Anyons と自発的磁場生成

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I. Anyons と Chern-Simons 理論

Non-relativistic quantum mechanics of anyons and Chern–Simons gauge theory are "almost", but not quite equivalent.[1] We show it by several steps.

a. エニオン

Under a clockwise or counter-clockwise interchange of two identical particles, say a and b, a Schröinger wave function picks up a phase

$$\begin{aligned} &P_\pm(a,b)\Phi(1,\cdots,N) = -e^{\pm i\theta_s}\Phi(1,\cdots,N) \\ &\Phi(1,\cdots,N)\big|_{\mathbf{x}_a=\mathbf{x}_b} = 0 \quad \text{for} \quad -e^{\pm i\theta_s} \neq 1 \quad . \end{aligned} \tag{1.1}$$

The second equation says that Pauli's principle holds for anyons except for the case of bosons. Free anyons are described by a Hamiltonian

$$H = \sum_{a=1}^{N} -\frac{\hbar^2}{2m}\nabla_a^2 \tag{1.2}$$

with the boundary condition (1.1)

b. フェルミオン表示

We perform a singular gauge transformation:

$$\begin{aligned} \Phi^{\mathrm{f}} &= \Omega_{\mathbf{singular}}\Phi \\ \Omega_{\mathbf{singular}} &= \exp\left\{ i\frac{\theta_s}{\pi}\sum_{a<b}\tan^{-1}\frac{(x_a-x_b)_1}{(x_a-x_b)_2} \right\} \end{aligned} \tag{1.3}$$

Φ^{f} satisfies

$$\begin{aligned} &i\frac{\partial}{\partial t}\Phi^{\mathrm{f}} = \sum_{a=1}^{N} -\frac{\hbar^2}{2m}\left\{ \nabla_a^j + i\frac{\theta_s}{\pi}\sum_{a\neq b}\epsilon^{jk}\frac{(\mathbf{x}_a-\mathbf{x}_b)_k}{(\mathbf{x}_a-\mathbf{x}_b)^2} \right\}^2 \Phi^{\mathrm{f}} \\ &P_\pm(a,b)\,\Phi^{\mathrm{f}}(1,\cdots,N) = -\Phi^{\mathrm{f}}(1,\cdots,N) \quad , \end{aligned} \tag{1.4}$$

i.e. Φ^{f} is a wave function in the fermion representation.

Non-trivial Hamiltonian in (1.5) implies that *free anyons are not free*, and that they are equivalent to a fermion system with a particular interaction.

c. Aharonov-Bohm 効果

The above interaction is interpreted as an Aharonov-Bohm effect. Suppose that each particle carries both a charge e and a magnetic flux μ. Then a vector potential $\vec{A}$ at the location of particle a generated by other particles is

$$\frac{e}{\hbar c} A^j(\mathbf{x}_a) = -\frac{e\mu}{2\pi\hbar c} \sum_{b\neq a} \epsilon^{jk} \frac{(\mathbf{x}_a - \mathbf{x}_b)_k}{(\mathbf{x}_a - \mathbf{x}_b)^2} \quad .$$

Upon identifying

$$\theta_s = \frac{e\mu}{2\hbar c} \tag{1.5}$$

one sees that the interaction is nothing but an Aharonov-Bohm (charge-flux) interaction. Hence one can phrase that

$$\text{anyon} = \begin{cases} \text{charge} & 1 \\ \text{flux} & 2\hbar c\theta_s \end{cases} \quad . \tag{1.6}$$

However it's not exactly a Maxwell interaction. There is no charge-charge interaction.

d. Chern-Simons ゲイジ理論

Consider a nonrelativisitic matter field ψ coupled to Chern-Simons gauge fields:

$$\mathcal{L} = -\frac{\kappa}{4\pi}\epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + i\psi^\dagger D_0 \psi - \frac{1}{2m}(D_k\psi)^\dagger (D_k\psi) \tag{1.7}$$

where $D_0 = \partial_0 + ia_0$ and $D_k = \partial_k - ia^k$. ($\hbar = c = 1$ hereafter.) Solving the Euler equation $-(\kappa/4\pi)\epsilon^{\mu\nu\rho} f_{\nu\rho} = j^\mu$ in the radiation gauge $\partial_k a^k = 0$, one can eliminate the Chern-Simons fields. The resultant theory of nonrelativistic (fermionic) matter field is described by

$$\begin{aligned} i\frac{\partial}{\partial t}\psi &= [\psi, H] \\ H[\psi,\psi^\dagger] &= \int d\mathbf{x}\ \frac{1}{2m}(D_k\psi)^\dagger (D_k\psi) \\ \text{where} \quad a^j(x) &= -\frac{1}{\kappa}\int d\mathbf{y}\ \epsilon^{jk}\frac{(\mathbf{x}-\mathbf{y})_k}{(\mathbf{x}-\mathbf{y})^2}\cdot \psi^\dagger\psi(y) \end{aligned} \tag{1.8}$$

The connection to quantum mechanics of a finite number of particles is made by identifying the Schrödinger wave function in the fermion representation as a matrix element of a string of field operators:

$$\Phi^{\mathrm{f}}(1,\cdots,N) = \langle 0|\psi(1)\cdots\psi(N)|\Psi_N\rangle \tag{1.9}$$

where $|0\rangle$ and $|\Psi_N\rangle$ are the vacuum and N-particle state.

The equation satisfied by the above wave function is

$$
\begin{aligned}
i\frac{\partial}{\partial t}\Phi^{\mathrm{f}} &= \sum_{a=1}^{N}\langle 0|\psi(1)\cdots[\psi(a),H]\cdots\psi(N)|\Psi_N\rangle \\
&\ \ \vdots \\
&= \sum_{a=1}^{N} -\frac{\hbar^2}{2m}\left\{\nabla_a^j + \frac{i}{\kappa}\sum_{a\neq b}\epsilon^{jk}\frac{(\mathbf{x}_a-\mathbf{x}_b)_k}{(\mathbf{x}_a-\mathbf{x}_b)^2}\right\}^2 \Phi^{\mathrm{f}} \quad .
\end{aligned} \tag{1.10}
$$

This is exactly the same as eq. (1.4), provided that

$$
\kappa = \frac{\theta_s}{\pi} \quad . \tag{1.11}
$$

e. ほとんど等価なこと

We have shown a series of equivalence

$$
\begin{array}{lll}
\mathcal{L}^{\mathrm{CS}}[a_\mu,\psi,\psi^\dagger] \quad (1.7) & & \\
\qquad \Updownarrow & a_\mu\text{の除去} & \\
H[\psi,\psi^\dagger] \quad (1.8) & & \\
\qquad \Updownarrow & \theta_s = \dfrac{\pi}{\kappa}\ ,\ \Phi^{\mathrm{f}} = \langle 0|\psi(1)\cdots\psi(N)|\Psi_N\rangle & \qquad (\text{Equiv}) \\
\text{Fermion 表示の QM} \quad (1.4) & & \\
\qquad \Updownarrow & \Omega_{\mathrm{singular}} & \\
\text{自由な anyon 系} \quad (1.1)\ ,\ (1.2) & &
\end{array}
$$

However, this does not necessarily mean that the role of Chern-Simons fields is just to change the statistics which is defined *mod* 2π.

To illustrate it, let us compare two models:

$$
\begin{aligned}
\mathcal{L}_0 &= i\psi^\dagger\dot{\psi} - \frac{1}{2m}|(\partial_k - ieA^k_{\mathrm{ext}})\psi|^2 \\
\mathcal{L}_p &= -\frac{1}{4\pi p}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + i\psi^\dagger(\partial_0 + ia_0)\psi - \frac{1}{2m}|(\partial_k - ia^k - ieA^k_{\mathrm{ext}})\psi|^2 \quad .
\end{aligned} \tag{1.12}
$$

In the latter, statistics phase is changed by $\theta_s = p\pi$ so that fermions are transformed into fermions for an even integer $p = 2n$. Are the two theories $\mathcal{L}_0$ and $\mathcal{L}_{2n}$ really equivalent? In the literature in QHE the equivalence is implicit.

Subtlety lies in the boundary condition. The corresponding Schrödinger wave functions in the fermion representation are related by

$$
\Phi^{\mathrm{f}}_{p=\pm 2n}(1,\cdots,N) = \prod_{a<b}\left(\frac{z_a - z_b}{\bar{z}_a - \bar{z}_b}\right)^{\pm n}\cdot\Phi^{\mathrm{f}}_0(1,\cdots,N) \tag{1.13}
$$

where $z_a = x_a + iy_a$. Although both $\Phi^{\rm f}$'s are single-valued, the transformation factor is singular under differentiation. Hence, even if $\Phi^{\rm f}_0$ belongs to the Hilbert space in the $\mathcal{L}_0$ theory, the corresponding $\Phi^{\rm f}_{\pm 2n}$ may not do so in the $\mathcal{L}_{\pm 2n}$ theory.

This subtlety is more profound when fermions are transformed into bosons, as in the case of $\mathcal{L}_p$ with an odd integer p. Although transformed particles are bosons, their wave functions must vanish when coordinates of two particles coincide. In other words resultant bosons have hard cores.

To conclude, Chern-Simons gague theory is equivalent to naive anyon quantum mechanics up to the subtlety in the boundary condition on wave functions.

II. 磁場の自発的生成

In relativistic theory the existence of a bare Chern-Simons term can induce spontaneous magnetization.[2] To be more specific, we shall show that in a model described by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \sum \overline{\psi} \{ \gamma^\mu (i\partial_\mu + qA_\mu) - m \} \psi \ , \tag{2.1}$$

a magnetic field $B \neq 0$ is spontaneously generated if a certain condition is satisfied. It implies that the Lorentz invariance also is spontaneously broken.

There are two types of two-component Dirac fermions in 2+1 dimensions. They are characterized by "chirality" defined by $\eta = \frac{i}{2} {\rm Tr}\, \gamma^0\gamma^1\gamma^2 = \pm 1$. Hence the fermion content is specified with $\{\eta_a, q_a, m_a\}$.

In the presence of a spontaneously generated uniform magnetic field B, there appear Landau levels whose energies are quantized as $E^2 = m^2 + (2n/l^2)$ where $n = 0, 1, 2, \cdots$ and l is the magnetic length ($|qB| = 1/l^2$). There results asymmetry at the lowest Landau level ($n = 0$). For $qB > 0$ ($qB < 0$) there are only positive (negative) energy solutions at $n = 0$. This asymmetry is responsible for the non-vanishing charge density

$$\langle j^0 \rangle = \sum_a \frac{1}{2\pi} \eta_a q_a^2 \left(\nu_a - \frac{1}{2}\right) B \tag{2.1}$$

where ν_a is the filling factor at the lowest Landau level. ν_a=0 or 1 if the level is empty or completely filled, respectively.

A variational ground state is specified by B and $\{\nu_a\text{=0 or 1}\}$. We denote its energy density by $\mathcal{E}(B, \{\nu_a\})$, whereas the energy density of the perturbative vacuum is given by $\mathcal{E}_0$.

One of the Euler equations implies that

$$\kappa = \sum_a \frac{1}{2\pi} \eta_a q_a^2 \left(\nu_a - \frac{1}{2}\right) = \Pi_1(0) \quad \text{in order for} \quad B \neq 0 \ . \tag{2.2}$$

Here $\Pi_1(p)$ is one of the invariant functions appearing in the vaccum polarization tensor. $-\Pi_1(0)$ represents the induced Chern-Simons term. In other words, to have a non-vanishing B, the bare Chern-Simons term must be exactly cancelled by the induced Chern-Simons term. It can be shown that the condition (2.2) is related to the Nambu-Goldstone theorem associated with the spontaneous breaking of the Lorentz invariance.

As an example we consider a chirally symmetric model in which all fermions have $q_a = e$ and $m_a = 0$. We prepare $2N$ species of fermions, a half of which have η=+, and the other half of which have η=−. Further we suppose that the bare CS coefficient is $\kappa = Ne^2/2\pi$. We pick a variational ground state in which ν_a=1 (0) for η=+ (−) so that the condition (2.2) is satisfied.

A detailed computation of quantum fluctuations shows that

$$\mathcal{E}(B, \{\nu_a\}) - \mathcal{E}_0 = -\frac{N_f e^2}{2\pi^3} \tan^{-1} \frac{4}{\pi} \cdot |eB| + \mathrm{O}(|B|^{3/2}) \tag{2.3}$$

for small $|B|$. The sign in the linear term is negative. For large B the Maxwell energy $\frac{1}{2}B^2$ dominates. There develops a minimum at $B \neq 0$.

A major factor in reducing the energy density by $B \neq 0$ is in the shift in zero-point energies of photons. In perturbation theory a photon is originally topologically massive ($m_{\mathrm{ph}} = \kappa$). In the $B \neq 0$ state the condition (2.2) implies that $m_{\mathrm{ph}}(\vec{p} = 0) = 0$. One can show that $m_{\mathrm{ph}}(\vec{p})$ changes from 0 to κ with the crossover around $|\vec{p}| \sim l^{-1}$.

To summarize we have found a model in which a non-vanishing magnetic field is spontaneously generated. This is probably the first consistent renormalizable theory where the Lorentz invariance is spontaneously broken.

References

[1] Y. Hosotani, *Int. J. Mod. Phys.* **B7** (1993) 2219 - 2323.

[2] Y. Hosotani, *"Spontaneously broken Lorentz invariance in three-dimensional gauge theories"*, UMN-TH-1211/93, to appear in *Phys. Lett.* **B**.