QUANTUM INFORMATION THEORY AND ITS APPLICATIONS TO IRREVERSIBLE PROCESSES

Masanori Ohya

Department of Information Sciences
Science University of Tokyo
Noda City, Chiba 278, Japan

§ 1 Mathematical Description of CDS and QDS

Let fix the notations used throughout this paper. Let μ be a probability measure on a measureble space $(\Omega,\mathfrak{F})$, $P(\Omega)$ be the set of all probability measures on Ω and $M(\Omega)$ be the set of all measurable functions on Ω. We denote the set of all bounded linear operators on a Hilbert space $\mathcal{H}$ by $B(\mathcal{H})$, and the set of all density operators on $\mathcal{H}$ by $\mathfrak{S}(\mathcal{H})$. Moreover, let $\mathfrak{S}(\mathcal{A})$ be the set of all states on $\mathcal{A}$ (C*-algebra or von Neumann algebra). Therefore the descriptions of classical dynamical systems, quantum dynamical systems and general quantum dynamical systems are given in the following Table:

	CDS	QDS	GQDS
obs.	real r.v.f in $M(\Omega)$	Hermitian op. A on $\mathcal{H}$ (s.a. op. in $B(\mathcal{H})$)	self-adjoint A in C*-algebra $\mathcal{A}$
state	prob. meas. $\mu \in P(\Omega)$	density op. ρ on $\mathcal{H}$	p.l.fnal $\varphi \in \mathfrak{S}$ with $\varphi(I)=1$
expec -tation	$\int_\Omega f d\omega$	$tr\rho A$	$\varphi(A)$

Table. 1.1 Description of CDS, QDS and GQDS

§2 Classical Information Theory

2.1 Discrete Case (Shannon's Theory)

A state in a discrete classical system is given by a probability distribution such that

$$\Delta_n = \left\{ p = \{p_i\}_{i=1}^n ; \sum_i p_i = 1, p_i \geq 0 \right\}$$

The entropy of a state $p = \{p_i\} \in \Delta_n$ is

$$S(\rho) = -\sum_i p_i \log p_i$$

The relative uncertainty (relative entropy) is defined by Kullback-Leibler as

$$S(p,q) = \begin{cases} \sum_i p_i \log \frac{p_i}{q_i} & (p << q) \\ \infty & (p \not< q) \end{cases}$$

for any $p, q \in \Delta_n$. Onece a state p is changed through a channel Λ^*, the information transmitted from a initial state p to a final state $q \equiv \Lambda^* p$ is described by the mutual entropy defined by

$$I(p; \Lambda^*) = S(r, p \otimes q) = \sum_{ij} r_{ij} \log \frac{r_{ij}}{p_i q_j}$$

where $\Lambda^* : \Delta_n \to \Delta_m$; $q = \Lambda^* p$ is a channel (e.g., $\Lambda = (p(j \mid i))$ transition matrix), $r_{ij} = p(j \mid i) p_i$ and $p \otimes q = \{p_i q_j\}$. The fundamental inequality of Shannon is

$$0 \leq I(p; \Lambda^*) \leq \min\{S(p), S(q)\}$$

According to this inequality, the ratio

$$r(p; \Lambda^*) = \frac{I(p; \Lambda^*)}{S(p)}$$

represents the efficiency of the channel transmission

2.2 Continuous Case

In classical continuous systems, a state is described by a probability measure μ. Let $(\Omega, \mathfrak{F}, P(\Omega))$ be an input probability space and $(\overline{\Omega}, \overline{\mathfrak{F}}, P(\overline{\Omega}))$ be an output probability space. A channel is a map Λ^* from $P(\Omega)$ to $P(\overline{\Omega})$, in particular, Λ^* is a Markov type if it is given by

$$\Lambda^*\varphi(Q)=\int_\Omega \lambda(x,Q)\varphi(dx),\ \varphi\in P(\Omega),\ Q\in\overline{\mathfrak{F}}$$

where $\lambda:\Omega\times\overline{\mathfrak{F}}\to R^+$ with (i) $\lambda(x,\bullet)\in P(\overline{\Omega})$, (ii) $\lambda(\bullet,Q)\in M(\Omega)$. In continuous case, the entropies are defined as follows: Let $F(\Omega)$ be the set of all finite partitions $\{A_k\}$ of Ω. For any $\varphi\in P(\Omega)$, the entropy is defined by

$$S(\varphi)=\sup\left\{-\sum_k \varphi(A_k)\log\varphi(A_k);\ \{A_k\}\in F(\Omega)\right\},$$

which is often infinite. For any $\varphi,\psi\in P(\Omega)$, the relative entropy is given by

$$S(\varphi,\psi)=\sup\left\{\sum_k \varphi(A_k)\log\frac{\varphi(A_k)}{\psi(A_k)};\ \{A_k\}\in F(\Omega)\right\}$$

$$=\begin{cases}\int_\Omega \log\left(\dfrac{d\varphi}{d\psi}\right)d\psi & (\varphi<<\psi)\\ +\infty & (\varphi\not<<\psi)\end{cases}$$

Let Φ,Φ_0 be two compound states (measures) defined as follows:

$$\Phi(Q_1,Q_2)=\int_\Omega \lambda(x,Q_2)\varphi(dx),\ Q_1\in\mathfrak{F},\ Q_2\in\overline{\mathfrak{F}}$$
$$\Phi_0(Q_1,Q_2)=(\varphi\otimes\Lambda^*\varphi)(Q_1,Q_2)=\varphi(Q_1)\Lambda^*\varphi(Q_2)$$

For $\varphi\in P(\Omega)$ and a channel Λ^*, the mutual entropy is given by

$$I(\varphi;\Lambda^*)=S(\Phi,\Phi_0).$$

§3 Quantum Information Theory

3.1 Entropies for density operators

A state in quantum systems is described by a density operator on a Hilbert space $\mathcal{H}$. The entropies are defined as follows: For a state $\rho\in\mathfrak{S}(\mathcal{H})$, the entropy [N.1] is given by

$$S(\rho)=-tr\rho\log\rho.$$

If $\rho=\sum_k p_kE_k$ (Schatten decomposition, $\dim E_n=1$), then

$$S(\rho)=-\sum_k p_k\log p_k.$$

For two states $\rho,\sigma\in\mathfrak{S}(\mathcal{H})$, the relative entropy [U.2, L.1] is given by

$$S(\rho,\sigma)=\begin{cases}tr\rho(\log\rho-\log\sigma) & (\rho<<\sigma)\\ +\infty & (\rho\not<<\sigma)\end{cases}$$

where $\rho << \sigma \Leftrightarrow$ for any $A \geq 0$, $tr\sigma A = 0 \Rightarrow tr\rho A = 0$. Let $\Lambda^* : \mathfrak{S}(\mathcal{H}) \to \mathfrak{S}(\overline{\mathcal{H}})$ be a channel and set

$$\sigma = \Lambda^*\rho,\ \theta_E = \sum_k p_k E_k \otimes \Lambda^* E_k, \theta_0 = \rho \otimes \Lambda^*\rho.$$

The mutual entropy [O.1] is given by

$$\begin{aligned} I(\rho;\Lambda^*) &= \sup\{S(\theta_E,\theta_0);\ E = \{E_k\}\} \\ &= \sup\left\{\sum_k p_k S(\Lambda^* E_k, \Lambda^*\rho);\ E = \{E_k\}\right\} \end{aligned}$$

for any state $\rho \in \mathfrak{S}(\mathcal{H})$ and any channel Λ^*. When the decomposition of ρ is fixed such that $\rho = \sum_k \lambda_k \rho_k$, then

$$I(\rho;\Lambda^*) = \sum_k \lambda_k S(\Lambda^*\rho_k, \Lambda^*\rho).$$

where $\theta_\lambda = \sum_k \lambda_k \rho_k \otimes \Lambda^*\rho_k$. The fundamental inequality of Shannon type is obtained:

$$0 \leq I(\rho;\Lambda^*) \leq \min\{S(\rho), S(\Lambda^*\rho)\}.$$

3.2 Channeling Transformations

A cahnnel $\Lambda^* : \mathfrak{S}(\mathcal{H}) \to \mathfrak{S}(\overline{\mathcal{H}})$ contains several physical transformations as special cases. First give the mathematical definitions of channels.

[Definition]

(i) Λ^* is linear if $\Lambda^*(\lambda\rho + (1-\lambda)\sigma) = \lambda\Lambda^*\rho + (1-\lambda)\Lambda^*\sigma$ for any $\lambda \in [0,1]$.

(ii) Λ^* is completely positive (C.P.) if Λ^* is linear and its dual $\Lambda : B(\overline{\mathcal{H}}) \to B(\mathcal{H})$ satisfies

$$\sum_{i,j=1}^{n} A_i^* \Lambda(\overline{A}_i^* \overline{A}_j) A_j \geq 0$$

for any $n \in N$ and any $\overline{A}_i \in B(\mathcal{H})$, $A_i \in B(\mathcal{H})$.

(iii) Λ^* is Schwarz type if $\Lambda(\overline{A}^*) = \Lambda(\overline{A})^*$ and $\Lambda(\overline{A})^*\Lambda(\overline{A}) \leq \Lambda(\overline{A}^*\overline{A})$.

(iv) Λ^* is stationary if $\Lambda \circ \alpha_t = \overline{\alpha}_t \circ \Lambda$ for any $t \in R$.

(v) Λ^* is ergodic if Λ^* is stationary and $\Lambda^*(exl(\alpha)) \subset exl(\overline{\alpha})$.

(vi) Λ^* is orthogonal if any two orthogonal states $\rho_1, \rho_2 \in \mathfrak{S}(\mathcal{H})$ (denoted by $\rho_1 \perp \rho_2$) implies $\Lambda^*\rho_1 \perp \Lambda^*\rho_2$.

(vii) Λ^* is deterministic if Λ is orthogonal and bijection.

(viii) For a subset S of $\mathfrak{S}(\mathcal{H})$, Λ^* is chaotic for S if $\Lambda^*\rho_1 = \Lambda^*\rho_2$ for any $\rho_1, \rho_2 \in S$.

(ix) Λ^* is chaotic if Λ^* is chaotic for $\mathfrak{S}(\mathcal{H})$.

Most of channels appeared in physical processes are C.P. channels. Examples of channels are the followings [O.2, D.1]:

(1) Unitary evolution :

$\rho \to \Lambda_t^* \rho = AdU_t(\rho) \equiv U_t^* \rho U_t,\ t \in R,\ U_t = \exp(itH)$

(2) Semigroup evolution :

$\rho \to \Lambda_t^* \rho = V_t^* \rho V_t,\ t \in R^+$, where $(V_t; t \in R^+)$ is a one parameter semigroup on $\mathcal{H}$

(3) Measurement :

When we measure an obserbable $A = \sum_n a_n P_n$ (spectral decomposition) in a state ρ, the state ρ changes to a state $\Lambda^* \rho$ by this measurement such as

$$\rho \to \Lambda^* \rho = \sum_n P_n \rho P_n$$

(4) Reduction :

If a system Σ_1 interacts with an external system Σ_2 described by another Hilbert space $\mathcal{K}$ and the initial states of Σ_1 and Σ_2 are ρ and σ, respectively, then the combined state θ_t of Σ_1 and Σ_2 at time t after the interaction between two systems is given by

$$\theta_t = U_t^* (\rho \otimes \sigma) U_t,$$

where $U_t = \exp(itH)$ with the total Hamiltonian H of Σ_1 and Σ_2. A channel is obtained by taking the partial trace w.r.t. $\mathcal{K}$ such as

$$\rho \to \Lambda_t^* \rho = tr_{\mathcal{K}} \theta_t.$$

3.3 Continuous Case

The entropy theory in general quantum dynamical systems has been studied by several researchers such as Araki [A.1, A.2], Uhlmann [U.1], Connes-Narnhofer-Thirring [C.1], Ohya [O.2]. These discussions are heavily dependent on operator algebraic setting, so that we here omit their details. See the reference [O.2] and [O.3] .

§ 4 Applications to Irreversible Processes

Irreversible phenomena can be treated by several different methods: One of them is due to the entropy change. However, it is difficult to explain the entropy change from reversible

equations of motion such as Schrodinger equation, Liouville equation. Therefore we need some modifications :

(i) QM + "α"

α = effect of noise, coarse graining etc.

(ii) Construct new theory including QM as a special case.

(iii) Develope the theory of entropy and find a suitable concept for irreversiblity.

We have consider (i) and (iii) above. Let ρ be a state and Λ^*, Λ_t^* be some channels. Then we ask

(1) $\rho \to \bar{\rho} = \Lambda^* \rho \Rightarrow S(\rho) \le S(\bar{\rho})$?

(2) $\rho \to \rho_t = \Lambda_t^* \rho \underset{t\to\infty}{\Rightarrow} \bar{\rho} \quad \Rightarrow \exists \lim_{t\to\infty} S(\rho_t) \le S(\bar{\rho})$?

(3) Consider the change of $I(\rho;\Lambda_t^*)$? ($I(\rho;\Lambda_t^*)$ should be decreasing!)

4.1 Entropy Change in Linear Response Dynamics

Let H be a lower bdd Hamiltonian and take

$$U_t = \exp(itH),\ \alpha_t(A) = U_t A U_{-t}$$

For a KMS state φ given by a density operator ρ such that $\varphi(\bullet) = tr\rho\bullet$ and for a perturbation λV ($V = V^* \in \mathcal{A}$, $\lambda \in [0,1]$), the perturbed time evolution is defined by a Dyson series:

$$\alpha_t^V(A) = \sum_{n\ge 0}(i\lambda)^n \int dt_1 \cdots \int_{0\le t_1\le\cdots\le t_n\le t} dt_n \left[\alpha_{t_1}(V), \left[\cdots, \left[\alpha_{t_n}(V), \alpha_t(A)\right]\cdots\right]\right]$$

and the perturbed state is

$$\varphi_t^V(A) = \frac{\varphi(W^*AW)}{\varphi(W^*W)}$$

where

$$W = \sum_{n\ge 0}(-\lambda)^n \int dt_1 \cdots \int_{0\le t_1\le\cdots\le t_n\le t} dt_n\, \alpha_{it_1}(V)\cdots\alpha_{it_n}(V)$$

The linear response time evolution and the linear response perturbed state are given by

$$\alpha_t^{V,1}(A) = \alpha_t(A) + i\lambda\int_0^t ds\left[\alpha_s(V), \alpha_t(A)\right]$$

$$\varphi^{V,1}(A) = \varphi_t(A) - \lambda\int_0^1 ds\,\varphi(A\alpha_{is}(V)) + \lambda\varphi(A)\varphi(V)$$

This linear response purturbed state $\varphi^{V,1}$ is written as

$$\varphi^{V,1}(A) = tr\rho^{V,1}A,$$

where

$$\rho^{V,1} = \left(I - \lambda\int_0^1 \alpha_{is}(V)ds + tr\rho V\right)\rho$$

The linear response time development state is

$$\rho^{V,1}(t) = \alpha_t^{V,1*}(\rho)$$
$$= \left(I - i\lambda\int_0^t \alpha_{-s}(V)ds + i\lambda\int_0^t \alpha_{-s+i}(V)ds\right)\rho$$

Put

$$\theta(t) \equiv \frac{|\rho^{V,1}(t)|}{tr|\rho^{V,1}(t)|}, \theta \equiv \frac{|\rho^{V,1}|}{tr|\rho^{V,1}|}$$
$$S(\rho^{V,1}(t)) \equiv S(\theta(t)), \quad S(\rho^{V,1}) \equiv S(\theta)$$

The change of the linear response entropy $S(\rho^{V,1}(t))$ is shown in the following theorem.

Theorem [O.2]

If $\rho^{V,1}(t)$ goes to $\rho^{V,1}$ as $t \to \infty$ and $S(\rho) < +\infty$,
then $S(\rho^{V,1}(t)) \to S(\rho^{V,1})$ as $t \to \infty$.

Remark: Even when $\alpha_t^{V*}(\rho) \to \rho^V$ $(t \to \infty)$, we have always

$$S(\alpha_t^{V*}(\rho)) = S(\rho) \neq S(\rho^V).$$

4.2 Entropy Change in Exact Dynamics

Concerning the entropy change in exact dynamics, we have the following general result:

Theorem [O.2]

Let Λ^*: $\mathfrak{S}(\mathcal{H}) \to \mathfrak{S}(\mathcal{K})$ be a channel satisfying $tr\Lambda\rho = tr\rho$ for any $\rho \in \mathfrak{S}(\mathcal{H})$. Thus $S(\rho) \leq S(\Lambda^*\rho)$.

4.3 Time Development of The Mutual Entropy

Assume that $\mathcal{A} = \overline{\mathcal{A}}$ is a von Neumann algebra (i.e., $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, $\mathcal{A}'' = \mathcal{A}$) and $\Lambda(R^+)$ is a dynamical semigroup (i.e., $\Lambda_t(A) \xrightarrow{w} A$ $(t \to 0)$, $\Lambda_{t+s}(A) = \Lambda_t(\Lambda_s(A))$ $(t \to \infty)$) on $\mathcal{A}$ having at least one faithful normal stationary state θ (i.e., $\Lambda_t^*\theta = \theta$ for any $t \in R^+$).
For this $\Lambda(R^+)$, put

$$\mathcal{A}_\Lambda = \{A \in \mathcal{A}\,;\, \Lambda_t(A) = A, t \in R^+\}$$

and

$$\mathcal{A}_C = \{A \in \mathcal{A}\,;\, \Lambda_t(A^*A) = \Lambda_t(A^*)\Lambda_t(A), t \in R^+\}.$$

Then $\mathcal{A}_\Lambda$ is a von Neumann subalgebra of $\mathcal{A}$ and there exists a conditional expectation $\mathcal{E}$ from $\mathcal{A}$ to $\mathcal{A}_\Lambda$.

Theorem [O.2]

When $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and $\mathcal{A}_\Lambda = \mathcal{A}_C$ holds, we have the followings :

(1) $I(\rho;\Lambda_t^*)$ decreases to $I(\rho;\mathcal{E}^*)$ as $t \to \infty$.

(2) There exists only one stationary state iff $I(\rho;\mathcal{E}^*) = 0$ for all ρ.

This theorem tells that the mutual entropy decreases with respect to time if the system is dissipartive, so that the mutual entropy can be a measure for the irreversibility.

The use of the mutual entropy in quantum optical communiaction is discussed in [B.1].

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