Equation of Motion for Interacting Pulses in the KdV Equation with Dissipation

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1. Introduction

The purpose of this paper is to develop a systematic theory of interaction pulses, which is capable of dealing with both dissipative and dispersive systems in a unified way. We emphasize that arbitrary parameters contained in a pulse solution play a central role in the coarse-grained description of pulse dynamics. A simple but non-trivial example in a continuum system is the position of a localized solution. Its specification violates the translational symmetry of the system and hence the position is a kind of Goldstone mode. Thus when we consider weak deformations of a localized solution, the position is a relevant slow variable. This is the basic idea in the theory of interface and/or phase dynamics [1].

We consider the Kortweg-deVries (KdV) equation with dissipative terms:

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} + a\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4}\right) = 0 \tag{1.1}$$

where a is a positive constant. Equation (1.1) was first derived and studied by Benney [2] and is called sometimes Benney equation. As is well known, Eq. (1.1) is completely integrable when the dissipative terms are absent, i.e., a = 0. The KdV equation admits propagating pulse (soliton) solutions. Because of the integrability, the collision of a pair of pluses can be analyzed in a rigorous manner. However, a completely integrable system is quite exceptional in Nature.

Some dissipation such as (1.1) is not avoidable in any realistic systems. Therefore to develop a systematic method of deriving the pulse interaction is necessary for a perturbed KdV equation which is not integrable any more.

There are several previous results for the pulse interaction of KdV equation and its modified version. The interaction for a dispersive KdV equation with a fifth derivative term was performed in Refs.[3] and [4] . Kawahara and Takaoka [5] derived the equation of motion for pulses of (1.1) with $a \neq 0$. However it seems to us that the validity of their results is questionable since the equation of motion obtained has a time-reversal symmetry despite the fact that the starting equation (1.1) does not.

The pulse interaction in a dissipative system with no Lyapunov functional has been studied by Yamada and Nozaki [6]. They have considered the FizHugh-Nagumo equation which is a model equation for pulse propagation along a nerve axon. Although their method is close somehow to ours, they have not applied it to dispersive systems.

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2. Pulse Interaction in KdV equation

In this section, we derive the pulse interaction of Eq. (1.1) without the dissipative terms, i.e., $a = 0$. In this case, one pulse solution is well known and is given by

$$u(x, t) = V(c, x - ct) = \frac{c}{2}\ \mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(x - ct)\right] \tag{2.1a}$$

where $V(c, x - ct)$ satisfies

$$-c\frac{\partial V}{\partial x} + 6V\frac{\partial}{\partial x}V + \frac{\partial^3 V}{\partial x^3} = 0 \tag{2.1b}$$

It should be noted that the velocity of the propagating pulse denoted by c is not specified but an arbitrary positive constant.

We consider the interaction of two pulses located at $x = x_1(t)$ and $x_2(t)$ propagating with almost identical velocities. The distance $x_2 - x_1$ is assumed to be much larger than the pulse width $1/\sqrt{c}$. The solution $u(x, t)$ can be written as

$$u(x, t) = V(c + \dot{x}_1, x - ct - x_1) + V(c + \dot{x}_2, x - ct - x_2) + b(x - ct, t) \tag{2.2}$$

We assume that $\dot{x}_1$ and $\dot{x}_2$ which arise from the interaction are sufficiently small compared to c. This will be checked self-consistently in the final results. Substituting (2.2) into (1.1) with a = 0, one obtains up to O(b)

$$b_t = Mb + g \tag{2.3a}$$

where

$$g = -\sum_{i=1,2} [\ddot{x}_i \frac{\partial V_i}{\partial c} - (c + \dot{x}_i)\frac{\partial V_i}{\partial x} + \frac{\partial^3 V_i}{\partial x^3}] - 6(V_1 + V_2)\frac{\partial}{\partial x}(V_1 + V_2) \tag{2.3b}$$

and $V_i = V(c + \dot{x}_i, x - ct - x_i)$, (i = 1, 2). The operator M is given by

$$M = c\frac{\partial}{\partial x} - \frac{\partial^3}{\partial x^3} - 6\sum_{i=1,2} [\frac{\partial V_i}{\partial x} + V_i \frac{\partial}{\partial x}] \tag{2.4a}$$

Now we examine the property of the operator M. In the vicinity of the position $x = x_1$, the pulse solution V_2 is sufficiently small so that one may ignore V_2 in (2.4a). Thus the operator M is simplified as

$$M = c\frac{\partial}{\partial x} - \frac{\partial^3}{\partial x^3} - 6[\frac{\partial V}{\partial x} + V\frac{\partial}{\partial x}] \tag{2.4b}$$

Here and in what follows, we occasionally omit the suffix 1 in V_1 when no confusion arises. One of the zero-eigenfunctions of M is given, as in the case of the TDGL equation, by

$$\Phi_1 = \frac{\partial V}{\partial x} \tag{2.5a}$$

It is emphasized, however, that there is another zero-eigenfunction for M, which is given by

$$\Phi_2 = \frac{\partial V}{\partial c} \tag{2.5b}$$

In fact, it is readily shown that $M\Phi_2 = -\Phi_1$ and hence $M^2\Phi_2 = -M\Phi_1 = 0$. This relation can be obtained from (2.1b) by differentiating it with respect to c. This property is a consequence of the fact that the speed c in the pulse solution (2.1) is arbitrary in the KdV equation. The degeneracy of the zero-eigenstate requires a caution in applying the solvability condition for (2.3) as will be shown below and in Appendix.

We need to introduce the adjoint operator M^+ of M:

$$\begin{aligned} M^+ &= -c\frac{\partial}{\partial x} + \frac{\partial^3}{\partial x^3} + 6(V_1 + V_2)\frac{\partial}{\partial x} \\ &\approx -c\frac{\partial}{\partial x} + \frac{\partial^3}{\partial x^3} + 6V_1\frac{\partial}{\partial x} \end{aligned} \tag{2.6}$$

The zero eigenfunction Ψ of M^+ is given by

$$\Psi = \mathrm{sech}^2\left(\frac{\sqrt{c}}{2}z\right) = V(c, z) \tag{2.7}$$

Note that corresponding to Φ_2 there is another zero-eigen function $\tilde{\Psi}$ such that

$$M^+\tilde{\Psi} = \Psi \tag{2.8}$$

Now we derive the equation for x_1. Since the operator M is not self-adjoint and has degenerate zero-states, it is not *a priori* obvious whether or not the orthogonality condition for Ψ and the inhomogeneous term in (2.3a) is the proper solvability condition. What one should require is that the solution b in (2.3a) must be bounded for $t \to \infty$. We can prove, however, that this is indeed equivalent with the condition:

$$(g, \Psi) = 0 \tag{2.9}$$

This leads us after some manupilations to the equation for x_1

$$\ddot{x}_1 = -16c^{5/2}\exp[-\sqrt{c}\,(x_2 - x_1)] \tag{2.10}$$

The equation for x_2 is given by changing $\ddot{x}_1$ by $-\ddot{x}_2$. Thus the pulse interaction is found to be repulsive in this case.

3. Pulse interaction in the Benney equation

When the KdV equation has a dissipative term as (1.6), the interaction among pulses is expected to be modified qualitatively. Here we explore this problem. Throughout this section, we assume that the parameter a in (1.1), which is a measure of the strength of the dissipative terms is sufficiently small, $0 < a << 1$.

One of the most important difference for a finite value of a is that the pulse velocity is uniquely determined asymptotically [8,9]. The asymptoticl velocity c* can be obtained by a singular perturbation method for small a [7].

Suppose that there is a single pulse which obeys the pure KdV equation with a=0. We switch on the dissipative terms at some instant. The pulse profile as well as its velocity changes gradually to the asymptotic form with c = c*. We are concerned with the interaction between these asymptotic pulses. We put two asymptotic pulses at $x=x_1$ and x_2 and see how these pulses interact each other. Since the speed c(a) satisfying c(0) = c* is uniquely determined, we may write the solution u(x, t) of Eq. (1.1) as

$$u(x, t) = V_a(z - x_1) + V_a(z - x_2) + b(z, t) \tag{3.1}$$

with $z = x - c(a)t$. The one-pulse solution is denoted by V_a emphasizing the finiteness of a. It satisfies

$$\left[-c(a)\frac{d}{dz} + \frac{d^3}{dz^3} + 6V_a\frac{d}{dz} + a\left(\frac{d^2}{dz^2} + \frac{d^4}{dz^4}\right)\right]V_a = 0 \tag{3.2}$$

By the same method as in section 2, we obtain the equation for x_1

$$\gamma \dot{x}_1 + 6(\frac{\partial}{\partial z}(V_{a1}V_{a2}), \Psi_a) = 0 \tag{3.3}$$

where

$$\gamma = -(\Psi_a, \frac{dV_a}{dz}) \tag{3.4}$$

The function Ψ_a obeys the equation similar to (4.8)

$$[-c(a)\frac{d}{dz} + \frac{d^3}{dz^3} + 6V_a\frac{d}{dz} - a(\frac{d^2}{dz^2} + \frac{d^4}{dz^4})]\Psi_a = 0 \tag{3.5}$$

It should be noted that the $\ddot{x}$ term does not exist in (3.3) since the velocity has been fixed to be the terminal one $c = c(a)$.

In the limit $a \to 0$, the coefficient γ vanishes identically as shown in section 2. We can prove that this constant is positive for nonzero values of a. After substantial manupilations we obtain

$$\gamma = \frac{a}{c^*}(\frac{\partial V}{\partial z}, \frac{\partial V}{\partial z}) + O(a^2) \tag{3.6}$$

Thus the equation of motion is given by

$$\dot{x}_1 = -\frac{120}{a}c^{*3/2}\exp[-\sqrt{c^*}(x_2 - x_1)] \tag{3.7}$$

The result (3.7) indicates that the interaction between two pulses turns out to be repulsive under the dissipation. This implies that in a system having many pulses the distance between adjacent pulses tend to be equal asymptotically due to the interaction.

4. Discussions

The analysis in section 3 fully relies on the assumption that a stable pulse solution exists asymptotically after the dissipation is switched on. At first sight, this seems unlikely because when a is positive the uniform solution $u = 0$ of eq. (1.1) is linearly unstable for long wavelength perturbations and many pulses are formed. However because the most rapid grwoth occurs at a finite wavenumber and there is a conservation $\int d r u =$ const. a periodic array of the asymptotic pulses is not entirely impossible. This does not contradict our result that the pulse interaction is repulsive in the dissipative case.

The discrepancy between the present results and those in Ref. [5] may be due to the difference of the time regime concerned such that Kawahara et. al. considered the intermediate regime while we deal with the final steady state.

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