# NONLINEAR RESPONSE THEORY BY USING THE UNIQUE FLOQUET DECOMPOSITION THEORY

T. Shimizu and G. Sauermann*
Graduate School of Engineering, Kokushikan University, Tokyo 154, Japan
*Theoretische Festkörperphysik, Technische Hochschule Darmstadt, D-64289 Darmstadt, Germany

Response theory for periodically driven systems has been studied by many authors in various subjects. The procedure of the linear response theory (Kubo theory) can, in principle, be generalized to calculations of higher order responses. In such a treatment, however, we have only higher harmonics of the external frequency in the stationary state, which is clear from the derivation of response functions, and we cannot discuss the stability of the total system. On the other hand, it is well known that in nonlinear systems the small change of the bifurcation parameter often yields successive bifurcations, which leads to chaos. Then we can observe the apperance of subharmonics. The appearance of subharmonics is also well known in Mathieu's equation , although it is a linear system with periodic frequency modulation. It is now clear that the usual response theory is not sufficient to discuss such phenomena or the stability of the stationary state. The main purpose of this study is to construct a general response theory, with which we can treat phenomena mentioned above. As a simple application of the general theory, a quantum version of Mathieu's equation is discussed to understand the whole structure of the theory and the appearance of subharmonics.

Let us consider a quantal system whose statistical operator $\rho(t)$ is governed by the equation

$$\frac{\partial}{\partial t}\rho(t) = L(t,h)\rho(t), \tag{1}$$

where $L(t,h)$ is the Liouville operator defined by

$$L(t,h) = L_0 + hL_1(t), \qquad L_0\cdots = -\frac{i}{\hbar}[H_0.\cdots], \qquad L_1(t)\cdots = -\frac{i}{\hbar}[H_1(t),\cdots]. \tag{2}$$

Here $H_0$ is the time-independent Hamiltonian of the system and $H_1(t)$ is the external field with the period $\tau$:$H_1(t+\tau) = H_1(t)$ and $h$ is the amplitude of the external field.

If we expand the statistical operator $\rho(t)$ in $h$, and substitute it in (1), we can get the result of the usual response theory. In this paper, however, we try to solve (1) in another way. We can apply the usual Floquet decomposition theory to (1), because $L(t,h)$ is linear and has the periodicity $\tau$. This would lead to $\rho(t) = P(t)e^{\Lambda t}\rho(0)$, where $P(t)$ is periodic with $\tau$ and $\Lambda$ is a Floquet exponent operator. As has been pointed out by Maricq and by Sauermann and Zhang, however, such a decomposition is not a unique decomposition into a periodic factor and a remaining factor. In principle it is the question, which branches of $\ln(\exp(\Lambda\tau))$ should define the operator $\Lambda$. To avoid the difficulties, which originate in the non-uniquness of decomposition, we replace $h$ by $he^{\eta t}$, where $\eta$ is a small positive number, according to Sauermann and Zhang.

We first make calculations keeping $\eta \neq 0$ and at a final stage of calculations we take the limit of $\eta \to 0$ appropriately. By using this procedure we can treat both the case of out-of-resonance and the case of in-resonance correctly by one algorithm and get a unique Floquet decomposition. Therefore we start with the following time evolution equation instead of (1),

$$\frac{\partial}{\partial t}\rho(t) = L(t,h,\eta)\rho(t), \qquad L(t,h,\eta) \equiv L_0 + he^{\eta t}L_1(t). \tag{3}$$

Noting the periodicity of $H_1(t)$, we find the property of the operator $L(t,h,\eta)$: $L(t+\tau,h,\eta) = L(t,he^{\eta\tau},\eta)$, which will be called the quasi-periodic relation. If we write a formal solution of (3) in terms of the propagator $\Psi(t,h,\eta)$ as $\rho(t) = \Psi(t,h,\eta)\rho(0)$, the propagator satisfies the same time evolution equation as (3). Let us next introduce the operator $\Xi(h,\eta)$, which plays the most important role in this theory. The operator $\Xi(h,\eta)$ is defined by

$$\Xi(h,\eta) \equiv \lim_{t\to-\infty} e^{-L_0 t}\Psi(t,h,\eta). \tag{4}$$

The we can show that the propagator $\Psi(t,h,\eta)$ can be decomposed into a quasi-periodic factor $\Pi(t,h,\eta)$, i.e., $\Pi(t+\tau,h,\eta) = \Pi(t,he^{\eta\tau},\eta)$ and a remainig factor $\Gamma(t,h,\eta)$: $\Psi(t,h,\eta) = \Pi(t,h,\eta)\Gamma(t,h,\eta)$, where

$$\Gamma(t,h,\eta) = \Xi(he^{\eta t},\eta)^{-1}e^{L_{nr}t}\Xi(h,\eta), \qquad \Pi(t,h,\eta) = \Psi(t,h,\eta)\Gamma(t,h,\eta)^{-1}. \tag{5}$$

Here $L_{nr}$ is the non-resonance part of $L_0$.

In the limit of $\eta \to 0$ the operator $\Gamma(t,h,\eta)$ must coincide with one of the usual Floquet exponential form with the Floquet operator $\Lambda(h)$, and finally we get

$$\rho(t) = P(t,h)e^{\Lambda(h)t}\rho(0). \tag{6}$$

Here we can show that the unique Floquet operator $\Lambda(h)$ can be described in terms of two terms $\Lambda_s(h)$ and $\Lambda_p(h)$ as $\Lambda(h) = \Lambda_s(h) + \Lambda_p(h)$, where

$$\Lambda_s(h) \equiv \lim_{\eta\to 0}(\frac{\partial}{\partial t}\Xi(he^{\eta t},\eta)^{-1})\Xi(he^{\eta t},\eta), \qquad \Lambda_p(h) \equiv \lim_{\eta\to 0}\Xi(he^{\eta t},\eta)^{-1}L_{nr}\Xi(he^{\eta t},\eta), \tag{7}$$

and the periodic factor $P(t,h)$ is defined by $P(t,h) \equiv \lim_{\eta\to 0}\Pi(t,h,\eta)$. The two terms $\Lambda_p(h)$ and $\Lambda_s(h)$ commute with each other. Since the eigenvalues of $\Lambda_p(h)$ describe simple oscillations independent of $h$, the stability of the system can be determined from the eigenvalues of $\Lambda_s(h)$. Therefore it turns out that $\Xi(h,\eta)$ provides us with a criterion of the stability of the system.

By expanding $P(t,h)$ and $\Lambda(h)$ of (6) in $h$, we can get an appropriate lowest order response to discuss the appearance of subharmonics and the stability of the system.

We applied the above general theory to a quantum version of Mathieu's equation, whose Hamiltonian is given by

$$H = \hbar\omega_0(a^\dagger a + \frac{1}{2}) + h\cos\omega t\frac{\hbar}{4\omega_0}(a^\dagger a^\dagger + aa + aa^\dagger + a^\dagger a). \tag{8}$$

Since we are interested in the subharmonic oscillation with the frequency $\omega/2$, we put $\omega_0 = \omega/2 + hB$, where $B = (\omega_0 - \omega/2)/h \sim O(1)$.

REFERENCES

[1] T.Shimizu and G.Sauermann, J. Korean Phys. Soc., **28** (1995) S354.

[2] T.Shimizu and G.Sauermann, Physica **A 218** (1995) 375.