = |0(0)\rangle$, is specified by

$$a|0\rangle = f\tilde{a}^\dagger|0\rangle, \tag{4.7}$$

where $f = n/(1+n)$ with $n = n(0)$. This can be expressed in terms of d and $\tilde{d}^\dagger$, which are introduced in (4.11) below, as

$$d|0\rangle = (n - \bar{n})\,\tilde{d}^\dagger|0\rangle. \tag{4.8}$$

The attractive expression (4.6), which was obtained first in [39], led us to the notion of a mechanism named the *spontaneous creation of dissipation* [13, 14], [40]-[42]. We can obtain the result (4.6) only by algebraic manipulations. This technical convenience of the operator algebra in NETFD, which is very much similar to that of the usual quantum mechanics, enables us to treat open systems in far-from-equilibrium state simpler and more transparent [43]-[48].

The hat-Hamiltonian (4.3) can be also written in the form

$$\hat{H} = \omega\left(d^\dagger d - \tilde{d}^\dagger \tilde{d}\right) - i\kappa\left(d^\dagger d + \tilde{d}^\dagger \tilde{d}\right) \tag{4.9}$$

$$= \omega\left(\gamma^{\ddagger}\gamma_t - \tilde{\gamma}^{\ddagger}\tilde{\gamma}_t\right) - i\kappa\left(\gamma^{\ddagger}\gamma_t + \tilde{\gamma}^{\ddagger}\tilde{\gamma}_t + 2\left[n(t) - \bar{n}\right]\gamma^{\ddagger}\tilde{\gamma}^{\ddagger}\right), \tag{4.10}$$

where $d^{\mu=1} = d,\ d^{\mu=2} = \tilde{d}^\dagger$ and $\bar{d}^{\mu=1} = d^\dagger,\ \bar{d}^{\mu=2} = -\tilde{d}$ are defined by

$$d^\mu = Q^{-1\mu\nu}a^\nu, \qquad \bar{d}^\mu = \bar{a}^\nu Q^{\nu\mu}, \tag{4.11}$$

with

$$Q^{\mu\nu} = \begin{pmatrix} 1 & \bar{n} \\ 1 & 1+\bar{n} \end{pmatrix}. \tag{4.12}$$

The annihilation and creation operators, $\gamma^{\mu=1} = \gamma_t,\ \gamma^{\mu=2} = \tilde{\gamma}^{\ddagger}$ and $\bar{\gamma}^{\mu=1} = \gamma^{\ddagger},\ \bar{\gamma}^{\mu=2} = -\tilde{\gamma}_t$, are defined through the relation

$$\gamma(t)^\mu = \hat{S}^{-1}(t)\gamma^\mu\hat{S}(t), \qquad \bar{\gamma}(t)^\mu = \hat{S}^{-1}(t)\bar{\gamma}^\mu\hat{S}(t). \tag{4.13}$$

It is easy to see from the diagonalized form (4.9) of $\hat{H}$ that

$$d(t) = \hat{S}^{-1}(t)\ d\ \hat{S}(t) = d\ e^{-(i\omega+\kappa)t}, \qquad \tilde{d}^{\dagger\dagger}(t) = \hat{S}^{-1}(t)\ \tilde{d}^\dagger\ \hat{S}(t) = \tilde{d}^\dagger\ e^{-(i\omega-\kappa)t}. \tag{4.14}$$

On the other hand, it is easy to see from the normal product form (4.10) of $\hat{H}$ that it satisfies Tool 7 since the annihilation and creation operators satisfy

$$\gamma_t|0(t)\rangle = 0, \qquad \langle 1|\tilde{\gamma}^{\ddagger} = 0. \tag{4.15}$$

The difference between the operators which diagonalize $\hat{H}$ and the ones which make $\hat{H}$ in the form of normal product is one of the features of NETFD, and shows the point that the formalism is quite different from usual quantum mechanics and quantum field theory. This is a manifestation of the fact that the hat-Hamiltonian is a time-evolution generator for irreversible processes.

The Heisenberg equation of motion for a coarse grained operator $A(t)$ is given by

$$\frac{d}{dt}A(t) = i[\hat{H}(t),\ A(t)], \tag{4.16}$$

with

$$\hat{H}(t) = \hat{S}^{-1}(t)\hat{H}\hat{S}(t). \tag{4.17}$$

We would like to emphasize here that the existence of the Heisenberg equation of motion (4.16) for coarse grained operators is one of the notable features of NETFD. This enabled us to construct a *canonical formalism of the dissipative quantum field theory*, where the coarse grained operator $a(t)$ etc. in the Heisenberg representation satisfies the equal-time canonical commutation relation

$$[a(t),\ a^\dagger(t)] = 1, \qquad [\tilde{a}(t),\ \tilde{a}^\dagger(t)] = 1. \tag{4.18}$$

For the present model, we have

$$\frac{d}{dt}a(t) = -i\omega a(t) - \kappa\left[(1+2\bar{n})\,a(t) - 2\bar{n}\tilde{a}^\dagger(t)\right], \tag{4.19}$$

$$\frac{d}{dt}a^\dagger(t) = i\omega a^\dagger(t) + \kappa\left[(1+2\bar{n})\,a^\dagger(t) - 2\,(1+\bar{n})\,\tilde{a}(t)\right]. \tag{4.20}$$

We see that the equation of motion for $a^\dagger(t)$ is not the hermite conjugate of the one for $a(t)$.

The two-point function for the model is given by (3.13) with the replacement of $\omega(t)$ and $\kappa(t)$ by the time-independent quantities ω and κ, respectively.

Let us check here the irreversibility of the system. The entropy of the system is given by

$$S(t) = -\left\{n(t)\ln n(t) - [1+n(t)]\ln[1+n(t)]\right\}, \tag{4.21}$$

whereas the heat change of the system is given by

$$d'Q = \omega dn. \tag{4.22}$$

Thermodynamics tells us that

$$dS = dS_e + dS_i, \qquad dS_e = d'Q/T_R, \tag{4.23}$$

$$dS_i \geq 0. \tag{4.24}$$

The latter inequality (4.24) is the second law of thermodynamics. Putting (4.21) and (4.22) in (4.23), for dS and dS_e, respectively, we have a relation for the entropy production rate [4]

$$\frac{dS_i}{dt} = \frac{dS}{dt} - \frac{dS_e}{dt} = 2\kappa\,[n(t) - \bar{n}]\ln\frac{n(t)[1+\bar{n}]}{\bar{n}[1+n(t)]} \geq 0. \tag{4.25}$$

It is easy to see that the expression on the right-hand side of the second equality satisfies the last inequality which is consistent with (4.24). The equality realizes either for the thermal equilibrium state, $n(t) = \bar{n}$, or for the quasi-stationary process, $\kappa \longrightarrow 0$.

5 Generating Functional Method

Let us consider a master equation [17]

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}_{tot}|0(t)\rangle, \tag{5.1}$$

with

$$\hat{H}_{tot} = \hat{H} + \hat{H}_{I,t}, \tag{5.2}$$

where $\hat{H}$ is given by (4.3), and $\hat{H}_{I,t}$ is defined by

$$\hat{H}_{I,t} = \bar{K}(t)^\mu a^\mu + \bar{a}^\mu K(t)^\mu = \bar{K}_\gamma(t)^\mu \gamma^\mu + \bar{\gamma}^\mu K_\gamma(t)^\mu. \tag{5.3}$$

The thermal doublet notation for the c-number external fields has been introduced by $K(t)^{\mu=1} = K(t)$, $K(t)^{\mu=2} = \tilde{K}^*(t)$ and $\bar{K}(t)^{\mu=1} = K(t)^*$, $\bar{K}(t)^{\mu=2} = -\tilde{K}(t)$. We see the relation

$$K_\gamma(t)^\mu = B(t)^{\mu\nu}K(t)^\nu, \qquad \bar{K}_\gamma(t) = \bar{K}(t)^\nu B^{-1}(t)^{\nu\mu}, \tag{5.4}$$

with (3.11) for $B(t)^{\mu\nu}$.

The generating functional for the system is defined by [17]

$$Z[K, \tilde{K}] = \langle 1|\hat{U}(\bar{t})|0\rangle, \tag{5.5}$$

where $\hat{U}(t)$ satisfies

$$\frac{d}{dt}\hat{U}(t) = -i\hat{H}_I(t)\hat{U}(t), \tag{5.6}$$

with the initial condition $\hat{U}(0) = 1$. The hat-Hamiltonian $\hat{H}_I(t)$ is given by

$$\hat{H}_I(t) = \hat{S}^{-1}(t)\hat{H}_{I,t}\hat{S}(t). \tag{5.7}$$

Taking the functional derivative of the generating functional (5.5), we have

$$\delta \ln Z[K, \tilde{K}] = -i\int_0^{\bar{t}} dt \left[\delta\bar{K}_\gamma(t)^\mu\langle\gamma(t)^\mu\rangle + \langle\bar{\gamma}(t)^\mu\rangle\delta K_\gamma(t)^\mu\right], \tag{5.8}$$

where $\langle\gamma(t)^\mu\rangle$ and $\langle\bar{\gamma}(t)^\mu\rangle$ are defined by

$$\langle\gamma(t)^\mu\rangle = i\frac{\delta}{\delta\bar{K}_\gamma(t)^\mu}\ln Z[K, \tilde{K}] = \langle 1|T\left[\hat{U}(\bar{t})\gamma(t)^\mu\right]|0\rangle, \tag{5.9}$$

$$\langle\bar{\gamma}(t)^\mu\rangle = i\frac{\delta}{\delta K_\gamma(t)^\mu}\ln Z[K, \tilde{K}] = \langle 1|T\left[\hat{U}(\bar{t})\bar{\gamma}(t)^\mu\right]|0\rangle. \tag{5.10}$$

The equation of motion for $\langle\gamma(t)^\mu\rangle$ [17]:

$$\frac{d}{dt}\langle\gamma(t)^\mu\rangle = -\left(i\omega\delta^{\mu\nu} + \kappa\tau_3^{\mu\nu}\right)\langle\gamma(t)^\nu\rangle - iK_\gamma(t)^\mu, \tag{5.11}$$

with the boundary conditions

$$\begin{aligned}\langle\gamma(0)^{\mu=1}\rangle &= \langle\gamma(0)\rangle = 0, & \langle\gamma(\bar{t})^{\mu=2}\rangle &= \langle\tilde{\gamma}^{\ddagger}(\bar{t})\rangle = 0,\\ \langle\bar{\gamma}(\bar{t})^{\mu=1}\rangle &= \langle\gamma^{\ddagger}(\bar{t})\rangle = 0, & \langle\bar{\gamma}(0)^{\mu=2}\rangle &= -\langle\tilde{\gamma}(0)\rangle = 0,\end{aligned} \tag{5.12}$$

can be solved in the form

$$\langle\gamma(t)^{\mu}\rangle = \int_0^{\bar{t}} dt'\ \mathcal{G}(t,t')^{\mu\nu}K_{\gamma}(t')^{\nu}, \tag{5.13}$$

where $\mathcal{G}(t,t')^{\mu\nu}$ is given by (3.14). The boundary conditions in (5.12) are derived by the thermal state conditions (3.12). The matrix $\tau_3^{\mu\nu}$ is defined by $\tau_3^{11} = -\tau_3^{22} = 1,\ \tau_3^{12} = \tau_3^{21} = 0$.

Substituting (5.13) into (5.8), we finally obtain [17]

$$\begin{aligned} Z[K,\tilde{K}] &= \exp\left[-i\int_0^{\bar{t}}dt\int_0^{\bar{t}}dt'\bar{K}_{\gamma}(t)^{\mu}\mathcal{G}(t,t')^{\mu\nu}K_{\gamma}(t')^{\nu}\right]\\ &= \exp\left[-i\int_0^{\bar{t}}dt\int_0^{\bar{t}}dt'\bar{K}(t)^{\mu}G(t,t')^{\mu\nu}K(t')^{\nu}\right]. \end{aligned} \tag{5.14}$$

This expression for an open system was derived first by Schwinger by means of the closed-time path method [18] (see also [19, 20]).

The derivation of the generating functional shown in this section reveals the relation between the quantum operator formalism of dissipative fields and their path integral formalism [18]. Note that the existence of a quantum operator formalism for dissipative fields had never been realized before NETFD was constructed.

6 Stochastic Semi-Free Hamiltonian

The general form of the stochastic semi-free hat-Hamiltonian $\hat{\mathcal{H}}_{f,t}$, appeared in the stochastic Liouville equation (2.8) of the Ito type, and the correlation of the random force operators can be derived under the following basic requirements:

A1. The stochastic semi-free operators are defined by

$$a(t) = \hat{S}_f^{-1}(t)a\hat{S}_f(t), \qquad \tilde{a}^{\dagger\!\dagger}(t) = \hat{S}_f^{-1}(t)\tilde{a}^{\dagger}\hat{S}_f(t), \tag{6.1}$$

where

$$d\hat{S}_f(t) = -i\hat{\mathcal{H}}_{f,t}dt\ \hat{S}_f(t), \tag{6.2}$$

with $\hat{S}_f(0) = 1$. Here, it is assumed that, at $t = 0$, the relevant system starts to contact with the irrelevant system representing the stochastic process described by the random force operators $dF(t)$, etc. defined in **A3** below.[‖] The stochastic operators $a,\ a^{\dagger}, \tilde{a}$ and $\tilde{a}^{\dagger}$ in the Schrödinger representation satisfy the canonical commutation relation:

$$[a,\ a^{\dagger}] = 1, \qquad [\tilde{a},\ \tilde{a}^{\dagger}] = 1. \tag{6.3}$$

‖Within the formalism, the random force operators $dF(t)$ and $dF^{\dagger}(t)$ are assumed to commute with any relevant system operator A in the Schrödinger representation: $[A,\ dF(t)] = [A,\ dF^{\dagger}(t)] = 0$.

The semi-free operators (6.1) keep the equal-time canonical commutation relation:

$$[a(t),\ a^{\dagger\!\dagger}(t)] = 1, \qquad [\tilde{a}(t),\ \tilde{a}^{\dagger\!\dagger}(t)] = 1. \tag{6.4}$$

The tildian nature of Tool 6 for $\hat{\mathcal{H}}_{f,t}dt$ is consistent with the definition (6.1) of the semi-free operators. Since the tildian hat-Hamiltonian $\hat{\mathcal{H}}_{f,t}dt$ is not necessarily hermite, we introduced the symbol $\dagger\!\dagger$ in order to distinguish it from the hermite conjugation $\dagger$. However, we will use $\dagger$ instead of $\dagger\!\dagger$, for simplicity, unless it is confusing. We use here the same notation $a(t)$ etc. for the stochastic semi-free operators as those for the coarse grained semi-free operators. We expect that there will be no confusion between them.

A2. The stochastic semi-free operators satisfy Tool 2:

$$\langle 1|a^{\dagger\!\dagger}(t) = \langle 1|\tilde{a}(t). \tag{6.5}$$

A3. The random force operators are of the Wiener process whose first and second cumulants are given by real c-numbers:

$$\langle dF(t)\rangle = \left\langle dF^{\dagger}(t)\right\rangle = 0, \tag{6.6}$$

$$\langle dF(t)dF(t)\rangle = \left\langle dF^{\dagger}(t)dF^{\dagger}(t)\right\rangle = 0, \tag{6.7}$$

$$\left\langle dF(t)dF^{\dagger}(t)\right\rangle = [\text{a real } c\text{-number}], \qquad \left\langle dF^{\dagger}(t)dF(t)\right\rangle = [\text{a real } c\text{-number}], \tag{6.8}$$

where $\langle\cdots\rangle = \langle|\cdots|\rangle$ represents the random average referring to the random force operators $dF(t)$.

A4. The random force operators satisfy Tool 2:

$$\langle|dF^{\dagger}(t) = \langle|d\tilde{F}(t). \tag{6.9}$$

A5. The stochastic semi-free operators and the random force operators satisfy the causality

$$\left\langle a(t)d\mathcal{F}^{\dagger}(t)\right\rangle = 0, \quad \text{etc.}, \tag{6.10}$$

where the random force operator $d\mathcal{F}^{\dagger}(t)$ in the *Heisenberg* representation** is defined by

$$d\mathcal{F}^{\dagger}(t) = \hat{S}_f^{-1}(t)dF^{\dagger}(t)\hat{S}_f(t). \tag{6.11}$$

The results are (see Appendix D for the derivation)

$$\begin{aligned}\hat{\mathcal{H}}_{f,t}dt &= \omega(t)(a^{\dagger}a - \tilde{a}^{\dagger}\tilde{a})dt - i\kappa(t)\left[(a^{\dagger}-\tilde{a})(\mu a + \nu\tilde{a}^{\dagger}) + \text{t.c.}\right]dt \\ &\quad + i\left\{2\kappa(t)\left[n(t)+\nu\right] + \frac{d}{dt}n(t)\right\}(a^{\dagger}-\tilde{a})(\tilde{a}^{\dagger}-a) \\ &\quad + i\left[(a^{\dagger}-\tilde{a})dW(t) + \text{t.c.}\right],\end{aligned} \tag{6.12}$$

**It can be the interaction representation when one includes non-linear terms in the hat-Hamiltonian, and performs a perturbational calculation. As we are dealing with only the bi-linear case in this lecture, we call the representation as the Heisenberg one.

and

$$\left\langle dW(t)d\tilde{W}(s)\right\rangle = \left\langle d\tilde{W}(s)dW(t)\right\rangle$$
$$= \mu\left\langle dF^{\dagger}(s)dF(t)\right\rangle + \nu\left\langle dF(t)dF^{\dagger}(s)\right\rangle \tag{6.13}$$
$$= \left\{2\kappa(t)\left[n(t)+\nu\right] + \frac{d}{dt}n(t)\right\}\delta(t-s)dtds, \tag{6.14}$$

(see Appendices D and F for the derivation) where the random force operator $dW(t)$ is defined by

$$dW(t) = \mu dF(t) + \nu d\tilde{F}^{\dagger}(t), \tag{6.15}$$

with $\mu+\nu=1$. The stochastic process here is a non-stationary Gaussian white one.

The correlations of $dF(t)$ and $dF^{\dagger}(t)$ are given by (see Appendix F)

$$\left\langle dF^{\dagger}(t)dF(s)\right\rangle = \left[2\kappa(t)n(t) + \frac{d}{dt}n(t)\right]\delta(t-s)dtds, \tag{6.16}$$
$$\left\langle dF(t)dF^{\dagger}(s)\right\rangle = \left\{2\kappa(t)\left[n(t)+1\right] + \frac{d}{dt}n(t)\right\}\delta(t-s)dtds. \tag{6.17}$$

The semi-free hat-Hamiltonian $\hat{H}_{f,t}dt$, appeared in the stochastic Liouville equation (2.9) of the Stratonovich type, is given by (see Appendix F)

$$\hat{H}_{f,t}dt = \omega(t)(a^{\dagger}a - \tilde{a}^{\dagger}\tilde{a})dt - i\kappa(t)\left[(a^{\dagger}-\tilde{a})(\mu a + \nu\tilde{a}^{\dagger}) + \text{t.c.}\right]dt$$
$$+i\left[(a^{\dagger}-\tilde{a})dW(t) + \text{t.c.}\right]. \tag{6.18}$$

7 Quantum Stochastic Differential Equations

Let us consider the case corresponding to (4.1), i.e. the stationary stochastic process (F.15) and (F.16). Then, the random force operator $dW(t)$ satisfies

$$\langle dW(t)\rangle = \langle d\tilde{W}(t)\rangle = 0, \tag{7.1}$$
$$\langle dW(t)dW(s)\rangle = \langle d\tilde{W}(t)d\tilde{W}(s)\rangle = 0, \tag{7.2}$$
$$\langle dW(t)d\tilde{W}(s)\rangle = \langle d\tilde{W}(s)dW(t)\rangle = 2\kappa(\bar{n}+\nu)\delta(t-s)dt\,ds. \tag{7.3}$$

The stochastic Liouville equations and the Langevin equations both of Ito and of Stratonovich types are constructed being compatible with the Fokker-Planck equation (4.5) with the hat-Hamiltonian (4.3).

The quantum stochastic Liouville equation of the Stratonovich type is given by [25]-[29]

$$d|0_f(t)\rangle = -i\hat{H}_{f,t}dt \circ |0_f(t)\rangle, \tag{7.4}$$

with the stochastic time-evolution generator:

$$\hat{H}_{f,t}dt = \hat{H}_S dt + \left[(a^\dagger - \tilde{a})\left\{id(\mu a + \nu\tilde{a}^\dagger) + [\hat{H}_S,\ \mu a + \nu\tilde{a}^\dagger]dt\right\} - \text{t.c.}\right], \tag{7.5}$$

where[††]

$$\hat{H}_S = H_S - \tilde{H}_S, \qquad H_S = \omega a^\dagger a, \tag{7.6}$$

and the flow operators da and $d\tilde{a}^\dagger$ are specified by[‡‡]

$$da = i[\hat{H}_S,\ a]dt - \kappa[(\mu - \nu)a + 2\nu\tilde{a}^\dagger]dt + dW(t), \tag{7.7}$$

$$d\tilde{a}^\dagger = i[\hat{H}_S,\ \tilde{a}^\dagger]dt - \kappa[2\mu a - (\mu - \nu)\tilde{a}^\dagger]dt + dW(t). \tag{7.8}$$

The quantum stochastic Liouville equation of the Ito type is given by

$$d|0_f(t)\rangle = -i\hat{\mathcal{H}}_{f,t}dt\ |0_f(t)\rangle, \tag{7.9}$$

with

$$\hat{\mathcal{H}}_{f,t}dt = \hat{H}_{f,t}dt + i(a^\dagger - \tilde{a})(\tilde{a}^\dagger - a)dW(t)d\tilde{W}(t) \tag{7.10}$$

$$= \hat{H}dt + i\left\{(a^\dagger - \tilde{a})dW(t) + \text{t.c.}\right\}, \tag{7.11}$$

where $\hat{H}_{f,t}dt$ and $\hat{H}$ are given, respectively, by (7.5) and (4.3). Here, we used the properties of the random force operators

$$dW(t)dW(s) = d\tilde{W}(t)d\tilde{W}(s) = 0, \tag{7.12}$$

$$dW(t)d\tilde{W}(s) = d\tilde{W}(s)dW(t) = 2\kappa[\bar{n} + \nu]\delta(t - s)dtds, \tag{7.13}$$

within the stochastic convergence, which can be derived from (7.2) and (7.3), and the fact that $dW(t)dt$ etc. can be neglected as higher orders.

Taking the random average of the Ito stochastic Liouville equation (7.9) with (7.11), we obtain the corresponding Fokker-Planck equation (4.5) by the process:

$$\begin{aligned}\left\langle d|0_f(t)\rangle\right\rangle &= d|0(t)\rangle \\ &= -i\hat{H}dt\left\langle|0_f(t)\rangle\right\rangle + \left\langle\left\{(a^\dagger - \tilde{a})dW(t) + \text{t.c.}\right\}|0_f(t)\rangle\right\rangle \\ &= -i\hat{H}dt\ |0(t)\rangle,\end{aligned} \tag{7.14}$$

with $|0(t)\rangle = \left\langle|0_f(t)\rangle\right\rangle$. Here, we used the properties

$$\left\langle dW(t)\hat{S}_f(t)\right\rangle = 0, \quad \text{etc.}, \tag{7.15}$$

[††]The following formulation is valid for the cases where H_S has non-linear terms.

[‡‡]The flow equations (7.7) and (7.8) read $d(\mu a + \nu\tilde{a}^\dagger) = i[\hat{H}_S,\ \mu a + \nu\tilde{a}^\dagger]dt - \kappa(\mu a + \nu\tilde{a}^\dagger)dt + dW(t)$, $d(a - \tilde{a}^\dagger) = i[\hat{H}_S,\ a - \tilde{a}^\dagger]dt + \kappa(a - \tilde{a}^\dagger)dt$.

which are the characteristics of the Ito multiplication. The Fokker-Planck equation (4.5) can be derived also by taking random average of the Stratonovich stochastic Liouville equation (7.4) with (7.5) (see Appendix G).

For the dynamical quantity

$$A(t) = \hat{S}_f^{-1}(t) A \hat{S}_f(t), \tag{7.16}$$

the quantum Langevin equation of the Stratonovich type is given as the Heisenberg equation of motion [27, 29]:

$$\begin{aligned} dA(t) &= i[\hat{H}_f(t)dt \,\circ\!\!,\; A(t)] && (7.17)\\ &= i[\hat{H}_S(t),\ A(t)]dt \\ &\quad +\kappa\left\{\left[\left(a^\dagger(t) - \tilde{a}(t)\right)\left(\mu a(t) + \nu\tilde{a}^\dagger(t)\right),\ A(t)\right]\right. \\ &\quad + \left.\left[\left(\tilde{a}^\dagger(t) - a(t)\right)\left(\mu\tilde{a}(t) + \nu a^\dagger(t)\right),\ A(t)\right]\right\} dt \\ &\quad - \left\{[a^\dagger(t) - \tilde{a}(t),\ A(t)] \circ dW(t) + [\tilde{a}^\dagger(t) - a(t),\ A(t)] \circ d\tilde{W}(t)\right\}, && (7.18) \end{aligned}$$

where

$$\hat{H}_f(t) = \hat{S}_f^{-1}(t)\hat{H}_{f,t}\hat{S}_f(t), \tag{7.19}$$

$$[X(t) \,\circ\!\!,\; Y(t)] = X(t) \circ Y(t) - Y(t) \circ X(t), \tag{7.20}$$

for arbitrary operators $X(t)$ and $Y(t)$, and use has been made of the fact that

$$\hat{S}_f^{-1}(t) dW(t) \hat{S}_f(t) = dW(t), \tag{7.21}$$

since the random force operator $dW(t)$ is commutative with $\hat{S}_f(t)$ due to the property (D.9) and (D.15). Note that, using (7.18), we can readily verify that

$$d[A(t)B(t)] = dA(t) \circ B(t) + A(t) \circ dB(t), \tag{7.22}$$

for arbitrary relevant system operators A and B. This fact proves that the quantum stochastic differential equation (7.18) is of the Stratonovich type.

The quantum Langevin equation of the Stratonovich type (7.18) is also derived by the algebraic identity

$$dA(t) = d\hat{S}_f^{-1}(t) \circ A\hat{S}_f(t) + \hat{S}_f^{-1}(t)A \circ d\hat{S}_f(t), \tag{7.23}$$

with the help of

$$d\hat{S}_f(t) = -i\hat{H}_{f,t}dt \circ \hat{S}_f(t), \qquad d\hat{S}_f^{-1}(t) = i\hat{S}_f^{-1}(t) \circ \hat{H}_{f,t}dt. \tag{7.24}$$

When $dY(t)$ is $dW(t)$, and $X(t)$ is constituted by the relevant operators satisfying the quantum Langevin equation (7.18) of the Stratonovich type, the connection formula (E.5) reduces to

$$X(t) \circ dW(t) = X(t)dW(t) - \kappa(\bar{n} + \nu)\left[\tilde{a}^\dagger(t) - a(t),\ X(t)\right] dt. \tag{7.25}$$

In deriving (7.25), we used the properties (7.12) and (7.13), and the fact that $dW(t)dt$ etc. can be neglected as higher orders.

By means of the connection formula (7.25) between the Ito and the Stratonovich products, we can derive the quantum Langevin equation of the Ito type from that of the Stratonovich type (7.18) as

$$\begin{aligned}
dA(t) &= i[\hat{\mathcal{H}}_f(t)dt,\ A(t)] \\
&\quad + \Big\{\Big(a^\dagger(t) - \tilde{a}(t)\Big)[\tilde{a}^\dagger(t) - a(t),\ A(t)] \\
&\quad + \Big(\tilde{a}^\dagger(t) - a(t)\Big)[a^\dagger(t) - \tilde{a}(t),\ A(t)]\Big\} dW(t)d\tilde{W}(t) \qquad (7.26)\\
&= i[\hat{H}_S(t),\ A(t)]dt \\
&\quad + \kappa\Big\{\Big[\Big(a^\dagger(t) - \tilde{a}(t)\Big)\Big(\mu a(t) + \nu\tilde{a}^\dagger(t)\Big),\ A(t)\Big] \\
&\quad + \Big[\Big(\tilde{a}^\dagger(t) - a(t)\Big)\Big(\mu\tilde{a}(t) + \nu a^\dagger(t)\Big),\ A(t)\Big]\Big\} dt \\
&\quad + 2\kappa(\bar{n} + \nu)[\tilde{a}^\dagger(t) - a(t),\ [a^\dagger(t) - \tilde{a}(t),\ A(t)]]dt \\
&\quad - \Big\{[a^\dagger(t) - \tilde{a}(t),\ A(t)]dW(t) + [\tilde{a}^\dagger(t) - a(t),\ A(t)]d\tilde{W}(t)\Big\}, \qquad (7.27)
\end{aligned}$$

where

$$\hat{\mathcal{H}}_f(t)dt = \hat{S}_f^{-1}(t)\hat{\mathcal{H}}_{f,t}dt\hat{S}_f(t), \qquad (7.28)$$

(see Appendix D for another derivation of (7.27)). With the help of (7.27), we can find the product $dA(t)dB(t)$ has the expression

$$\begin{aligned}
dA(t)dB(t) = 2\kappa(\bar{n} + \nu)\Big\{&[a^\dagger(t) - \tilde{a}(t),\ A(t)][\tilde{a}^\dagger(t) - a(t),\ B(t)] \\
&+ [\tilde{a}^\dagger(t) - a(t),\ A(t)][a^\dagger(t) - \tilde{a}(t),\ B(t)]\Big\} dt, \qquad (7.29)
\end{aligned}$$

which leads to the calculus rule of the Ito type

$$d[A(t)B(t)] = dA(t)\cdot B(t) + A(t)\cdot dB(t) + dA(t)dB(t), \qquad (7.30)$$

for arbitrary relevant stochastic operators A and B. This proves that the quantum stochastic differential equation (7.27) is in fact of the Ito type. Furthermore, since (7.27) is the time-evolution equation for any relevant stochastic operator $A(t)$, it is *Ito's formula* for quantum systems as will be proven in section 8.

Putting a and $\tilde{a}^\dagger$ for A, we see that (7.18) and (7.27) reduce to

$$da(t) = i[\hat{H}_S,\ a(t)]dt - \kappa[(\mu - \nu)a(t) + 2\nu\tilde{a}^\dagger(t)]dt + dW(t), \qquad (7.31)$$

$$d\tilde{a}^\dagger(t) = i[\hat{H}_S,\ \tilde{a}^\dagger(t)]dt - \kappa[2\mu a(t) - (\mu - \nu)\tilde{a}^\dagger(t)]dt + dW(t), \qquad (7.32)$$

whose formal structures are the same as (7.7) and (7.8), respectively.

In the Langevin equation approach, the dynamical behavior of systems is specified when one characterizes the correlations of random forces. The quantum Langevin equation is the equation in the Heisenberg representation, therefore the characterization of random force operators should be performed in this representation. This cannot be done in terms of $d\mathcal{F}(t)$ etc., since the information of the stochastic process is masked by the dynamics generated by $\hat{H}_f(t)$ in these operators. Whereas, the specification of the correlation between $dW(t)$ etc. directly characterizes the stochastic process because of the relations in (7.21).

Taking the random average and the vacuum expectation of (7.27), we obtain the equation of motion for the expectation value of an arbitrary operator $A(t)$ of the relevant system as

$$\begin{aligned}\frac{d}{dt}\langle\!\langle A(t)\rangle\!\rangle = i\,&\langle\!\langle [H_S(t),\ A(t)]\rangle\!\rangle \\ &+\kappa\left(\langle\!\langle a^\dagger(t)[A(t),\ a(t)]\rangle\!\rangle + \langle\!\langle [a^\dagger(t),\ A(t)]a(t)\rangle\!\rangle\right) \\ &+2\kappa\bar{n}\langle\!\langle [a^\dagger(t),\ [A(t),\ a(t)]]\rangle\!\rangle,\end{aligned} \tag{7.33}$$

where $\langle\!\langle\cdots\rangle\!\rangle = \langle 1|\langle\cdots\rangle|0\rangle$, which means to take both random average and vacuum expectation. This is the exact equation of motion for systems with linear-dissipative coupling to reservoir, which can be also derived by means of Fokker-Planck equation (4.5). Here, we used the property

$$\langle a(t)dW(t)\rangle = 0, \quad \text{etc.}, \tag{7.34}$$

which are the characteristics of the Ito multiplication [37]. Note that (7.33) was derived for general $\hat{H}_S$ including non-linear interaction terms.

8 Phase-Space Method

Mapping (4.5) to the one in phase-space by means of the coherent state representation for coarse grained operators, which is constructed just the same process as (H.1)–(H.13) with respect to $|0(t)\rangle$ and $P^{(\mu,\nu)}(z,t)$, we obtain the Fokker-Planck equation in phase-space [34]

$$\frac{d}{dt}P^{(\mu,\nu)}(z,t) = -i\Omega^{(\mu,\nu)}(z)\ P^{(\mu,\nu)}(z,t), \tag{8.1}$$

with

$$P^{(\mu,\nu)}(z,t) = \langle P_f^{(\mu,\nu)}(z,t)\rangle, \tag{8.2}$$

and the coarse-grained generator

$$\Omega^{(\mu,\nu)}(z) = (-\partial z + \partial_* z^*)\,\hat{E}^{(\mu,\nu)}(z,\partial) + i\kappa\,(\partial z + \partial_* z^*) + i2\kappa(\bar{n}+\nu)\partial\partial_*. \tag{8.3}$$

Note that the expression (8.1) for $\mu = 1,\ \nu = 0$ is the same as (A.9) obtained by mapping the master equation (A.1) in the density operator method by means of the coherent representation in the Liouville space (see e.g. [9]).

The quantum stochastic Liouville equation (7.4) of the Stratonovich type is mapped as [34]

$$dP_f^{(\mu,\nu)}(z,t) = -i\Omega_f^{(\mu,\nu)}(z,t)dt \circ P_f^{(\mu,\nu)}(z,t), \tag{8.4}$$

with the stochastic time-evolution generator

$$\Omega_f^{(\mu,\nu)}(z,t)dt = (-\partial z + \partial_* z^*)\,\hat{E}^{(\mu,\nu)}(z,\partial)dt \\ +i\kappa\,(\partial z + \partial_* z^*)\,dt - i\,[\partial \circ dW(t) + \partial_* \circ dW^*(t)]\,, \tag{8.5}$$

mapped from $\hat{H}_{f,t}dt$, where $(-\partial z + \partial_* z^*)\hat{E}^{(\mu,\nu)}(z,\partial)$ is defined for $\hat{H}_S$ by means of (H.10) and (H.11) with the property

$$-z\hat{E}^{(\mu,\nu)}(z,\partial)\partial + z^*\hat{E}^{(\mu,\nu)}(z,\partial)\partial_* = (-\partial z + \partial_* z^*)\,\hat{E}^{(\mu,\nu)}(z,\partial). \tag{8.6}$$

Here, we are confining ourselves to the case where H_S has the structure like $\sum_n g_n(a^\dagger)^n a^n$, which leads us to

$$\hat{E}^{(\mu,\nu)}(z,\partial) = \sum_{\substack{p,q,m,n \\ p+q=m+n}} g_{p,q,m,n}\,[z^p\,(z^*)^q\,\partial^m\partial_*^n + (z^*)^p\,z^q\partial_*^m\partial^n]\,, \tag{8.7}$$

with real parameters $g_{p,q,m,n}$. For a harmonic oscillator with frequency ω, $\hat{E}^{(\mu,\nu)}(z,\partial) = \omega$.

The quantum stochastic Liouville equation (7.4) of the Ito type is mapped to the one in phase-space as

$$dP_f^{(\mu,\nu)}(z,t) = -i\Omega_f^{(\mu,\nu)}(z,t)dt\; P_f^{(\mu,\nu)}(z,t), \tag{8.8}$$

with

$$\Omega_f^{(\mu,\nu)}(z,t)dt = \Omega^{(\mu,\nu)}(z)dt - i\,[\partial\; dW(t) + \partial_*\; dW^*(t)]\,. \tag{8.9}$$

It is easily seen that (8.8) reduces to the Fokker-Planck equation (8.1) when the random average is taken.

The quantum Langevin equation (7.18) of the Stratonovich type is transformed to [34]

$$dA^{(\nu,\mu)}(t) = i\left[-\check{E}^{(\mu,\nu)}(z(t),\partial(t))z(t)\partial(t) + \check{E}^{(\mu,\nu)}(z(t),\partial(t))z^*(t)\partial_*(t)\right]A^{(\nu,\mu)}(t)dt \\ -\kappa\,[z(t)\partial(t) + z^*(t)\partial_*(t)]\,A^{(\nu,\mu)}(t)dt \\ +\left\{\left[\partial(t)A^{(\nu,\mu)}(t)\right]\circ dW(t) + \left[\partial_*(t)A^{(\nu,\mu)}(t)\right]\circ dW^*(t)\right\}, \tag{8.10}$$

where $\partial(t) = \partial/\partial z(t)$ and $\partial_*(t) = \partial/\partial z^*(t)$, and $\check{E}^{(\mu,\nu)}(z,\partial)$ is the adjoint differential operator function defined by

$$\int_z f_1(z)\left[-z\hat{E}^{(\mu,\nu)}(z,\partial)\partial + z^*\hat{E}^{(\mu,\nu)}(z,\partial)\partial_*\right]f_2(z) \\ = \int_z f_2(z)\left[\partial\check{E}^{(\mu,\nu)}(z,\partial)z - \partial_*\check{E}^{(\mu,\nu)}(z,\partial)z^*\right]f_1(z), \tag{8.11}$$

and use has been made of the property

$$-\partial\check{E}^{(\mu,\nu)}(z,\partial)z + \partial_*\check{E}^{(\mu,\nu)}(z,\partial)z^* = -\check{E}^{(\mu,\nu)}(z,\partial)z\partial + \check{E}^{(\mu,\nu)}(z,\partial)z^*\partial_*. \tag{8.12}$$

Using the connection formula between the Ito and Stratonovich products in phase-space which has the same structure as (7.25) for quantum stochastic operators, we can derive the Langevin equation of the Ito type as

$$
\begin{aligned}
dA^{(\nu,\mu)}(t) = & \; i\left[-\check{E}^{(\mu,\nu)}(z(t),\partial(t))z(t)\partial(t) + \check{E}^{(\mu,\nu)}(z(t),\partial(t))z^*(t)\partial_*(t)\right]A^{(\nu,\mu)}(t)dt \\
& -\kappa\left[z(t)\partial(t) + z^*(t)\partial_*(t)\right]A^{(\nu,\mu)}(t)dt + 2\kappa(\bar{n}+\nu)\partial(t)\partial_*(t)A^{(\nu,\mu)}(t)dt \\
& + \left\{\left[\partial(t)A^{(\nu,\mu)}(t)\right]dW(t) + \left[\partial_*(t)A^{(\nu,\mu)}(t)\right]dW^*(t)\right\}.
\end{aligned} \tag{8.13}
$$

This can be obtained also by mapping the quantum Langevin equation (7.27) of the Ito type into the one in phase-space.

By making use of (8.10) or (8.13) for $z(t)$, we have

$$
dz(t) = -i\check{E}^{(\mu,\nu)}(z(t),\partial(t))z(t)dt - \kappa z(t)dt + dW(t). \tag{8.14}
$$

With the help of (8.14), we can rewrite (8.13) in the form

$$
dA^{(\nu,\mu)}(t) = dz(t)\partial(t)A^{(\nu,\mu)}(t) + dz^*(t)\partial_*(t)A^{(\nu,\mu)}(t) + dz(t)dz^*(t)\partial(t)\partial_*(t)A^{(\nu,\mu)}(t), \tag{8.15}
$$

where we used the relation

$$
dz(t)dz^*(t) = dW(t)dW^*(t) = 2\kappa(\bar{n}+\nu)dt, \tag{8.16}
$$

which is proven within the stochastic convergence with (8.14) and the properties (H.15)–(H.17). The equation (8.15) is nothing but the well known Ito's formula for complex stochastic variable $z(t)$.

It is worthy to note that, with the definition of flow:

$$
dz_t = -i\check{E}^{(\mu,\nu)}(z,\partial)zdt - \kappa zdt + dW(t), \tag{8.17}
$$

being in the same structure as (8.14), the stochastic time-evolution generator (8.5) can be expressed in the form [34]

$$
\begin{aligned}
\Omega_f^{(\mu,\nu)}(z,t)dt = & \; -i\left(\partial\, dz_t + \partial_*\, dz_t^*\right) \\
& -\check{E}^{(\mu,\nu)}(z,\partial)\left(-\partial z + \partial_* z^*\right)dt + \left(-\partial z + \partial_* z^*\right)\hat{E}^{(\mu,\nu)}(z,\partial)dt.
\end{aligned} \tag{8.18}
$$

The latter two terms on the right hand side represent quantum effects. This is an extension of Kubo's generator for the stochastic Liouville equation [8] to quantum systems.

Taking average of (8.13) with respect to both the initial distribution $P_f^{(\mu,\nu)}(z)$ and the random forces, we obtain the equation of motion for the expectation value of an arbitrary observable operator $A(t)$ of the relevant system as

$$
\begin{aligned}
\frac{d}{dt}\langle\langle A(t)\rangle\rangle = & \left\langle \int_z A^{(\nu,\mu)}(z)\left[-i\left(-\partial z + \partial_* z^*\right)\hat{E}^{(\mu,\nu)}(z,\partial)\right.\right. \\
& \left.\left. +\kappa\left(\partial z + \partial_* z^*\right) + 2\kappa(\bar{n}+\nu)\partial\partial_*\right]P_f^{(\mu,\nu)}(z,t)\right\rangle \\
= & \int_z A^{(\nu,\mu)}(z)\left[-i\left(-\partial z + \partial_* z^*\right)\hat{E}^{(\mu,\nu)}(z,\partial)\right. \\
& \left. +\kappa\left(\partial z + \partial_* z^*\right) + 2\kappa(\bar{n}+\nu)\partial\partial_*\right]P^{(\mu,\nu)}(z,t),
\end{aligned} \tag{8.19}
$$

where

$$\langle\!\langle A(t)\rangle\!\rangle = \left\langle \int_z A^{(\nu,\mu)}(z)P_f^{(\mu,\nu)}(z,t)\right\rangle = \int_z A^{(\nu,\mu)}(z)P^{(\mu,\nu)}(z,t), \tag{8.20}$$

(see (H.13)). Here, we used the properties

$$\langle z(t)dW(t)\rangle = 0, \quad \text{etc.}, \tag{8.21}$$

which are the characteristics of the Ito multiplication [37]. The averaged equation of motion can also be derived by making use of the Fokker-Planck equation (8.1), as can be seen in the second expression of (8.19).

We showed that the framework, including both the quantum Fokker-Planck equation and the quantum stochastic differential equations constructed within NETFD, is compatible with the one of the classical Fokker-Planck equation and of the classical stochastic differential equations. It was done by mapping the entire framework of NETFD to the c-number phase-space by means of the phase-space method in thermal space [49]. Note that the mapped framework in phase-space keeps the information of quantum effects. The success of the formulation of the stochastic differential equations for quantum systems within a canonical formalism of dissipative quantum fields may be a lesson for those attempts trying to construct it based on the Schrödinger equation or the equivalent [50]-[58].

9 Discussions

In Fig. 1, we put the structure of the methods dealt in this lecture note. The approaches of the Langevin equation (*III*) and of the stochastic Liouville equation (*IV*) are *microscopic* ones in the sense that they take into account thermal effect as a random process, whereas the approach of the Fokker-Planck equation (*I* and *II*) is coarse grained one. A unified formalism for quantum systems covering whole the aspects, *I* to *IV* in Table 1, was realized first by means of the framework of NETFD.

The relation between the Langevin equation and the stochastic Liouville equation is the same as the one between the Heisenberg equation and the Schrödinger equation in quantum mechanics and in quantum field theory. Since they are the stochastic differential equations, there are two types of stochastic multiplication, i.e. the Ito and the Stratonovich types. The Langevin equation (7.17) of the Stratonovich type has the same structure as the Heisenberg equation of motion for analytical quantities. Whereas, the Ito type (7.26) contains an extra term proportional to $dW(t)d\tilde{W}(t)$ due to the difference of stochastic differentiations. Although the stochastic Liouville equations both of the Stratonovich and Ito types, (7.4) and (7.9), have the same form, the latter is more convenient than the former to get the corresponding Fokker-Planck equation (4.5) by taking random average. It is due to the characteristics of the Ito multiplication. The equation of motion for the dynamical variables taken both the random average and the vacuum expectation value can be obtained by two paths, i.e. the one from the Langevin equation directly by taking both the random average and the vacuum expectation, the other from the Fokker-Planck equation by taking the vacuum expectation of the operators corresponding to the dynamical variables. It should be noted that the discovery of the stochastic

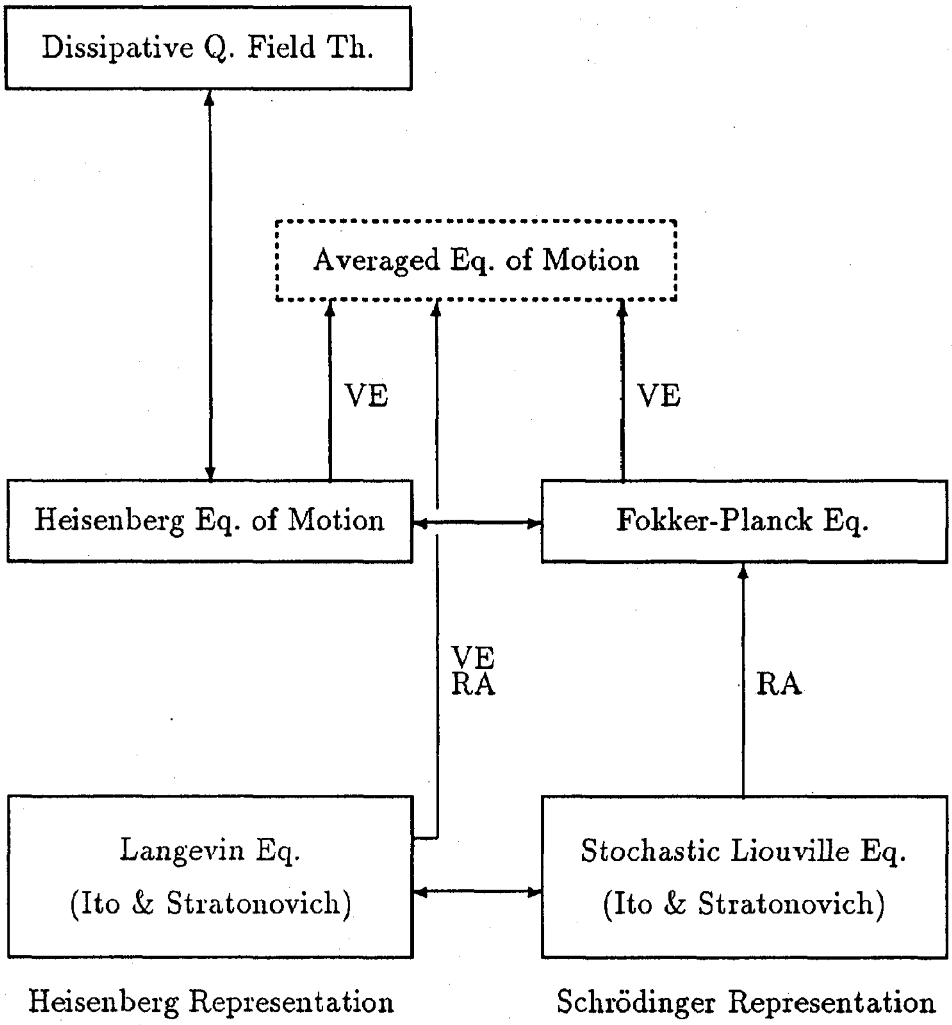

Figure 1: Structure of the Formalism. RA stands for the random average. VE stands for the vacuum expectation.

Liouville equation is the key point for the construction of whole the unified quantum canonical formalism.

With the help of the hat-Hamiltonian for the Fokker-Planck equation, we can construct the Heisenberg equation for coarse grained operators. As was mentioned before, the existence of the Heisenberg equation of motion for coarse grained operators enabled us to construct the canonical formalism of the dissipative quantum fields. It is quite interesting that for somewhat artificial values of μ, ν, i.e. $\mu = 1 + \bar{n}$, $\nu = -\bar{n}$, we can obtain the coarse grained equation of motion (4.16) directly by taking the random average of the Langevin equations (7.18) and (7.27). For this case, (7.25) tells us that the Stratonovich and the Ito multiplications are identical, and (7.3) gives

$$\langle dW(t) d\tilde{W}(s) \rangle = \langle d\tilde{W}(s) dW(t) \rangle = 0. \tag{9.1}$$

The latter indicates that

$$\bar{n} \left\langle dF(t) dF^{\dagger}(s) \right\rangle = (1 + \bar{n}) \left\langle dF^{\dagger}(s) dF(t) \right\rangle, \tag{9.2}$$

as can be seen by (6.13). This is nothing but the KMS-condition [59, 60]. The physical meaning of this artificial case is still to be investigated.

As was claimed by Kubo [61], there had been several deficiencies in the theories of quantum Langevin equation. The first one is that the representation space of the Langevin equation should be an extended Hilbert space which is constituted by both the one for the relevant system and the one for the random force (an irrelevant system). However, usually the equation of motion for the random force operator is not considered. The second is that the correlations of random force operators for *thermal ensemble* do not satisfy KMS-condition [59, 60] in the case of the *white* process for quantum systems. The third claim was how one can obtain the correlation of random force operators for the Langevin equation which is compatible with the master equation derived by the non-conventional treatment of the damping theory, where the effect of non-linearity within a relevant system on its relaxation behavior is taken into account. The last claim was resolved by NETFD (see refs. [28]-[32], [48, 44] and the references therein for detail). As for the first and the second problems, the readers are expected to consider by themselves how they are solved within the present unified formalism of NETFD (see [27]-[29]).

Let us close this lecture note by mentioning about those which were not included in the above sections. It was shown that the *divisor* method of the canonical quantum field theory can be generalized to the present dissipative quantum field theory [15, 16]. The derivation of the generalized kinetic equation within NETFD were studied [21]. Note that most of the studies by means of NETFD were those in the *kinetic stage*. Thermal processes in the *hydrodynamical stage* has started to be investigated by means of NETFD [23]. There, the concept of non-equilibrium thermodynamics, especially that of the *local equilibrium*, is tried to be interpreted in terms of the concept of quantum field theory. There were several applications of NETFD to optical systems [43]-[47] and spin relaxation [48]. Dynamical rearrangements of vacuum in the thermal space were investigated [39] for the boson transformation and the BCS model. The cases of fermion were not investigated. It is somewhat straightforward to extend whole the framework to the case of fermion fields.

Acknowledgement

It has passed just ten years since I started to construct the framework of NETFD in 1983. In this occasion, I would like to thank all the co-workers of the research (including my ex- and present graduate students) whose names are found in the references.

Appendix A Density Operator Method

Here, we show how we had been dealing with the model within the density operator formalism before NETFD was constructed. The master equation for a damped harmonic oscillator is given by [9]

$$\partial_t \rho_S(t) = -i \left(H_S^{\times} + i\Pi \right) \rho_S(t), \tag{A.1}$$

with the symbol $H_S^{\times} X = [H_S,\ X]$, where H_S is the Hamiltonian of the system we are interested in:

$$H_S = \omega a^{\dagger} a, \qquad \omega = \epsilon - \mu, \tag{A.2}$$

with ϵ and μ being the one-particle energy and the chemical potential, respectively, and where Π is the damping operator:

$$\Pi X = \kappa \left\{ [aX,\ a^{\dagger}] + [a,\ Xa^{\dagger}] \right\} + 2\kappa\bar{n}[a,\ [X,\ a^{\dagger}]], \tag{A.3}$$

with $\bar{n}$ being given by (4.2), and

$$\kappa = \Re\mathrm{e}\ g^2 \int_0^{\infty} dt \sum_{\mathbf{k}} \left\langle [R_{\mathbf{k}}(t),\ R_{\mathbf{k}}^{\dagger}(0)] \right\rangle_R e^{i\omega t}. \tag{A.4}$$

Here, we have introduced the average, $\langle \cdots \rangle_R = \mathrm{tr}_R \cdots \rho_R$, where the density operator for a reservoir is given by $\rho_R = Z_R^{-1} e^{-\beta H_R}$, $Z_R = \mathrm{tr}_R\ e^{-\beta H_R}$. The coupling constant g represents the strength of the interaction between the damped harmonic oscillator and the reservoir whose temperature is $T = \beta^{-1}$. We see that the one-particle distribution function, defined by $n(t) = \mathrm{tr}\ a^{\dagger} a \rho_S(t)$, satisfies the Boltzmann equation (4.1).

The above master equation (A.1) can be obtained by projecting out the reservoir by means of the damping theory [9]-[11], starting with the Liouville equation:

$$\frac{\partial}{\partial t} \rho(t) = -iH^{\times} \rho(t), \tag{A.5}$$

with the model given by the Hamiltonian

$$H = H_S + H_R + H_I, \tag{A.6}$$

where H_I is the Hamiltonian describing the interaction between the system and the reservoir:

$$H_I = g \sum_{\mathbf{k}} \left(a R_{\mathbf{k}}^\dagger + \text{h.c.} \right), \tag{A.7}$$

with $R_{\mathbf{k}}^\dagger$ and $R_{\mathbf{k}}$ being the operators of the reservoir, and H_R is the Hamiltonian of the reservoir the explicit form of which needs not be specified to get the master equation (A.1). The coarse-grained density operator $\rho_S(t)$ is defined by $\rho_S(t) = \text{tr}_R\, \rho(t)$.

Introducing the boson coherent state representation of the anti-normal ordering [62]-[64] through

$$\rho_S(t) = \int \frac{d^2 z}{\pi} f_S(t) |z\rangle\langle z|, \tag{A.8}$$

with the boson coherent state $|z\rangle$, defined by $a|z\rangle = z|z\rangle$, we can map the master equation (A.1) into a partial differential equation for the c-number function $f_S(t)$ as [9]

$$\partial_t f_S(t) = \left[-i\omega \left(\partial_* z^* - \text{c.c.} \right) + \kappa \left(\partial_* z^* + \text{c.c.} \right) + 2\kappa \bar{n} \partial_* \partial \right] f_S(t), \tag{A.9}$$

where we have introduced the abbreviation, $\partial = \partial/\partial z$, $\partial_* = \partial/\partial z^*$. This is nothing but a Fokker-Planck equation.

The Fokker-Planck equation (A.9) is transformed into

$$\partial_t F(t) = 2\kappa \left(\xi \partial_\xi + \frac{1}{2} + \bar{n} \partial_\xi \xi \partial_\xi \right) F(t), \tag{A.10}$$

with the help of the relation

$$F(t) = \mathrm{e}^{it\omega(\partial_* z^* - \partial z)} f_S(t), \tag{A.11}$$

where $\xi = |z|^2$, and $\partial_\xi = \partial/\partial\xi$. We can solve (A.10) in the form

$$F(t) = \frac{1}{n(t)} \mathrm{e}^{-\xi/n(t)}, \tag{A.12}$$

with the initial condition $F(0) = f_S(0) = \frac{1}{n}\mathrm{e}^{-\xi/n}$. Here, $n(t)$ in (A.12) satisfies the Boltzmann equation (4.1). In deriving the solution (A.12), we have used the Laguerre polynomials

$$L_\ell(\xi) = \frac{1}{\ell!} \mathrm{e}^{\xi} (-\partial_\xi \xi \partial_\xi)^\ell \mathrm{e}^{-\xi}, \tag{A.13}$$

and the relation

$$\sum_{\ell=0}^{\infty} \frac{1}{\ell!} L_\ell(\xi) x^\ell = \frac{\exp -\xi(x/(1-x))}{1-x}. \tag{A.14}$$

Substituting (A.12) into (A.11), and putting the obtained $f_S(t)$ into (A.8), we have

$$\rho_S(t) = \frac{1}{n(t)} \int \frac{d^2 z}{\pi} \mathrm{e}^{-|z|^2/n(t)} |z\rangle\langle z|. \tag{A.15}$$

This density operator contains the same information as the thermal ket-vacuum $|0(t)\rangle$ given by (4.6).

It may be worthwhile to note that the relation of the operator algebra for a harmonic oscillator within quantum mechanics to the Hermite polynomials is very much similar to the relation of the operator algebra for a damped harmonic oscillator within NETFD to the Laguerre polynomials.

Appendix B The Principle of Correspondence

With the principle of correspondence [65, 1, 2]:

$$\rho_S(t) \longleftrightarrow |0(t)\rangle, \tag{B.1}$$

$$A_1\rho_S(t)A_2 \longleftrightarrow A_1\tilde{A}_2{}^{\dagger}|0(t)\rangle, \tag{B.2}$$

the master equation (A.1) reduces to the Schrödinger equation (4.5) with the hat-Hamiltonian (4.3). It was noticed first by Crawford [66] that the introduction of two kinds of operators for each operator enables us to handle the Liouville equation as the Schrödinger equation.

Appendix C General Form of Hat-Hamiltonian

The hat-Hamiltonian of the semi-free field is bi-linear in $(a, \tilde{a}, a^{\dagger}, \tilde{a}^{\dagger})$, and is invariant under the phase transformation $a \longrightarrow ae^{i\theta}$:

$$\hat{H}_t = g_1(t)a^{\dagger}a + g_2(t)\tilde{a}^{\dagger}\tilde{a} + g_3(t)a\tilde{a} + g_4(t)a^{\dagger}\tilde{a}^{\dagger} + g_0(t), \tag{C.1}$$

where $g(t)$'s are time-dependent c-number complex functions.

Tool 6 makes (C.1) tildian:

$$\hat{H}_t = \omega(t)(a^{\dagger}a - \tilde{a}^{\dagger}\tilde{a}) + i\hat{\Pi}_t, \tag{C.2}$$

with

$$\hat{\Pi}_t = c_1(t)(a^{\dagger}a + \tilde{a}^{\dagger}\tilde{a}) + c_2(t)a\tilde{a} + c_3(t)a^{\dagger}\tilde{a}^{\dagger} + c_4(t), \tag{C.3}$$

where $\omega(t) = \Re\mathrm{e}\ g_1(t) = -\Re\mathrm{e}\ g_2(t)$, $c_1(t) = \Im\mathrm{m}\ g_1(t) = \Im\mathrm{m}\ g_2(t)$, $c_2(t) = \Im\mathrm{m}\ g_3(t)$, $c_3(t) = \Im\mathrm{m}\ g_4(t)$ and $c_4(t) = \Im\mathrm{m}\ g_0(t)$.

Tool 2 and 7 give us relations

$$2c_1(t) + c_2(t) + c_3(t) = 0, \qquad c_3(t) + c_4(t) = 0, \tag{C.4}$$

which reduce (C.3) to

$$\hat{\Pi}_t = c_1(t)(a^{\dagger}a + \tilde{a}^{\dagger}\tilde{a}) + c_2(t)a\tilde{a} - [2c_1(t) + c_2(t)]\, a^{\dagger}\tilde{a}^{\dagger} + [2c_1(t) + c_2(t)]\,. \tag{C.5}$$

The equation of motion for $n(t) = \langle 1|a^\dagger(t)a(t)|0\rangle$ becomes

$$\frac{d}{dt}n(t) = -2[c_1(t) + c_2(t)]n(t) - [2c_1(t) + c_2(t)] \tag{C.6}$$
$$= -2\kappa(t)n(t) + i\Sigma^<(t), \tag{C.7}$$

where we used Tool 2, and introduced $\kappa(t)$ and $\Sigma^<(t)$ defined, respectively, by

$$\kappa(t) = c_1(t) + c_2(t), \tag{C.8}$$
$$\Sigma^<(t) = i[2c_1(t) + c_2(t)]. \tag{C.9}$$

Solving (C.6) and (C.8) with respect to $c_1(t)$ and $c_2(t)$, and substituting them into (C.5), we arrive at the expression (3.1) of the semi-free hat-Hamiltonian.

Appendix D　Stochastic Hat-Hamiltonian

The hat-Hamiltonian for the *stochastic semi-free* field is bi-linear in a, $a^\dagger$, $dF(t)$, $dF^\dagger(t)$ and their tilde conjugates, and is invariant under the phase transformation $a \to ae^{i\theta}$, and $dF(t) \to dF(t)\, e^{i\theta}$:

$$\begin{aligned}\hat{\mathcal{H}}_{f,t}dt = \hat{H}_t dt + i\Big\{&h_1 a^\dagger dF(t) + h_2 a^\dagger d\tilde{F}^\dagger(t) + h_3\tilde{a}\; dF(t) + h_4\tilde{a}\; d\tilde{F}^\dagger(t) \\ &+h_5\tilde{a}^\dagger d\tilde{F}(t) + h_6\tilde{a}^\dagger dF^\dagger(t) + h_7 a\; d\tilde{F}(t) + h_8 a\; dF^\dagger(t)\Big\},\end{aligned} \tag{D.1}$$

where $\hat{H}_t$ is given by (3.1), and h's are time-independent c-number quantities. The time-dependence of h's has been put on the time-dependence of the random force operators $dF(t)$ etc. in the Schrödinger representation. We used the fact that the bi-linear terms with respect to $dF(t)$, $dF^\dagger(t)$ and their tilde conjugates are c-number quantities within the stochastic convergence (see **A3**).

A2 makes $\hat{\mathcal{H}}_{f,t}dt$ tildian, and **A4** gives us the relations

$$h_1 + h_3 = 0, \qquad h_2 + h_4 = 0. \tag{D.2}$$

Then (D.1) reduces to

$$\hat{\mathcal{H}}_{f,t}dt = \hat{H}_t dt + i\Big\{(a^\dagger - \tilde{a})dW(t) + \text{t.c.}\Big\}, \tag{D.3}$$

where $\hat{H}_t$ is given by (C.2), i.e.

$$\hat{H}_t = \omega(t)(a^\dagger a - \tilde{a}^\dagger\tilde{a}) + i\hat{\Pi}_t, \tag{D.4}$$

with

$$\hat{\Pi}_t = -\kappa(t)\left[(a^\dagger - \tilde{a})(\xi a + \eta\tilde{a}^\dagger) + \text{t.c.}\right] + \left\{2\kappa(t)\,[n(t) + \eta] + \frac{d}{dt}n(t)\right\}(a^\dagger - \tilde{a})(\tilde{a}^\dagger - a). \tag{D.5}$$

The expression of $\hat{\Pi}_t$ was arranged, by introducing real c-numbers ξ and η satisfying

$$\xi + \eta = 1, \tag{D.6}$$

so as to be written down by the canonical operators $a - \tilde{a}^\dagger$ and $\xi a^\dagger + \eta\tilde{a}$ satisfying

$$[a - \tilde{a}^\dagger,\ \xi a^\dagger + \eta\tilde{a}] = 1. \tag{D.7}$$

Here, we introduced random force operator $dW(t)$ defined by

$$dW(t) = h_1 dF(t) + h_2 d\tilde{F}^\dagger(t). \tag{D.8}$$

whose cumulants are given by

$$\langle dW(t)\rangle = \langle d\tilde{W}(t)\rangle = 0, \qquad \langle dW(t)dW(t)\rangle = \left\langle d\tilde{W}(t)d\tilde{W}(t)\right\rangle = 0, \tag{D.9}$$

$$\left\langle dW(t)d\tilde{W}(t)\right\rangle = (h_1 + h_2)\left\{h_1^*\left\langle dF^\dagger(t)dF(t)\right\rangle + h_2^*\left\langle dF(t)dF^\dagger(t)\right\rangle\right\}, \tag{D.10}$$

$$\left\langle d\tilde{W}(t)dW(t)\right\rangle = (h_1^* + h_2^*)\left\{h_1\left\langle dF^\dagger(t)dF(t)\right\rangle + h_2\left\langle dF(t)dF^\dagger(t)\right\rangle\right\}, \tag{D.11}$$

where we used **A4**.

The requirement Tool 2 of the commutativity, $\left\langle dW(t)d\tilde{W}(t)\right\rangle = \left\langle d\tilde{W}(t)dW(t)\right\rangle$, gives us the relations

$$(h_1 + h_2)h_1^* = (h_1^* + h_2^*)h_1, \qquad (h_1 + h_2)h_2^* = (h_1^* + h_2^*)h_2. \tag{D.12}$$

which reduce to

$$h_1^* h_2 = h_1 h_2^* = (h_1^* h_2)^*, \tag{D.13}$$

and allow us to put

$$h_1 = \mu e^{i\theta}, \quad h_2 = \nu e^{i\theta}, \tag{D.14}$$

where $\mu = |h_1|$ and $\nu = |h_2|$. Then, (D.10) and (D.11) reduce to

$$\begin{aligned}\left\langle dW(t)d\tilde{W}(s)\right\rangle &= \left\langle d\tilde{W}(s)dW(t)\right\rangle \\ &= (\mu + \nu)\left\{\mu\left\langle dF^\dagger(s)dF(t)\right\rangle + \nu\left\langle dF(t)dF^\dagger(s)\right\rangle\right\}.\end{aligned} \tag{D.15}$$

This shows that $dW(t)$ and $d\tilde{W}(s)$ are commutative even for $t \neq s$, as well as for $t = s$, within the stochastic convergence. The vector $\langle|dW(t)$ is calculated as

$$\langle|dW(t) = (\mu + \nu)e^{i\theta}\langle|dF(t), \tag{D.16}$$

by using **A4**.

The further requirement that the norm of $\langle|dW(t)$ should be equal to that of $\langle|dF(t)$, i.e.

$$\|\langle|dW(t)\| = \|\langle|dF(t)\|, \tag{D.17}$$

leads us to the relation

$$\mu + \nu = 1. \tag{D.18}$$

This requirement indicates that the intensities of the random force operators $dW(t)$ and $dF(t)$ are same. By putting the phase factor $e^{i\theta}$ on $dF(t)$ and $d\tilde{F}^{\dagger}(t)$, (D.15) and (D.8) reduce, respectively, to (6.13) and (6.15). Using (F.18) below, we see that (D.3) with (D.4) and (D.5) gives (6.12).

The quantum Langevin equation (7.27) of the Ito type can be derived also by using the calculus rule of the Ito type:

$$dA(t) = d\hat{S}_f^{-1}(t) \cdot A\hat{S}_f(t) + \hat{S}_f^{-1}(t)A \cdot d\hat{S}_f(t) + d\hat{S}_f^{-1}(t) \cdot A \cdot d\hat{S}_f(t), \tag{D.19}$$

with

$$d\hat{S}_f^{-1}(t) = i\hat{S}_f^{-1}(t)\left[\hat{\mathcal{H}}_{f,t}dt - 2i(a^{\dagger} - \tilde{a})(\tilde{a}^{\dagger} - a)dW(t)d\tilde{W}(t)\right]. \tag{D.20}$$

The latter was derived by the identity

$$0 = d\left[\hat{S}_f^{-1}(t)\hat{S}_f(t)\right] = d\hat{S}_f^{-1}(t) \cdot \hat{S}_f(t) + \hat{S}_f^{-1}(t) \cdot d\hat{S}_f(t) + d\hat{S}_f^{-1}(t) \cdot d\hat{S}_f(t), \tag{D.21}$$

with the help of the properties

$$\begin{aligned} \hat{\mathcal{H}}_{f,t}dt\ \hat{\mathcal{H}}_{f,t}dt &= i\left\{(a^{\dagger} - \tilde{a})dW(t) + \text{t.c.}\right\} i\left\{(a^{\dagger} - \tilde{a})dW(t) + \text{t.c.}\right\} \\ &= -2(a^{\dagger} - \tilde{a})(\tilde{a}^{\dagger} - a)dW(t)d\tilde{W}(t), \end{aligned} \tag{D.22}$$

and

$$\underbrace{\hat{\mathcal{H}}_{f,t}dt\ \hat{\mathcal{H}}_{f,t}dt\ \cdots\ \hat{\mathcal{H}}_{f,t}dt}_{\text{more than 3 times}} = 0, \tag{D.23}$$

within the stochastic convergence (see (D.9) and (D.15)).

Appendix E　Ito and Stratonovich Multiplications

The definitions of the Ito [37] and the Stratonovich [38] multiplications are given, respectively, by

$$X^{(H)}(t) \cdot dY^{(H)}(t) = X^{(H)}(t)\left[Y^{(H)}(t+dt) - Y^{(H)}(t)\right], \tag{E.1}$$

$$dX^{(H)}(t) \cdot Y^{(H)}(t) = \left[X^{(H)}(t+dt) - X^{(H)}(t)\right]Y^{(H)}(t), \tag{E.2}$$

and

$$X^{(H)}(t) \circ dY^{(H)}(t) = \frac{X^{(H)}(t+dt) + X^{(H)}(t)}{2}\left[Y^{(H)}(t+dt) - Y^{(H)}(t)\right], \tag{E.3}$$

$$dX^{(H)}(t) \circ Y^{(H)}(t) = \left[X^{(H)}(t+dt) - X^{(H)}(t)\right]\frac{Y^{(H)}(t+dt) + Y^{(H)}(t)}{2}, \tag{E.4}$$

for arbitrary stochastic operators $X^{(H)}(t)$ and $Y^{(H)}(t)$ in the Heisenberg representation. From (E.1), (E.2) and (E.3), (E.4), we have the formulae which connect the Ito and the Stratonovich products in the differential form

$$X^{(H)}(t) \circ dY^{(H)}(t) = X^{(H)}(t)dY^{(H)}(t) + \frac{1}{2}dX^{(H)}(t) \cdot dY^{(H)}(t), \tag{E.5}$$

$$dX^{(H)}(t) \circ Y^{(H)}(t) = dX^{(H)}(t) \cdot Y^{(H)}(t) + \frac{1}{2}dX^{(H)}(t) \cdot dY^{(H)}(t), \tag{E.6}$$

for the operators in the Heisenberg representation, i.e. $X^{(H)}(t) = \hat{S}_f^{-1}(t)X^{(S)}(t)\hat{S}_f(t)$ with operator $X^{(S)}(t)$ in the Schrödinger representation, and $dX^{(H)}(t) = \hat{S}_f^{-1}(t)dX^{(S)}(t)\hat{S}_f(t)$ with the flow operator $dX^{(S)}(t)$ etc..

The connection formulae for the stochastic operators in the Schrödinger representation are given, in the same form as (E.5) and (E.6), by

$$X^{(S)}(t) \circ dY^{(S)}(t) = X^{(S)}(t)dY^{(S)}(t) + \frac{1}{2}dX^{(S)}(t) \cdot dY^{(S)}(t), \tag{E.7}$$

$$dX^{(S)}(t) \circ Y^{(S)}(t) = dX^{(S)}(t) \cdot Y^{(S)}(t) + \frac{1}{2}dX^{(S)}(t) \cdot dY^{(S)}(t). \tag{E.8}$$

Appendix F Correlation of Random Force Operators

Applying the connection formula (E.8) to the multiplications, for example $dW(t)\hat{S}_f(t)$, in the right hand side of the equation (6.2), we have the equation of motion for the time-evolution generator of the Stratonovich type as

$$d\hat{S}_f(t) = -i\hat{H}_{f,t}dt \circ \hat{S}_f(t), \tag{F.1}$$

where $\hat{H}_{f,t}dt$ is the stochastic semi-free hat-Hamiltonian of the Stratonovich type defined by

$$\begin{aligned}
\hat{H}_{f,t}dt &= \hat{H}_t dt - i(a^\dagger - \tilde{a})(\tilde{a}^\dagger - a)dW(t)d\tilde{W}(t) + i\left[(a^\dagger - \tilde{a})dW(t) + \text{t.c.}\right] \\
&= \omega(t)(a^\dagger a - \tilde{a}^\dagger \tilde{a})dt - i\kappa(t)\left[(a^\dagger - \tilde{a})(\xi a + \eta \tilde{a}^\dagger) + \text{t.c.}\right]dt \\
&\quad + i\left[(a^\dagger - \tilde{a})dW(t) + \text{t.c.}\right].
\end{aligned} \tag{F.2}$$

In deriving the expression (F.2), we demanded that the Stratonovich time-evolution generator should not depend on the diffusion terms, which leads to

$$dW(t)d\tilde{W}(t) = \left\{2\kappa(t)\left[n(t) + \eta\right] + \frac{d}{dt}n(t)\right\}dt. \tag{F.3}$$

This expression is compatible with the assumption that the process is white. Let us put the subscript F on $\Sigma^<(t)$ in the Boltzmann equation (3.2) in order to remember that it is due to the interaction with the random force $dF(t)$:

$$\frac{d}{dt}n(t) = -2\kappa(t)n(t) + i\Sigma_F^<(t). \tag{F.4}$$

Making use of (F.3) and (F.4), we have

$$
\begin{aligned}
i\Sigma_F^{<}(t)dt &= -2\kappa(t)\eta dt + dW(t)d\tilde{W}(t) \\
&= -2\kappa(t)\eta dt + \left\langle dF^\dagger(t)dF(t)\right\rangle + \nu\left\{\left\langle dF(t)dF^\dagger(t)\right\rangle - \left\langle dF^\dagger(t)dF(t)\right\rangle\right\}. \qquad \text{(F.5)}
\end{aligned}
$$

where the property (6.13) has been used within the stochastic convergence, and μ has been erased with the help of (D.18).

It is reasonable to assume that the quantity η may depend on ν, i.e. $\eta = \eta(\nu)$, and that the physical quantities $\kappa(t)$, $i\Sigma_F^{<}(t)$, $\left\langle dF^\dagger(t)dF(t)\right\rangle$, and $\left\langle dF(t)dF^\dagger(t)\right\rangle$ may *not* depend on ν. Then, differentiating (F.5) with respect to ν, we have

$$
0 = -2\kappa(t)\frac{\partial\eta}{\partial\nu}dt + \left\langle dF(t)dF^\dagger(t)\right\rangle - \left\langle dF^\dagger(t)dF(t)\right\rangle. \qquad \text{(F.6)}
$$

This leads to

$$
\frac{\partial\eta}{\partial\nu} = k(t), \qquad \text{(F.7)}
$$

which is solved as

$$
\eta = k(t)\nu + l(t), \qquad \text{(F.8)}
$$

where $k(t)$ and $l(t)$ are real numbers independent of ν. Substituting (F.7) into (F.6), we have

$$
\left\langle dF(t)dF^\dagger(t)\right\rangle - \left\langle dF^\dagger(t)dF(t)\right\rangle = 2\kappa(t)k(t)dt. \qquad \text{(F.9)}
$$

By means of (F.8) and (F.9), (F.5) becomes

$$
i\Sigma_F^{<}(t)dt = -2\kappa(t)l(t)dt + \left\langle dF^\dagger(t)dF(t)\right\rangle, \qquad \text{(F.10)}
$$

which leads to

$$
\begin{aligned}
\left\langle dF^\dagger(t)dF(t)\right\rangle &= \left[i\Sigma_F^{<}(t) + 2\kappa(t)l(t)\right]dt \\
&= \left\{2\kappa(t)\left[n(t) + l(t)\right] + \frac{d}{dt}n(t)\right\}dt, \qquad \text{(F.11)}
\end{aligned}
$$

where we have used (F.4) at the second equality. The substitution of (F.11) into (F.9) gives us

$$
\begin{aligned}
\left\langle dF(t)dF^\dagger(t)\right\rangle &= \left\{i\Sigma_F^{<}(t) + 2\kappa(t)\left[k(t) + l(t)\right]\right\}dt \\
&= \left\{2\kappa(t)\left[n(t) + k(t) + l(t)\right] + \frac{d}{dt}n(t)\right\}dt. \qquad \text{(F.12)}
\end{aligned}
$$

For the system specified by the Boltzmann equation (4.1), (F.11) and (F.12) reduce, respectively, to

$$
\left\langle dF^\dagger(t)dF(t)\right\rangle = 2\kappa\left[\bar{n} + l(t)\right], \qquad \text{(F.13)}
$$

$$
\left\langle dF(t)dF^\dagger(t)\right\rangle = 2\kappa\left[\bar{n} + k(t) + l(t)\right]dt. \qquad \text{(F.14)}
$$

Since the Boltzmann equation (4.1) is compatible with the stationary process specified by

$$\left\langle dF^\dagger(t)dF(t)\right\rangle = 2\kappa\bar{n}dt, \tag{F.15}$$

$$\left\langle dF(t)dF^\dagger(t)\right\rangle = 2\kappa(\bar{n}+1)dt, \tag{F.16}$$

we know that

$$l(t) = 0, \quad k(t) = 1, \tag{F.17}$$

which lead to

$$\eta = \nu \quad (\xi = \mu). \tag{F.18}$$

Substituting (F.17) into (F.11) and (F.12), we obtain (6.16) and (6.17). We also get (6.14) by putting (F.18) into (F.3).

Appendix G Derivation of Fokker-Planck Eq. from Stratonovich Stochastic Liouville Eq.

The Fokker-Planck equation (4.5) can be also derived, systematically, from the stochastic Liouville equation (7.4) of the Stratonovich type by means of the method [25, 26]:

$$\begin{aligned}\left\langle d\,|0_f(t)\rangle\right\rangle &= d|0(t)\rangle \\ &= -i\left\langle \hat{H}_{f,t}dt \circ |0_f(t)\rangle\right\rangle \\ &= -i\hat{H}dt|0(t)\rangle,\end{aligned} \tag{G.1}$$

with

$$-i\hat{H} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t}\hat{K}(t_1)dt_1, \tag{G.2}$$

where

$$\hat{K}(t)dt = \sum_{n=1}^{\infty}\hat{K}_n(t)dt, \tag{G.3}$$

with

$$\hat{K}_n(t)dt = (-i)^n\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-2}}\langle\hat{H}_{f,t}dt\circ\hat{H}_{f,t_1}dt_1\circ\cdots\circ\hat{H}_{f,t_{n-1}}dt_{n-1}\rangle_{o.c.}. \tag{G.4}$$

The symbol $\langle\cdots\rangle_{o.c.}$ indicates the *ordered cumulants* [67, 10] defined, for example, by

$$\langle X(t)\rangle_{o.c.} = \langle X(t)\rangle, \tag{G.5}$$

$$\langle X(t)X(t_1)\rangle_{o.c.} = \langle X(t)X(t_1)\rangle - \langle X(t)\rangle\langle X(t_1)\rangle, \tag{G.6}$$

$$\begin{aligned}\langle X(t)X(t_1)X(t_2)\rangle_{o.c.} &= \langle X(t)X(t_1)X(t_2)\rangle - \langle X(t)X(t_1)\rangle\langle X(t_2)\rangle \\ &\quad -\langle X(t)X(t_2)\rangle\langle X(t_1)\rangle - \langle X(t)\rangle\langle X(t_1)X(t_2)\rangle \\ &\quad +\langle X(t)\rangle\langle X(t_1)\rangle\langle X(t_2)\rangle + \langle X(t)\rangle\langle X(t_2)\rangle\langle X(t_1)\rangle,\end{aligned} \tag{G.7}$$

for any operator $X(t)$.

Substituting (7.7) and (7.8) into (7.5), and using the properties (7.1)–(7.3) for the Wiener process, we obtain the Fokker-Planck generator $\hat{H}$ in (4.5) as

$$\hat{H} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_t^{t+\Delta t} \left\{ \langle \hat{H}_{f,t_1} dt_1 \rangle - i \int_0^{t_1} \langle \hat{H}_{f,t_1} dt_1 \circ \hat{H}_{f,t_2} dt_2 \rangle_{o.c.} \right\}. \tag{G.8}$$

Appendix H Coherent State Representation in NETFD

We introduce a phase-space method for NETFD by means of a generalized coherent state representation [49].

1. The *probability distribution function* $P_f^{(\mu,\nu)}(z,t)$ corresponding to $|0_f(t)\rangle$ is defined by

$$|0_f(t)\rangle = \int_z P_f^{(\mu,\nu)}(z,t) |\Delta^{(\mu,\nu)}(z)\rangle, \tag{H.1}$$

with

$$|\Delta^{(\mu,\nu)}(z)\rangle = \int_\alpha \mathrm{e}^{s|\alpha|^2/2} \mathrm{e}^{z\alpha^* - z^*\alpha} |D(\alpha)\rangle, \tag{H.2}$$

where $|D(\alpha)\rangle$ is specified by

$$(a - \tilde{a}^\dagger)|D(z)\rangle = z|D(z)\rangle, \quad (\mu a + \nu \tilde{a}^\dagger)|D(z)\rangle = -\partial_* |D(z)\rangle, \tag{H.3}$$

and

$$\langle 1|D(z)\rangle = \pi \delta^{(2)}(z), \qquad \delta^{(2)}(z) = \delta(\Re\mathrm{e}(z))\delta(\Im\mathrm{m}(z)). \tag{H.4}$$

Here, we introduced abbreviations $\int_z = \int d^2z/\pi$, and $\partial = \partial/\partial z$, $\partial_* = \partial/\partial z^*$. The parameter $s = \nu - \mu$ specifies the ordering of operators, e.g. $s = 1$ for normal ordering, $s = 0$ for anti-normal ordering and $s = 1/2$ for Weyl ordering. Equation (H.1) shows the correspondence between thermal space and phase-space as

$$(\mu a + \nu \tilde{a}^\dagger)|0_f(t)\rangle \longleftrightarrow z P_f^{(\mu,\nu)}(z,t), \quad (a - \tilde{a}^\dagger)|0_f(t)\rangle \longleftrightarrow \partial_* P_f^{(\mu,\nu)}(z,t). \tag{H.5}$$

Note that the tilde invariance, $|0_f(t)\rangle^\sim = |0_f(t)\rangle$, reads

$$P_f^{(\mu,\nu)}(z,t)^* = P_f^{(\mu,\nu)}(z,t), \tag{H.6}$$

and that $a - \tilde{a}^\dagger$ and $\mu\tilde{a} + \nu a^\dagger$ are canonical operators satisfying the canonical commutation relation

$$[a - \tilde{a}^\dagger,\ \mu\tilde{a} + \nu a^\dagger] = 1. \tag{H.7}$$

2. The phase-space quantity $G^{(\mu,\nu)}(z_1, z_1^*, z_2, z_2^*)$ for the operator $G(a, a^\dagger, \tilde{a}^\dagger, \tilde{a})$ in the thermal space is defined through

$$G(a, a^\dagger, \tilde{a}^\dagger, \tilde{a}) = \int_{z_1} \int_{z_2} G^{(\mu,\nu)}(z_1, z_1^*, z_2, z_2^*) \Delta^{(\mu,\nu)}(z_1) \tilde{\Delta}^{(\mu,\nu)}(z_2), \tag{H.8}$$

with

$$\Delta^{(\mu,\nu)}(z) = \int_\alpha e^{s|\alpha|^2/2} e^{z\alpha^* - z^*\alpha} D(\alpha), \quad D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}. \tag{H.9}$$

Then, for the state

$$G(a, a^\dagger, \tilde{a}^\dagger, \tilde{a})|0_f(t)\rangle = \int_z F^{(\mu,\nu)}(z, z^*, t)|\Delta^{(\mu,\nu)}(z)\rangle, \tag{H.10}$$

we obtain

$$\begin{aligned} F^{(\mu,\nu)}(z, z^*, t) &= e^{\nu\partial^1\partial^2_* - \mu\partial^1_*\partial^2} \\ &\quad \times G^{(\mu,\nu)}(z_1 + \nu\partial_*, z_1^* - \mu\partial, z_2 - \mu\partial_*, z_2^* + \nu\partial) P_f^{(\mu,\nu)}(z, t)\Big|_{\substack{z_1 = z_2 = z \\ z_1^* = z_2^* = z^*}} \end{aligned} \tag{H.11}$$

3. The expectation value of the observable operator

$$G(a, a^\dagger) = \int_z F^{(\mu,\nu)}(z, z^*)\Delta^{(\mu,\nu)}(z), \tag{H.12}$$

is given by

$$\langle 1|G(a, a^\dagger)|0_f(t)\rangle = \int_z F^{(\nu,\mu)}(z, z^*) P_f^{(\mu,\nu)}(z, t). \tag{H.13}$$

4. As for the random force operators $dW(t)$ and $d\tilde{W}(t)$, we cast the mapping correspondence between thermal space and phase-space as

$$dW(t) \longleftrightarrow dW(t), \quad d\tilde{W}(t) \longleftrightarrow dW^*(t). \tag{H.14}$$

The stochastic process for these random forces in phase-space are specified by (7.1)–(7.3) with the replacement of the operators according to the correspondence (H.14). Namely,

$$\langle dW(t)\rangle = \langle dW^*(t)\rangle = 0, \tag{H.15}$$

$$\langle dW(t)dW(s)\rangle = \langle dW^*(t)dW^*(s)\rangle = 0, \tag{H.16}$$

$$\langle dW(t)dW^*(s)\rangle = 2\kappa(\bar{n} + \nu)\delta(t - s)dt\, ds. \tag{H.17}$$

References

[1] T. Arimitsu and H. Umezawa, Prog. Theor. Phys. **74** (1985) 429.

[2] T. Arimitsu and H. Umezawa, Prog. Theor. Phys. **77** (1987) 32.

[3] H. Umezawa, H. Matsumoto and M. Tachiki, *Thermo Field Dynamics and Condensed States* (North-Holland 1982).

[4] T. Arimitsu, Mathematical Sciences [*Sūri Kagaku*], June (1990) 22, in Japanese.

[5] R. Kubo, J. Phys. Soc. Japan **9** (1954) 935.

[6] P. W. Anderson, J. Phys. Soc. Japan **9** (1954) 316.

[7] R. Kubo, J. Math. Phys. **4** (1963) 174.

[8] R. Kubo, M. Toda and N. Hashitsume, *Statistical Physics II* (Springer, Berlin 1985).

[9] F. Haake, *Springer Tracts in Modern Physics*, vol. 66 (Springer-Verlag, 1973) 98.

[10] F. Shibata and T. Arimitsu, J. Phys. Soc. Japan **49** (1980) 891, and references therein.

[11] T. Arimitsu, J. Phys. Soc. Japan **51** (1982) 1720.

[12] T. Arimitsu and H. Umezawa, Prog. Theor. Phys. **77** (1987) 53.

[13] T. Arimitsu, M. Guida and H. Umezawa, Europhys. Lett. **3** (1987) 277.

[14] T. Arimitsu, M. Guida and H. Umezawa, Physica **A148** (1988) 1.

[15] T. Arimitsu and H. Umezawa, J. Phys. Soc. Japan **55** (1986) 1475.

[16] T. Arimitsu, H. Umezawa and Y. Yamanaka, J. Math. Phys. **28** (1987) 2741.

[17] T. Arimitsu, J. Pradko and H. Umezawa, Physica **A135** (1986) 487.

[18] J. Schwinger, J. Math. Phys. **2** (1961) 407.

[19] L. V. Keldysh, Sov. Phys. JETP **20** (1965) 1018.

[20] K. Chou, Z. Su, B. Hao and L. Yu, Phys. Rep. **118** (1985) 1.

[21] T. Arimitsu, Physica **A148** (1988) 427.

[22] T. Arimitsu, Physica **A158** (1989) 317.

[23] T. Arimitsu, J. Phys. A: Math. Gen. **24** (1991) L1415.

[24] D. N. Zubarev and M. V. Tokarchuk, Theor. Math. Phys. **88** (1991) 876.

[25] T. Arimitsu, *Thermal Field Theories*, eds. H. Ezawa, T. Arimitsu and Y. Hashimoto (North-Holland, 1991) 207.

[26] T. Arimitsu, Phys. Lett. **A153** (1991) 163.

[27] T. Saito and T. Arimitsu, Modern Phys. Lett. B **6** (1992) 1319.

[28] T. Arimitsu and T. Saito, Bussei Kenkyu **59-2** (1992) 213, in Japanese.

[29] T. Arimitsu and T. Saito, *A Unified Framework of Quantum Stochastic Differential Equations*, Proceedings of the Conference on Field Theory and Collective Phenomena (1993) in press.

[30] T. Arimitsu and T. Saito, Vistas in Astronomy (Pergamon Press, 1992) in press.

[31] T. Arimitsu, M. Ban and T. Saito, Physica **A177** (1991) 329.

[32] T. Arimitsu, M. Ban and T. Saito, in *Structure: from Physics to General Systems*, eds. M. Marinaro and G. Scarpetta (World Scientific, 1991) 163.

[33] T. Saito and T. Arimitsu, *Quantum Stochastic Equations for A Non-Linear Damped Oscillator*, Modern Phys. Lett. B (1993) in press.

[34] T. Arimitsu and T. Saito, *Quantum Stochastic Differential Equations in Phase-Space Methods* (1993) submitted.

[35] T. Saito and T. Arimitsu, *Quantum Stochastic Liouville Equation of Ito Type*, Modern Phys. Lett. B (1993) in press.

[36] T. Arimitsu and T. Saito, *General Structure of the Time-Resolved Generator for Quantum Stochastic Liouville Equation* (1993) in preparation to submit.

[37] K. Ito, Proc. Imp. Acad. Tokyo **20** (1944) 519.

[38] R. Stratonovich, J. SIAM Control **4** (1966) 362.

[39] T. Arimitsu, Y. Sudo and H. Umezawa, Physica **A146** (1987) 433.

[40] T. Arimitsu and H. Umezawa, in: *Advances on Phase Transitions and Disordered Phenomena*, eds. G. Busiello, L. De Cesare, F. Mancini and M. Marinaro (World Scientific, Singapore 1987) pp. 483-504.

[41] H. Umezawa and T. Arimitsu, in: *Foundation of Quantum Mechanics — In the Light of New Technology*, eds. M. Namiki, Y. Ohnuki, Y. Murayama and S. Nomura (Physical Society of Japan, Tokyo 1987) pp. 79-90.

[42] T. Arimitsu, H. Umezawa, Y. Yamanaka and P. Papastamatiou, Physica **A148** (1988) 27.

[43] M. Ban and T. Arimitsu, Physica **A146** (1987) 89.

[44] T. Tominaga, T. Arimitsu, J. Pradko and H. Umezawa, Physica **A150** (1988) 97.

[45] I. Iwasaki, T. Arimitsu and F. H. Willeboordse, *Thermal Field Theories*, eds. H. Ezawa, T. Arimitsu and Y. Hashimoto (North-Holland, 1991) 459.

[46] T. Arimitsu F. H. Willeboordse and T. Iwasaki, Physica A **182** (1991) 214.

[47] T. Arimitsu and N. Arimitsu, *A Formulation of Time-Resolved Optical Spectrum* (1993) preprint, U of Tsukuba.

[48] T. Tominaga, M. Ban, T. Arimitsu, J. Pradko and H. Umezawa, Physica **A149** (1988) 26.

[49] T. Arimitsu (1993) in preparation to submit.

[50] I. R. Senitzky, Phys. Rev. **119** (1960) 670.

[51] M. Lax, Phys. Rev. **145** (1966) 110.

[52] H. Haken, *Optics. Handbuch der Physik* vol. XXV/2c (1970), [*Laser Theory* (Springer, Berlin, 1984)], and the references therein.

[53] R. F. Streater, J. Phys. A: Math. Gen. **18** (1982) 1477.

[54] C. W. Gardiner and M. J. Collett, Phys. Rev. A **31** (1985) 3761.

[55] H. Hasegawa, J. R. Klauder and M. Lakshmanan, J. Phys. A: Math. Gen. **18** (1985) L123.

[56] R. L. Hudson and K. R. Parthasarathy, Commun. Math. Phys. **83** (1984) 301.

[57] K. R. Parthasarathy, Rev. Math. Phys. **1** (1989) 89.

[58] L. Accardi, Rev. Math. Phys. **2** (1990) 127.

[59] R. Kubo, J. Phys. Soc. Japan **12** (1957) 570.

[60] P. C. Martin and J. Schwinger, *Phys. Rev.* **115** (1959) 1342.

[61] R. Kubo, J. Phys. Soc. Japan **26** Suppl. (1969) 1.

[62] E. C. G. Sudarshan, Phys. Rev. Lett. **10** (1963) 277.

[63] R. J. Glauber, Phys. Rev. **131** (1963) 2766.

[64] G. S. Agarwal and E. Wolf, Phys. Rev. D **2** (1970) 2161, 2187 and 2206.

[65] M. Schmutz, Z. Phys. **B 30** (1978) 97.

[66] J. A. Crawford, Nuovo Cim. **10** (1958) 698.

[67] R. Kubo, J. Phys. Soc. Japan **17** (1962) 1100.