A TOPOLOGICAL CHARACTERIZATION OF STRANGE ATTRACTORS

H. Yamazaki, T. Motoike* and T. Arimitsu
Institute of Physics, University of Tsukuba, Ibaraki 305, JAPAN
*Dept. of Electronics and Informatics, North-Shore College
428 Nurumizu, Atsugi, Kanagawa 243, JAPAN

We will show that the topological nature of a chaotic orbit is determined only by that of the periodic orbits in the sequence of period-doubling bifurcations leading to the strange attractor containing the chaotic orbit.

A new template, named (ξ,η)-template, was introduced [1, 2] by extending Holmes' horseshoe template [3] to represent the sequence of period-doubling orbits generated by a saddle-node bifurcation originated with a p-period primary orbit having the crossing number c_0. The (ξ,η)-template is characterized by a set of integers (ξ,η). The even number ξ and the odd number η represent the number of twists along two manifolds in the template.

The linking number, $\ell_{n+1,n}$, between the 2^n-period and 2^{n+1}-period orbits is given by [2]

$$\ell_{n+1,n} = 2\cdot 4^n c_0 + \xi\kappa_n + \eta\kappa_{n+1}, \quad \text{with} \quad \kappa_n = (4^n-(-1)^n)/5. \tag{1}$$

We can get the values of (ξ,η) from *local crossing number* [1, 2] which counts the number of half-twists of a period-doubled orbit along the tubular neighborhood of the orbit just before the bifurcation. It was shown that the local crossing number can be extracted by inspecting the power spectrum of period-doubling orbits [1, 2].

A strange attractor is supposed to be constituted by an assembly of unstable periodic orbits. Therefore, we will characterize a chaotic orbit (or strange attractor) by specifying some topological nature of and among the unstable periodic orbits by means of the set of template matrices

$$\begin{bmatrix} \xi & 2\ell(x,y) \\ 2\ell(y,x) & \eta \end{bmatrix}, \qquad (0,m), \tag{2}$$

which were introduced by Mindlin et al. [4]-[6]. The linking number among the unstable periodic orbits, which consist of the strange attractor, can be expressed by the elements of the template matrices: $\ell(x,y)$, m and (ξ,η). For example, the linking number $\ell(xy,y)$ between period 2 and 1, and $\ell(xy^3,xy)$ between period 4 and 2 are given by

$$\ell(xy,y) = \ell(x,y) + (\eta-m)/2, \qquad \ell(xy^3,xy) = 4\ell(x,y) + 3(\eta-m)/2 + \xi/2. \tag{3}$$

Now we make an assumption that the unstable 2^n-period orbits which were generated at the $(n+1)$th period-doubling bifurcation point preserve their topological nature even in the chaotic region which start from the accumulation point of the period-doubling bifurcation.

Then, putting $\ell_{2,1} = \ell(xy,y)$ and $\ell_{4,2} = \ell(xy^3,xy)$, and equating the two (ξ,η), one for the (ξ,η)-template of the period-doubling bifurcation and the other for the template matrices, we finally obtain the elements $\ell(x,y)$ and m of the template matrices (2) by the expressions

$$\ell(x,y) = 2c_0 + \xi/2, \qquad m = \xi - \eta. \tag{4}$$

The above investigation will be checked by a numerical experiment [7, 8] of laser model [9] wihch is specified by

$$\dot{u} = -u\left(\delta\cos(\omega_f t+\phi) - v\right), \quad \dot{v} = -\epsilon_1 v - u - \epsilon_2 uv + 1, \tag{5}$$

where δ is a control parameter of the system, ϵ_1 and ϵ_2 are small constants and $\omega_f = 2\pi/T$. We studied two chaotic orbits ($\delta = 1.67$, and 1.80) which belong to the chaotic region originated from the $p = 1$ primary saddle-node bifurcation (PSNB1), and extracted unstable periodic orbits associated with each chaotic orbits [7, 8]. Inspecting the period-doubling bifurcation leading to the chaotic reagion under consideration, we get $\xi = 2$ and $\eta = 1$. Then, using our new relation (4), we can derive $\ell(x, y) = 1$ and $m = 1$ which give us the template in Fig. 1.

We see that the observed linking numbers $L(p_1, p_2)$ between two unstable periodic orbits (p_1-period and p_2-period) are consistent with those predicted by the template in Fig. 1 [7, 8]. Note that the diagonal element $L(p_1, p_1)$ represents the global crossing number (the self-linking number) of an unstable periodic orbit of a period p_1.

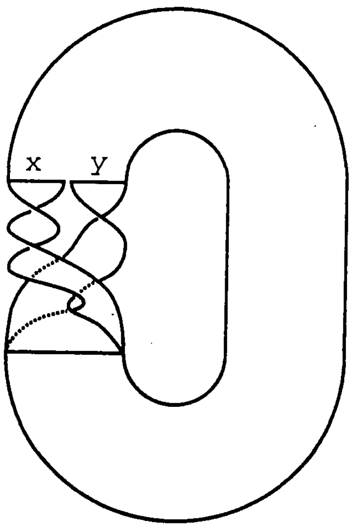

Figure 1: Template

In summarizng, the template matrix was determined by the quantities which were given by a universality of period-doubling bifurcations: ξ, η and c_0. The induced template can reproduce the linking number among any period unstable orbits in the chaotic region, in addition to the 2^n-period unstable orbits. Therefore, we conclude that the topological nature of the chaotic orbits is determined only by that of the periodic orbits in the sequence of period-doubling bifurcations leading to the strange attractor.

We are now verifying the template model with the help of a piecewise linear system by inspecting the structure of the unstable manifold of one periodic orbit. The unstable manifold has two characteristics, i.e., a twisting along the one periodic orbit and a folding.

References

[1] T. Motoike, T. Arimitsu and H. Konno, Phys. lett. A **182** (1993) 373.

[2] T. Arimitsu and T. Motoike, Physica D **84** (1994) 290.

[3] P. Holmes, Physica **21D** (1986) 7.

[4] G.B. Mindlin, X.J. Hou, H.G. Solari, R. Gilmore and N.B. Tufillaro, Phys. Rev. Lett. **64** (1990) 2350.

[5] G.B. Mindlin, H.G. Solari, M.A. Natiello, R. Gilmore and X.J. Hou, J. Nonlinear Sci. **1** (1991) 147.

[6] G.B. Mindlin and R. Gilmore, Physica D **58** (1992) 229.

[7] H. Yamazaki, M. Thesis (in Japanese), University of Tsukuba (1995) unpublished.

[8] H. Yamazaki and T. Arimitsu, in preparation to publish.

[9] J.R. Tredicce, F.T. Arecchi, G.P. Puccioni, A. Poggi and W. Gadomski, Phys. Rev. A **34** (1986) 2073.