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On the Landau Levels and Quantum Group

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ABSTRACT

We find a quantum group structure in two-dimensional motion of nonrelativistic electrons in a uniform magnetic field. The quantum algebra is relevalnt to the degenerate Landau level and the deformation parameter of the algebra is given by a Landau-level filling factor.

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In this note we report a quantum group structure [1] in two-dimensional motion of nonrelativistic N_e electrons in a uniform magnetic field. We consider the N_e-electron systems on a square plane and torus of side Ll_B (l_B is called the magnetic length defined by $\sqrt{\hbar c/eB}$) and derive a quantum group algebra acting within each Landau level. We show that the wavefunctions form a representation basis of the quantum algebra $\mathcal{U}_q(sl(2))$ and the deformation parameter q is given by the filling factor $\nu = 1/m$ (m odd).

The generators of the quantum algebra $\mathcal{U}_q(sl(2))$ are realized by the magnetic translation operators T_α

$$T_{(\alpha_1,\alpha_2)} = exp(\frac{i}{\hbar}\alpha \cdot \beta)\,, \tag{1}$$

where

$$\beta_i = p_i - \frac{e}{c}A_i - \frac{eB}{c}\epsilon_{ij}x^j\,, \tag{2}$$

$$\epsilon_{11} = \epsilon_{22} = 0, \qquad \epsilon_{12} = -\epsilon_{21} = 1, \tag{3}$$

and the gauge potential is

$$A_i = -\frac{1}{2}B\epsilon_{ij}x^j - \partial_i\Lambda. \tag{4}$$

For N_e electrons, we define T_α by the product of each magnetic translation

$$T_{(\alpha_1,\alpha_2)} = \prod_{i=1}^{N_e} T^{(i)}_{(\alpha_1,\alpha_2)}. \tag{5}$$

The following combinations [4] of magnetic translations

$$E^+ = \frac{T_{(\Delta,\Delta)} - T_{(-\Delta,\Delta)}}{q-q^{-1}} \tag{6}$$

$$E^- = \frac{T_{(-\Delta,-\Delta)} - T_{(\Delta,-\Delta)}}{q-q^{-1}} \tag{7}$$

$$k = T_{(\Delta,0)}. \tag{8}$$

satisfy the defining relations of the $\mathcal{U}_q(sl(2))$

$$[E^+, E^-] = \frac{k^2 - k^{-2}}{q - q^{-1}}, \qquad kE^{\pm}k^{-1} = q^{\pm 1}E^{\pm} \tag{9}$$

with the identification

$$q = exp(i\Delta^2 l_B^{-2} N_e). \tag{10}$$

From here on, we choose the scalar function to be $\Lambda = \frac{1}{2}xy$ and assume that the total flux N_s through the surface satisfies the condition

$$N_s = \frac{L^2}{2\pi}. \tag{11}$$

Now we operate the generators (6)-(8) on the following wavefunctions:

(a) for one particle on the square plane [2]

$$\psi_l = exp\{2\pi il\frac{x}{Ll_B} - \frac{1}{2l_B^2}(y - y_0)^2\}H_n(\frac{y - y_0}{l_B}) \tag{12}$$

where

$$y_0 = -2\pi l_B \frac{l}{L} \tag{13}$$

and we have imposed the periodic boundary condition;

$$p_x = 2\pi\hbar\frac{l}{Ll_B}. \tag{14}$$

$H_n(x)$ is the Hermite polynomial. We have ignored ortho-normalization factor for (12). When the possible range of l is given as $-j \le l \le j$, the number of degenerate states N_s amounts to the odd integer $N_s = 2j + 1$.

(b) for N_e particles on the square torus with $N_s = mN_e$ (m odd integer) in the lowest Landau level [3]

$$\psi_l = exp[-\frac{1}{2l_B^2}\sum_{i=1}^{N_e} y_i^2] f_l(z_1, \ldots, z_{N_e}), \tag{15}$$

$$f_l(z_1, \ldots, z_{N_e}) = \prod_{j<k}^{N_e} \left[\frac{\Theta_1(z_j - z_k|i)}{\Theta_1(0|i)}\right]^m \Theta \begin{bmatrix} \frac{l}{m} \\ 0 \end{bmatrix} (\frac{m}{L}\sum_j z_j | im), \tag{16}$$

where $z_j = (x_j + iy_j)/l_B$ ($j = 1, \ldots, N_e$) and

$$\Theta \begin{bmatrix} a \\ b \end{bmatrix} (w|\tau) = \sum_{n \in \mathrm{Z}} exp\{i\pi\tau(n+a)^2 + 2\pi i(n+a)(w+b)\}. \tag{17}$$

Acting the quantum group generators on the wave functions (12) and (15), we obtain the relations in both cases (a) and (b) [4,5]

$$E^{\pm}\psi_l = [\frac{1}{2} \pm l]_q \psi_{l\pm 1}, \qquad k\psi_l = q^l \psi_l, \tag{18}$$

when we choose

$$\Delta = 2\pi\frac{l_B}{L}. \tag{19}$$

The notation $[x]_q$ means

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{20}$$

The value of Δ is nothing but the deviation of the coordinate y_0 such that it changes the quantum number l by one. We note that k measures the quantum number l and E^+ (E^-) raises (lowers) the l and that our quantum algebra is associated with only the quantum number l, namely the degeneracy of the Landau levels. The energy level is invariant under the action of the quantum algebra. This is the difference from the case of the $su(2)$ angular momentum algebra.

Substituting (19) into (10), we find that the deformation parameter q is related to the filling factor ν

$$q = exp(2\pi i\nu), \qquad \nu = \frac{N_e}{N_s}. \tag{21}$$

We finally have a remark on the representaion of the planer case. If we put $\nu = 1/(2j+1)$ and $-j \leq l \leq j$, the matrix representation of the equation (18) become

$$\begin{aligned} \rho(E^+) &= diag^+([-1]_q, [-2]_q, \ldots, [-j]_q, [j]_q, [j-1]_q, \ldots, [1]_q), \\ \rho(E^-) &= diag^-([1]_q, [2]_q, \ldots, [j]_q, [-j]_q, [-(j-1)]_q, \ldots, [-1]_q), \\ \rho(k) &= diag(q^j, q^{j-1}, \ldots, q^{-j}). \end{aligned} \tag{22}$$

It is easy to verify that the representation (22) satisfies the relations (9). It turns out that (22) coincides with the spin-j representation π by making use of the relation (21);

$$\begin{aligned} \pi(E^+) &= diag^+[[2j]_q, [2j-1]_q, \ldots, [1]_q], \\ \pi(E^-) &= diag^-[[1]_q, [2]_q, \ldots, [2j]_q], \\ \pi(k) &= diag[q^j, q^{j-1}, \ldots, q^{-j}]. \end{aligned} \tag{23}$$

In this report, we discussed the quantum group structure of two-dimensional motion of the non-relativistic electrons in a uniform magnetic field. We showed that the deformation parameter q is given by the filling $\nu = 1/m$ (m odd) and pointed out that the matrix representation (22) is connected with (23) by the relation (21). It is interesting to speculate the relation between other rational values of filling factor and the quantum group structure. This would allow us to expect a new approach to the quantized Hall effect utilizing the representation theory of quantum groups.

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