Kolmogorov - Sinai 型エントロピーとその応用

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(日本語要旨)

主に電流や電波を搬送波(情報を乗せて運ぶ媒体)に用いる通信過程において，どれほど通信が効率よく行われているかを調べるためにチャネルの数理構造の定式化とエントロピーや相互エントロピーという重要な尺度が Shannon 等によって導入され，これらを用いることによって古典系の通信理論は数理的に厳密に定式化されている．ところで，1960 年にメイマンらによって光(レーザ)を搬送波に用いる技術が開発され，現在，様々な通信システムにおいて用いられている．このような光通信過程で用いられる光は最も基本的な素粒子であるということから，光信号は量子力学的対象として取り扱うことが本来必要であり，従って，光を信号とする通信過程を数理的に厳密に記述するためには，量子力学系における通信理論(量子通信理論)の厳密な定式化が必要となってくる．このような量子系におけるエントロピー理論の研究は，1932 年に von Neumann によって導入された von Neumann エントロピーにさかのぼることができる．また，量子系の相対エントロピーは Umegaki や Lindblad によって，古典系の Shannon 型の相対エントロピーの対応で研究され，その後，Araki や Uh lmann などによって，より一般の量子系へ拡張する研究が行われてきた．また，量子通信理論を定式化するうえで重要となる量子系におけるチャネルの研究は，Ohya，Li ndblad，Holevo などによってなされており，特に，1981 年の Ohya による量子力学的チャネルの数理的な定式化や，リフティングなどとの関わりの研究などをあげることができる．ところで，通信過程のエントロピー解析において最も重要な尺度が，相互エントロピーである．エントロピーそのものは，入力状態のもつ情報の量を表しているが，相互エントロピーは，入力の情報量のうち，チャネルを通して出力系に正しく伝達された情報の量を表しており，通信過程の情報伝送の効率を調べる上で必要不可欠なものとなっている．この量子系における相互エントロピーは 1983 年，Ohya によって定式化された．この定式化によって，初めて，光通信過程のエントロピー解析が可能となったのである．さらには，相転移や無限自由度の問題などを取り扱う場合に必要となる一般の量子力学系 (C*-力学系) において，S エントロピーと S - 相互エントロピーが Ohya によって導入され，これらの一般の量子力学系における S エントロピーと S - 相互エントロピーをベースとして，Ohya は，K-S 型平均エントロピーと平均相互エントロピーを状態空間の上に新たに定式化したのである．これらのエントロピーの定式化は量子情報理論おいて符号化の定理を与える上できわめて重要なものである．

本論文では，Ohya によって新たに定式化された K-S 型平均エントロピーと平均相互エントロピーについて説明し，それらを，光変調方式に対して具体的に計算した結果について述べる．

まず，第 1 章では，量子系のチャネル，リフティングと一般の量子力学系の S-エントロピー・S-相互エントロピーについて簡単に復習する．次に，第 2 章では，一般の量子力学系において K-S 型平均エントロピーと平均相互エントロピーの新たな定式化について説明する．第 3 章では，これらの K-S 型平均エントロピーと平均相互エントロピーを光変調方式 (PAM，PPM) によって用意された量子状態に対して具体的に計算する．

New Treatment of Kolmogorov - Sinai Type Entropies and Its Applications

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Abstracts

We review some techniques and notions for quantum information theory established by one of the present authors. New treatment of Kolmogorov - Sinai type entropies is discussed and a numerical computations of these entropies are carried for modulated states.

Introduction

In §1, we briefly review quantum channels and liftings and the formulations of quantum entropy and quantum mutual entropy for general quantum states. In §2, we explain new formulations of quantum entropy and quantum mutual entropy of K-S (Kolmogorov - Sinai) type. In §3, we show some numerical computations of these K-S type entropies for optical modulated states.

§1. Channels and Entropies

When we send information carried by a state, the state is caused to change under some effects from physical devices or the outside of the system. This state change is expressed by the concepts of channels and liftings [A.1,O.1,O.2,O.6].

In order to define the channel in quantum dynamical systems, we need two dynamical systems; input system denoted by $(\mathcal{A}, \mathfrak{S}(\mathcal{A}), \alpha(G))$ and an output system by $(\overline{\mathcal{A}}, \overline{\mathfrak{S}}(\overline{\mathcal{A}}), \overline{\alpha}(\overline{G}))$, where $\mathcal{A}$ (resp. $\overline{\mathcal{A}}$) is a C^*-algebra or the set all bounded operators $B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ (resp. $\overline{\mathcal{H}}$), $\mathfrak{S}(\mathcal{A})$ (resp. $\overline{\mathfrak{S}}(\overline{\mathcal{A}})$) is the set of all states on $\mathcal{A}$ (resp. $\overline{\mathcal{A}}$) or the set of all density operators in $\mathcal{H}$ (resp. $\overline{\mathcal{H}}$) and $\alpha(G)$ (resp. $\overline{\alpha}(\overline{G})$) is an automorphism of $\mathcal{A}$ representing the dynamics of the input (resp. output) system.

A map $\Lambda^* : \mathfrak{S}(\mathcal{A}) \to \overline{\mathfrak{S}}(\overline{\mathcal{A}})$ is called a **channel**.

(1) Λ^* is a **linear channel** if Λ^* is affine.

(2) Λ^* is a **completely positive (CP) channel** if its dual map $\Lambda : \overline{\mathcal{A}} \to \mathcal{A}$ satisfies

$$\sum_{i,j=1}^{n} B_i^* \Lambda(A_i^* A_j) B_j \geq 0$$

for any $\{B_j\} \subset \mathcal{A}, \{A_i\} \subset \overline{\mathcal{A}}$ and any $n \in N$.

(3) Λ^* is **Schwarz type** if $\Lambda\left(\bar{A}^*\right) = \Lambda\left(\bar{A}\right)^*$ and $\Lambda\left(\bar{A}\right)^* \Lambda\left(\bar{A}\right) \leq \Lambda\left(\bar{A}^* \bar{A}\right)$.

Take $\mathcal{A} = \overline{\mathcal{A}} = B(\mathcal{H})$, hence $\mathfrak{S}(\mathcal{A}) = \overline{\mathfrak{S}}(\overline{\mathcal{A}}) = \mathfrak{S}(\mathcal{H})$ is the set of all density operators on a Hilbert space $\mathcal{H}$, and let $\mathcal{B}$ be a C*-algebra on another Hilbert space $\mathcal{K}$. Then the state space of the total system is described by $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$. As is discussed in [O.4,O.5,O.8], we have several examples of channels encountered in physics and engineering, among which we show one example used in §3.

Quantum communication process is described by the following scheme [O.2].

$$\begin{array}{ccc}
 & \nu \in \mathfrak{S}(\mathcal{K}) & \\
 & \downarrow & \\
\mathfrak{S}(\mathcal{H}) \ni \rho & \longrightarrow & \overline{\rho} = \Lambda^*\rho \in \mathfrak{S}(\mathcal{H}) \\
 & \downarrow & \\
 & \text{Loss} &
\end{array}$$

$$\begin{array}{ccc}
\mathfrak{S}(\mathcal{H}) & \xrightarrow{\Lambda^*} & \mathfrak{S}(\mathcal{H}) \\
\gamma^* \downarrow & & \uparrow a^* \\
\mathfrak{S}(\mathcal{H}\otimes\mathcal{K}) & \underset{\pi^*}{\longrightarrow} & \mathfrak{S}(\mathcal{H}\otimes\mathcal{K})
\end{array}$$

The above maps γ^*, a^* are given as

$$\begin{aligned}
\gamma^*(\rho) &= \rho\otimes\nu, \quad \rho\in\mathfrak{S}(\mathcal{H}), \\
a^*(\theta) &= tr_{\mathcal{K}}\theta, \quad \theta\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{K}),
\end{aligned}$$

where ν is a noise coming from the outside of the system. The map π^* is a certain channel determined by physical properties of the combined system. Hence the channel for the above process are given as

$$\begin{aligned}
\mathcal{E}^*\rho &\equiv \pi^*(\rho\otimes\nu) = (\pi^*\circ\gamma^*)(\rho), \\
\Lambda^*\rho &\equiv tr_{\mathcal{K}}\pi^*(\rho\otimes\nu) = (a^*\circ\pi^*\circ\gamma^*)(\rho).
\end{aligned}$$

Based on the above setting, the attenuation channel Λ^* for an input state ρ is defined as follows:

$$\Lambda^*(\rho) = \mathrm{tr}_{\mathcal{K}}V(\rho\otimes\nu)V^*, \tag{1.1}$$

where ν is the vacuum noise state expressed by $\nu = |y_0 >< y_0|$ and V is given by

$$V|x_n\otimes y_0> = \sum_{j=0}^{n} C_j^n |x_j\otimes y_{n-j}>,$$

$$C_j^n = \sqrt{\frac{n!}{(n-j)!j!}\eta^j(1-\eta)^{n-j}}$$

and η is the transmission ratio for the channel.

Let us discuss the entropy in C* systems introduced in [O.4]. The formulation of quantum entropy was presented by von Neumann [N.1] about 1930, 20 years ahead of Shannon, and it now becomes a fundamental tool in analysing physical phenomena.

For a density operator $\rho \in \mathfrak{S}(\mathcal{H})$, the von Neumann entropy is defined by

$$\mathrm{S}(\rho) = -tr\rho\log\rho. \tag{1.2}$$

Now, the spectral set of ρ is discrete, so that we write the spectral decomposition of ρ as

$$\rho = \sum_n \lambda_n P_n,$$

where λ_n is an eigenvalue of ρ and P_n is the projection from $\mathcal{H}$ onto the eigenspace associated with λ_n. Therefore, if every eigenvalue λ_n is non-degenerate, then the dimension of the range of P_n is one (we denote this by $dimP_n = 1$). If a certain eigenvalue, say λ_n, is degenerate, then P_n can be further decomposed into one dimensional projections:

$$P_n = \sum_{j=1}^{dimP_n} E_j^{(n)}.$$

where $E_j^{(n)}$ is a one-dimensional projection expressed by $E_j^{(n)} = |x_j^{(n)} >< x_j^{(n)}|$ with the eigenvector $x_j^{(n)}$ $(j = 1, 2, \cdots, \dim P_n)$ for λ_n. By relabelling the indices j, n of $\{E_j^{(n)}\}$, we write

$$\rho = \sum_n \lambda_n E_n \tag{1.3}$$

with

$$\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n \geq \cdots,$$
$$E_n \perp E_m (n \perp m).$$

We call this decomposition the Schatten decomposition. Then the entropy $\mathrm{S}(\rho)$ can be expressed as

$$\mathrm{S}(\rho) = -\sum_n \lambda_n \log \lambda_n.$$

The relative entropy of two states was introduced in [U.2] for σ-finite and semifinite von Neumann algebras. For two density operators ρ and σ it is defined as

$$\mathrm{S}(\rho, \sigma) = tr\rho(\log\rho - \log\sigma). \tag{1.4}$$

Umegaki [U.2] and Lindblad [L.1] studied some fundamental properties of this relative entropy corresponding to those of classical Shannon's type relative entropy. There were several trials to extend the relative entropy to more general quantum systems and apply it to some other fields [A.3, U.1, O.6].

Another important entropy is the mutual entropy, which was discussed by Shannon to study the information transmission in classical systems and the quantum mutual entropy was introduced in [O.2,O.3], and its fully general quantum version was formulated in [O.4].

Since the Schatten decomposition of ρ is not unique unless every eigenvalue λ_n is nondegenerate, the compound state σ_E is given by

$$\sigma_E = \sum_n \lambda_n E_n \otimes \Lambda^* E_n, \tag{1.5}$$

where E represents a Schatten decomposition $\{E_n\}$. The compound state σ_E expresses the correlation between the input state ρ and the output state $\Lambda^*\rho$. Then the mutual entropy for ρ and the channel Λ^* is given by

$$\mathrm{I}(\rho;\Lambda^*) = \sup\{\mathrm{I}_E(\rho;\Lambda^*);\ E = \{E_n\} \text{ of } \rho\}, \tag{1.6}$$

with

$$\mathrm{I}_E(\rho;\Lambda^*) = \mathrm{S}(\sigma_E, \sigma_0) = tr\sigma_E(\log\sigma_E - \log\sigma_0). \tag{1.7}$$

where $\sigma_0 = \rho \otimes \Lambda^*\rho$.

Since every Schaten decomposition is discrete and orthogonal, for a state ρ, we have the following fundamental inequality [O.2,O.4].

Theorem 1.1

$$0 \leq \mathrm{I}(\rho;\Lambda^*) \leq \min\{\mathrm{S}(\rho), \mathrm{S}(\Lambda^*\rho)\}.$$

This theorem tells us that the information correctly transmitted from the input system to the output system is less than that carried by the initial state. When we send an information through a channel, we have to consider the efficiency of the communication. This efficiency is measurcd by the mutual entropy; namely, we ask for which channel is the mutual entropy becomes larger (see Theorem 5.5 of [O.4]).

In order to discuss some physical phenomena, for instance, phase transitions, we had better start without Hilbert space, so that we need to formulate the entropy of a state in a C* system [O.6].

Let $(\mathcal{A}, \mathfrak{S}(\mathcal{A}), \alpha(R))$ be a C*-dynamical system and $\mathcal{S}$ be a weak* compact and convex subset of $\mathfrak{S}(\mathcal{A})$. For instance, $\mathcal{S} = \mathfrak{S}(\mathcal{A})$; $\mathcal{S} = I(\alpha)$, the set of all invatiant states for α; $\mathcal{S} = K(\alpha)$, the set of all KMS states.

Every state $\varphi \in \mathcal{S}$ has a maximal measure μ pseudosupported on $ex\mathcal{S}$ (the set of all extreme points of $\mathcal{S}$) such that

$$\varphi = \int_{\mathcal{S}} \omega d\mu. \tag{1.8}$$

The measure μ giving the above decomposition is not unique unless $\mathcal{S}$ is a Choquet simplex, so that we denote the set of all such measures by $M_\varphi(\mathcal{S})$. Put

$$D_\varphi(\mathcal{S}) = \{M_\varphi(\mathcal{S}); \exists\mu_k \subset R^+ \text{ and } \{\varphi_k\} \subset ex\mathcal{S}$$
$$\text{s.t. } \sum_k \mu_k = 1, \mu = \sum_k \mu_k \delta(\varphi_k)\},$$

where $\delta(\varphi)$ is the Dirac measure concentrated on an state φ, and define

$$H(\mu) = -\sum_k \mu_k \log \mu_k$$

for a measure $\mu \in D_\varphi(\mathcal{S})$. Then the **entropy of a state** $\varphi \in \mathcal{S}$ w.r.t. $\mathcal{S}$ (**$\mathcal{S}$-mxinig entropy**) is defined by

$$\mathrm{S}^{\mathcal{S}}(\varphi) = \begin{cases} \inf\{H(\mu); & \mu \in D_\varphi(\mathcal{S})\}, \\ +\infty & if D_\varphi(\mathcal{S}) = \emptyset. \end{cases} \tag{1.9}$$

This entropy is an extension of von Neumann's entropy [N.1], and it depends on the set $\mathcal{S}$ chosen. Hence it represents the uncertainty of the state measured from the reference system $\mathcal{S}$. When $\mathcal{S} = \mathfrak{S}(\mathcal{A})$, we simply denote $\mathrm{S}^{\mathfrak{S}(\mathcal{A})}(\varphi)$ by $\mathrm{S}(\varphi)$ in the sequel.

When $\mathcal{A}$ is the full algebra $B(\mathcal{H})$, any normal state φ is described by a density operator ρ such as $\varphi(A) = tr\rho A$ for any $A \in \mathcal{A}$. Then C*-entropy $\mathrm{S}(\varphi)$ defined by (1.9) is equal to that of von Neumann: $\mathrm{S}(\varphi)$ $(= \mathrm{S}(\rho)) = -tr\rho\log\rho$. Every Schatten decomposition $\rho = \sum_n \lambda_n E_n$, $E_n = |x_n><x_n|$ provides every orthogonal measure in $D_\varphi(\mathfrak{S}(\mathcal{A}))$ defining the entropy $\mathrm{S}(\varphi)$.

For an initial state $\varphi \in \mathcal{S}$ and a channel $\Lambda^* : \mathfrak{S}(\mathcal{A}) \to \mathfrak{S}(\mathcal{B})$, two fundamental compound states are

$$\Phi_\mu^{\mathcal{S}} = \int_{\mathcal{S}} \omega \otimes \Lambda^*\omega d\mu \tag{1.10}$$

$$\Phi_0 = \varphi \otimes \Lambda^*\varphi. \tag{1.11}$$

The mutual entropy w.r.t. $\mathcal{S}$ and μ is

$$\mathrm{I}_\mu^{\mathcal{S}}(\varphi\,;\,\Lambda^*) = \mathrm{S}\left(\Phi_\mu^{\mathcal{S}},\,\Phi_0\right) \tag{1.12}$$

and the mutual entropy w.r.t. $\mathcal{S}$ is defined as

$$\mathrm{I}^{\mathcal{S}}(\varphi\,;\Lambda^*) = \sup\left\{\mathrm{I}_\mu^{\mathcal{S}}(\varphi\,;\Lambda^*)\ ;\ \mu \in M_\varphi(\mathcal{S})\right\}. \tag{1.13}$$

When a state φ is expressed as $\varphi = \sum_k \mu_k\varphi_k$ (fixed), the mutual entropy is given by

$$\mathrm{I}(\varphi\,;\Lambda^*) = \sum_k \mu_k \mathrm{S}(\Lambda^*\omega, \Lambda^*\varphi). \tag{1.14}$$

The CNT entropy [C.1] for a subalgebra $\mathfrak{N}$ of $\mathcal{A}$ is

$$H_\varphi(\mathfrak{N}) = \sup\Big\{\sum_k \lambda_k \mathrm{S}(\omega_k \mid \mathfrak{N},\, \varphi \mid \mathfrak{N})\,; \\ \varphi = \sum_k \lambda_k\omega_k \text{ finite decomposition of } \varphi\Big\}. \tag{1.15}$$

where $\varphi \mid \mathfrak{N}$ is the restriction of a state φ to $\mathfrak{N}$ and $\mathrm{S}(\cdot,\,\cdot)$ is the relative entropy for C^*-algebra.

The relations between $\mathrm{S}^{\mathcal{S}}(\varphi)$ and $H_\varphi(\mathfrak{N})$ were discussed in [M.1] and it was shown that $\mathrm{S}^{\mathcal{S}}(\varphi)$ distinguishes states more sharply than $H_\varphi(\mathfrak{N})$.

§2. Quantum mean mutual entropy of K-S type

2.1 A new formulation of quantum mean mutual entropy of K-S type

In quantum information theory, a stationary information source is described by a C^*triple $(\mathcal{A}, \mathfrak{S}(\mathcal{A}), \theta_{\mathcal{A}})$ with a stationary state φ with respect to $\theta_{\mathcal{A}}$; that is, $\mathcal{A}$ is a unital C^*-algebra, $\mathfrak{S}(\mathcal{A})$ is the set of all states on $\mathcal{A}$, $\theta_{\mathcal{A}}$ is an automorphism of $\mathcal{A}$, and $\varphi \in \mathfrak{S}(\mathcal{A})$ is a state over $\mathcal{A}$ with $\varphi \circ \theta_{\mathcal{A}} = \varphi$.

Let an output C^*-dynamical system be the triple $(\mathcal{B}, \mathfrak{S}(\mathcal{B}), \theta_{\mathcal{B}})$, and $\Lambda^* : \mathfrak{S}(\mathcal{A}) \to \mathfrak{S}(\mathcal{B})$ be a covariant c.p. channel: $\Lambda : \mathcal{B} \to \mathcal{A}$ such that $\Lambda \circ \theta_{\mathcal{B}} = \theta_{\mathcal{A}} \circ \Lambda$.

In this section we explain new functionals $\mathrm{S}_\mu^{\mathcal{S}}(\alpha^M)$, $\mathrm{S}^{\mathcal{S}}(\alpha^M)$, $\mathrm{I}_\mu^{\mathcal{S}}(\alpha^M, \beta^N)$ and $\mathrm{I}^{\mathcal{S}}(\alpha^M, \beta^N)$ introduced in [O.7,M.2] for a pair of finite sequences of $\alpha^M = (\alpha_1, \alpha_2, \cdots, \alpha_M)$, $\beta^N = (\beta_1, \beta_2, \cdots, \beta_N)$ of completely positive unital maps $\alpha_m : \mathcal{A}_m \to \mathcal{A}$, $\beta_n : \mathcal{B}_n \to \mathcal{B}$ where $\mathcal{A}_m$ and $\mathcal{B}_n$ ($m = 1, \cdots, M$, $n = 1, \cdots, N$) are finite dimensional unital C^*-algebras.

For a given finite sequences of completely positive unital maps $\alpha_m : \mathcal{A}_m \to \mathcal{A}$ and a given measure μ of $\varphi \in M_\varphi(\mathcal{S})$, the compound state of $\alpha_1^*\varphi, \alpha_2^*\varphi, \cdots, \alpha_M^*\varphi$ on the tesor product algebra $\otimes_{m=1}^M \mathcal{A}_m$ is given by [O.7,M.2]

$$\Phi_\mu^{\mathcal{S}}(\alpha^M) = \int_{S(\mathcal{A})} \bigotimes_{m=1}^{M} \alpha_m^* \omega \; d\mu(\omega). \tag{2.1}$$

Furthermore $\Phi_\mu^{\mathcal{S}}(\alpha^M \cup \beta^N)$ is a compound state of $\Phi_\mu^{\mathcal{S}}(\alpha^M)$ and $\Phi_\mu^{\mathcal{S}}(\beta^N)$ with $\alpha^M \cup \beta^N \equiv (\alpha_1, \alpha_2, \cdots, \alpha_M, \beta_1, \beta_2, \cdots, \beta_N)$ constructed as

$$\Phi_\mu^{\mathcal{S}}\left(\alpha^M \cup \beta^N\right) = \int_{S(\mathcal{A})} \left(\bigotimes_{m=1}^{M} \alpha_m^* \omega\right) \left(\bigotimes_{n=1}^{N} \beta_n^* \omega\right) d\mu. \tag{2.2}$$

For any pair (α^M, β^N) of finite sequences $\alpha^M = (\alpha_1, \cdots, \alpha_M)$ and $\beta^N = (\beta_1, \cdots, \beta_N)$ of completely positive unital maps (c.p.u. maps for short) $\alpha_m : \mathcal{A}_m \to \mathcal{A}$, $\beta_n : \mathcal{B}_n \to \mathcal{A}$ from finite dimensional unital C^*-algebras and any extremal decomposition measure μ of φ, the entropy functional S_μ and the mutual entropy functional I_μ are defined by [O.7,M.2]

$$\mathrm{S}_\mu^{\mathcal{S}}(\varphi; \alpha^M) = \int_{S(\mathcal{A})} \mathrm{S}\left(\otimes_{m=1}^M \alpha_m^* \omega, \Phi_\mu^{\mathcal{S}}(\alpha^M)\right) d\mu(\omega), \tag{2.3}$$

$$\mathrm{I}_\mu^{\mathcal{S}}(\varphi; \alpha^M, \beta^N) = \mathrm{S}\left(\Phi_\mu^{\mathcal{S}}(\alpha^M \cup \beta^N), \Phi_\mu^{\mathcal{S}}(\alpha^M) \otimes \Phi_\mu^{\mathcal{S}}(\beta^N)\right), \tag{2.4}$$

where $S(\cdot,\cdot)$ is the relative entropy for a finite algebra.

For a given pair of finite sequences of c.p.u. maps $\alpha^M = (\alpha_1, \cdots, \alpha_M)$, $\beta^N = (\beta_1, \cdots, \beta_N)$, the functional $\mathrm{S}^{\mathcal{S}}(\varphi; \alpha^M)$ (resp. $\mathrm{I}^{\mathcal{S}}(\varphi; \alpha^M, \beta^N)$) is given by taking the supremum of $\mathrm{S}_\mu^{\mathcal{S}}(\varphi; \alpha^M)$ (resp. $\mathrm{I}_\mu^{\mathcal{S}}(\varphi; \alpha^M, \beta^N)$) for all possible extremal decompositions μ's of φ:

$$\mathrm{S}^{\mathcal{S}}(\varphi; \alpha^M) = \sup\{\mathrm{S}_\mu^{\mathcal{S}}(\varphi; \alpha^M);\; \mu \in M_\varphi(\mathcal{S})\}, \tag{2.5}$$

$$\mathrm{I}^{\mathcal{S}}(\varphi; \alpha^M, \beta^N) = \sup\{\mathrm{I}_\mu^{\mathcal{S}}(\varphi; \alpha^M, \beta^N);\; \mu \in M_\varphi(\mathcal{S})\}. \tag{2.6}$$

Let $\mathcal{A}$ (resp. $\mathcal{B}$) be a unital C^*-algebra with a fixed automorphism $\theta_{\mathcal{A}}$ (resp. $\theta_{\mathcal{B}}$), Λ be a covariant c.p.u. map from $\mathcal{B}$ to $\mathcal{A}$, and φ be an invariant state over $\mathcal{A}$, i.e., $\varphi \circ \theta_{\mathcal{A}} = \varphi$.

$$\alpha^N \equiv \left(\alpha, \theta_{\mathcal{A}} \circ \alpha, \cdots, \theta_{\mathcal{A}}^{N-1} \circ \alpha\right), \tag{2.7}$$

$$\beta_\Lambda^N \equiv \left(\Lambda \circ \beta, \Lambda \circ \theta_{\mathcal{B}} \circ \beta, \cdots, \Lambda \circ \theta_{\mathcal{B}}^{N-1} \circ \beta\right). \tag{2.8}$$

For each c.p.u. map $\alpha : \mathcal{A}_0 \to \mathcal{A}$ (resp. $\beta : \mathcal{B}_0 \to \mathcal{B}$) from a finite dimensional unital C^*-algebra $\mathcal{A}_0$ (resp. $\mathcal{B}_0$) to $\mathcal{A}$ (resp. $\mathcal{B}$), $\tilde{\mathrm{S}}^{\mathcal{S}}(\varphi; \theta_{\mathcal{A}}, \alpha)$, $\tilde{\mathrm{I}}^{\mathcal{S}}(\varphi; \Lambda^*, \theta_{\mathcal{A}}, \theta_{\mathcal{B}}, \alpha, \beta)$ are given by

$$\tilde{\mathrm{S}}^{\mathcal{S}}(\varphi; \theta_{\mathcal{A}}, \alpha) = \limsup_{N\to\infty} \frac{1}{N} \mathrm{S}^{\mathcal{S}}(\varphi; \alpha^N), \tag{2.9}$$

$$\tilde{\mathrm{I}}^{\mathcal{S}}(\varphi; \Lambda^*, \theta_{\mathcal{A}}, \theta_{\mathcal{B}}, \alpha, \beta) = \limsup_{N\to\infty} \frac{1}{N} \mathrm{I}^{\mathcal{S}}(\varphi; \alpha^N, \beta_\Lambda^N). \tag{2.10}$$

The functionals $\tilde{S}^{S}(\varphi;\theta_{\mathcal{A}})$ and $\tilde{I}^{S}(\varphi;\Lambda^*,\theta_{\mathcal{A}},\theta_{\mathcal{B}})$ are defined by taking the supremum for all possible $\mathcal{A}_0$'s, α's, $\mathcal{B}_0$'s, and β's:

$$\tilde{S}^{S}(\varphi;\theta_{\mathcal{A}}) = \sup_{\alpha} \tilde{S}^{S}(\varphi;\theta_{\mathcal{A}},\alpha), \tag{2.11}$$

$$\tilde{I}^{S}(\varphi;\Lambda^*,\theta_{\mathcal{A}},\theta_{\mathcal{B}}) = \sup_{\alpha,\beta} \tilde{I}^{S}(\varphi;\Lambda^*,\theta_{\mathcal{A}},\theta_{\mathcal{B}},\alpha,\beta). \tag{2.12}$$

The fundamental inequality in information theory holds for $\tilde{S}^{S}(\varphi;\theta_{\mathcal{A}})$ and $\tilde{I}^{S}(\varphi;\Lambda^*,\theta_{\mathcal{A}},\theta_{\mathcal{B}})$.

Theorem 2.1 $0 \le \tilde{I}^{S}(\varphi;\Lambda^*,\theta_{\mathcal{A}},\theta_{\mathcal{B}}) \le \min\{\tilde{S}^{S}(\varphi;\theta_{\mathcal{A}}), \tilde{S}^{S}(\Lambda^*\varphi;\theta_{\mathcal{B}})\}$.

Our formulations $\tilde{S}^{S}(\varphi;\theta_{\mathcal{A}})$ and $\tilde{I}^{S}(\varphi;\Lambda^*,\theta_{\mathcal{A}},\theta_{\mathcal{B}})$ are K-S type entropies and generalize both classical and quantum K-S entropies. In particular, we have the following propositions.

Proposition 2.2 *If $\mathcal{A}_k, \mathcal{A}$ are abelian C^*-algebras and each α_k is an embedding, then our functionals concide with classical K-S entropies:*

$$S_{\mu}^{\mathfrak{S}(\mathcal{A})}(\varphi;\alpha^M) = S_{\mu}^{classical}\left(\bigvee_{m=1}^{M} \tilde{A}_m\right),$$

$$I_{\mu}^{\mathfrak{S}(\mathcal{A})}(\varphi;\alpha^M,\beta_{id}^N) = I_{\mu}^{classical}\left(\bigvee_{m=1}^{M} \tilde{A}_m, \bigvee_{n=1}^{N} \tilde{B}_n\right)$$

for any finite partitions $\tilde{A}_m, \tilde{B}_n$ of a certain probability space.

In general quantum systems, we have the following theorems [O.7,M.2].

Thorem 2.3 *Let α_m be a sequence of c.p.u. maps $\alpha_m : \mathcal{A}_m \to \mathcal{A}$ such that there exist c.p.u. maps $\alpha'_m : \mathcal{A} \to \mathcal{A}_m$ satisfying $\alpha_m \circ \alpha'_m \to id_{\mathcal{A}}$ in the pointwise topology. Then*

$$\tilde{S}^{S}(\varphi;\theta_{\mathcal{A}}) = \lim_{m\to\infty} \tilde{S}^{S}(\varphi;\theta_{\mathcal{A}},\alpha_m).$$

Theorem 2.4 *Let α_m and β_m be sequences of c.p. maps $\alpha_m : \mathcal{A}_m \to \mathcal{A}$ and $\beta_m : \mathcal{B}_m \to \mathcal{B}$ such that there exist c.p.u. maps $\alpha'_m : \mathcal{A} \to \mathcal{A}_m$ and $\beta'_m : \mathcal{B} \to \mathcal{B}_m$ satisfying $\alpha_m \circ \alpha'_m \to id_{\mathcal{A}}$ and $\beta_m \circ \beta'_m \to id_{\mathcal{B}}$ in the pointwise topology. Then*

$$\tilde{I}^{S}(\varphi;\Lambda^*,\theta_{\mathcal{A}},\theta_{\mathcal{B}}) = \lim_{m\to\infty} \tilde{I}^{S}(\varphi;\Lambda^*,\theta_{\mathcal{A}},\theta_{\mathcal{B}},\alpha_m,\beta_m).$$

This theorem is a Kolmogorov-Sinai type convergence thorem for the mutual entropy.

Based on the above construction, we rewrite the mean mutual entropy in terms of density operators.

Let $B(\mathcal{H}_0)$ (resp. $B(\overline{\mathcal{H}}_0)$) be the set of all bounded linear operators on a Hilbert space $\mathcal{H}_0$ (resp. $\overline{\mathcal{H}}_0$) and $\mathcal{A}_0$ (resp. $\mathcal{B}_0$) be a finite subset in $B(\mathcal{H}_0)$ (resp. $B(\overline{\mathcal{H}}_0)$). Let $\mathcal{A}$ (resp. $\mathcal{B}$) be an infinite tensor product space of $B(\mathcal{H}_0)$ (resp. $B(\overline{\mathcal{H}}_0)$) represented by

$$\mathcal{A} \equiv \bigotimes_{i=-\infty}^{\infty} B(\mathcal{H}_0)$$
$$\mathcal{B} \equiv \bigotimes_{i=-\infty}^{\infty} B(\overline{\mathcal{H}}_0)$$

Moreover, let $\theta_{\mathcal{A}}$ (resp. $\theta_{\mathcal{B}}$) be a shift transformations on $\mathcal{A}$ (resp. $\mathcal{B}$) defined by

$$\theta_{\mathcal{A}}(\bigotimes_{i=-\infty}^{\infty} A_i) = \bigotimes_{i'=-\infty}^{\infty} A_{i'} \quad (i' = i-1) \qquad \text{for any } \bigotimes_{i=-\infty}^{\infty} A_i \in \mathcal{A},$$
$$\theta_{\mathcal{B}}(\bigotimes_{j=-\infty}^{\infty} B_j) = \bigotimes_{j'=-\infty}^{\infty} B_{j'} \quad (j' = j-1) \qquad \text{for any } \bigotimes_{j=-\infty}^{\infty} B_j \in \mathcal{B},$$

Let α (resp. β) be the embedding map from $\mathcal{A}_0$ to $\mathcal{A}$, (resp. $\mathcal{B}_0$ to $\mathcal{B}$) given by

$$\alpha(A) = \cdots I \otimes I \otimes A \otimes I \otimes \cdots \in \mathcal{A}, \qquad \text{for any } A \in \mathcal{A}_0$$
$$\beta(B) = \cdots I \otimes I \otimes B \otimes I \otimes \cdots \in \mathcal{B}, \qquad \text{for any } B \in \mathcal{B}_0$$

We denote the set of all density operators on $\mathcal{H}_0$ (resp. $\overline{\mathcal{H}}_0$) by $\mathfrak{S}_0$ (resp. $\overline{\mathfrak{S}}_0$), and let $\mathfrak{S}$ (resp. $\overline{\mathfrak{S}}$) be the set of all states $\rho \in \bigotimes_{i=-\infty}^{\infty} \mathfrak{S}_0$ (resp. $\overline{\rho} \in \bigotimes_{i=-\infty}^{\infty} \overline{\mathfrak{S}}_0$).
The dual maps $\theta_{\mathcal{A}}^*, \theta_{\mathcal{B}}^*, \alpha^*, \beta^*$ of $\theta_{\mathcal{A}}, \theta_{\mathcal{B}}, \alpha, \beta$ are obtained as follows:
(1) $\theta_{\mathcal{A}}^*$ is a map from $\mathfrak{S}$ to $\mathfrak{S}$ satisfying

$$\theta_{\mathcal{A}}^*(\bigotimes_{i=-\infty}^{\infty} \rho_i) = \bigotimes_{i'=-\infty}^{\infty} \rho_{i'} \quad (i' = i+1), \qquad \text{for any } \bigotimes_{i=-\infty}^{\infty} \rho_i \in \mathfrak{S},$$

(2) $\theta_{\mathcal{B}}^*$ is a map from $\overline{\mathfrak{S}}$ to $\overline{\mathfrak{S}}$ satisfying

$$\theta_{\mathcal{B}}^*(\bigotimes_{j=-\infty}^{\infty} \overline{\rho}_j) = \bigotimes_{j'=-\infty}^{\infty} \overline{\rho}_{j'} \quad (j' = j+1), \qquad \text{for any } \bigotimes_{j=-\infty}^{\infty} \overline{\rho}_j \in \mathfrak{S},$$

(3) α^* is a map from $\mathfrak{S}$ to $\mathfrak{S}_0$ such as

$$\alpha^*(\bigotimes_{i=-\infty}^{\infty} \rho_i) \equiv tr_{i\neq 0}(\bigotimes_{i=-\infty}^{\infty} \rho_i) = \rho_0, \qquad \text{for any } \bigotimes_{i=-\infty}^{\infty} \rho_i \in \mathfrak{S},$$

(4) β^* is a map from $\overline{\mathfrak{S}}$ to $\overline{\mathfrak{S}}_0$ such as

$$\beta^*(\bigotimes_{j=-\infty}^{\infty} \overline{\rho}_j) \equiv tr_{j\neq 0}(\bigotimes_{j=-\infty}^{\infty} \overline{\rho}_j) = \overline{\rho}_0, \qquad \text{for any } \bigotimes_{j=-\infty}^{\infty} \overline{\rho}_j \in \mathfrak{S},$$

where $tr_{i\neq 0}$ means to take a partial trace except $i = 0$.
Under the above settings, we rewrite the mean mutual entropy of K-S type for density operators as follows:

Take

$$\alpha^N \equiv (\alpha, \theta_{\mathcal{A}} \circ \alpha, \cdots, \theta_{\mathcal{A}}^{N-1} \circ \alpha),$$
$$\beta_\Lambda^N \equiv (\Lambda \circ \beta, \Lambda \circ \theta_{\mathcal{B}} \circ \beta, \cdots, \Lambda \circ \theta_{\mathcal{B}}^{N-1} \circ \beta),$$

where $\Lambda^* \equiv \underset{i=-\infty}{\overset{\infty}{\otimes}} \Lambda^*$ is a channel from $\mathfrak{S}$ to $\overline{\mathfrak{S}}$. For any $\rho = \underset{i=-\infty}{\overset{\infty}{\otimes}} \rho_i \in \mathfrak{S}$, an input compound state $\Phi_\mu^S(\rho;\alpha^N)$ with respect to $\alpha^*(\rho),\cdots,\alpha^*\circ\theta_{\mathcal{A}}^{*N-1}(\rho)$ is defined as

$$\Phi_E(\rho;\alpha^N) \equiv \sum_{n=1}^{M} \lambda_n \overset{N-1}{\underset{i=0}{\otimes}} \alpha^*\circ\theta_{\mathcal{A}}^{*i}\Big(\underset{i=-\infty}{\overset{\infty}{\otimes}} E_n^{(i)}\Big) = \overset{N-1}{\underset{i=0}{\otimes}} \rho_i$$

When a Schatten decomposition of $\rho_i \in \mathfrak{S}_0$ $(i=0,\cdots,N-1)$ is given by

$$\rho_i = \sum_{n_i=1}^{M_i} \lambda_{n_i} E_{n_i}, \quad \Big(\sum_{n_i=1}^{M_i} \lambda_{n_i} = 1, \quad 0 \le \lambda_{n_i} \le 1\Big),$$

the compound state $\Phi_E(\rho;\alpha^N)$ is expressed as

$$\Phi_E(\rho;\alpha^N) = \sum_{n_0=1}^{M} \cdots \sum_{n_{N-1}=1}^{M} \Big(\prod_{k=0}^{N-1} \lambda_{n_k}\Big)\Big(\overset{N-1}{\underset{i=0}{\otimes}} E_{n_i}\Big)$$

By relabelling the indices $n_0,\cdots,n_{N-1}$ of $\{E_{n_i}\}$, the above state can be written as

$$\Phi_E(\alpha^N) = \sum_{n=1} \lambda_n \Big(\overset{N-1}{\underset{i=0}{\otimes}} E_n^{(i)}\Big). \tag{2.13}$$

For an initial state $\rho \in \mathfrak{S}$, an output compound state $\Phi_E(\rho;\beta_\Lambda^N)$ with respect to $\beta^*\circ\Lambda^*(\rho),\cdots,\beta^*\circ\theta_B^{*N-1}\circ\Lambda^*(\rho)$ is defined as

$$\Phi_E(\rho;\beta_\Lambda^N) \equiv \overset{N-1}{\underset{i=0}{\otimes}} \beta^*\circ\theta_B^{*i}\circ\Lambda^*(\rho) = \overset{N-1}{\underset{i=0}{\otimes}} \Lambda^*\rho_i, \tag{2.14}$$

For any state $\rho = \underset{i=-\infty}{\overset{\infty}{\otimes}} \rho_i \in \mathfrak{S}$, the correlated compound state with respect to $\Phi_E(\rho;\alpha^N)$ and $\Phi_E(\rho;\beta_\Lambda^N)$ is given by

$$\Phi_E(\rho;\alpha^N)\otimes\Phi_E(\rho;\beta_\Lambda^N) = \Big(\overset{N-1}{\underset{i=0}{\otimes}} \rho_i\Big)\otimes\Big(\overset{N-1}{\underset{i=0}{\otimes}} \Lambda^*\rho_i\Big). \tag{2.15}$$

Using the decomposition of (2.13), the state $\Phi_E(\rho;\alpha^N\cup\beta_\Lambda^N)$ is written as

$$\Phi_E(\rho;\alpha^N\cup\beta_\Lambda^N) \equiv \sum_{n=1} \lambda_n \Big(\overset{N-1}{\underset{i=0}{\otimes}} E_n^{(i)}\Big)\otimes\Big(\overset{N-1}{\underset{i'=0}{\otimes}} \Lambda^* E_n^{(i')}\Big). \tag{2.16}$$

For any inital state $\rho = \underset{i=-\infty}{\overset{\infty}{\otimes}} \rho_i \in \mathfrak{S}$, the functionals $\mathrm{I}_E(\rho;\alpha^N,\beta_\Lambda^N)$, $\mathrm{I}(\rho;\alpha^N,\beta_\Lambda^N)$, $\mathrm{S}_E(\rho;\alpha^N)$ and $\mathrm{S}(\rho;\alpha^N)$ are given by

$$\mathrm{I}_E(\rho;\alpha^N,\beta_\Lambda^N) \equiv \mathrm{S}(\Phi_E(\rho;\alpha^N\cup\beta_\Lambda^N),\Phi_E(\rho;\alpha^N)\otimes\Phi_E(\rho;\beta_\Lambda^N))$$
$$\mathrm{I}(\rho;\alpha^N,\beta_\Lambda^N) \equiv \sup\{\mathrm{I}_E(\rho;\alpha^N,\beta_\Lambda^N); \quad E=\{E_n\}\}$$

where the supremum of $\mathrm{I}_E(\rho;\alpha^N,\beta_\Lambda^N)$ is taken over possible choices $E=\{E_n\}$ of the Schatten decompositions of ρ_i.

$$\mathrm{S}_E(\rho;\alpha^N)\equiv\sum_n\lambda_n\mathrm{S}(\overset{N-1}{\underset{i=0}{\otimes}}E_n^i,\overset{N-1}{\underset{i'=0}{\otimes}}\rho_{i'})$$
$$\mathrm{S}(\rho;\alpha^N)\equiv\sup\{\mathrm{S}_E(\rho;\alpha^N);\quad E=\{E_n\}\}$$

Then the mean mutual entropy and the mean entropy of K-S type are

$$\tilde{\mathrm{I}}(\rho;\Lambda^*,\theta_{\mathcal{A}},\theta_{\mathcal{B}},\alpha,\beta)\equiv\lim_{N\to\infty}\frac{1}{N}\mathrm{I}(\rho;\alpha^N,\beta_\Lambda^N),$$
$$\tilde{\mathrm{S}}(\rho;\theta_{\mathcal{A}},\alpha)\equiv\lim_{N\to\infty}\frac{1}{N}\mathrm{S}(\rho;\alpha^N).$$

when the above limits exist.

§3. Computation of K-S entropies for modulated states

We here introduce the concept of ideal modulator and give some examples of the ideal modulator (IPAM, IPPM).

Let $\{a_1,\cdots,a_N\}$ be an alphabet set constructing input signals and $\{E_1,\cdots,E_N\}$ be the set of all one dimensional projections on certain a Hilbert space $\mathcal{H}_0$ satisfying
(1) $E_n\perp E_m\quad(n\neq m)$,
(2) E_n corresponds to the alphabet a_n.
We denote the set of all density operators on $\mathcal{H}_0$ by

$$\mathfrak{S}_0\equiv\{\rho_0=\sum_{n=1}^N\lambda_nE_n;\rho_0\geq0,tr\rho_0=1\},$$

whose element represents a state of the quantum input system. A state is transmitted from the quantum input system to a quantum modulator in order to send information effectively, whose transmitted state is called the quantum modulated state. Ideal quantum modulated states are considered as follows: A modulator M is said to ideal if the set of some modulated states $\{\rho_1^{(M)},\cdots,\rho_N^{(M)}\}$ satisfies $\rho_i^{(M)}\perp\rho_j^{(M)}$ when $\rho_i\perp\rho_j$. We denote the set of all modulated states by

$$\mathfrak{S}_0^{(M)}\equiv\{\rho_0^{(M)}=\sum_{n=1}^N\lambda_nE_n^{(M)};\rho_0^{(M)}\geq0,tr\rho_0^{(M)}=1\}.$$

There are many expressions for the modulations. In this paper, we construct the modualted states by photon number states.

Let γ_M^* be a map from $\mathfrak{S}_0$ to $\mathfrak{S}_0^{(M)}$ satisfying that γ_M is a completely positive unital map from $\mathcal{A}_0$ to $\mathcal{A}$. The map γ_{IM}^* is called ideal if $\gamma_{IM}^*(E_n)\perp\gamma_{IM}^*(E_m)$ for any orthogonal $\{E_n\}\subset\mathfrak{S}_0$. Some examples of ideal modulator are given as follows:

(1) For any $E_n\in\mathfrak{S}_0$, the IPAM (Ideal Pulse Amplitude Modulator) is defined by

$$\gamma_{IPAM}^*(E_n)\equiv E_n^{(IPAM)}=|n><n|,$$

where $|n><n|$ is the n photon number state on $\mathcal{H}$.
(2) For any $E_n \in \mathfrak{S}_0$, the IPPM (Ideal Pulse Position Modulator) is defined by

$$\begin{aligned}\gamma^*_{IPPM}(E_n) &\equiv E_n^{(IPPM)}\\ &= E_0^{(IPAM)} \otimes \cdots \otimes E_0^{(IPAM)} \otimes E_d^{(IPAM)} \otimes E_0^{(IPAM)} \otimes \cdots \otimes E_0^{(IPAM)}\end{aligned}$$

where $E_0^{(IPAM)}$ is a vacuum state and $E_d^{(IPAM)} = |d><d|$.

Some ideal modulators were used to compute the transmission efficiency by the mutual entropy ratio [W.1].

Now we calculate the mean mutual entropy of K-S type for some ideal modulated states (IPAM, IPPM) expressed by the photon number states as above.

The maps $\alpha^N_{(IM)}, \beta^N_{\tilde{\Lambda}(IM)}$ are given by

$$\begin{aligned}\alpha^N_{(IM)} &\equiv (\alpha \circ \tilde{\gamma}_{(IM)}, \theta_{\mathcal{A}} \circ \alpha \circ \tilde{\gamma}_{(IM)}, \cdots, \theta_{\mathcal{A}}^{N-1} \circ \alpha \circ \tilde{\gamma}_{(IM)}),\\ \beta^N_{\tilde{\Lambda}(IM)} &\equiv (\tilde{\gamma}_{(IM)} \circ \tilde{\Lambda} \circ \beta, \tilde{\gamma}_{(IM)} \circ \tilde{\Lambda} \circ \theta_{\mathcal{B}} \circ \beta, \cdots, \tilde{\gamma}_{(IM)} \circ \tilde{\Lambda} \circ \theta_{\mathcal{B}}^{N-1} \circ \beta),\end{aligned}$$

where we took a special channel and modulator such that $\tilde{\Lambda} \equiv \bigotimes_{i=-\infty}^{\infty} \Lambda$ and $\tilde{\gamma}_{(IM)} \equiv \bigotimes_{i=-\infty}^{\infty} \gamma_{(IM)}$.

(I) IPAM Take a stationary initial state

$$\rho = \sum_m \mu_m \bigotimes_{i=-\infty}^{\infty} \rho_m^{(i)} \in \mathfrak{S}. \tag{3.1}$$

Let $\rho_m^{(i)} = \sum_{n_i=1}^{M} \lambda_{n_i}^{(m)} E_{n_i} \in \mathfrak{S}_0$ be a Schatten decomposition of $\rho_m^{(i)}$. Then we have

$$\Phi_E(\alpha^N_{(IPAM)}) = \sum_{n_0=1}^{M} \cdots \sum_{n_{N-1}=1}^{M} \Big(\sum_m \mu_m \prod_{k=0}^{N-1} \lambda_{n_k}^{(m)}\Big) \Big(\bigotimes_{i=0}^{N-1} E_{n_i}^{(IPAM)}\Big), \tag{3.2}$$

$$\Phi_E(\beta^N_{\tilde{\Lambda}(IPAM)}) = \sum_{n_0=1}^{M} \cdots \sum_{n_{N-1}=1}^{M} \Big(\sum_m \mu_m \prod_{k=0}^{N-1} \lambda_{n_k}^{(m)}\Big) \Big(\bigotimes_{i=0}^{N-1} \Lambda^* E_{n_i}^{(IPAM)}\Big). \tag{3.3}$$

When Λ^* is an attenuation channel, we have [O.2]

$$\Lambda^* E_{n_i}^{(IPAM)} = \sum_{j_i=0}^{n_i} |C_{j_i}^{n_i}|^2 F_{j_i}^{(IPAM)},$$

where $F_{j_i}^{(IPAM)} = |j_i><j_i|$ is the j_i-photon number state in the output space $\overline{\mathfrak{S}}_0$ and

$$|C_{j_i}^{n_i}|^2 = \frac{n_i!}{j_i!(n_i-j_i)!} \eta^{j_i} (1-\eta)^{(n_i-j_i)}$$

where η is the transmission ratio of the channel [0.2,0.4]. The compound states through a channel $\tilde{\Lambda}^*$ becomes

$$\begin{aligned}
&\Phi_E(\alpha^N_{(IPAM)} \cup \beta^N_{\tilde{\Lambda}(IPAM)}) \\
&= \sum_{n_0=1}^{M} \cdots \sum_{n_{N-1}=1}^{M} \Big(\sum_m \mu_m \prod_{k=0}^{N-1} \lambda^{(m)}_{n_k}\Big) \sum_{j_0=0}^{n_0} \cdots \sum_{j_{N-1}=0}^{n_{N-1}} \Big(\prod_{k'=0}^{N-1} |C^{n_{k'}}_{j_{k'}}|^2\Big) \\
&\times \Big(\bigotimes_{i=0}^{N-1} E^{(IPAM)}_{n_i}\Big)\Big(\bigotimes_{i'=0}^{N-1} F^{(IPAM)}_{j_{i'}}\Big),
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&\Phi_E(\alpha^N_{(IPAM)}) \otimes \Phi_E(\beta^N_{\tilde{\Lambda}(IPAM)}) \\
&= \sum_{n_0=1}^{M} \cdots \sum_{n_{N-1}=1}^{M} \Big(\sum_m \mu_m \prod_{k=0}^{N-1} \lambda^{(m)}_{n_k}\Big) \sum_{n'_0=1}^{M} \cdots \sum_{n'_{N-1}=1}^{M} \Big(\sum_{m'} \prod_{k'=0}^{N-1} \lambda^{(m')}_{n'_{k'}}\Big) \\
&\times \sum_{j'_0=0}^{n'_0} \cdots \sum_{j'_{N-1}=0}^{n'_{N-1}} \Big(\prod_{k"=0}^{N-1} |C^{n'_{k"}}_{j'_{k"}}|^2\Big)\Big(\bigotimes_{i=0}^{N-1} E^{(IPAM)}_{n_i}\Big)\Big(\bigotimes_{i'=0}^{N-1} F^{(IPAM)}_{j'_{i'}}\Big).
\end{aligned} \tag{3.5}$$

Lemma 3.1 *For an initial state ρ in (3.1), we have*

$$\begin{aligned}
&\mathrm{I}_E(\rho; \alpha^N_{(IPAM)}, \beta^N_{\tilde{\Lambda}(IPAM)}) \\
&= \sum_{j_0=0}^{M} \cdots \sum_{j_{N-1}=0}^{M} \sum_{n_0=J_0}^{M} \cdots \sum_{n_{N-1}=J_{N-1}}^{M} \Big(\sum_m \mu_m \prod_{k=0}^{N-1} \lambda^{(m)}_{n_k}\Big)\Big(\prod_{k'=0}^{N-1} |C^{n_{k'}}_{j_{k'}}|^2\Big) \\
&\times \log \frac{\prod_{k=0}^{N-1} |C^{n_k}_{j_k}|^2}{\sum_{n'_0=J_0}^{M} \cdots \sum_{n'_{N-1}=J_{N-1}}^{M} \Big(\sum_{m'} \mu_{m'} \prod_{k'=0}^{N-1} \lambda^{(m')}_{n'_{k'}}\Big)\Big(\prod_{k"=0}^{N-1} |C^{n'_{k"}}_{j_{k"}}|^2\Big)}.
\end{aligned}$$

Under the above lemma, we obtain the following theorem.

Theorem 3.2 (1) *For an initial state ρ in (3.1), we have the lower bound of $\tilde{S}(\rho;\, \theta_{\mathcal{A}},\, \alpha_{(IPAM)})$ such as*

$$\tilde{\mathrm{S}}(\rho; \theta_{\mathcal{A}}, \alpha_{(IPAM)}) \geq \sum_m \mu_m \mathrm{S}(\rho^{(0)}_m).$$

(2) *Let $\mu_0 = 1, \mu_k = 0$ ($\forall k \geq 1$) and $\lambda^{(0)}_n = \lambda_n$ in (3.1). Then we have the following equalities:*

$$\tilde{\mathrm{S}}(\rho; \theta_{\mathcal{A}}, \alpha_{(IPAM)}) = -\sum_{n=1}^{M} \lambda_n \log \lambda_n.$$

$$\tilde{\mathrm{I}}(\rho; \Lambda^*, \theta_{\mathcal{A}}, \theta_{\mathcal{B}}, \alpha_{(IPAM)}, \beta_{(IPAM)}) = \sum_{j=0}^{M} \sum_{n=J}^{M} \lambda_n |C^n_j|^2 \log \frac{|C^n_j|^2}{\sum_{n'=J}^{M} \lambda_{n'} |C^{n'}_j|^2}.$$

(II) IPPM For an initial state ρ in (3.1), we have the following compound states:

$$\Phi_E(\alpha^N_{(IPPM)}) = \sum_{n_0=1}^{M}\cdots\sum_{n_{N-1}=1}^{M}\Big(\sum_m \mu_m \prod_{k=0}^{N-1}\lambda^{(m)}_{n_k}\Big)\Big(\bigotimes_{i=0}^{N-1} E^{(IPPM)}_{n_i}\Big)$$
$$= \sum_{n_0=1}^{M}\cdots\sum_{n_{N-1}=1}^{M}\Big(\sum_m \mu_m \prod_{k=0}^{N-1}\lambda^{(m)}_{n_k}\Big)\Big(\bigotimes_{i=0}^{N-1}\Big(\bigotimes_{j=0}^{N-1} E^{(IPAM)}_{d\delta_{j,n_i}}\Big)\Big). \quad (3.6)$$

$$\Phi_E(\beta^N_{\Lambda(IPPM)}) = \sum_{n_0=1}^{M}\cdots\sum_{n_{N-1}=1}^{M}\Big(\sum_m \mu_m \prod_{k=0}^{N-1}\lambda^{(m)}_{n_k}\Big)\Big(\bigotimes_{i=0}^{N-1}\Lambda^* E^{(IPPM)}_{n_i}\Big)$$
$$= \sum_{n_0=1}^{M}\cdots\sum_{n_{N-1}=1}^{M}\Big(\sum_m \mu_m \prod_{k=0}^{N-1}\lambda^{(m)}_{n_k}\Big)$$
$$\times \sum_{\ell_0=0}^{d\delta_{1,n_0}}\cdots\sum_{\ell_{N-1}=0}^{d\delta_{M,n_{N-1}}} |C^{d\delta_{1,n_0}}_{\ell_0}|^2\cdots|C^{d\delta_{M,n_{N-1}}}_{\ell_{N-1}}|^2 \times \Big(\bigotimes_{i=0}^{N-1}\Big(\bigotimes_{j=1}^{M} F^{(IPAM)}_{\ell_{iM+j}}\Big)\Big). \quad (3.7)$$

$$\Phi_E(\alpha^N_{(IPPM)}\cup\beta^N_{\Lambda(IPPM)})$$
$$= \sum_{n_0=1}^{M}\cdots\sum_{n_{N-1}=1}^{M}\Big(\sum_m \mu_m \prod_{k=0}^{N-1}\lambda^{(m)}_{n_k}\Big)\Big(\bigotimes_{i=0}^{N-1}\Big(\bigotimes_{j=1}^{M} E^{(IPAM)}_{d\delta_{j,n_i}}\Big)\Big)\Big(\bigotimes_{i'=0}^{N-1}\Big(\bigotimes_{j'=0}^{M}\Lambda^* E^{(IPAM)}_{d\delta_{j',n_{i'}}}\Big)\Big)$$
$$= \sum_{n_0=1}^{M}\cdots\sum_{n_{N-1}=1}^{M}\Big(\sum_m \mu_m \prod_{k=0}^{N-1}\lambda^{(m)}_{n_k}\Big)\sum_{\ell_1=0}^{d\delta_{1,n_0}}\cdots\sum_{\ell_{N-1}=0}^{d\delta_{M,n_{N-1}}} |C^{d\delta_{1,n_0}}_{\ell_1}|^2\cdots|C^{d\delta_{M,n_{N-1}}}_{\ell_{N-1}}|^2$$
$$\times \Big(\bigotimes_{i=0}^{N-1}\Big(\bigotimes_{j=1}^{M} E^{(IPAM)}_{d\delta_{j,n_i}}\Big)\Big)\Big(\bigotimes_{i'=0}^{N-1}\Big(\bigotimes_{j'=0}^{M} F^{(IPAM)}_{\ell_{i'M+j'}}\Big)\Big). \quad (3.8)$$

$$\Phi_E(\alpha^N_{(IPPM)})\otimes\Phi_E(\beta^N_{\Lambda(IPPM)})$$
$$= \sum_{n_0=1}^{M}\cdots\sum_{n_{N-1}=1}^{M}\Big(\sum_m \mu_m \prod_{k=0}^{N-1}\lambda^{(m)}_{n_k}\Big)\Big(\bigotimes_{i=0}^{N-1}\Big(\bigotimes_{j=1}^{M} E^{(IPAM)}_{d\delta_{j,n_i}}\Big)\Big)$$
$$\times \sum_{n'_0=1}^{M}\cdots\sum_{n'_{N-1}=1}^{M}\Big(\sum_{m'} \mu_{m'} \prod_{k'=0}^{N-1}\lambda_{n'_{k'}}\Big)\Big(\bigotimes_{i'=0}^{N-1}\Big(\bigotimes_{j'=1}^{M}\Lambda^* E^{(IPAM)}_{d\delta_{j',n'_{i'}}}\Big)\Big)$$
$$= \sum_{n_0=1}^{M}\cdots\sum_{n_{N-1}=1}^{M}\sum_{n'_0=1}^{M}\cdots\sum_{n'_{N-1}=1}^{M}\Big(\sum_m \mu_m \prod_{k=0}^{N-1}\lambda^{(m)}_{n_k}\Big)$$
$$\times \sum_{\ell_1=0}^{d\delta_{1,n_0}}\cdots\sum_{\ell_{N-1}=0}^{d\delta_{M,n_{N-1}}} |C^{d\delta_{1,n_0}}_{\ell_1}|^2\cdots|C^{d\delta_{M,n_{N-1}}}_{\ell_{N-1}}|^2 \sum_{n'_0=1}^{M}\cdots\sum_{n'_{N-1}=1}^{M}\Big(\sum_{m'} \mu_{m'} \prod_{k'=0}^{N-1}\lambda_{n'_{k'}}\Big)$$
$$\times \Big(\bigotimes_{i=0}^{N-1}\Big(\bigotimes_{j=1}^{M} E^{(IPAM)}_{d\delta_{j,n_i}}\Big)\Big)\Big(\bigotimes_{i'=0}^{N-1}\Big(\bigotimes_{j'=1}^{M} F^{(IPAM)}_{\ell_{i'M+j'}}\Big)\Big). \quad (3.9)$$

Lemma 3.3 *For an initial state ρ in* (3.1), *we have*

$$
\begin{aligned}
&\mathrm{I}_E(\rho;\alpha^N_{(IPPM)},\beta^N_{\tilde{\Lambda}(IPPM)}) \\
&= -\sum_{n_0=1}^{M}\cdots\sum_{n_{N-1}=1}^{M}\Big(\sum_m \mu_m \prod_{k=0}^{N-1}\lambda^{(m)}_{n_k}\Big)\sum_{p=1}^{N}\sum_{\{q_1,\cdots,q_p\}\subset\{1,2,\cdots,N\}} \\
&\times \sum_{\ell_1=1}^{d}\cdots\sum_{\ell_p=1}^{d}|C^d_{\ell_1}|^2\cdots|C^d_{\ell_p}|^2(1-\eta)^{N-p}\log\Big(\sum_{m'}\mu_{m'}\prod_{k'=0}^{p}\lambda^{(m')}_{n_{q_{k'}}}\Big).
\end{aligned}
$$

Theorem 3.4 (1) *For an initial state ρ in* (3.1), *we have the lower bound of $\tilde{S}(\rho;\ \theta_{\mathcal{A}},\ \alpha_{(IPPM)})$ such as*

$$\tilde{\mathrm{S}}(\rho;\theta_{\mathcal{A}},\alpha_{(IPPM)}) \geq \sum_m \mu_m \mathrm{S}(\rho^{(0)}_m).$$

(2) *Let $\mu_0 = 1, \mu_k = 0$ ($\forall k \geq 1$) and $\lambda^{(0)}_n = \lambda_n$ in* (3.1). *Then we have the following equalities:*

$$\tilde{\mathrm{S}}(\rho;\theta_{\mathcal{A}},\alpha_{(IPPM)}) = -\sum_{n=1}^{M}\lambda_n\log\lambda_n.$$

$$\tilde{\mathrm{I}}(\rho;\Lambda^*,\theta_{\mathcal{A}},\theta_{\mathcal{B}},\alpha_{(IPPM)},\beta_{(IPPM)}) = (1-(1-\eta)^d)\tilde{\mathrm{S}}(\rho;\theta_{\mathcal{A}},\alpha_{(IPPM)}).$$

The detail computation of the quantum K-S entropies will be shown in [O.9,O.10]. The computation of quantum K-S entropy in quantum Markov chain is studied for some simple models [A.2].

References

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