

Fomin's* Lecture Note on Low Temperature Phases of ^3He

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§1 Introduction

^3He at low temperatures is a good object for the application and testing of basic ideas of condensed matter physics. At low pressures and above $\sim 1\text{mK}$ ^3He is a normal Fermi liquid which is described by the Landau theory. Below $\sim 1\text{mK}$ it is a superfluid with unconventional Cooper pairing and at pressures above ~ 30 bar it is a quantum solid, forming magnetically ordered phases at temperatures $\sim 1\text{mK}$. In these lectures there will be discussed mainly the liquid properties of ^3He at low-temperatures; normal phase and superfluid phases with the emphasis on their magnetic properties. For introduction of basic ideas, notations and approaches, it will be convenient to start with the simplest Fermi system; that is, an ideal Fermi gas.

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** 本稿は京大 理学専攻大学院で「低温物理学特論」として Fomin 教授が行なった講義の一部を、堀田尚二 (京都大学理学部物理学第一教室 大学院生) が整理しまとめたものである。この後、超流動 ^3He についての一般的な解説の後、スピン動力学についての講義があった。

§2. Ideal Fermi-gas

2.1 Distribution function

Let us consider a collection of noninteracting identical particles with spin $s = 1/2$. If there is no external field one can choose as a basis of the states with a given momentum \mathbf{p} and projection of spin $\sigma = \pm 1/2$ (or \uparrow, \downarrow):

$$\psi_{\mathbf{p},\sigma} = \frac{\chi_{\sigma}}{\sqrt{V}} e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}$$

where V is the volume occupied by the gas, \hbar is Planck's constant, and χ_{σ} is the spin part of the wave function. In thermodynamics we are dealing with properties of the gas which are averaged over an equilibrium ensemble of systems. More generally the ensemble is characterized by the average occupation numbers, or a distribution function, which formally can be written as

$$n(\mathbf{p},\sigma) = \langle \hat{a}_{\mathbf{p}\sigma}^{\dagger} \hat{a}_{\mathbf{p}\sigma} \rangle , \quad (1)$$

where $\hat{a}_{\mathbf{p}\sigma}^{\dagger}$ and $\hat{a}_{\mathbf{p}\sigma}$ are correspondingly operators of creation and annihilation of a particle in a state (\mathbf{p},σ) , and brackets denote averaging over a given ensemble. A distribution function is a convenient macroscopic characteristic of a state of a gas. Many other macroscopic characteristics of a gas can be expressed in terms of $n(\mathbf{p},\sigma)$ in a straightforward way. These are the total energy

$$E = \sum_{\mathbf{p},\sigma} \varepsilon(\mathbf{p},\sigma) n(\mathbf{p},\sigma) , \quad (2)$$

total momentum

$$\mathbf{P} = \sum_{\mathbf{p},\sigma} \mathbf{p} n(\mathbf{p},\sigma) , \quad (3)$$

total z -projection of spin

$$S_z = \frac{\hbar}{2} \sum_{\mathbf{p},\sigma} \sigma n(\mathbf{p},\sigma) = \frac{\hbar}{2} \sum_{\mathbf{p}} \{n_{\uparrow}(\mathbf{p}) - n_{\downarrow}(\mathbf{p})\} . \quad (4)$$

For a more complete description of a spin-dependent properties one would have to

consider the 2×2 density matrix

$$n_{\alpha\beta}(\mathbf{p}) = \langle \hat{a}_{p\alpha}^\dagger \hat{a}_{p\beta} \rangle \quad (5)$$

Then for all three projections of spin we have,

$$\mathbf{S} = \frac{\hbar}{2} \sum_{\mathbf{p}, \alpha\beta} \boldsymbol{\sigma}_{\alpha\beta} n_{\alpha\beta}(\mathbf{p}) \quad (6)$$

where $\boldsymbol{\sigma}_{\alpha\beta} = (\sigma_{\alpha\beta}^x, \sigma_{\alpha\beta}^y, \sigma_{\alpha\beta}^z)$ is a vector of Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The entropy S of an ideal gas can be also expressed in terms of $n(\mathbf{p}, \sigma)$ via the standard formula

$$S = - \sum_{\mathbf{p}, \sigma} \{ n \ln n + (1 - n) \ln(1 - n) \} \quad (7)$$

For practical purposes the most interesting properties of a system are those near thermodynamic equilibrium at given conditions. According to the general prescription of thermodynamics an equilibrium state minimizes the thermodynamic potential, according to a given condition. In particular, to find the equilibrium distribution function $n_{eq}(\mathbf{p}, \sigma)$ at a given temperature T and a given total number of particles N in a volume V , one would have to minimize a free energy $F = E - TS$ with respect to an arbitrary variation of n at the constraint of $N = \text{constant}$. Minimization with this constraint is technically inconvenient. The usual way to avoid it is to consider ensemble of systems with different total number of particles but with the given average number $\langle N \rangle$. In that case minimization has to be carried out at given temperature and chemical potential μ . The potential to be minimized is then

$$\Omega = F - \mu N = E - \mu N - TS \quad (8)$$

Combining expressions for E (Eq.(2)), N, S (Eq.(7)) in terms of n we obtain for Ω :

$$\Omega = \sum_{\mathbf{p}, \sigma} \{ [\varepsilon(\mathbf{p}, \sigma) - \mu]n + T[n \ln n + (1 - n)\ln(1 - n)] \} . \quad (9)$$

The equilibrium distribution function n_{eq} is found from the condition that at arbitrary variation of $\delta n = n - n_{eq}$ the terms of first order in δn in a variation of $\delta\Omega$ vanish. The first order terms in $\delta\Omega$ can be easily found with the aid of formula (9). They are

$$\delta\Omega = \sum_{\mathbf{p}, \sigma} \left\{ (\varepsilon - \mu) + T \ln \frac{n}{1 - n} \right\} \delta n_{\mathbf{p}, \sigma} \quad (10)$$

One can see that for $\delta\Omega = 0$ we need

$$n = n_{eq} = \frac{1}{e^{(\varepsilon - \mu)/T} + 1} \quad (11)$$

i.e. the Fermi distribution function. A chemical potential μ is defined from a requirement that average number of particles in a volume V is fixed

$$\sum_{\mathbf{p}, \sigma} n(\mathbf{p}, \sigma) = N \quad (12)$$

Transforming the summation into integration over a phase space, we have

$$\int n \cdot 2V d\tau = \int n \cdot 2V \frac{d^3 p}{(2\pi\hbar)^3} = N \quad (13)$$

Equation (13) can be solved explicitly with respect to μ in the limit $T \rightarrow 0$. In that limit the Fermi function (11) is a step function $\theta(-\varepsilon + \mu)$, i.e. all states up to $\varepsilon = \mu$ are occupied with probability equal to unity and all states with $\varepsilon > \mu$ are empty. If we define a Fermi momentum as $p_F^2/2m = \mu$, then formula (13) gives

$$2V \frac{4}{3} \frac{\pi p_F^3}{(2\pi\hbar)^3} = N$$

or

$$p_F = \hbar \left(\frac{3\pi^2 N}{V} \right)^{1/3} \quad (14)$$

$$\mu_0 = \frac{p_F^2}{2m} = \frac{\hbar^2}{2m} \left(\frac{3\pi^2 N}{V} \right)^{2/3}$$

2.2 Low-temperature properties

The usual way to investigate the properties of a gas, liquid, or any other system in condensed matter physics is to study reaction of the system to a perturbation, i.e. on a small change of external conditions. Expression of Ω in terms of the distribution function makes it possible to discuss these reactions in a unique way. Let us consider a state with a small deviation of its distribution function $n_{\mathbf{p}\sigma}$ from equilibrium and expand the corresponding change in Ω : $\Omega - \Omega_{eq}$ in this deviation $\bar{n} = n - n_{eq}$. Such an expansion with the condition (10) starts with the terms of the second order in \bar{n} .

$$\begin{aligned}\Omega - \Omega_{eq} &= \frac{1}{2}T \sum_{\mathbf{p},\sigma} \frac{(\bar{n}_{\mathbf{p}\sigma})^2}{n_{eq}(1 - n_{eq})} = \frac{1}{2}TV \sum_{\sigma} \int \frac{(\bar{n}_{\mathbf{p}\sigma})^2}{n_{eq}(1 - n_{eq})} d\tau \\ &= \frac{V}{2} \sum_{\sigma} \int \frac{(\bar{n}_{\mathbf{p}\sigma})^2}{\left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right)} d\tau = \frac{1}{2} \sum_{\mathbf{p},\sigma} \frac{(\bar{n}_{\mathbf{p}\sigma})^2}{\left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right)}\end{aligned}\quad (15)$$

So, $\Omega - \Omega_{eq}$ is represented by a sum of energies of harmonic oscillators, $\bar{n}_{\mathbf{p}\sigma}$ being normal coordinates. Now it is easy to find the reaction of the gas to an external perturbation if this perturbation is also expressed in terms of $\bar{n}_{\mathbf{p}\sigma}$. Procedure is analogous to that for a one dimensional oscillator with coordinate variable X and energy $U = k\frac{X^2}{2}$. In the presence of an external force F the total energy \tilde{U} acquires an additional term $\delta U = -FX$, so that

$$\tilde{U} = \frac{kX^2}{2} - FX$$

Minimization of \tilde{U} with respect to X gives a shift of equilibrium $\Delta X = F/k$. As a result, any X -dependent function $f(X)$, characterizing properties of the oscillator, acquires an increment $\Delta f = (df/dX) \cdot \Delta X$. Following that procedure for an ideal Fermi gas we find its differential "susceptibilities".

1) Momentum density — reaction to the motion of a container.

Consider a gas in a tube, which moves with a velocity \mathbf{w} . The new equilibrium must be a minimum of the potential $\Omega - \mathbf{w} \cdot \mathbf{P}$, where \mathbf{P} is the total momentum of the gas. We use the fact that in the original equilibrium state $\mathbf{P} = 0$ and

$\mathbf{P} = \sum_{\mathbf{p}, \sigma} \mathbf{p} \bar{n}_{\mathbf{p}\sigma}$. Then we have to minimize

$$\Omega - \Omega_{eq} - \mathbf{w} \cdot \mathbf{P} = \frac{1}{2} \sum_{\mathbf{p}, \sigma} \frac{(\bar{n}_{\mathbf{p}\sigma})^2}{\left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right)} - \mathbf{w} \cdot \sum_{\mathbf{p}, \sigma} \mathbf{p} \bar{n}_{\mathbf{p}\sigma} \quad (16)$$

Taking the variation of eq.(16) with respect to all $\bar{n}(\mathbf{p}, \sigma)$, we get

$$\delta(\Omega - \Omega_{eq} - \mathbf{w} \cdot \mathbf{P}) = \sum_{\mathbf{p}, \sigma} \left[\frac{\bar{n}_{\mathbf{p}\sigma}}{\left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right)} - \mathbf{w} \cdot \mathbf{p} \right] \delta \bar{n}_{\mathbf{p}\sigma} \quad (17)$$

By setting $\delta(\Omega - \Omega_{eq} - \mathbf{w} \cdot \mathbf{P}) = 0$ with respect to all $\bar{n}(\mathbf{p}, \sigma)$, we can get the following expression ;

$$\bar{n}_{\mathbf{p}\sigma} = -\frac{\partial n_{eq}}{\partial \varepsilon} \mathbf{w} \cdot \mathbf{p} \quad (18)$$

Choosing the z -axis as the direction of \mathbf{w} : $\mathbf{w} = (0, 0, w)$, we can calculate the projection of the total momentum on the z -axis

$$\begin{aligned} P_z &= \sum_{\mathbf{p}, \sigma} p_z \bar{n} = \sum_{\mathbf{p}, \sigma} \left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right) w p_z^2 \\ &= \sum_{\sigma} \int \left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right) w p_z^2 \frac{V}{(2\pi\hbar)^3} d^3 p \quad (19) \end{aligned}$$

The integrand of eq.(19) is independent of the spin, and setting θ to be the angle between \mathbf{p} and the z -axis, we see

$$\begin{aligned} P_z &= 2wV \int \left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right) p^4 \cos^2 \theta \sin \theta \frac{1}{(2\pi\hbar)^3} dp d\theta d\phi \\ &= \frac{wV}{3} \int \left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right) p^4 \frac{dp}{\pi^2 \hbar^3} \quad (20) \end{aligned}$$

At low temperatures we can replace $\left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right)$ by $\delta(\varepsilon - \mu)$, and using the relations $\varepsilon = \frac{p^2}{2m}$ and $\mu = \frac{p_F^2}{2m}$, we can get

$$P_z = \frac{1}{3} V w p_F^2 \nu(\mu) \quad (21)$$

where p_F is the Fermi momentum and $\nu(\mu)$ is the density of states on the Fermi

surface :

$$\nu(\mu) = \frac{p_F m}{\pi^2 \hbar^3} \left(= 2 \times \frac{d\tau}{d\varepsilon} \Big|_{\varepsilon=\mu} \right) . \quad (22)$$

To derive the above expression for $\nu(\mu)$, we only need to consider a shell in momentum space, which has the thickness dp , then

$$\nu(\mu) = \frac{2 \times 4\pi p_F^2 \frac{dp}{d\varepsilon} \Big|_{\varepsilon=\mu}}{(2\pi\hbar)^3} = \frac{8\pi p_F^2 \frac{1}{v_F}}{(2\pi\hbar)^3} = \frac{p_F m}{\pi^2 \hbar^3}$$

where $v_F = \frac{p_F}{m}$ is the Fermi velocity. Using eq.(14), we can reduce eq.(21) to a simple form.

$$P_z = \frac{1}{3} V w p_F^2 \nu(\mu) = w m \frac{V}{3} \frac{p_F^3}{\pi^2 \hbar^3} = N m w . \quad (23)$$

2) Sound velocity

Let us calculate the sound velocity at absolute zero, i.e. the quantity $u^2 = \left(\frac{\partial P}{\partial \rho}\right)$. When $T = 0, S = 0$ and therefore we don't need to distinguish adiabatic and isothermal quantities. Using the relation ;

$$d\mu = \frac{V}{N} dP - \frac{S}{N} dT = \frac{V}{N} dP \quad (24)$$

we can write

$$\frac{\partial P}{\partial \rho} = \frac{N}{V} \left(\frac{\partial \mu}{\partial \rho} \right) . \quad (25)$$

Since $\rho = \frac{Nm}{V}$, we can get the following expression :

$$\frac{\partial \mu}{\partial N} = \frac{\partial \mu}{\partial \rho} \frac{\partial \rho}{\partial N} = \frac{\partial \mu}{\partial \rho} \frac{m}{V} \quad (26)$$

Thus the sound velocity can be written as

$$u^2 = \frac{\partial P}{\partial \rho} = \frac{N}{V} \left(\frac{\partial \mu}{\partial \rho} \right) = \frac{N}{m} \frac{1}{\left(\frac{\partial N}{\partial \mu} \right)} . \quad (27)$$

To calculate the quantity $\left(\frac{\partial N}{\partial \mu}\right)$, we again consider the reaction of our system

due to a change in μ . By taking the variation of the difference of the thermodynamic potential from equilibrium, we get

$$\delta[\Omega - \Omega_{eq} - \Delta\mu N] = 0 . \quad (28)$$

where $\Delta\mu = \mu - \mu_{eq}$. Equation(28) can be written in terms of $\bar{n} = n - n_{eq}$ as

$$\delta \left[\frac{1}{2} \sum_{\mathbf{p}, \sigma} \frac{(\bar{n}_{\mathbf{p}\sigma})^2}{\left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right)} - \Delta\mu \sum_{\mathbf{p}, \sigma} (n_{eq} + \bar{n}_{\mathbf{p}\sigma}) \right] = 0 . \quad (29)$$

Carrying out the variation, we get

$$\sum_{\mathbf{p}, \sigma} \left[\frac{\bar{n}_{\mathbf{p}\sigma}}{\left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right)} - \Delta\mu \right] \delta \bar{n}_{\mathbf{p}\sigma} = 0 . \quad (30)$$

Then we obtain the deviation of the distribution function

$$\bar{n}_{\mathbf{p}\sigma} = -\frac{\partial n_{eq}}{\partial \varepsilon} \Delta\mu . \quad (31)$$

We can calculate the change of the total number ΔN due to a change in the chemical potential $\Delta\mu$.

$$\begin{aligned} \Delta N &= V \int \bar{n} \cdot 2 \cdot \frac{d\tau}{d\varepsilon} d\varepsilon \\ &= V \int \left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right) \Delta\mu \nu(\varepsilon) d\varepsilon \end{aligned} \quad (32)$$

In the limit of $T = 0$, we can replace $\left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right)$ by $\delta(\varepsilon - \mu)$, so we obtain $\Delta N = V \Delta\mu \nu(\mu)$, and so

$$\frac{\partial N}{\partial \mu} = V \nu(\mu) . \quad (33)$$

Substituting eq.(33) into eq.(27), we get

$$u^2 = \frac{N}{m} \frac{1}{\left(\frac{\partial N}{\partial \mu}\right)} = \frac{N}{V} \frac{1}{m \nu(\mu)} = \frac{p_F^3}{3\pi^2 \hbar^3} \frac{\pi^2 \hbar^3}{m^2 p_F m} = \left(\frac{p_F}{m}\right)^2 \cdot \frac{1}{3} .$$

Since the Fermi velocity is equal to $\left(\frac{p_F}{m}\right)$, we obtain the simple result.

$$u = \frac{1}{\sqrt{3}} v_F . \quad (34)$$

3) Spin susceptibility

Let us assume that our system is in an external magnetic field \mathbf{H} . In this case we have to minimize $\Omega - \Omega_{eq} - \mathbf{M} \cdot \mathbf{H}$ in order to find the new equilibrium distribution function. If we assume that the magnetic field is along the z -axis, then the last term becomes

$$\mathbf{M} \cdot \mathbf{H} = g\mathbf{S} \cdot \mathbf{H} = gS_z H \quad (35)$$

where g is the gyromagnetic ratio. S_z can be written as

$$S_z = \frac{\hbar}{2} \sum_{\mathbf{p}} \left\{ n_{\uparrow}(\mathbf{p}) - n_{\downarrow}(\mathbf{p}) \right\} = \frac{\hbar}{2} \sum_{\mathbf{p}} \left\{ \bar{n}_{\uparrow}(\mathbf{p}) - \bar{n}_{\downarrow}(\mathbf{p}) \right\} . \quad (36)$$

Expressing $\Omega - \Omega_{eq}$ in terms of \bar{n} , \bar{n} must satisfy

$$\delta \left[\sum_{\mathbf{p}, \sigma} \frac{1}{2} \frac{(\bar{n}_{\mathbf{p}\sigma})^2}{\left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right)} - \frac{g\hbar H}{2} \sum_{\mathbf{p}} (\bar{n}_{\mathbf{p}\uparrow} - \bar{n}_{\mathbf{p}\downarrow}) \right] = 0 . \quad (37)$$

Carrying out the variation, we get

$$\sum_{\mathbf{p}} \left[\frac{\bar{n}_{\mathbf{p}\uparrow}}{\left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right)} - \frac{g\hbar H}{2} \right] \delta \bar{n}_{\mathbf{p}\uparrow} = 0 , \quad \sum_{\mathbf{p}} \left[\frac{\bar{n}_{\mathbf{p}\downarrow}}{\left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right)} + \frac{g\hbar H}{2} \right] \delta \bar{n}_{\mathbf{p}\downarrow} = 0 . \quad (38)$$

From eq.(38), we obtain

$$\bar{n}_{\uparrow} = \left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right) \frac{g\hbar H}{2} , \quad \bar{n}_{\downarrow} = -\left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right) \frac{g\hbar H}{2} . \quad (39)$$

The difference of these two is

$$\bar{n}_{\uparrow} - \bar{n}_{\downarrow} = \left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right) g\hbar H . \quad (40)$$

Then S_z becomes

$$\begin{aligned} S_z &= \frac{\hbar}{2} V \int (\bar{n}_{\uparrow} - \bar{n}_{\downarrow}) \frac{d\tau}{d\varepsilon} d\varepsilon \\ &= \left(\frac{\hbar}{2}\right)^2 gHV \int \left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right) \nu(\varepsilon) d\varepsilon . \end{aligned} \quad (41)$$

Here we can replace $(-\frac{\partial n_{eq}}{\partial \varepsilon})$ by $\delta(\varepsilon - \mu)$ in the limit of $T = 0$. Then

$$S_z = \left(\frac{\hbar}{2}\right)^2 gHV\nu(\mu) . \quad (42)$$

Finally we get the following expression for the spin susceptibility:

$$\chi = \frac{1}{V} \frac{M}{H} = \left(\frac{g\hbar}{2}\right)^2 \nu(\mu) . \quad (43)$$

4) Specific heat

The specific heat c is defined as

$$c = \frac{1}{V} T \left(\frac{\partial S}{\partial T}\right) . \quad (44)$$

Let us calculate the change of the entropy due to a change in temperature. From eq.(7), we may write

$$\Delta S = - \sum_{\mathbf{p}, \sigma} \ln \frac{n_{eq}}{1 - n_{eq}} \bar{n}_{\mathbf{p}\sigma} , \quad \bar{n}_{\mathbf{p}\sigma} = \frac{\partial n_{eq}}{\partial T} \Delta T . \quad (45)$$

Using the relations

$$\frac{\partial n_{eq}}{\partial T} = \frac{\partial n_{eq}}{\partial z} \left(-\frac{\varepsilon - \mu}{T^2}\right) , \quad \frac{\partial n_{eq}}{\partial \varepsilon} = \frac{\partial n}{\partial z} \frac{1}{T} , \quad (46)$$

where $z = \frac{\varepsilon - \mu}{T}$, \bar{n} can be written as

$$\bar{n} = \frac{\partial n_{eq}}{\partial T} \Delta T = -\frac{\partial n_{eq}}{\partial \varepsilon} \frac{\varepsilon - \mu}{T} \Delta T . \quad (47)$$

Substituting eq.(47) into eq.(45), we get

$$\Delta S = \sum_{\mathbf{p}, \sigma} \left(\frac{\varepsilon - \mu}{T}\right)^2 \left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right) \Delta T = \sum_{\mathbf{p}, \sigma} z^2 \left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right) \Delta T \quad (48)$$

from which we can calculate $(\frac{\partial S}{\partial T})$;

$$\frac{\partial S}{\partial T} = V \int z^2 \left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right) 2 \frac{d\varepsilon}{d\varepsilon} d\varepsilon = V \int z^2 \left(-\frac{\partial n_{eq}}{\partial \varepsilon}\right) \nu(\varepsilon) d\varepsilon . \quad (49)$$

The main contribution of $z^2(-\frac{\partial n_{eq}}{\partial \varepsilon})$ to the integral is at $\varepsilon = \mu$, so we can replace $\nu(\varepsilon)$ by $\nu(\mu)$, and changing variables from z to ε means the integration limits are

$-\infty$ and $+\infty$. Then we get

$$\frac{\partial S}{\partial T} = V\nu(\mu) \int_{-\infty}^{\infty} z^2 \left(-\frac{\partial n_{eq}}{\partial z} \right) dz .$$

Because the integral is an even function, the right-hand-side becomes

$$\begin{aligned} &= 2V\nu(\mu) \int_0^{\infty} z^2 \left(-\frac{\partial n_{eq}}{\partial z} \right) dz \\ &= 2V\nu(\mu) \left[-n_{eq}z^2 \Big|_0^{\infty} + 2 \int_0^{\infty} n_{eq}z dz \right] \\ &= 4V\nu(\mu) \int_0^{\infty} \frac{z}{e^z + 1} dz = \frac{\pi^2}{3} \nu(\mu) V . \end{aligned} \quad (50)$$

Finally we obtain the specific heat

$$c = \frac{\pi^2}{3} \nu(\mu) T \quad (51)$$

As we have discussed, all quantities of u , c and χ are essentially described by one quantity i.e. $\nu(\mu)$. This is not surprising since $\bar{\Omega} \equiv \Omega - \Omega_{eq}$ can be described as a sum of independent harmonic oscillators, which have the same coefficients $\frac{1}{(-\partial n_{eq}/\partial \varepsilon)}$. In principle, if we measure one quantity, for example, the specific heat c , we can predict the values of the other quantities.

§3. Landau Fermi-liquid theory

Next let us consider the application of the discussion about the ideal system to the real systems, in particular, systems of electrons in metals or liquid- ^3He at low temperatures. These systems have strong interactions, so we can't expect the ideal Fermi gas model to be valid.

3.1 Quasiparticles

In the Landau theory, we assume that at low temperatures, if the interaction between particles is switched on gradually and adiabatically in the ideal Fermi system, we can obtain the real Fermi liquid system, and that the classification of the quantum states of particles are specified by \mathbf{p} and σ similarly to the ideal system. Then we can introduce the quasiparticles as excitations and describe the system of the Fermi-liquid in terms of quasiparticles.

If there is no interaction, the states, both the ground state and excited states, are exactly stationary, but once we introduce the interactions, only the ground state is stationary and the excited states are not. Because of the interactions a certain transition from one excited state to another state becomes possible. So an excited state described by quasiparticles has a lifetime τ . This can also be regarded as the lifetime of the quasiparticles. The lifetime τ is related to the uncertainty of the energy of the quasiparticle : $\Delta\varepsilon \sim \frac{\hbar}{\tau}$. Thus when $\Delta\varepsilon$ is much smaller than the energy of the quasiparticle, the description in terms of quasiparticles becomes meaningful.

$$\Delta\varepsilon \sim \frac{\hbar}{\tau} \ll v_F(|\mathbf{p}| - p_F) \sim \varepsilon(\mathbf{p}) . \quad (52)$$

Here, the energy of the quasiparticle is measured from the Fermi energy. Let us estimate the lifetime τ . We consider an excited state such that only one quasiparticle exists outside the Fermi surface, and the Fermi sphere is filled with quasiparticles. The quasiparticle outside the Fermi surface, which has momentum \mathbf{p}_1 , can collide with all other quasiparticles, \mathbf{p}_2 , inside the Fermi surface, and two quasiparticles, \mathbf{p}_3 and \mathbf{p}_4 , are created outside the Fermi surface. Through this transition, the total energy and the total momentum are conserved.

$$(\mathbf{p}_1) + (\mathbf{p}_2) \rightarrow (\mathbf{p}_3) + (\mathbf{p}_4)$$

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4 = \mathbf{P} \quad ; \quad p_1, p_3, p_4 > p_F, \quad p_2 < p_F$$

$$\varepsilon_1 + \varepsilon_2 = \varepsilon_3 + \varepsilon_4 \quad ; \quad \varepsilon_1, \varepsilon_3, \varepsilon_4 > \mu, \quad \varepsilon_2 < \mu$$

Then we estimate the total probability of all possible transitions using the Fermi Golden Rule. Assuming that each elementary transition has the same amplitude a , we can write the total probability w as

$$\begin{aligned} w &\sim |a|^2 \int \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) d\mathbf{p}_2 d\mathbf{p}_3 d\mathbf{p}_4 \\ &= |a|^2 \int_{\mathbf{p}_4 = \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3} \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) d\mathbf{p}_2 d\mathbf{p}_3 . \end{aligned}$$

For a given \mathbf{p}_2 , setting θ to be the angle between \mathbf{p}_3 and $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$, we can

write

$$\begin{aligned} w &\sim |a|^2 \int d\mathbf{p}_2 \int \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) p_3^2 dp_3 d(-\cos\theta) d\phi \\ &\sim |a|^2 \int d\mathbf{p}_2 \int \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) p_3^2 dp_3 d(-\cos\theta) \end{aligned}$$

From the relation

$$p_4^2 = (\mathbf{p} - \mathbf{p}_3)^2 = p^2 - 2pp_3 \cos\theta + p_3^2$$

we can replace $d(-\cos\theta)$ by dp_4 ;

$$d(-\cos\theta) = \frac{p_4}{pp_3} dp_4 .$$

Then the total probability w becomes

$$\begin{aligned} w &\sim |a|^2 \int d\mathbf{p}_2 \int \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) p_3^2 dp_3 \frac{p_4}{pp_3} dp_4 \\ &\sim |a|^2 \int d\mathbf{p}_2 \frac{1}{p} \int \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) p_3 p_4 dp_3 dp_4 \end{aligned}$$

Recalling that $\varepsilon = v_F(p - p_F)$, we can replace dp by $d\varepsilon$,

$$w \sim |a|^2 \int d\mathbf{p}_2 \frac{1}{p} \int \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \frac{p_3 p_4}{v_F^2} d\varepsilon_3 d\varepsilon_4 .$$

Since p_1 is very close to p_F at low temperatures, we can replace p_3 and p_4 by p_F .

Thus we get

$$\begin{aligned} w &\sim \left(\frac{p_F}{v_F}\right)^2 |a|^2 \int d\mathbf{p}_2 \frac{1}{p} \int \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) d\varepsilon_3 d\varepsilon_4 \\ &= \left(\frac{p_F}{v_F}\right)^2 |a|^2 \int d\mathbf{p}_2 \frac{1}{p} \int_{\substack{\varepsilon_3 > 0 \\ \varepsilon_4 = \varepsilon_1 + \varepsilon_2 - \varepsilon_3 > 0}} d\varepsilon_3 \\ &= \left(\frac{p_F}{v_F}\right)^2 |a|^2 \int d\mathbf{p}_2 \frac{1}{p} (\varepsilon_1 + \varepsilon_2) . \end{aligned}$$

Again setting θ to be the angle between \mathbf{p}_1 and \mathbf{p}_2 , $d\mathbf{p}_2$ can be replaced by

$$\begin{aligned} d\mathbf{p}_2 &= p_2^2 dp_2 d(-\cos\theta) d\phi \\ &= -p_2^2 dp_2 \frac{p}{p_1 p_2} dp d\phi \end{aligned}$$

then w becomes

$$\begin{aligned} w &\sim \left(\frac{p_F}{v_F}\right)^2 |a|^2 \int dp_2 \frac{p_2}{p_1} (\varepsilon_1 + \varepsilon_2) \int_{p_1 - p_F < p < p_1 + p_F} dp \\ &\sim \left(\frac{p_F}{v_F}\right)^3 |a|^2 \int_{-\varepsilon_1 < \varepsilon_2 < 0} d\varepsilon_2 (\varepsilon_1 + \varepsilon_2) \\ &\sim \left(\frac{p_F}{v_F}\right)^3 |a|^2 \varepsilon_1^2 \sim |a|^2 (p_1 - p_F)^2 . \end{aligned}$$

Therefore we can conclude

$$w \sim |a|^2 (p_1 - p_F)^2 \quad (53)$$

w is equal to the inverse of τ , so that we have to satisfy

$$\frac{1}{\tau} \sim |a|^2 (p - p_F)^2 \ll v_F (p - p_F) . \quad (54)$$

If p is sufficiently close to the Fermi momentum p_F , this condition can be always satisfied.

3.2 Distribution function and energy of a quasiparticle

Let us formulate Landau Fermi-liquid theory. We assume that in the ground state the Fermi sphere is filled with quasiparticles, similarly to the ideal system, and that the number of quasiparticles is equal to that of particles. Then the density of quasiparticles is

$$\frac{N}{V} = \frac{8\pi p_F^3}{(2\pi\hbar)^3} . \quad (55)$$

The total energy of the system can't be expressed in a simple form because of the interaction of quasiparticles, but it should be described as a functional of the

distribution function of quasiparticles $n(\mathbf{p}, \sigma)$;

$$E = E\{n\} .$$

Then we consider the change of the total energy E due to the small change of the distribution function of the quasiparticles, and define the energy of a quasiparticle as the functional derivative of E with respect to the distribution function;

$$\delta E = \sum_{\sigma} V \int \varepsilon(\mathbf{p}, \sigma) \delta n(\mathbf{p}, \sigma) d\tau . \quad (56)$$

Although interactions change the total energy, the total momentum and the total number of quasiparticles are not changed through interactions, so that we can write

$$\begin{aligned} \mathbf{P} &= \sum_{\sigma} V \int \mathbf{p} n(\mathbf{p}, \sigma) d\tau \\ N &= \sum_{\sigma} V \int n(\mathbf{p}, \sigma) d\tau . \end{aligned} \quad (57)$$

Next we consider the equilibrium distribution of quasiparticles. As in the ideal case, we have to minimize the thermodynamic potential $\Omega = E - \mu N - TS$ with respect to the distribution function. Because the energy levels of quasiparticles in the Fermi-liquid and those of particles in the Fermi-gas are classified in the same manner, we can use the same formula for the entropy S ;

$$S = - \sum_{\mathbf{p}, \sigma} \{n \ln n + (1 - n) \ln(1 - n)\} . \quad (58)$$

Then taking the variation with respect to n , we get

$$\delta \Omega = \sum_{\sigma} V \int (\varepsilon - \mu) \delta n d\tau + T \sum_{\sigma} V \int \ln\left(\frac{n}{1 - n}\right) \delta n d\tau . \quad (59)$$

Setting $\delta \Omega$ to be zero, we obtain the same expression of the distribution function

of quasiparticles ;

$$n_{eq} = \frac{1}{e^{(\varepsilon-\mu)/T} + 1} . \quad (60)$$

However, here ε is a functional of the distribution function because of the interaction of quasiparticles ;

$$\varepsilon = \varepsilon\{n\} .$$

So, eq.(60) is quite a complicated implicit equation for n .

At $T = 0$, n takes the simple Fermi step function ;

$$n = \begin{cases} 1 & \varepsilon < \mu , \\ 0 & \varepsilon > \mu . \end{cases}$$

We can not write ε in an explicit form, so we consider the change in ε due to a small change in n ;

$$\delta\varepsilon(\mathbf{p}, \sigma) = \sum_{\sigma'} \int f(\mathbf{p}, \sigma; \mathbf{p}', \sigma') \delta n(\mathbf{p}', \sigma') d\tau' . \quad (61)$$

3.3 Low temperature properties and effective mass

Now we have introduced the concepts of Landau theory ; the excitations, the lifetime and the energy of quasiparticles. Then let us consider the thermodynamic properties in the same way as for the ideal Fermi-gas. First we introduce the deviation of the distribution function in the vicinity of equilibrium

$$\bar{n} = n - n_{eq} .$$

We have to expand the thermodynamic potential Ω up to second order in \bar{n} since the first order term disappears in equilibrium. In the case of a Fermi-liquid, we have to add the second variation of E to the expression for the ideal case. $\bar{\Omega}$ can then be written as

$$\bar{\Omega} = \Omega - \Omega_{eq} = \frac{1}{2} \sum_{\substack{\mathbf{p}, \mathbf{p}' \\ \sigma, \sigma'}} f(\mathbf{p}, \sigma; \mathbf{p}', \sigma') \bar{n}_{\mathbf{p}\sigma} \bar{n}_{\mathbf{p}'\sigma'} + \frac{T}{2} \sum_{\mathbf{p}, \sigma} \frac{(\bar{n}_{\mathbf{p}\sigma})^2}{n_{eq}(1 - n_{eq})} . \quad (62)$$

The second term is already in diagonal form, but the first term is not. It has the form of the sum of products of different \mathbf{p} and σ . To take the same procedure

of deriving thermodynamic quantities, we have to diagonalize both terms. To do so, at first we split f and \bar{n} into a spin symmetric and a spin antisymmetric part. Because f should depend on the relative orientation of spin, f can be written as

$$f_{\uparrow\uparrow} = f_{\downarrow\downarrow} = f^{(s)} + f^{(a)} \quad , \quad f_{\uparrow\downarrow} = f_{\downarrow\uparrow} = f^{(s)} - f^{(a)} \quad . \quad (63)$$

Similarly we split \bar{n} into two terms ;

$$\bar{n}_{\uparrow} = \bar{n}_s + \bar{n}_a \quad , \quad \bar{n}_{\downarrow} = \bar{n}_s - \bar{n}_a \quad . \quad (64)$$

Substituting eq.(64) into the first term of $\bar{\Omega}$, we can write

$$\begin{aligned} & \frac{1}{2} \sum_{\substack{\mathbf{p}, \mathbf{p}' \\ \sigma, \sigma'}} f(\mathbf{p}, \sigma; \mathbf{p}' \sigma') \bar{n}_{\mathbf{p}\sigma} \bar{n}_{\mathbf{p}'\sigma'} \\ &= \frac{1}{2} \sum_{\mathbf{p}, \mathbf{p}'} \{ f_{\uparrow\uparrow} [(\bar{n}_s + \bar{n}_a)(\bar{n}_{s'} + \bar{n}_{a'}) + (\bar{n}_s - \bar{n}_a)(\bar{n}_{s'} - \bar{n}_{a'})] \\ &+ f_{\uparrow\downarrow} [(\bar{n}_s + \bar{n}_a)(\bar{n}_{s'} - \bar{n}_{a'}) + (\bar{n}_s - \bar{n}_a)(\bar{n}_{s'} + \bar{n}_{a'})] \} \\ &= \frac{1}{2} \sum_{\mathbf{p}, \mathbf{p}'} 2 \cdot 2 [f^{(s)} \bar{n}_s(\mathbf{p}) \bar{n}_s(\mathbf{p}') + f^{(a)} \bar{n}_a(\mathbf{p}) \bar{n}_a(\mathbf{p}')] \end{aligned}$$

In the second term we replace $T \frac{1}{n_{eq}(1-n_{eq})}$ by $\frac{1}{(-\partial n_{eq} / \partial \epsilon)}$;

$$\begin{aligned} \frac{T}{2} \sum_{\mathbf{p}, \sigma} \frac{(\bar{n}_{\mathbf{p}\sigma})^2}{n_{eq}(1-n_{eq})} &= \frac{1}{2} \sum_{\mathbf{p}} \frac{1}{\left(-\frac{\partial n_{eq}}{\partial \epsilon}\right)} [(\bar{n}_s + \bar{n}_a)^2 + (\bar{n}_s - \bar{n}_a)^2] \\ &= \frac{1}{2} \sum_{\mathbf{p}} 2 \frac{\bar{n}_s^2(\mathbf{p}) + \bar{n}_a^2(\mathbf{p})}{\left(-\frac{\partial n_{eq}}{\partial \epsilon}\right)} \quad . \quad (65) \end{aligned}$$

Therefore $\bar{\Omega}$ is

$$\begin{aligned} \bar{\Omega} &= \frac{1}{2} V \int 2d\tau \frac{\bar{n}_s^2(\mathbf{p}) + \bar{n}_a^2(\mathbf{p})}{\left(-\frac{\partial n_{eq}}{\partial \epsilon}\right)} + \frac{1}{2} V \int 2d\tau 2d\tau' [f^{(s)}(\mathbf{p}, \mathbf{p}') \bar{n}_s(\mathbf{p}) \bar{n}_s(\mathbf{p}') \\ &+ f^{(a)}(\mathbf{p}, \mathbf{p}') \bar{n}_a(\mathbf{p}) \bar{n}_a(\mathbf{p}')] \quad . \quad (66) \end{aligned}$$

Since the change of the distribution function is concentrated in the vicinity of the

Fermi surface, we define η and ζ by

$$\bar{n}_s = \left(-\frac{\partial n_{eq}}{\partial \varepsilon} \right) \eta(\hat{p}) \quad , \quad \bar{n}_a = \left(-\frac{\partial n_{eq}}{\partial \varepsilon} \right) \zeta(\hat{p}) \quad (67)$$

where η and ζ depend only on the direction of \mathbf{p} . For the same reason, $f(\mathbf{p}, \mathbf{p}')$ depends only on the angle between \mathbf{p} and \mathbf{p}' . Then $\bar{\Omega}$ can be written as

$$\begin{aligned} \frac{\bar{\Omega}}{V} = & \frac{1}{2} \int 2d\tau \left(-\frac{\partial n_{eq}}{\partial \varepsilon} \right) (\eta(\hat{p})^2 + \zeta(\hat{p})^2) + \frac{1}{2} \int 2d\tau 2d\tau' \frac{\partial n_{eq}}{\partial \varepsilon} \frac{\partial n_{eq}}{\partial \varepsilon'} \\ & \times [f^{(s)}(\hat{p}, \hat{p}') \eta(\hat{p}) \eta(\hat{p}') + f^{(a)}(\hat{p}, \hat{p}') \zeta(\hat{p}) \zeta(\hat{p}')] \quad . \quad (68) \end{aligned}$$

Remember that we can replace $(-\frac{\partial n_{eq}}{\partial \varepsilon})$ by $\delta(\varepsilon - \mu)$, and that $\nu(\mu) = [2 \times \frac{d\tau}{d\varepsilon}]_{\varepsilon=\mu} = \frac{2 \times 4\pi p_F^2}{(2\pi\hbar)^3} \frac{dp}{d\varepsilon} |_{\varepsilon=\mu}$, we get

$$\begin{aligned} \frac{\bar{\Omega}}{V} = & \frac{1}{2} \nu(\mu) \left[\int \frac{d\Omega}{4\pi} (\eta(\hat{p})^2 + \zeta(\hat{p})^2) + \int \frac{d\Omega}{4\pi} \frac{d\Omega'}{4\pi} [F^{(s)}(\hat{p}, \hat{p}') \eta(\hat{p}) \eta(\hat{p}') \right. \\ & \left. + F^{(a)}(\hat{p}, \hat{p}') \zeta(\hat{p}) \zeta(\hat{p}')] \right] \quad , \quad (69) \end{aligned}$$

where $F^{(s,a)}(\hat{p}, \hat{p}')$ is the dimensionless quantity defined by

$$F^{(s,a)}(\hat{p}, \hat{p}') = \nu(\mu) f^{(s,a)}(\hat{p}, \hat{p}') \quad . \quad (70)$$

Setting Θ to be the angle between \hat{p} and \hat{p}' , we can expand $F^{(s,a)}$ in a series of Legendre polynomials

$$F^{(s,a)}(\cos \Theta) = \sum_{\ell=0}^{\infty} F_{\ell}^{(s,a)} P_{\ell}(\cos \Theta) \quad . \quad (71)$$

Similarly, expressing the direction of \hat{p} by θ and ϕ , η and ζ can be expanded in a series of normalized spherical harmonics, such as

$$\begin{aligned} \eta(\hat{p}) = \eta(\theta, \phi) &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \eta_{\ell m} Y_{\ell m}(\theta, \phi) \\ \zeta(\hat{p}) = \zeta(\theta, \phi) &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \zeta_{\ell m} Y_{\ell m}(\theta, \phi) \end{aligned} \quad (72)$$

$Y_{\ell m}(\theta, \phi)$ is normalized by

$$\int \frac{d\theta}{4\pi} Y_{\ell m}(\theta, \phi) Y_{\ell' m'}^*(\theta, \phi) = \delta_{\ell\ell'} \delta_{mm'}$$

Using the addition theorem for spherical harmonics

$$P_\ell(\cos \Theta) = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta, \phi) Y_{\ell, -m}(\theta', \phi')$$

we can obtain the following expression for $\bar{\Omega}$

$$\frac{\bar{\Omega}}{V} = \frac{1}{2} \nu(\mu) \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[\left(1 + \frac{F_\ell^s}{2\ell+1} \right) |\eta_{\ell m}|^2 + \left(1 + \frac{F_\ell^a}{2\ell+1} \right) |\zeta_{\ell m}|^2 \right]. \quad (73)$$

This shows that $\bar{\Omega}$ is described by a sum of independent oscillators, with all the oscillators having different coefficients $1 + \frac{F_\ell^{(s,a)}}{2\ell+1}$. For the equilibrium state to be stable, a deviation of the distribution function should increase the energy. Therefore the coefficients $1 + \frac{F_\ell^{(s,a)}}{2\ell+1}$ must be positive.

One more important difference from the Fermi-gas is related to the density of states $\nu(\mu)$. Of course $\nu(\mu)$ itself has the same form as that of the ideal Fermi-gas;

$$\nu(\mu) = \left[2 \frac{d\tau}{d\varepsilon} \right]_{\varepsilon=\mu} = \frac{2 \times 4\pi p_F^2}{(2\pi\hbar)^3} \frac{dp}{d\varepsilon} \Big|_{\varepsilon=\mu} = \frac{1}{\pi^2 \hbar^3} p_F \frac{p_F}{v_F}$$

In the case of the ideal Fermi-gas, $\frac{p_F}{v_F}$ is equal to the mass of the particle, the “bare mass”, but in the case of the Fermi-liquid it is not true. So if we introduce the “effective mass” m^* of the quasiparticle defined by

$$m^* = \frac{p_F}{v_F},$$

then we can write down $\nu(\mu)$ in the same form

$$\nu(\mu) = \frac{p_F m^*}{\pi^2 \hbar^3}. \quad (74)$$

m^* should be related to m in terms of the set of parameters $F_\ell^{(s,a)}$.

Now we apply the same procedure as in the case of the ideal Fermi-gas to derive the momentum, the sound velocity, the spin susceptibility and the specific heat of the Fermi-liquid.

1) Momentum density

We assume that the Fermi liquid system moves with a small velocity \mathbf{w} . Choosing the z -axis along the direction of \mathbf{w} , we can write

$$\begin{aligned} P_z &= \sum_{\sigma} V \int d\tau p_z \bar{n}(\mathbf{p}, \sigma) \\ &= V \int 2d\tau p_z \bar{n}_s(\mathbf{p}) \\ &= V \int 2d\tau p \cos \theta \left(-\frac{\partial n_{eq}}{\partial \varepsilon} \right) \eta(\theta, \phi) \end{aligned} \quad (75)$$

where θ is the angle between \mathbf{p} and the z -axis. Replacing $(\frac{\partial n_{eq}}{\partial \varepsilon})$ by $-\delta(\varepsilon - \mu)$, we can take $\nu(\mu)$ out of the integral as usual ;

$$\begin{aligned} P_z &= p_F \nu(\mu) V \int \cos \theta \eta(\theta, \phi) \frac{d\Omega}{4\pi} \\ &= p_F \nu(\mu) V \int P_1(\cos \theta) \left\{ \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \eta_{\ell m} Y_{\ell m}(\theta, \phi) \right\} \frac{d\Omega}{4\pi} \\ &= p_F \nu(\mu) V \frac{1}{\sqrt{3}} \eta_{10} . \end{aligned} \quad (76)$$

Thus we can express P_z with only η_{10} . We then minimize $\bar{\Omega} - \mathbf{w} \cdot \mathbf{P}$ with respect to η_{10} , that is

$$\frac{\partial}{\partial \eta_{10}} \left[\frac{1}{2} \nu(\mu) \left(1 + \frac{F_1^s}{3} \right) \eta_{10}^2 - w p_F \nu(\mu) \frac{1}{\sqrt{3}} \eta_{10} \right] = 0 \quad (77)$$

from which we get

$$\eta_{10} = \frac{1}{\sqrt{3}} \frac{1}{1 + \frac{F_1^s}{3}} w p_F . \quad (78)$$

Substituting eq.(78) into eq.(76), P_z can be written as

$$P_z = \frac{1}{3} V w p_F^2 \nu(\mu) \frac{1}{1 + \frac{F_1^s}{3}} . \quad (79)$$

Because the interaction does not change the total momentum, P_z should be equal to mNw , which is also true in the case of the ideal Fermi-gas. So comparing the

expression of P_z with that of the ideal case, we can get

$$\frac{\nu_{liq}(\mu)}{1 + \frac{F_1^s}{3}} = \nu_{gas}(\mu) . \quad (80)$$

Therefore,

$$\frac{\nu_{liq}}{\nu_{gas}} = 1 + \frac{F_1^s}{3} .$$

This also means

$$\frac{m^*}{m} = 1 + \frac{F_1^s}{3} . \quad (81)$$

The interaction changes the effective mass and also the density of states, but the ratios are described with only one parameter F_1^s .

2) Sound velocity

We consider a change in the chemical potential as the perturbation, and find the change in the number of particles as the response of the system. In this case we have to express $\Delta\mu N$ using the parameters η_{lm} or ζ_{lm} .

$$\Delta\mu N = \Delta\mu(N_{eq} + \Delta N) = \Delta\mu \sum_{\sigma} V \int (n_{eq} + \bar{n}) d\tau = \Delta\mu V \int 2d\tau (n_{eq} + \bar{n}_s) . \quad (82)$$

The change in the number of particles becomes

$$\Delta N = V \int 2d\tau \left(-\frac{\partial n_{eq}}{\partial \varepsilon} \right) \eta(\theta, \phi) = \nu(\mu) V \int \frac{d\Omega}{4\pi} \eta(\theta, \phi) = \nu(\mu) \eta_{00} V \quad (83)$$

from which we have to minimize $\bar{\Omega} - \Delta\mu N$ with respect to only η_{00} ;

$$\frac{\partial}{\partial \eta_{00}} \left[\frac{1}{2} \nu(\mu) (1 + F_0^s) \eta_{00}^2 - \Delta\mu \nu(\mu) \eta_{00} \right] = 0 . \quad (84)$$

Then we get

$$\eta_{00} = \frac{\Delta\mu}{1 + F_0^s} . \quad (85)$$

Substituting eq.(85) into eq.(83), ΔN can be written as

$$\Delta N = \frac{\nu(\mu)}{1 + F_0^s} \Delta\mu V . \quad (86)$$

Therefore

$$u^2 = \frac{N}{m \left(\frac{dN}{d\mu} \right)} = \frac{1}{V} \frac{N}{m\nu(\mu) \frac{1}{1+F_0^s}} = \frac{1}{V} \frac{N}{m\nu(\mu)} (1+F_0^s) \quad (87)$$

The velocity u can be also written with only one parameter F_0^s , but this is different from what we need to express P_z . Although the thermodynamic quantities of the Fermi-gas are described with only one quantity $\nu(\mu)$, in the case of the Fermi-liquid we cannot describe the system with only one quantity because of the different coefficients of oscillators.

3) Spin susceptibility

We assume that our system is in an external magnetic field \mathbf{H} . Choosing the z -axis along \mathbf{H} , M_z can be written as

$$\begin{aligned} M_z &= gS_z = gV \int d\tau \frac{\hbar}{2} (\bar{n}_\uparrow - \bar{n}_\downarrow) \\ &= \frac{g\hbar}{2} V \int 2d\tau \bar{n}_a \\ &= \left(\frac{g\hbar}{2} \right) V \nu(\mu) \int \frac{d\Omega}{4\pi} \zeta(\theta, \phi) \\ &= \left(\frac{g\hbar}{2} \right) V \nu(\mu) \zeta_{00} \end{aligned} \quad (88)$$

Then we have to minimize $\bar{\Omega} - \mathbf{M} \cdot \mathbf{H}$ with respect to only ζ_{00} ;

$$\frac{\partial}{\partial \zeta_{00}} \left[\frac{1}{2} \nu(\mu) (1+F_0^a) \zeta_{00}^2 - \frac{g\hbar}{2} \nu(\mu) \zeta_{00} H \right] = 0 \quad (89)$$

we get

$$\zeta_{00} = \left(\frac{g\hbar}{2} \right) H \frac{1}{1+F_0^a} \quad (90)$$

Therefore

$$M_z = M = \left(\frac{g\hbar}{2} \right)^2 V \nu(\mu) \frac{1}{1+F_0^a} H \quad (91)$$

The spin susceptibility is

$$\chi = \frac{1}{V} \frac{M}{H} = \left(\frac{g\hbar}{2} \right)^2 \nu(\mu) \frac{1}{1+F_0^a} \quad (92)$$

χ can be also expressed with only one parameter F_0^a .

4) Specific heat

To derive the specific heat, we have to know the change in the entropy due to a change in the temperature, but the entropy of the Fermi liquid is the same as that of the Fermi gas. So the derivation of the specific heat of the Fermi liquid is almost the same. Only one difference is that we have to replace m by m^* . That is,

$$C = \frac{\pi^2}{3} T \nu(\mu) \Big|_{m=m^*} .$$

3.4 Zero sound

So far we have discussed the static properties of the Fermi-liquid. In the case of the sound velocity, we also considered the static response of the system. This is true if the frequency of the sound ω satisfies

$$\omega \ll \frac{1}{\tau} ,$$

because during one period of oscillation the system has enough time to equilibrate through collisions.

However, since τ is proportional to $1/T^2$, any ω cannot satisfy the restriction $\omega\tau \ll 1$ at very low temperatures. So we have to change our argument about the sound propagation at very low temperatures. We have to consider the dynamic equation for the distribution function.

To derive it we consider the conservation law of particles. Let us start from the ideal Fermi-gas. The total number of particles is

$$N = \sum_{\sigma} \int n(\mathbf{p}, \sigma, \mathbf{r}, t) d^3r d\tau , \quad (93)$$

here n is a function of \mathbf{r} and t . Considering a small volume element in phase space $d^3p d^3r$, the conservation law of particles is expressed by

$$\frac{\partial n}{\partial t} + \text{div } \mathbf{j} = 0 \quad (94)$$

where \mathbf{j} is the current of particles. In this case divergence has six components, that is, the derivative with respect to \mathbf{r} and \mathbf{p} , and \mathbf{j} also has six components of

$n\dot{\mathbf{r}}$ and $n\dot{\mathbf{p}}$. So we can write

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \mathbf{r}}(n\dot{\mathbf{r}}) + \frac{\partial}{\partial \mathbf{p}}(n\dot{\mathbf{p}}) = 0 . \quad (95)$$

Using the Hamilton equations

$$\dot{\mathbf{r}} = \frac{\partial \epsilon}{\partial \mathbf{p}} , \quad \dot{\mathbf{p}} = -\frac{\partial \epsilon}{\partial \mathbf{r}}$$

we can derive the following transport equation for particles ;

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial \mathbf{r}} \cdot \frac{\partial \epsilon}{\partial \mathbf{p}} - \frac{\partial n}{\partial \mathbf{p}} \frac{\partial \epsilon}{\partial \mathbf{r}} = 0 . \quad (96)$$

The right-hand side of this equation should be zero if interaction of particles is exactly zero, but this is not true when interactions are present. Because of the interactions, quasiparticles change their momenta through collisions. This means that quasiparticles jump from a small volume element to another element in momentum space through collisions. So the right-hand side of the transport equation is not zero any longer. We need a collision integral term $I(n)$ in the right-hand side for the case of the Fermi-liquid. If we write n as

$$n(\mathbf{p}, \sigma, \mathbf{r}, t) = n_0(\mathbf{p}, \sigma) + \bar{n}(\mathbf{p}, \sigma, \mathbf{r}, t) , \quad (97)$$

where n_0 is the equilibrium distribution function in the ground state, then the order of $I(n)$ is $\sim -\frac{\bar{n}}{\tau}$. On the other hand the order of $\frac{\partial n}{\partial t}$ is $\sim \omega \bar{n}$, where ω is the frequency of the sound. So if ω is much larger than $\frac{1}{\tau}$, that is

$$\omega \tau \gg 1 , \quad (98)$$

then we can drop the collision integral. Since τ is proportional to $1/T^2$, we can neglect the collision integral at very low temperatures, and we can set the right-hand side of the transport equation to be zero.

Now let us derive the sound velocity of the Fermi liquid under the condition $\omega\tau \gg 1$; the velocity of zero sound. The transport equation for quasiparticles is

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial \mathbf{r}} \cdot \frac{\partial \varepsilon}{\partial \mathbf{p}} - \frac{\partial n}{\partial \mathbf{r}} \frac{\partial \varepsilon}{\partial \mathbf{r}} = 0 . \quad (99)$$

The energy of a quasiparticle ε can be written as

$$\varepsilon - \mu = v_F(p - p_0) + \sum_{\sigma'} \int f(\mathbf{p}, \sigma; \mathbf{p}', \sigma') \bar{n}(\mathbf{p}', \sigma', r, t) d\tau' . \quad (100)$$

Substituting eq.(100) into eq.(99), and collecting terms of the first order in \bar{n} , we get

$$\frac{\partial \bar{n}}{\partial t} + \mathbf{v} \cdot \frac{\partial \bar{n}}{\partial \mathbf{r}} - \frac{\partial n_0}{\partial \mathbf{p}} \cdot \left[\sum_{\sigma'} \int f(\mathbf{p}, \sigma; \mathbf{p}', \sigma') \frac{\partial \bar{n}}{\partial \mathbf{r}} d\tau' \right] = 0 . \quad (101)$$

If we divide f and \bar{n} into the spin symmetric and antisymmetric parts, we can get the independent equations

$$\begin{cases} \frac{\partial \bar{n}_s}{\partial t} + \mathbf{v} \cdot \frac{\partial \bar{n}_s}{\partial \mathbf{r}} - \frac{\partial n_0}{\partial \mathbf{p}} \int f^{(s)}(\mathbf{p}, \mathbf{p}') \frac{\partial \bar{n}_s}{\partial \mathbf{r}} 2d\tau' = 0 \\ \frac{\partial \bar{n}_a}{\partial t} + \mathbf{v} \cdot \frac{\partial \bar{n}_a}{\partial \mathbf{r}} - \frac{\partial n_0}{\partial \mathbf{p}} \int f^{(a)}(\mathbf{p}, \mathbf{p}') \frac{\partial \bar{n}_a}{\partial \mathbf{r}} 2d\tau' = 0 \end{cases} . \quad (102)$$

We treat only the spin symmetric equation below, but the same results can be derived from the spin antisymmetric equation. If we write \bar{n}_s as

$$\bar{n}_s = \left(-\frac{\partial n_0}{\partial \varepsilon} \right) \eta ,$$

the transport equation becomes

$$\frac{\partial \eta}{\partial t} + \mathbf{v} \cdot \frac{\partial \eta}{\partial \mathbf{r}} - \frac{\partial \varepsilon}{\partial \mathbf{p}} \int f^{(s)}(\mathbf{p}, \mathbf{p}') \frac{\partial n_0}{\partial \varepsilon} \frac{\partial \eta}{\partial \mathbf{r}} \cdot 2d\tau' = 0 . \quad (103)$$

$\frac{\partial n_0}{\partial \varepsilon}$ can be replaced by $-\delta(\varepsilon - \mu)$ and with account of definition of $F^{(s)}$, we can

write

$$\frac{\partial \eta}{\partial t} + \mathbf{v} \cdot \frac{\partial \eta}{\partial \mathbf{r}} + \mathbf{v} \cdot \int F^{(s)}(\cos \Theta) \frac{\partial \eta}{\partial \mathbf{r}} \frac{d\Omega'}{4\pi} = 0 \quad (104)$$

where Θ is the angle between \mathbf{p} and \mathbf{p}' . In the sound wave η changes like

$$\eta = \eta_0(\hat{\mathbf{p}}) e^{i(\mathbf{K} \cdot \mathbf{r} - \omega t)} \quad (105)$$

Then we can express the transport equation as

$$(\omega - \mathbf{K} \cdot \mathbf{v}) \eta_0(\hat{\mathbf{p}}) - \mathbf{K} \cdot \mathbf{v} \int F^{(s)}(\cos \Theta) \eta_0(\hat{\mathbf{p}}') \frac{d\Omega'}{4\pi} = 0 \quad (106)$$

that is

$$\eta_0(\hat{\mathbf{p}}) = \frac{\mathbf{K} \cdot \mathbf{v}}{\omega - \mathbf{K} \cdot \mathbf{v}} \int F^{(s)}(\cos \Theta) \eta_0(\hat{\mathbf{p}}') \frac{d\Omega'}{4\pi} \quad (107)$$

$F^{(s)}$ can be expanded in a series of Legendric polynomials

$$F^{(s)}(\cos \Theta) = \sum_{\ell} F_{\ell}^{(s)} P_{\ell}(\cos \Theta) \quad (108)$$

Similarly, we can expand η_0 like

$$\eta_0(\hat{\mathbf{p}}) = \eta_0(\theta, \phi) = \sum_{m=-\infty}^{\infty} \eta_0^m(\theta) e^{im\phi} \quad (109)$$

Here we discuss two simple models taking the initial one or two components of $F^{(s)}$. In the simplest model, we assume that the interaction $f^{(s)}(\mathbf{p}, \mathbf{p}')$ is constant independent of both \mathbf{p} and \mathbf{p}' . In this case the only component of $F^{(s)}$ is $F_0^{(s)}$. The transport equation becomes

$$\eta_0 = \frac{\cos \theta}{u - \cos \theta} \int F_0^{(s)} \eta_0^0(\cos \theta') \frac{1}{2} \sin \theta' d\theta' \quad (110)$$

where we assume that the direction of \mathbf{K} is along z -axis and u is defined by

$$u = \frac{\omega}{K v_F} \quad (111)$$

To solve eq.(110), we write

$$\eta_0 = \eta_0^0 = A \frac{\cos \theta}{u - \cos \theta} , \quad (112)$$

and substitute eq.(112) into eq.(110). Then we get

$$-\frac{1}{F_0^{(s)}} = 1 + \frac{u}{2} \ln \left(\frac{u-1}{u+1} \right) . \quad (113)$$

If $F_0^{(s)} > 0$, there is one real solution, such that $u > 1$. From the definition of u , we can write

$$\omega = uv_F K . \quad (114)$$

This means that uv_F is the sound velocity. This velocity is larger than v_F because $u > 1$. This sound is called zero sound since it exists at zero temperature. If we take one more component of $F^{(s)}$, that is

$$F^{(s)} = F_0^{(s)} + F_1^{(s)} \cos \Theta . \quad (115)$$

then we can obtain a different mode of the oscillation. Because $\cos \Theta$ can be expressed in terms of θ and ϕ as

$$\begin{aligned} \cos \Theta &= (\hat{p} \cdot \hat{p}') \\ &= \sin \theta \sin \theta' (\cos \phi \cos \phi' + \sin \phi \sin \phi') + \cos \theta \cos \theta' , \end{aligned} \quad (116)$$

we need only $m = 1$ terms of η_0 . So we can write η_0 as

$$\eta_0(\theta, \phi) = \eta_0^1(\theta) \cos \phi . \quad (117)$$

Substituting (115)~(117) into eq.(107), we get

$$\eta_0^1 \cos \phi = \frac{\cos \theta}{u - \cos \theta} F_1^{(s)} \int \sin \theta \sin \theta' \cos \phi \cos \phi' \eta_0^1(\theta') \cos \phi' \frac{d\Omega'}{4\pi} . \quad (118)$$

Eq.(118) can be reduced to

$$\eta_0^1 = \frac{\sin \theta \cos \theta}{u - \cos \theta} F_1^{(s)} \int \sin \theta' \eta_0^1(\theta') \frac{\sin \theta' d\theta'}{4} . \quad (119)$$

We can consider the solution in the same way as the last simplest case. We

therefore obtain

$$1 + \frac{u}{2} \ln \frac{u-1}{u+1} = \frac{6 - F_1^{(s)}}{3F_1^{(s)}(u^2 - 1)}. \quad (120)$$

As discussed before, if $F_1^{(s)} > 6$, we have a real solution.

So far we have discussed two models. In the case of $F^{(s)} = F_0^{(s)}$, η_0 is proportional to $\frac{\cos \theta}{u - \cos \theta}$. To illustrate this oscillation, we compare it with the small shift of the whole Fermi sphere along the z -axis. The displacement of the Fermi surface along the z -axis gives η , which is proportional to $\cos \theta$. In the case of zero sound $\eta_0 \sim \frac{\cos \theta}{u - \cos \theta}$ the situation is some what more complicated, but $\eta_0 \approx \frac{1}{u} \cos \theta$ when $u \gg 1$. On the other hand, in the case of $F^{(s)} = F_0^{(s)} + F_1^{(s)} \cos \Theta$, for the second of the models considered, η_0 is proportional to $\frac{\sin \theta \cos \theta}{u - \cos \theta} \cos \phi$. In the limit $u \gg 1$, it describes a shift of a Fermi sphere in the direction transverse to the propagation of the wave.