研究会報告

ORIGIN OF THE LONG TIME TAIL IN CHAOTIC DYNAMICS

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Abstract

One of the most striking phenomena in chaotic systems is the appearance of long time tails or $f^{-\nu}$ spectral fluctuations. The main purpose of this paper is to discuss the recent studies concerning the long time behaviors in non-stationary regime. In the first part, some ergodic-theoretical quantities and indecies are concretely determined by use of the modified Bernoulli shift. The origin of $f^{-\nu}$ spectral fluctuations and the stationary/non-stationary phase transition are characterized in terms of the large derivation theory. The break-up process not only in the law of large number, but also in the law of small number will be precisely described. In the second part, it will be shown that the nearly integrable Hamiltonian systems universally reveals a typical long time tail, and that the universal behavior can be theoretically explained by the stagnant layer theory based on the Nekhoroshev theorem. The importance of two key concepts – non-stationarity and multi-ergodicity will be emphasized. Finally, the origin of the $f^{-\nu}$ noises in quartz crystal experiments will be discussed from the view-point of the stagnant layer theory.

1. Introduction

The transition from the stationary chaos to the non-stationary chaos is one of the most interesting phenomena in dynamical systems. Typical examples of such transition are often observed in the intermittent chaos and the Arnold diffusion in Hamiltonian dynamics. The predictability of such chaos in the transition regime should be discussed from two different angles; deterministic and statistical ones. It is important to point out that both predictions are mutually complementary. In deterministic prediction, the unpredictability measure is computational errors, and on the other hand the statistical unpredictability is measured by an averaged quantity such as the standard deviation. The deterministic prediction is more useful than the statistical one only for the time stage $t < \lambda^{-1}$, but the statistical predictability recovers for $t > \lambda^{-1}$ where λ is the Lyapunov exponent. However, it is very essential that both deterministic and statistical predictions completely fail in the transition regime between stationary and non-stationary chaos.

In §2, the stationary/non-stationary phase transition is characterized in terms of the large deviation theory. Some anomalous behaviors in the Allan variance, the entropy function and the Lyapunov exponent distribution are discussed, and corresponding characteristic indecies are precisely determined by computer simulations. Results are compared with theoretical estimations[1,2].

In §3, recent developments of the stagnant layer theory for Hamiltonian systems are reviewed [3,4,5], and some typical phenomena such as the chaotic scattering [6] and the induction phenomenon in the crystal lattice vibration [4,7,8] are explained from the general viewpoint of the stagnant layer effect. The first passage time distribution P(T) universally obeys an inverse power law; $P(T) \propto T^{-1}$, so that the mean recurrence time becomes divergent. This is an essential feature of the non-stationarity in nearly integrable Hamiltonian systems, where the existence of an infinite number of invariant tori plays a significant role. Furthermore, the power spectral density (PSD) function $S(f)$ usually becomes multi-ergodic, that is to say, $S(f)$ has a lot of singularities. Finally, in the discussion §4 the quartz crystal experimental results will be explained by the simple lattice vibration model with a small dissipation, and that the $1/f$ spectra of phase noises, phonon number fluctuation and the dielectric constant (imaginary part) are successfully reproduced.

2. Stationary/Non-stationary Phase Transition in Modified Bernoulli Shift

Let us consider the modified Bernoulli map which is defined in a unit interval $x \in [0,1]$,

$$x_{n+1} = T(x_n) = \begin{cases} x_n + 2^{B-1}x_n^B, & \text{for } x_n \le 1/2 \\ x_n - 2^{B-1}(1-x_n)^B, & \text{for } x_n > 1/2 \end{cases}$$

where B is a positive parameter and n stands for discrete integer time. The Bernoulli shift corresponds to $B = 1$. The symbolic sequence σ_n is defined by,

$$\sigma_n = sgn(2x_n - 1) \tag{1}$$

The sequence reveals a remarkable long time tail, for instance the PSD $S(f;\sigma)$ obeys,

$$S(f,\sigma) \propto f^{-\nu} \text{for } f \ll 1 \tag{2}$$

with $\nu = 3 - \beta(\beta = B/(B-1))$, so that the transition from stationary to non-stationary occurs at B=2[1,2]. The parameter regime is classified as follows,

(i) $B < 3/2$	Gaussian stationary regime
(ii) $3/2 < B < 2$	Non-Gaussian stationary regime
(iii)$2 < B$	Non-stationary regime

The distribution function of the partial mean $Y = \frac{1}{N}\sum_{i=1}^{N}\sigma_i$ becomes,

$$P_N(Y) \simeq exp[-N^{\theta}\Phi(Y)] \tag{3}$$

where θ is a scaling index and $\Phi(Y)$ is the entropy function. In each regime results are classified as follows;

(i)$B < 3/2$

$$\begin{aligned} \theta &= 1 \\ \Phi(Y) &\propto \frac{Y^2}{1+Y^2} \end{aligned} \tag{4}$$

(ii)$3/2 \le B < 2$

$$\begin{aligned} \theta &= 2(2-B) \\ \Phi(Y) &\propto \frac{Y^2}{((1+Y)^{2\alpha} + (1-Y)^{2\alpha} - 2b(1-Y^2)^{\alpha})^{1/\alpha}} \end{aligned} \tag{5}$$

(iii)$2 \le B$

$$\begin{aligned} \theta &= 0 \\ \Phi(Y) &\propto \frac{(1-Y^2)^{\alpha-1}}{(1+Y)^{2\alpha} + (1-Y)^{2\alpha} - 2b(1-Y^2)^{\alpha}} \end{aligned} \tag{6}$$

where b is a parameter and $\alpha = \beta - 1$. Numerical calculations are well supporting these estimations (see Fig.1).

Next we consider the Allan variance which is defined by,

$$\sigma_A^2(N) = \frac{1}{2}\langle(\frac{1}{N}\sum_{i=1}^{N}\sigma_i - \frac{1}{N}\sum_{i=1}^{N}\sigma_{i+1})^2\rangle \tag{7}$$

and for large N the following scaling is satisfied,

$$\sigma_A^2(N) = O(N^{\gamma})$$

The relation $\gamma = \theta$ is obtained straightforwardly in the stationary regime. However, in the non-stationary regime the fluctuation of the Allan variance itself is always very large, so that the overall behavior should be considered $\gamma = 0$, though the scaling regime characterized by $\gamma = (\beta - 2)/(\beta - 1)$ comes to appear by introducing small perturbations[2]. The appearance of such pseudo-scaling regime can be analysed by the excess entropy function $\Delta\Phi$,

$$\Delta\Phi \propto O(N^{(\beta-2)/(\beta-1)}) \tag{8}$$

The estimation of the index γ is an important problem in various experiments of $1/f$ noises, since the index for the power spectral density ν is interrelated to γ as follows,

$$\nu = 1 - \gamma = 3 - \beta \tag{9}$$

then $\gamma = 0$ is called the flicker floor. Scaling indecies are summarized in Fig.2 [2].
The distribution of the local Lyapunov exponent is determined as follows ; putting $\lambda = \frac{1}{N}\sum_{i=1}^{N} \ln |T'(x_i)|$, the variance $\langle(\lambda - \lambda_0)^2\rangle$ is scaled as,

$$\langle(\lambda - \lambda_0)^2\rangle \simeq O(N^{\delta}) \tag{10}$$

where $\delta = -2(2 - B)$ in the stationary regime, and $\delta = (2 - B)/(B - 1)$ in the non-stationary regime. Furthermore, the distribution function obeys an inverse power law $P(\lambda) \propto \lambda^{\beta-3}$ for $N = 1$. Figure 3 shows the results of numerical and theoretical calculations[2]. The singularity of $P(\lambda)$ at $\lambda = 0$ implies the incompleteness of the probabilistic description due to the non-stationarity, for example if we consider the recurrence of a very rare event given by the symbolic sequence $\{\sigma_0\sigma_1\sigma_2\cdots.\sigma_N\} = \{1\ 0^N\}$, the recurrence time distribution $P(\tau)$ is analytically derived for large N,

$$P(\tau) \propto \begin{cases} exp[-\tau/\langle\tau\rangle], & \text{for } B < 2 \\ \tau^{-\beta}, & \text{for } B \geq 2 \end{cases}$$

The point is that the poissonian law or the law of small number breaks down as well in the non-stationary regime [9].

3. Long Time Tails in Hamiltonian Chaos

Universal aspects of the long time tail in Hamiltonian systems are clearly explained from the stagnant layer theory[3,4,5]. Here the basic idea and results of the stagnant layer theory will be quickly reviewed, and some evidence and applications of the theory will be given.

We consider the nearly integrable system, of which Hamiltonian is

$$H(p,q) = H_0(p) + \epsilon H_1(p,q) \tag{11}$$

where action-angle variables (p, q), and the small perturbation ϵH_1. When ϵ is small enough, almost all tori remain stably in the off-resonant region,

$$|(k) \cdot \frac{\partial H_0}{\partial P}| > |k|^{-a} \tag{12}$$

But in the resonant region non-integrable chaotic orbits come out together with the higher order resonance tori(so-called Poincare-Birkhoff chain). Then the characteristic time scale T for the chaotic diffusion was derived by N. N. Nekhoroshev[10],

$$T \simeq \epsilon^{-1} exp[\epsilon^{-b}] \tag{13}$$

This kind of slow motion is called "Arnold diffusion", and its origin and the detailed mechanism have been surveyed in the boundary layer between chaos and torus or the stagnant layer.

The distribution of the pausing time $P(T)$ in the stagnant layer was derived by a scaling-theoretical approach[3,4],

$$P(T) \propto \frac{1}{T \log T} \tag{14}$$

and the PSD function for typical dynamical variables $S(f)$ should reveal,

$$S(f) \propto f^{-\nu} \quad (\nu = 2) \tag{15}$$

These results are universal in every stagnant layer and for all Hamiltonian systems as well. The essential point is that the mean pausing time is divergent. In other words, the motion in the stagnant layer is non-stationary. Furthermore, the PSD function for globally diffusive motions in phase space usually becomes,

$$S(f) \propto \sum_k A_k (f - f_k)^{-\nu_k} \tag{16}$$

where f_k is the characteristic frequency for the k-th stagnant layer. A_k and ν_k stand for the amplitude and the index for each stagnant motion. Since each singularity in $S(f)$ denotes the metrical decomposability of phase space, the dynamical system with such singularities is called multi-ergodic. The large deviation theory discussed in Eq.(10) suggests that the amplitude A_k should be scaled as,

$$A_k \propto O(N^{\xi}) \tag{17}$$

Indeed, the relation is justified for the standard mapping [4].

The induction phenomenon in crystal lattice vibrations is a typical example which shows the remarkable effect of the stagnant layer[4,7,8]. Let us consider the following Hamiltonian system with a small nonlinearity β,

$$H(p,x) = \frac{1}{2m}\sum_{i=1}^{N} p_i^2 + \frac{k}{2}\sum_{i=1}^{N}(x_{i+1} - x_i)^2 + \frac{\beta}{4}\sum_{i=1}^{N}(x_{i+1} - x_i)^4 \tag{18}$$

The time course of a normal mode energy usually reveals a long stagnation period τ – so-called induction time before the equipartition low of energy would be realized[8]. The distribution of the induction time $P(\tau)$ obeys Eq.(14) and the Nekhoroshev bound is also clearly adjusted (see Fig.3).

The diversity of such induction phenomena has been pursued still now [11], and the most striking point is that the induction is not a typical phenomena only in the early stage, but also in the aged stage where the thermal equilibrium should be realized. This is an important point in order to extend the large deviation theory to the case of the long time tail in the local equilibrium state [12].

4. Application to the Quartz Oscillation System – Beyond the Stagnant Layer Theory

An appropriate canonical transformation rewrites the equation of motion into the eigen mode (I_i, θ_i) description,

$$\begin{aligned} \dot{I}_i &= -\epsilon\frac{\partial \tilde{H}_1}{\partial \theta_i} \\ \dot{\theta}_i &= \omega_i + \epsilon\frac{\partial \tilde{H}_1}{\partial I_i}, (i = 1, 2, \cdots, N) \end{aligned} \tag{19}$$

where ω_i's are the proper angular frequencies. According to the phonon-picture in crystals, the eigen frequency does not change and only the number of phonons fluctuates around an averaged quantity in thermal equilibrium. But in the nonlinear systems, both quantities should be changed. The fluctuation of the frequency is simply called "phase noise". The recent experiments carried out with quartz crystals reveal a very long time fluctuation generated even in the thermal equilibrium state, to say $1/f$ noise. The origin of the $1/f$ noise has not yet been completely elucidated, and many possible scenario have been proposed so far. Here I will propose a new interpretation of the mechanism to generate the $1/f$ noise in crystals.

In quartz oscillator experiments, the spectral indecies ν are usually smaller than 2, e.g. $0.8 \leq \nu \leq 1.5$. On the other hand, the stagnant layer theory predicts the relation $\nu = 2$ strictly. The difference between both indecies is very crucial, though the artifact is unavoidable in material experiments. Our idea is to introduce the mechanism leading to the free energy dissipation of crystals, and extend the lattice vibration model to include the small energy loss. The significance of the free energy dissipation is clearly indicated in the measurement of the imaginary part of dielectric constant for TGS crystal [13,14]. The detailed physical process for the dissipation has not yet been fully solved, but it is surmised that the cavity loss through some resonant modes (I_r, θ_r) can be described by,

$$\begin{aligned} \dot{I}_r &= 1\frac{\partial \bar{H}_1}{\partial \theta_r} + \gamma g(I,\theta) \\ \dot{\theta}_r &= \omega_r + \frac{\partial \bar{H}_1}{\partial I_r} + \gamma(I,\theta) \end{aligned} \tag{20}$$

where γ is a small damping coefficient. The detailed mechanisms of energy transfer are all included in the coupling functions g and h. Here we consider a very simple case of the Rayleigh's dissipation [12]. The phase noise is described by the fluctuation of $\dot{\theta}(t)$and the phonon number fluctuation by $I(t)$. Furthermore the imaginary part of the dielectric constant can be assumed to reflect the change of the Rayleigh's dissipation function $R(t)$ which is derived from Eq.(20). The PSD function for each variable $(S(f;I), S(f;\dot{\theta})$ and $S(f;R))$ is shown in Fig.4. The point is that the spectral index ν for each quantity is smaller than the value predicted by the stagnant layer theory, i.e., $1 < \nu < 1.5$, and that a certain scaling regime can be clearly obtained. The results obtained here are well reproducing the experimental data[12]. Theoretical studies for the dissipative model given by Eq.(20) have not yet been finished.

Further problems

The long time tail in Hamiltonian dynamics raises some significant problems concerning the foundation of statistical mechanics. One is the effect to the Boltzmann equation which describes the kinetic behaviors of gases, where the Boltzmann-Grad limit is usually assumed. However, the study on the chaotic scattering reveals the break-down of that limit, that is to say the collision time may become larger than the mean free time. Furthermore, the scattering cross-section may become a infinitely many valued function [6]. Then, it is surmised that these results induce the macroscopic fluctuations in local equilibrium state. Another is the effect to the transport equation of heat. By the use of lattice vibration models dynamical theory of heat transfer has been pursued, but the realistic interpretation has not yet been obtained from the microscopic level based on Hamiltonian dynamics, e.g., the FPU model with small nonlinearity does not give a reasonable estimation of thermal conductivity. On the other hand the diatomic Toda potential model can give a correct conductivity consistent with the Fourier's law. The time course of heat diffusion sensitively depends not only on the strength of the nonlinearity but also the effect of stagnant layers in Hamiltonian phase space [11]. These problems will be briefly discussed elsewhere.

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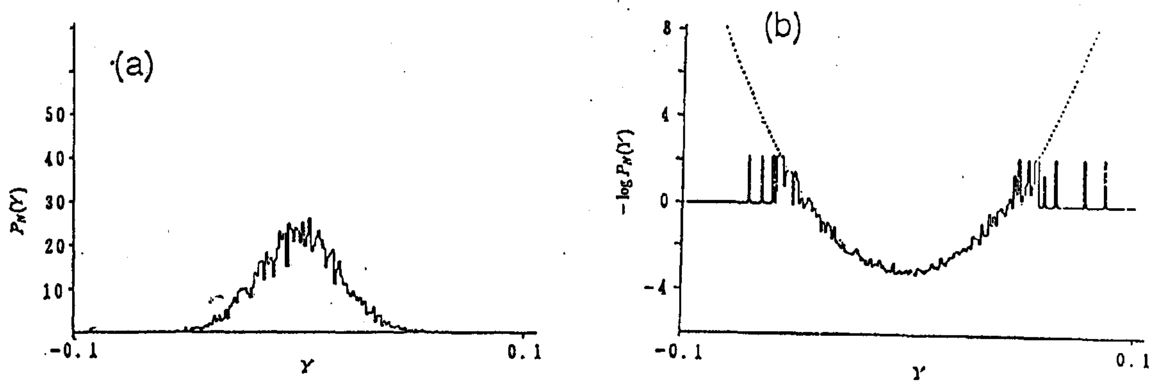

Fig.1 Large deviation properties (B=1.1); (a) distribution function and (b) entropy function. The theoretical result is shown by a dotted line.

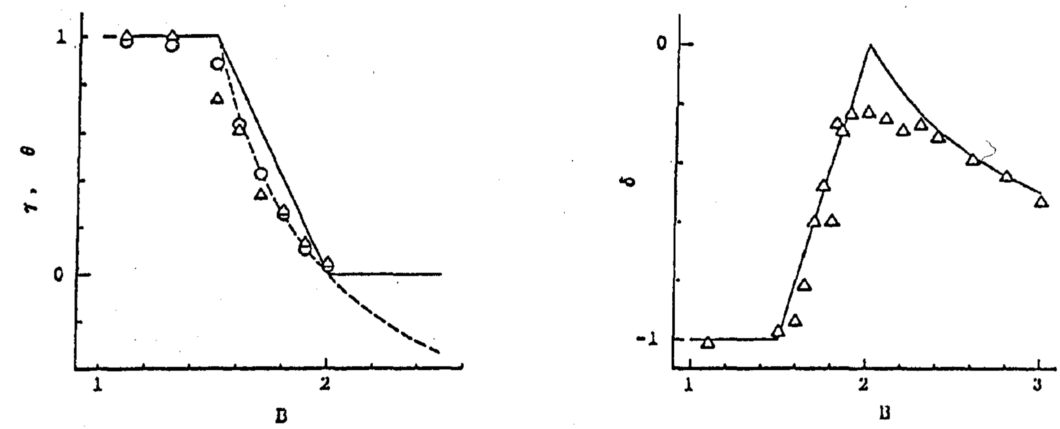

Fig.2 Scaling indices for the entropy function (△), the Allan variance (□) and the distribution of Lyapunov exponent (δ).

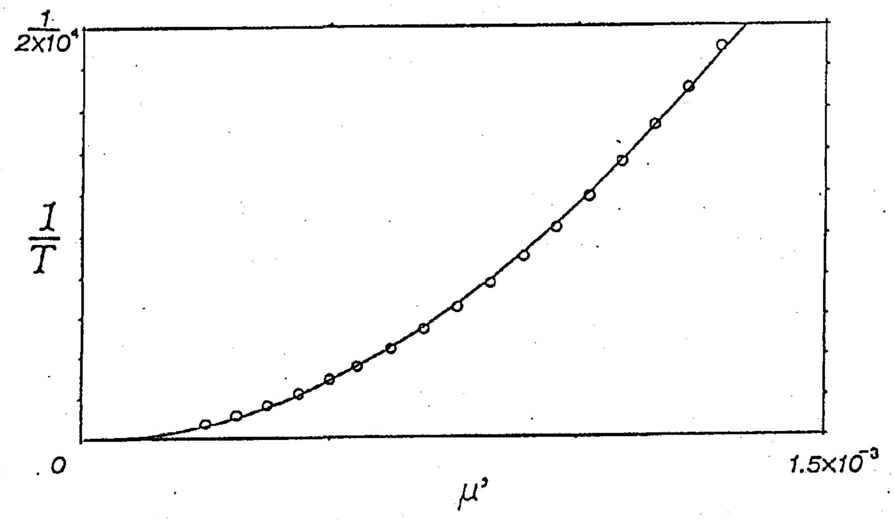

Fig.3 The Nekhoroshev bound for a lattice vibration model. μ' stands for effective nonlinear parameter [8].

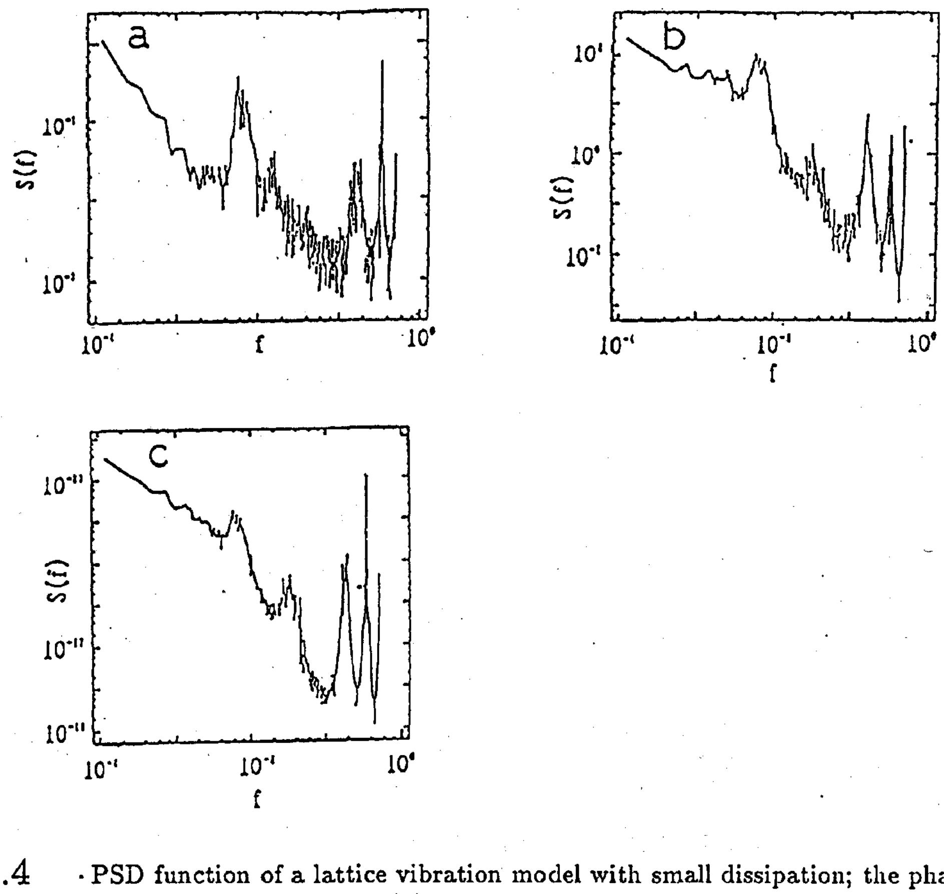

Fig.4 · PSD function of a lattice vibration model with small dissipation; the phase noise (a), the phonon number fluctuation (b) and the Rayleigh's function (c).