A Purely Spatial Map, and Spatial Chaos in a Coupled Map Lattice Model for Open Flow

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Abstract

It is shown that a well known coupled map lattice model for open flow exhibits spatial chaos under temporal periodicity. A purely spatial map corresponding to the coupled map lattice is introduced which is very efficient in reproducing its static patterns.

1 Introduction

Coupled map lattices (CML) have drawn much interest in recent years as a model for spatiotemporal chaos. They are models with a continuous state, discrete space and discrete time, which have become popular not only for their computational efficiency but also for their extremely rich phenomenology and the prospects of wide ranges of applications through the discovery of universality classes [1-3].

In this proceedings we investigate a CML which has a logistic map as its local element, and in which the coupling is only to the next element to the left, i.e., it is a one-way coupled logistic lattice. Due to its conceptual reminiscence with open fluid flow, it is generally referred to as an open flow model (OFM) and may be expressed as:

$$x_{n+1}^{i} = (1-\epsilon)f(x_n^i) + \epsilon f(x_n^{i-1}), \tag{1}$$

where the local element is the logistic map

$$f(x_n) = 1 - \alpha x_n^2. \tag{2}$$

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The parameters are the nonlinearity α and the coupling constant ϵ, while i is the index for the lattice sites, and n the discrete time.

Many interesting features of (1), like spatial period-doubling and selective amplification of noise, were already reported in [4] and [5], but those investigations did not take the full range of possible coupling constants ($0.0 \leq \epsilon \geq 1.0$) into account and were generally limited to $\epsilon \leq 0.5$. Recent studies of the diffusively coupled logistc lattice (DCLL), of which the OFM is in principle nothing but a version with a maximally asymmetric coupling, however, have shown that larger values of ϵ may yield unexpected phenomena like the traveling wave or suppression of supertransient chaos at high nonlinearity [6], [7].

In the OFM, for larger values of ϵ, we find the fascinating novelty of spatial chaos with temporal periodicity. With this discovery, all the basic possibilities of chaos in space and time now have a representative in a coupled logistic lattice. Spatio-temporal chaos has of course long been known to exist in the DCLL and (non-trivial) purely temporal chaos with spatial homogeneity was recently reported for a lattice with asymmetric coupling in [8].

The one-way coupling and the temporal periodicity make it possible to introduce a spatial map as an implicit equation of equal-time spatial variables. The spatial map can be used to reproduce a spatial pattern of the OFM and is thus most useful for its analysis. One application is the calculation of Lyapunov exponents. Another is the rapid generation of spatial patterns due to the absence of a transient time.

2 Spatial Chaos

The Chaos that was thus far known to exist in the open flow model (1) is of a spatio-temporal nature. Although it may occur in the entire chaotic region of the single logistic map including the periodic windows, it is limited to values of the coupling constant ϵ that are not too big. For the DCLL, it was shown [7] that for larger ϵ chaos is suppressed after a supertransient, even up to the maximum nonlinearity of $\alpha = 2.0$. Although the fully asymmetric coupling of the OFM is less likely to yield stability than the symmetric one, one might nevertheless expect at least some manifestation of a tendency to regularity.

Our new finding here is that for $\epsilon > \epsilon_{min}(\alpha)$ not only the spatio-temporal patterns of the OFM become temporally periodic, but also that they may be spatially chaotic. Two examples are given in Fig. 1, where a) shows the state of a lattice with at most a temporal periodicity of 32 after the transients have died out, b) the corresponding return map, c) a much more regularly appearing lattice with at most a temporal periodicity of 16, and d) the return map corresponding to c).

The temporal bifurcations, of course, immediately raise the question of whether the bifurcation sequence continues to chaos or whether it is truncated. In general, the answer cannot be given at this moment, but it was numerically verified that some very large lattices (N in the order of 10e4 – 10e5) are still temporally periodic. It also should be noted that even though the thermodynamic limit is not clear all the way downflow, the one-way coupling guarantees the validity of the results upflow regardless of the behaviour downflow.

Spatial chaos with temporal periodicity is not limited to a small region in parameter space, but exists in quite a large area as can be seen in the phase diagram Fig. 2, where it is marked as (SC). The other three basic types of patterns we would like to distinguish here, are: spatially (and temporally) periodic patterns (SP), spatially quasiperiodic patterns with temporal periodicity (SQP), and spatio-temporal patterns of various kinds including spatio-temporal chaos (STP). The region marked as 'ZZ' is the well-known zigzag pattern which is spatially and temporally periodic and therefore a special case of SP.

The various areas are separated by sharp lines in the phase diagram in order to give some impression of their location. In fact, however, the lines aren't very sharp at all and in many cases multiple attractors belonging to different basic patterns exist for the same parameters. In some cases this might be related to the dependence of the locations of the bifurcation points on the initial conditions, and thus mainly reflect the difference between up- and downflow. In other cases, however, different types of patterns may exits for the same temporal periodicity, indicating genuine multiple states. The diagram indicates only the most common patterns in a larger area of parameter space. That is to say that small SQP regions within the SC region etc. were not taken into consideration.

In figure 3, examples for SP and SQP are depicted. Besides the spatially period 4 pattern shown in Fig. 3a), a zigzag pattern is the most common in SP.

3 Spatial Map

The perfect temporal periodicity for larger values of the coupling constant ϵ makes its possible to define a spatial map as an implicit function. In principle this should be possible for any coupled map lattice, the special situation here however is the one-way coupling which opens up the possibility to successively generate lattice sites downflow based on the states of k lattice sites upflow, where k is the temporal periodicity of the OFM.

If we define

$$F(x_n^i) = (1-\epsilon)f(x_n^i) + \epsilon f(x_n^{i-1}), \tag{3}$$

then a lattice site x_n^i which has a temporal periodicity k must fulfill,

$$x_n^i = F^k(x_n^i), \tag{4}$$

where $F^k(x_n^i)$ is the k-the iterate of $F(x_n^i)$. In general, an equation like 4 may have a large number of solutions, especially if the periodicity is high. In a future publication, it will be shown, however, that for sufficiently large ϵ there is only one. Up to second order, the condition is

$$\epsilon \geq 1 - \frac{3}{4\alpha}. \tag{5}$$

This condition holds even for $k \to \infty$ and therefore may explain the large area of temporal periodicity in the phase diagram 2. Starting from the boundary upflow (necessarily with a temporal periodicity of one), the dynamical system naturally selects this solution for successive sites downflow.

Due to its iterative nature, Eq. 4 can in principle be implemented on a computer in a straightforward way, even for high periodicities k, although it might be rather tedious to keep track of all the necessary intermediate results. In the case of Eq. 4, however, one may employ the fact that in the course of calculation, every lattice site needs to be accessed only once when using the following recursive algorithm,

```
for(l=1;l≤k;l++){
    for(m=k;m≥l;m−−){
        y^{i−m} = F(y(i − m + 1));
    }
},
```

where y^i is temporary variable equal to x^i, and in which intermediate results of F are successively shifted to the right until finally the result $F^k(x_n^i) = y^{i-k}$ is obtained.

We can now define a spatial map corresponding to Eq. 4 as

$$G^k(x^i) = -x^i + F^k(x^i), \tag{6}$$

with which a lattice can be generated by successively finding the roots of Eq. 6 and incrementing the index i. Since Eq. 6 depends on k variables, $x^0, \ldots, x^k$ need to be supplied as initial conditions (note the absence of the time-index n).

In Fig. 4 some patterns generated by the spatial map are shown. The lines in (a) are the solutions for $k = 1, k = 2$ and $k = 4$, respectively, and in (b) the solution for $k = 16$ is given. The levels are at the same amplitudes as the corresponding ones in the OFM shown in Fig. 3. Of course the OFM has k phases and the spatial map doesn't have any. Depending on the initial conditions, however, the spatial map can yield all the possible phases. As can be seen in Fig. 4c), the return map of the pattern generated is also most similar to the one of the OFM in Fig. 3b). The pattern in (a) is an attractor and small inaccuracies in the calculation of the roots will not matter. Due to the sensitive dependence of (spatially) chaotic patterns on initial conditions, however, the spatial map cannot reproduce spatial chaos as exactly with regrad to particular sections of the pattern of the OFM. Nevertheless general characteristics should accurately be reproduced as is shown in (d) where indeed a pattern is generated very similar in appearance to the one in Fig. 1c).

4 Conclusions

Spatial chaos with temporal periodicity was shown to exist in a coupled map lattice model for open flow. It was furthermore shown that an implicit purely spatial map can be introduced which may efficiently reproduce many spatial patterns of the open flow model and which promises to be very useful for theoretical analysis.

Acknowledgements

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References

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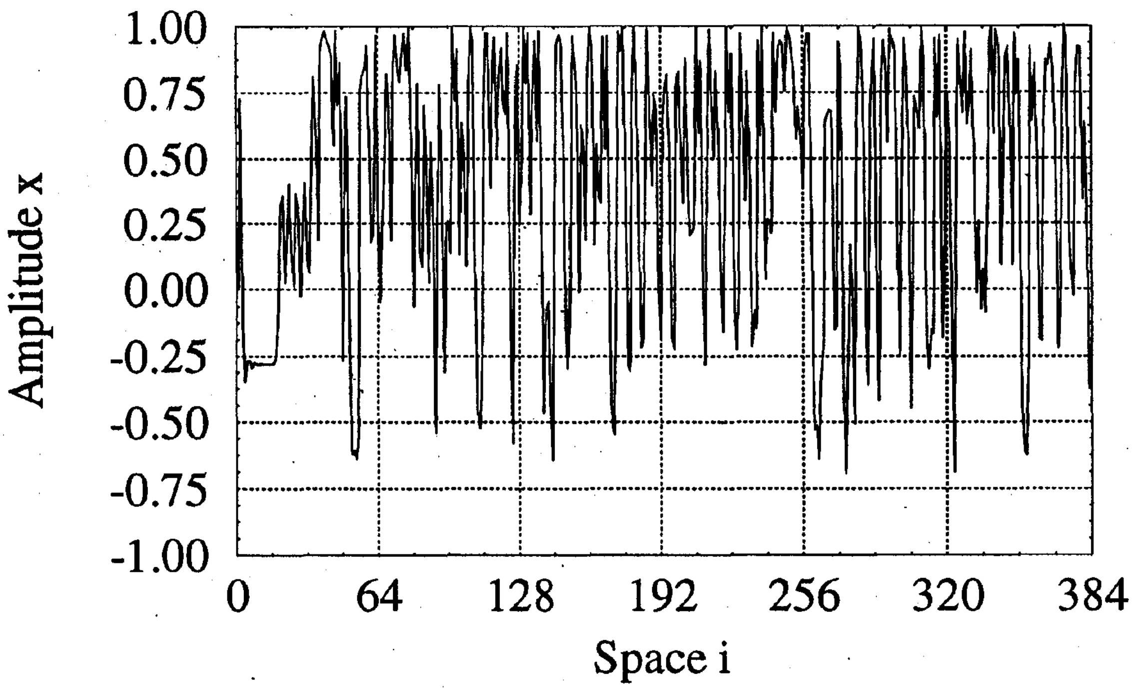
Figure Captions

Fig. 1. Spatial chaos with temporal periodicity. a) $\alpha = 1.7$, $\epsilon = 0.45$ and $N = 384$, the state of the lattice is depicted at time 19328 after all the transients have died out. The temporal periodicities are 1 for $N = 1-2$, 2 for $N = 3-16$, 4 for $N = 17-31$, 8 for $N = 32-78$, 16 for $N = 79-127$ and 32 for $N = 128-384$ respectively. b) The spatial return map corresponding to (a) for $N = 128-384$. c) $\alpha = 1.6$, $\epsilon = 0.5$ and $N = 1024$, the state of the lattice is depicted at time 6592 after all the transients have died out. Only the first 512 sites are depicted. The temporal periodicities are 1 for $N = 1-17$, 2 for $N = 18-47$, 4 for $N = 48-96$, 8 for $N = 97-173$ and 16 for $N = 174-1024$ respectively. d) The spatial return map corresponding to (c) for $N = 256-1024$.

Fig. 2. Phase diagram of the open flow model. The regions indicated are only meant to give a rough impression of the locations of the various patterns. The system size is $N = 383$. The abbreviations stand for spatially (and temporally) periodic (SP), spatially chaotic with temporal periodicity (SC), spatially quasiperiodic with temporal periodicity (SQP), spatiotemporal patterns including spatio-temporal chaos (STP), and zigzag pattern (ZZ).

Fig. 3. Temporally periodic patterns. a) Spatially periodic pattern (SP), $\alpha = 1.9$, $\epsilon = 0.9$ and $N = 1024$. Only the first 64 sites are shown, the remaining sites are the same. b) Return map corresponding to (a) for sites $N = 128-1024$. c) Spatially quasiperiodic pattern (SQP), $\alpha = 1.45$, $\epsilon = 0.5$ and $N = 1024$. Only the first 384 sites are shown, the remaining sites are the same. d) Return map corresponding to (c) for sites $N = 256-1024$.

Fig. 4. Patterns generated by the spatial map. a) $\alpha = 1.45$ and $\epsilon = 0.5$. The cases for $k = 1$, $k = 2$ and $k = 4$ are plotted together. b) same as (a) for $k = 16$. c) The return map of (b). d) $\alpha = 1.6$, $\epsilon = 0.5$ and $k = 16$.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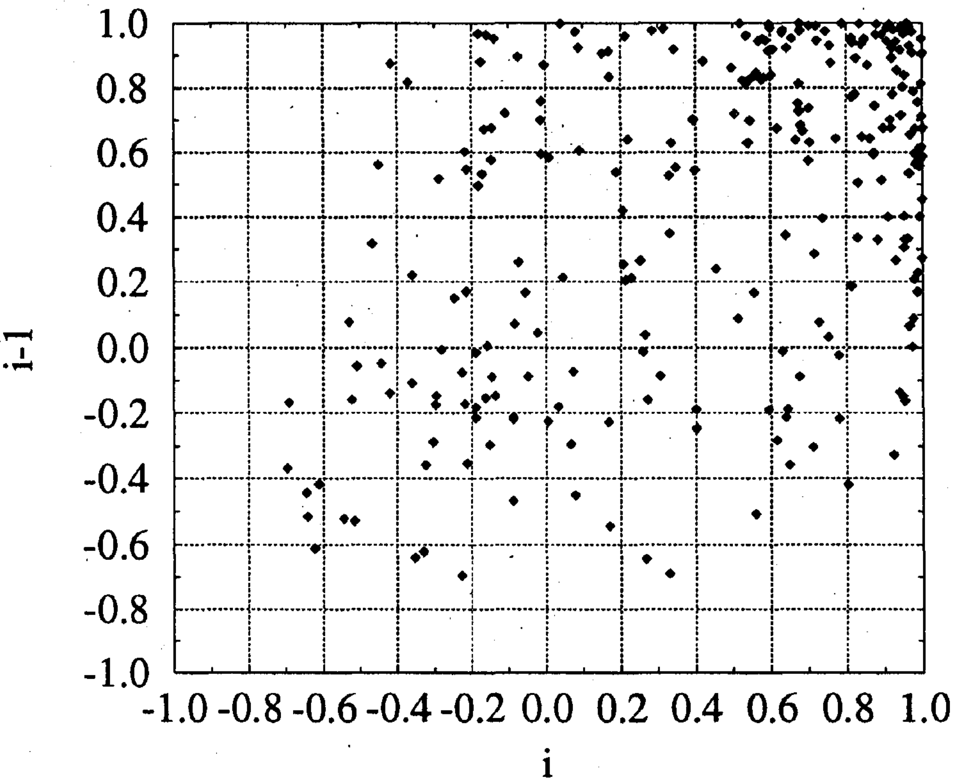
Space i

Fig. 1a.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i

Fig. 1b.

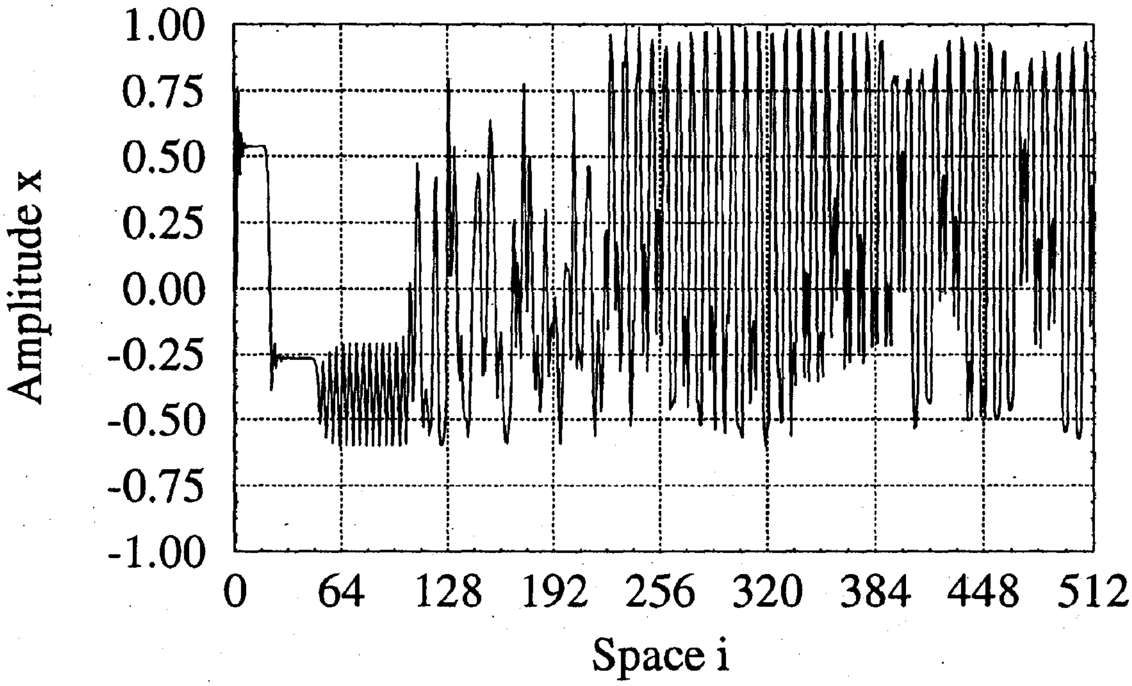
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Fig. 1c.

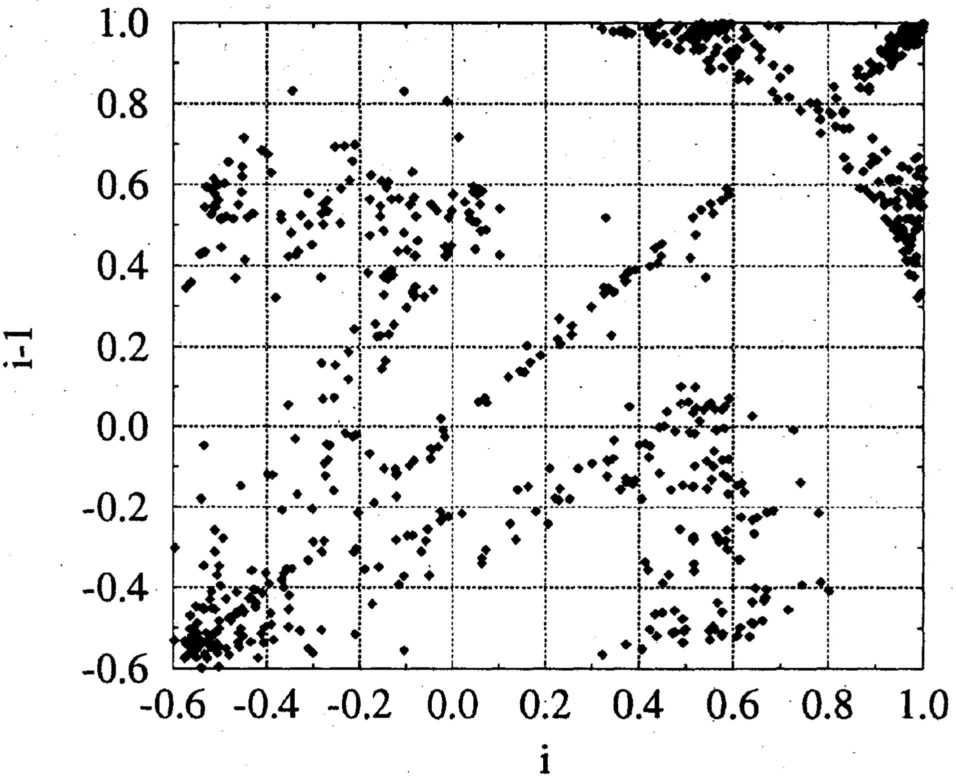
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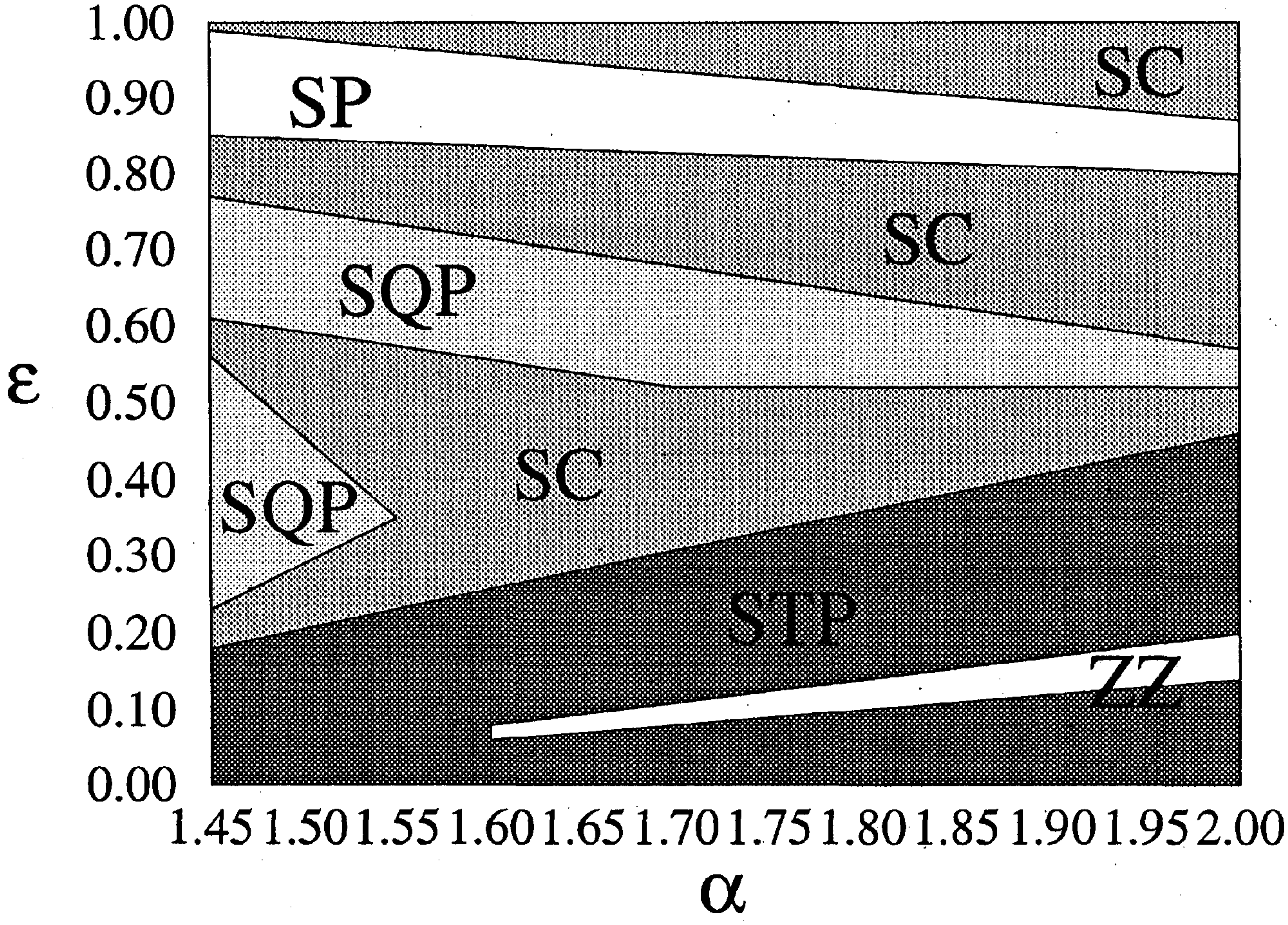
Fig. 1d.



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α

Fig. 2.

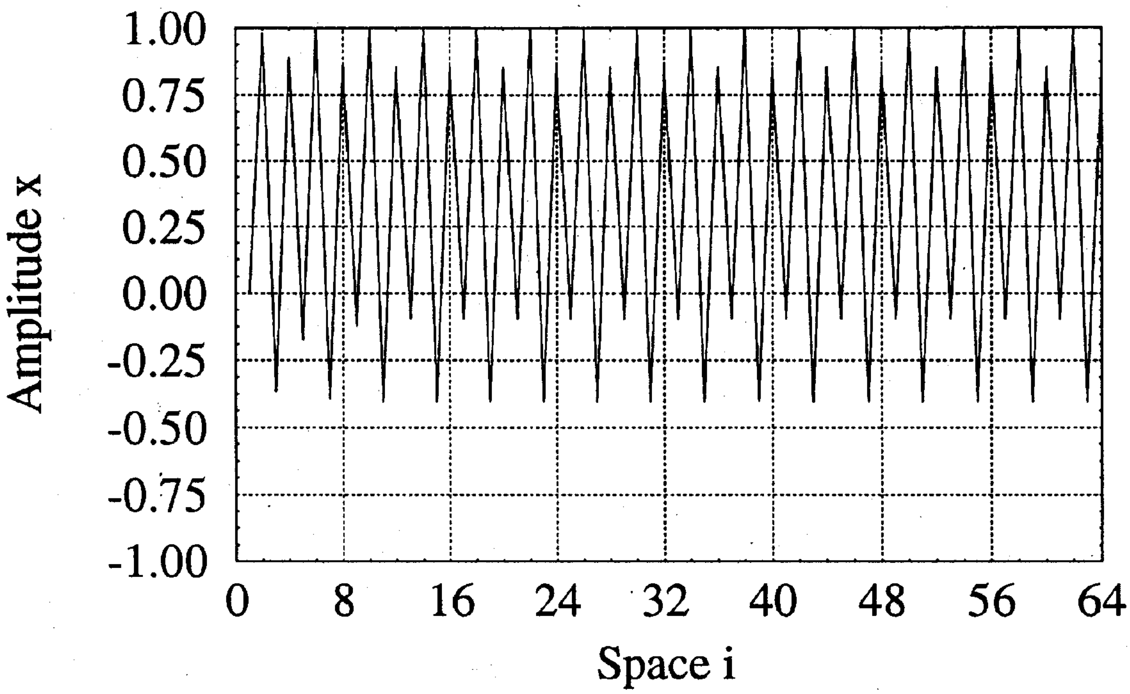
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Fig. 3a.

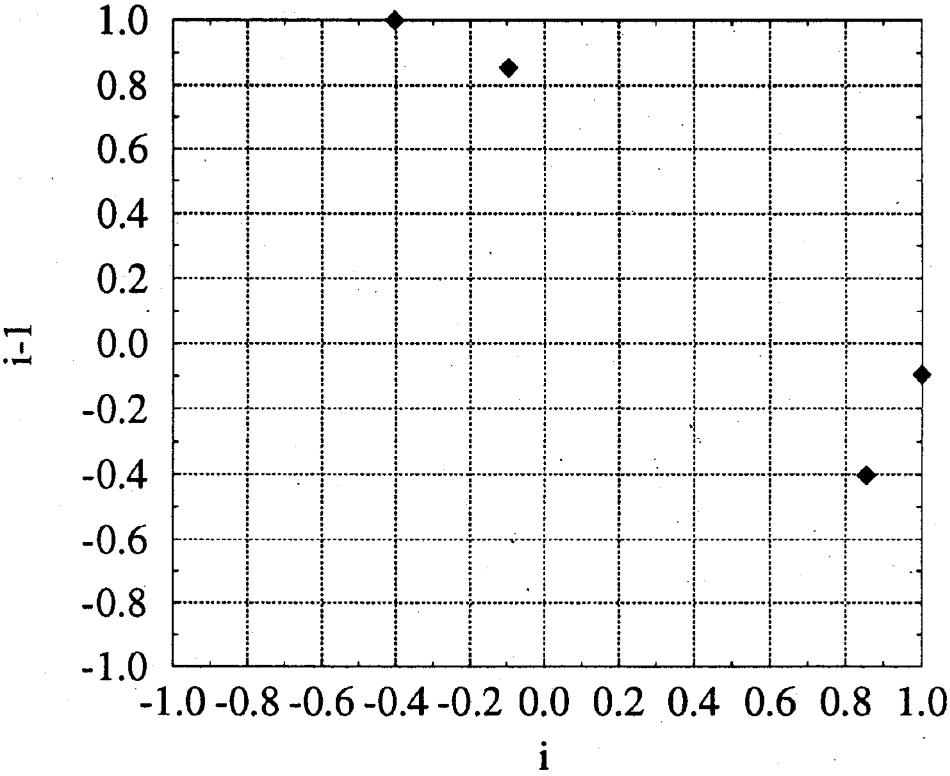
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Fig. 3b.

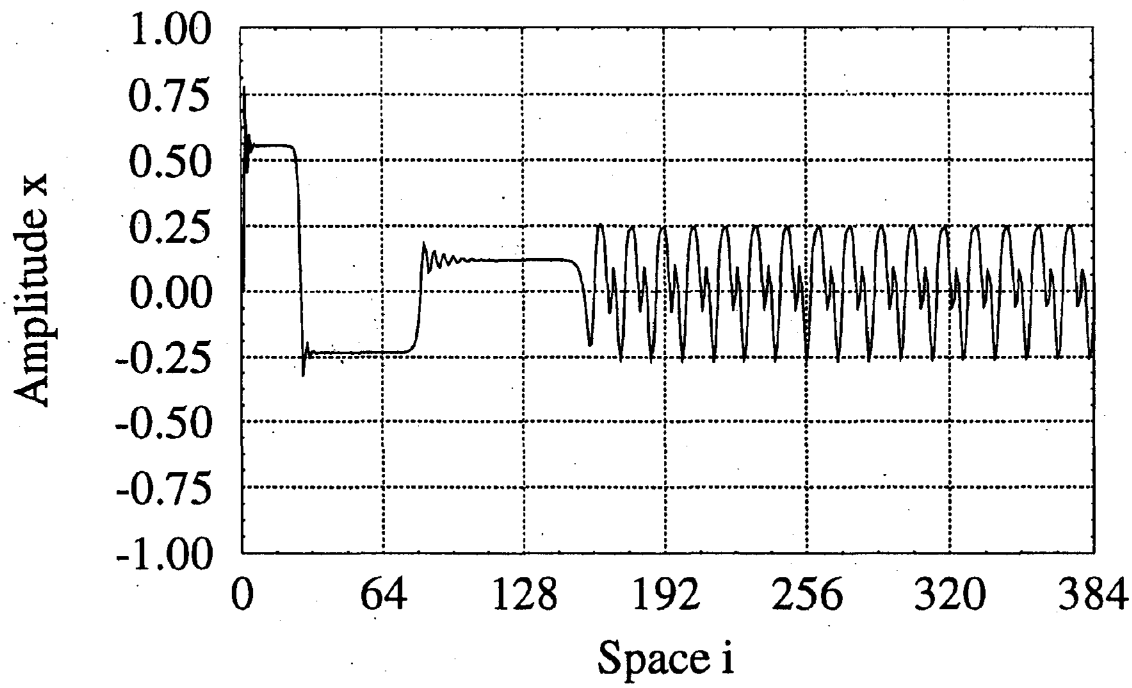
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Fig. 3c.

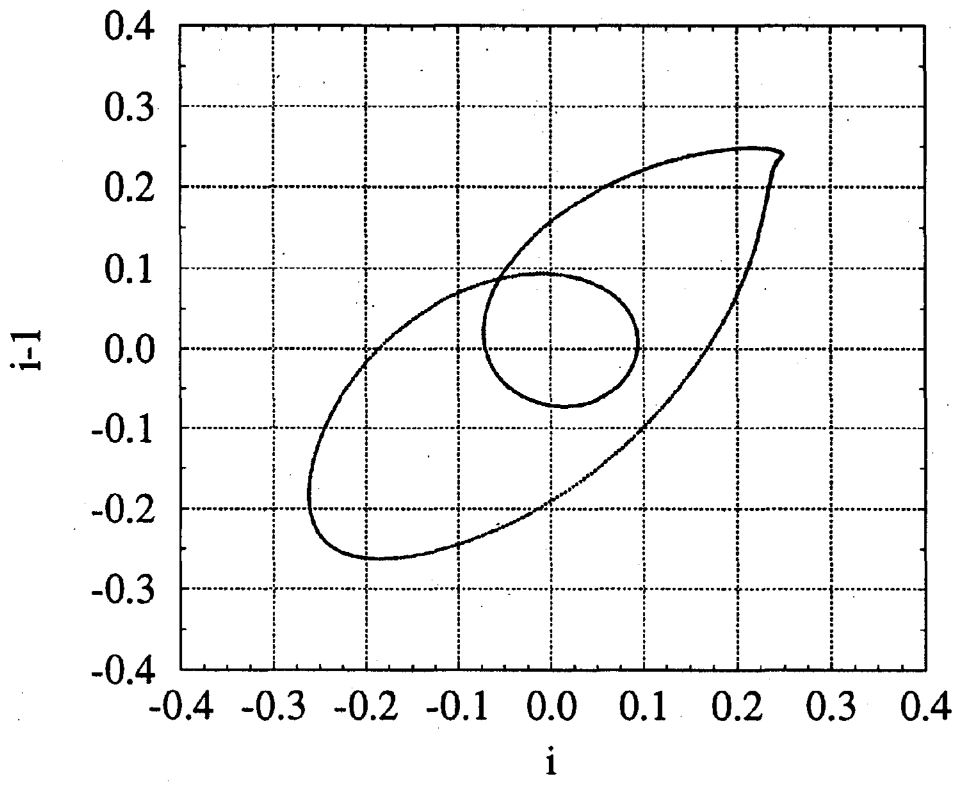
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Fig. 3d.

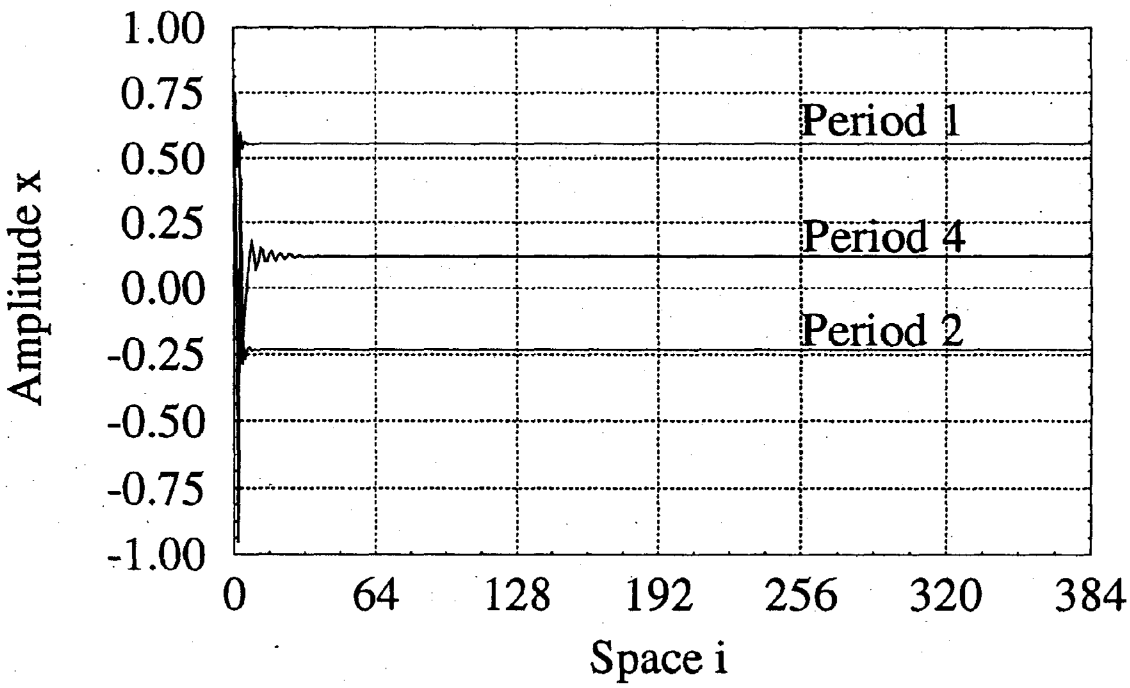
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Fig. 4a.

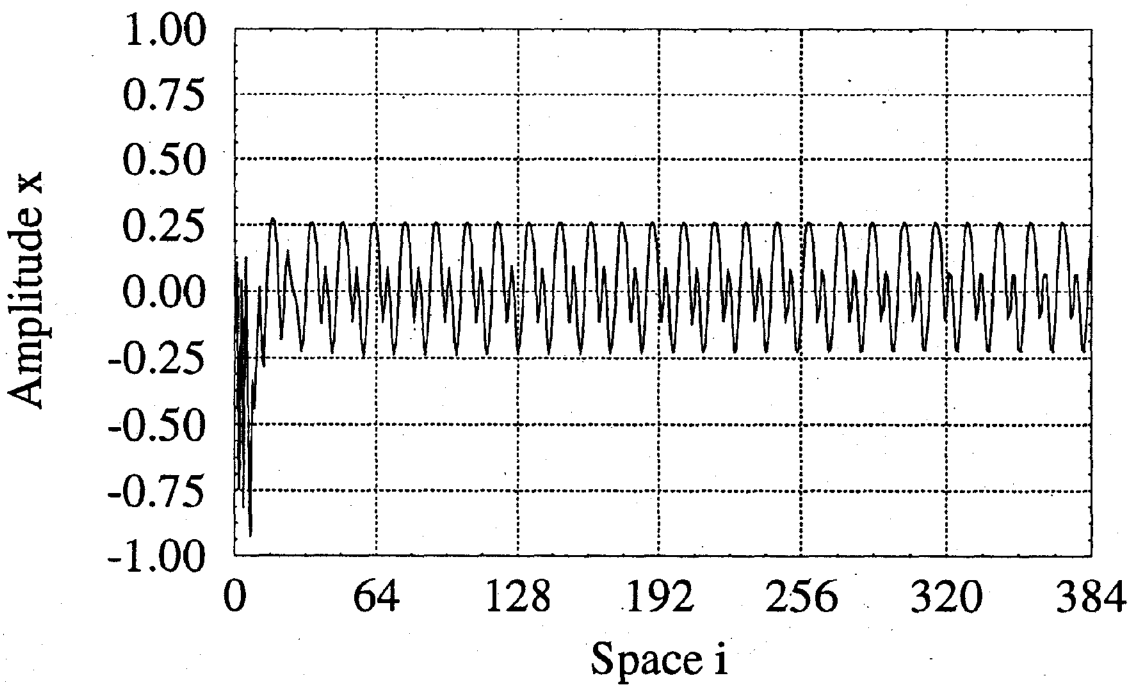
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Fig. 4b.

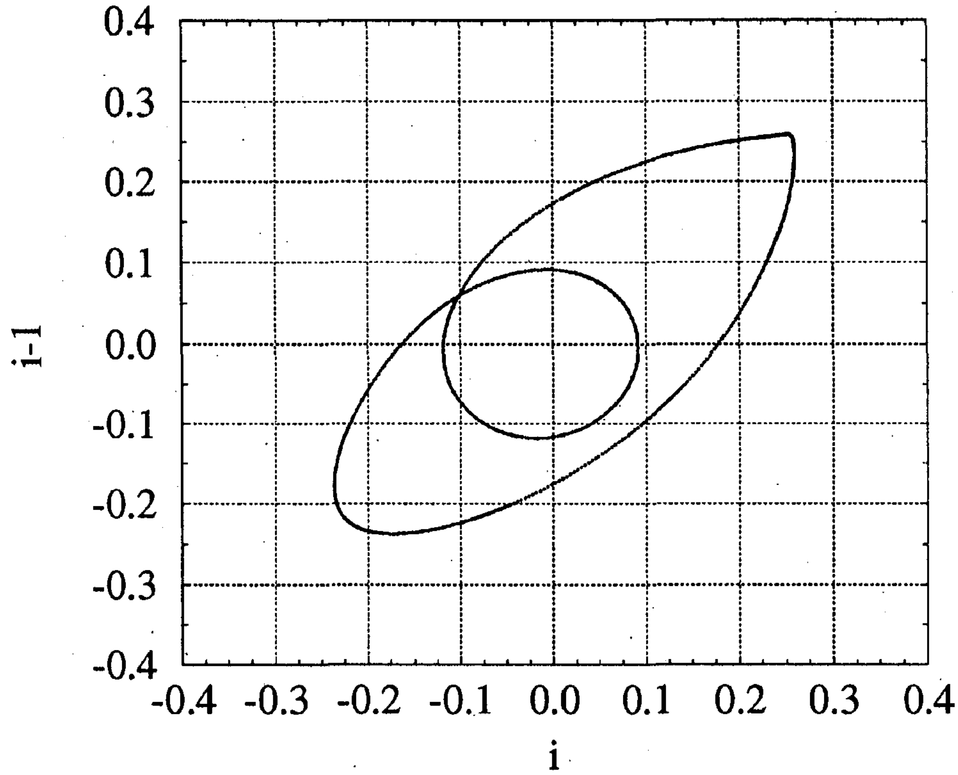
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Fig. 4c.

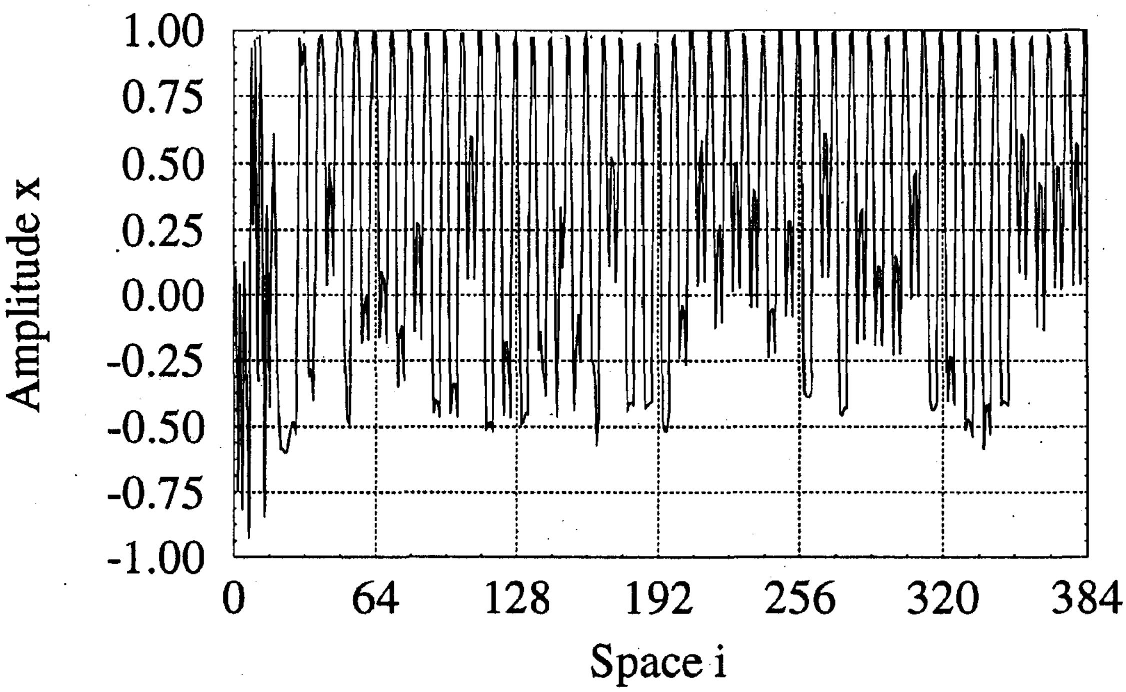
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Fig. 4d.