UNIVERSAL SCALING FUNCTIONS FOR PERCOLATION ON PLANAR LATTICES

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ABSTRACT

In this paper, we briefly review our recent results in the calculation of universal scaling functions for site and bond percolation on finite square, plane triangular, and honeycomb lattices. We find that, by choosing an aspect ratio for each lattice and a very small number of nonuniversal metric factors, all scaled data of the existance probability E_p and the percolation probability P fall on the same universal scaling functions. We also find that free and periodic boundary conditions share the same nonuniversal metric factors. When the aspect ratio of each lattice is reduced by the same factor, the nonouniversal metric factors remain the same. The probabilities for the appearance of n, $n = 1, 2, 3, \ldots$, percolating clusters for bond and site percolations on various planar lattices also have universal scaling functions. The implications of such results on numerical and experimental studies of critical phenomena are discussed.

1. INTRODUCTION

Universality and scaling are two important concepts in the theory of critical phenomena. According to the idea of universality [1, 2], different systems in the same spatial dimensionality and having the same Hamiltonian symmetry share the same set of critical exponents. However, such systems usually have different scaling functions. In 1984, in a paper on finite-size scaling, Privman and Fisher [3] proposed the concept of universal scaling functions and nonuniversal metric factors. Specifically, they proposed that, near the critical point $t = 0$, where $t = (T - T_c)/T_c$ with T_c being the critical temperature, the singular part of the free energy for a system of linear dimension L can be written as

$$F_s(t, L) \sim L^{-d} Y(DtL^{1/\nu}), \tag{1}$$

where d is the spatial dimensionality of the lattice, Y is a universal scaling function, ν is the correlation length exponent and D is an nonuniversal metric factor [3]. However, it seems that there have been no published results which show that many different systems in the same universality class [1, 2] share the same set of scaling functions.

In recent papers, we use a histogram Monte Carlo simulation method (HMCSM) [4, 5, 6, 7, 8, 9] to evaluate the existence probability $E_p(G,p)$ and the percolation probability $P(G,p)$ of bond and site percolations on the square (sq), the planar triangular (pt), and the honeycomb (hc) lattices. Here $E_p(G,p)$ is the probability that the system percolates. In the limit of $L \to \infty$, $E_p(G,p)$ approaches the step function $\theta(p-p_c)$ [10], where p_c is the critical probability. $P(G,p)$ is the fraction of lattice sites in the largest cluster in G, which is percolating; it is the order parameter of the system. $E_p(G,p)$ and $P(G,p)$ may be used in a percolation renormalization group method to calculate the critical point, critical exponents, and the thermodynamic order parameter for the percolation problem [4, 8]. We find that, by choosing an aspect ratio for each lattice and a very small number of nonuniversal metric factors, all scaled data of the existance probability E_p and the percolation probability P fall on the same scaling functions [11]. We also find that free and periodic boundary conditions share the same nonuniversal metric factors [11]. When the aspect ratio of each lattice is reduced by the same factor, the nonouniversal metric factors remain the same [12]. Hu and Lin find that the probabilities for the appearance of n, $n = 1,\ 2,\ 3, \ldots$, percolating clusters for bond and site percolations on various planar lattices have good scaling behavior and such scaling functions are universal [13, 14]. In section 2, we briefly our results. The implications of such results on numerical and experimental studies of critical phenomena are discussed in section 3.

2. UNIVERSAL SCALING FUNCTIONS

It has been found that for site and bond percolation on the sq lattice $E_p(G,p_c) = 0.5$ [15, 16] and it has been proposed [17] that for bond and site percolation on the pt lattice with aspect ratio $\sqrt{3}/2$ and on the hc lattice with aspect ratio $\sqrt{3}$, $E_p(G,p_c)$ is equal to that for the square lattice, i.e. 0.5. Therefore, we first use the HMCSM to calculate the $E_p(G,p)$ and $P(G,p)$ for site and bond percolations on a 512×512 sq lattice, a 433×500 pt lattice whose aspect ratio 433/500 is very close to $\sqrt{3}/2$, and a 433×250 hc lattice whose aspect ratio 433/250 is very close to $\sqrt{3}$. We consider both free and periodic boundary conditions.

We find that by choosing such aspect ratios [17] and appropriate values of nonuniversal metric factors D_1, D_2, and D_3 listed in [11], E_p as a function of $x = D_1(p-p_c)L^{1/\nu}$ and $D_3P/L^{\beta/\nu}$ as a function of $x = D_2(p-p_c)L^{1/\nu}$ for various models have universal scaling functions [11], where β is the critical exponent of the order parameter. We also find that within numerical uncertainty $D_1 = D_2$ and free and periodic boundary conditions share the same nonuniversal metric factors [11]. Thus, our results support, and generalize, Privman and Fisher's idea of universal scaling functions.

We have also studied site and bond percolations on a 256×512 sq lattice, a 216×250 hc lattice, and a 216×500 pt lattice; the aspect ratios of such lattices are about half of the lattices used in [11]. The calculated $E_p(G,p)$, $P(G,p)$, scaling function for $E_p(G,p)$: $F(G,z)$, scaling function for $P(G,p)$: $S(G,z)$, universal scaling function for $E_p(G,p)$: $F(x)$, and universal scaling function for $P(G,p)$: $S(x)$ are shown in Figs. 1-3 of [12], where $z = (p-p_c)L^{1/\nu}$, $x = D_1 z$ with D_1 being a nonuniversal metric factor. We have found that nonuniversal metric factors for each of these new lattices are consistent with those of the corresponding lattices considered in [11]. If other factor is used to reduce the aspect ratios, similar results could be expected.

In the low temperature experiment of quantum Hall effects, when the external magnetic field is increased from small values to large values, the conductivity σ_{xy} changes from one plateau with $\sigma_{xy} = \sigma_1$ to another plateau with the value $\sigma_{xy} = \sigma_2$ and the conductivity σ_{xx} has a maximum σ_{xx}^{max} in the transition region. It has been predicted that for an infinite sample [18] $\sigma_{xx}^{max} = \frac{1}{2}(\sigma_2 - \sigma_1)$. However, such prediction is not confirmed by experiments and it has been found that σ_{xx}^{max} is sample dependent. In a recent theory of quantum Hall effect, Ruzin, Cooper, and Halperin [18] proposed that $\sigma_{xx}^{max} = \frac{k}{2}(\sigma_2 - \sigma_1)$, where k is the number of percolating clusters in the sample. Therefore, it is valuable to calculate the number of percolating clusters in percolation problems.

Hu has used the HMCSM [4, 5, 6, 7] to evaluate the probability $W_n(L_1, L_2, p)$ for the appearance of n top-to-bottom percolating clusters of bond percolation on finite $L_1 \times L_2$ square lattices with a periodic boundary condition in horizontal L_1 direction and a free boundary condition in vertical L_2 direction. He has found that, for a given aspect ratio L_1/L_2 all scaled data of $W_n(L_1, L_2, p)$ fall on the same scaling function, i.e. $W_n(L_1, L_2, p)$ has good scaling behavior [13]. Therefore, the results obtained from small simulation systems could be applied to much larger experimental systems. Using the nonuniversal metric factor D_1 of [11], Hu and Lin have found that six bond and site percolation models considered in [11] have universal scaling functions for $W_n(L_1, L_2, p)$ [14].

3. FINAL COMMENTS

We expect that the features of universal scaling functions and nonuniversal metric factors found in this paper may be applied to a variety of critical systems, e.g. Ising-type spin models and hard-core particle models whose phase transitions are percolation transitions of the corresponding correlated percolation models [19, 20, 21]. We may extend the method of this paper to calculate universal scaling functions for $E_p(G, p)$ and $P(G, p)$ of such models.

It is also of interest to find universal scaling functions for dynamic critical phenomena and to calculate nonuniversal metric factors by conformal theory or renormalization group theory.

With the rapid progress of computing and experimental facilities, more and more results of critical systems may be obtained and analyzed by finite-size scalings. The results reviewed in this paper will greatly reduced the amount of jobs to obtain experimental or numerical data.

ACKNOWLEDGEMENTS

We are indebted to N. R. Cooper and B. I. Halperin for useful discussions. This work was supported by the National Science Council of the Republic of China (Taiwan) under grant numbers NSC 84-2112-M-001-048 and NSC 85-2112-M-001-007 Y. The authors thank the Computing Center of Academia Sinica (Taipei), the National Center of High-speed Computing in Taiwan, and the Department of Physics of Harvard University through NSF grant number DMR 94-16910 for providing the computing and research facilities.

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