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THE FIRST BIFURCATION & THE FIRST TANGENCY INSIDE THE HORSESHOE

SHIN KIRIKI

ABSTRACT. We introduce a general definition of first homoclinic tangency, and construct a 1-parameter family of planar diffeomorphisms from Smale's n-fold horseshoe diffeomorphism, which has the first homoclinic tangency that is a first bifurcation introduced by Palis and Takens. Moreover, the limit capacity of its maximal invariant set associated with this first bifurcation can be as small as is required.

1. INTRODUCTION: THE CONCEPT OF THE FIRST TANGENCY

Homoclinic tangencies play an important role in the study of bifurcations of a 1-parameter family of diffeomorphisms [2] [4] [7] [14]. Palis and Takens studied them extensively and presented many results and problems [8] [9]. Among them is the problem of clarifying the relation between the first bifurcation and the first homoclinic tangency inside the horseshoe. The purpose of this note is to give a brief description of this problem along with our motives, and give the results with the outlines of its proofs. Precise proofs are presented in [5] [6].

Here we define our general concept of the first tangency for a 1-parameter family of planar diffeomorphisms. Let \( \mathbb{R}^2 \) be a Euclidean 2-dimensional space, and \( \varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a \( C^1 \) diffeomorphism. For each integer \( n \geq 2 \), we say that \( \varphi \) has an affine type-\( n \) horseshoe, e.g. \( n = 3 \) for the left panel in Figure 1, if there exists a square \( Q \subset \mathbb{R}^2 \) mapped by \( \varphi \) such that \( \varphi(Q) \) passes through \( Q \) \( n \) times, 1, 2, 3 and 4 are mapped to 1', 2', 3' and 4', respectively, and that \( \varphi \) is affine and preserves both horizontal and vertical directions on \( \varphi^{-1}(Q) \cap Q \) [1] [12]. We denote the maximal invariant subset of \( Q \) under \( \varphi \) by \( \Lambda \), i.e. \( \Lambda = \bigcap_{n \in \mathbb{Z}} \varphi^n(Q) \), which is called type-\( n \) horseshoe. Specially, in the case of \( n = 2 \) it is called just a horseshoe. It is well known that \( \varphi|_{\Lambda} \) is topologically conjugate to the shift map of \( \Sigma_n \) which is the set of all doubly-infinite sequences of \( n \) symbols, and that \( \Lambda \) is a hyperbolic set for \( \varphi \) [11] [12].

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We now consider a 1-parameter family of $C^1$ diffeomorphisms on $\mathbb{R}^2$, denoted by $\{\varphi_\mu : \mu \in I\}$ where $I = [0, 1]$, such that $\varphi_0$ has an affine type-$n$ horseshoe. So $\varphi_0$ has $n$ saddles in the horseshoe [11]. For $n = 3$, if we restrict our attention to one of three saddles, which is denoted by $p_0$, then we obtain such a configuration of stable and unstable separatrices of $p_0$ as shown in the left panel in Figure 1, which are denoted by $W^s(p_0)$ and $W^u(p_0)$, respectively, see [3] [9]. In this situation, each intersection between $W^s(p_0)$ and $W^u(p_0)$ is transverse, which is called transverse homoclinic point. When stable and unstable separatrices are tangent to each other, it is called a homoclinic tangency [10] [13]. We also denote the maximal invariant subset of $Q$ under $\varphi_\mu$ by $\Lambda_\mu$, and assume that, for every $\mu \in (0, 1)$, there exists a saddle $p_\mu$ that is the continuation of the saddle $p_0$ [7]. It is clear that, for $\mu$ near 0, there exists a disk $D_\mu \subset Q$ as shown in the right panel in Figure 1 such that $D_\mu \supset \Lambda_\mu$, $\partial D_\mu \subset W^u(p_\mu) \cup W^s(p_\mu)$, where $\partial D_\mu$ is the boundary of $D_\mu$, and $\cap_{i \in \mathbb{Z}} \varphi_\mu^i(D_\mu)$ is equal to $\Lambda_\mu$. Let $\Omega(\varphi_\mu)$ be the nonwandering set of $\varphi_\mu$. We also assume that $\Lambda_\mu = \Omega(\varphi_\mu) \cap D_\mu$, and $\Omega(\varphi_\mu) \setminus \Lambda_\mu$ consists of a finite number of hyperbolic periodic points. In this situation, we say that $\{\varphi_\mu : \mu \in I\}$ has a first (homoclinic) tangency at $\mu = 1$, if the family satisfies the following conditions:

1. for any $\mu < 1$, all intersections of $W^s(p_\mu)$ and $W^u(p_\mu)$ are transversal;
2. For $\mu = 1$, it has an orbit of contact-breaking homoclinic tangency on the boundary of $D_\mu$ and there is no other orbit of homoclinic tangency.

When $n = 2$, i.e. $\{\varphi_\mu : \mu \in I\}$ has an affine type-2 horseshoe for $\mu = 0$ and a first tangency at $\mu = 1$, the configurations of its separatrices at
For a first tangency as defined in the previous subsection, Palis and Takens [9] posed the question:

"Whether this first tangency is a first bifurcation or not",

where the first bifurcation means that, for any $\mu < 1$, $\varphi_\mu$ satisfies Axiom A and the transversality condition, that is, it is globally stable. We have an answer to this problem for a special case:

**Theorem 2.1 ([5]).** There exists a special 1-parameter family of $C^1$ diffeomorphisms near type-2 horseshoe diffeomorphism such that its first tangency is a first bifurcation.

The above special 1-parameter family was obtained by locally modifying a planar $C^1$ diffeomorphism which has an affine type-2 horseshoe with the contracting eigenvalue of $(d\varphi)_p$ which is not too small, as
in the left panel in Figure 3. If the contracting eigenvalue is very small, as shown in the right panel in Figure 3, we do not know how to construct a 1-parameter family such that its first tangency is a first bifurcation.

We have some fractal dimensions available for the Cantor set which is defined as the intersection between the separatrix and the maximal invariant subset for the 1-parameter family, that is $W^*(p_\mu) \cap \Lambda_\mu$ or $W^u(p_\mu) \cap \Lambda_\mu$. Among them some numerical invariants distinguish between the left situation and the right situation in Figure 3, so that their values for the Cantor set in the left situation are greater than those for the right situation in Figure 3. Results by Palis and Takens [8] [9] suggest the importance of the limit capacity, which is one of such numerical invariants, of the Cantor set, for the study of the first bifurcation. For the Cantor set $K = W^*(p_\mu) \cap \Lambda_\mu$ or $W^u(p_\mu) \cap \Lambda_\mu$, the limit capacity is defined as $d(K) = \lim \sup_{\epsilon \to 0} \frac{\log N_\epsilon(K)}{-\log \epsilon}$, where $N_\epsilon(K)$ is the minimum number of balls of radius $\epsilon$ needed to cover $K$. Their results also suggest that the study of homoclinic bifurcation with small limit capacities is easier than that with large limit capacities. However, we were not able to obtain a first tangency with small limit capacities in [5]. In order to study about the bifurcations associated with the first tangency, we need such a first tangency. Therefore, we now have the motive:

"Is it possible to construct a 1-parameter family having the first tangency, which is also a first bifurcation, inside the maximal invariant subset with a very small limit capacity?"

To answer this question, we construct a special 1-parameter family, for $n \geq 3$, from a type-$n$ horseshoe diffeomorphism with very small limit capacities. Our second result now follows:

**Theorem 2.2** ([6]). For any $\epsilon > 0$, there is a real number $\delta(\epsilon) > 0$ such that, for each $0 < \delta \leq \delta(\epsilon)$, there exists a special 1-parameter family of planar $C^1$ diffeomorphism $\{\varphi_\mu ; \mu \in I\}$ depending on $\delta$, which satisfies the following two conditions:

1. it has the first tangency at $\mu = 1$, which is a first bifurcation;
2. $d(W^*(p_1) \cap \Lambda_1) + d(W^u(p_1) \cap \Lambda_1) < \epsilon$.

3. OUTLINES OF THE PROOFS

**Sketch of the proof of Theorem 2.1.** First, for $0 < \delta < 1/2$ and a unit square $D$, we take a precise form of type-2 horseshoe $C^1$ diffeomorphism $\varphi$ which has the saddle $p$ such that $\lambda = \frac{1}{2} - \delta$ where $\lambda$ is the
contracting eigenvalue of \((d\varphi)_\mu\), as shown in the left panel in Figure 4. We also define a \(C^1\) perturbation \(\Psi_\mu\) on \(V = V_1 \cup V_2\) as in Figure 4 such that it compresses the vertical lines in \(V_1\), and slides the points in \(V_2\) downward as \(\mu\) increases from 0 to 1. We denote \(\Psi_\mu \circ \varphi\) by \(\varphi_\mu\) which is called a special 1-parameter family, and denote the maximal invariant subset of \(D\) under \(\varphi_\mu\) by \(\Lambda_\mu\). From the construction of this family, it also depends on \(\delta\). We write, for any \(0 < \mu \leq 1\), \(V_\mu = \{x = (s, u) \in D \mid 0 \leq s \leq 1, 1 - \mu \delta \leq u \leq 1\}\) which is shown in Figure 5. In the next proposition, for \(0 < \mu < 1\), \(x \in \Lambda_\mu \cap V_\mu\) and \(n > 0\) such that \(\varphi_\mu^n(x) \in V_\mu\), we construct an unstable cone \(C_\mu(x) \subset T_x\mathbb{R}^2\) for \(0 < \delta \leq \delta_h\) such that \((d\varphi_\mu^n)_x C_\mu(x) \subset C_\mu(\varphi_\mu^n(x))\) and that the length of any nonzero vector in \(C_\mu(x)\) is expanded exponentially. The stable cone field can be constructed by taking the complement of the unstable cone field.

**Proposition 3.1 ([5]).** There exists a \(\delta_h\) such that, for \(0 < \delta \leq \delta_h\),

\[ C_\mu(x) \subset C_\mu(\varphi_\mu^n(x)) \]

\[ (d\varphi_\mu^n)_x C_\mu(x) \subset C_\mu(\varphi_\mu^n(x)) \]
there is a continuous unstable cone field satisfying the above properties.

Next, we construct a partition for the above $\Lambda_\mu$. The continuous cone fields over $\Lambda_\mu$ are obtained for this partition by using some locally linear interpolation of the angle of the cone. These cone fields assure the hyperbolicity of this family and the transversality of the stable and unstable separatrices for $0 \leq \mu < 1$. □

**Sketch of the proof of Theorem 2.2.** The special 1-parameter family $\varphi_\mu (= \Psi_\mu \circ \varphi)$ in this theorem is composed of the type-3 horseshoe $C^1$ diffeomorphism $\varphi$ and the $C^1$ perturbation $\Psi_\mu$ on the open set $V$ as shown in Figure 6. The components of $\varphi(D) \cap D$, where $D$ is a unit square, are denoted by $R_0$, $R_1$ and $R_2$ as in Figure 6. We assume that $\rho(R_0, R_1) > \rho(R_1, R_2)$, and write $\delta = \rho(R_1, R_2)/2$, where $\rho$ is the Euclidean metric. For some $\delta_h > 0$, similarly as in Theorem 2.1, we construct the 1-parameter family satisfying the hyperbolicity and the transversality condition. Moreover, we estimate the limit capacity of $W^s(\varphi_\mu) \cap \Lambda_\mu$ as

$$d_\mu^* = d(W^s(\varphi_\mu) \cap \Lambda_\mu) \leq \lim_{n \to 0} \frac{\log 3^n}{-\log(\lambda + \delta)\lambda^{n-1}} \leq \lim_{n \to 0} \frac{\log 3^n}{-\log(\lambda + \delta)^n} = \frac{\log 3}{-\log(\lambda + \delta)},$$

where $\lambda$ is the contracting eigenvalue of $(d\varphi_\mu)_{p_\mu}$. Similarly, $d_\mu^u = d(W^u(\varphi_\mu) \cap \Lambda_\mu)$ is estimated. Then we get the 1-parameter family such that, for any $\varepsilon > 0$, there is $\delta_l > 0$ depending on $\varepsilon$ such that $d_\mu^u + d_\mu^* < \varepsilon$ for each $0 < \delta \leq \delta_l$. We now write $\delta(\varepsilon) = \min\{\delta_h, \delta_l\}$, and get the result. □

![Figure 6](image-url)
We believe that the above two results will be extended to higher dimensions, e.g. a special 1-parameter family can probably be constructed by modifying a 3-dimensional horseshoe diffeomorphism as in Figure 7.

Professor F. Takens posed us a new problem of the relation between the homoclinic tangency and the first bifurcation as follows. Let \( \{\varphi_\mu : \mu \in \mathbb{R}\} \) be a 1-parameter family of diffeomorphisms on \( \mathbb{R}^2 \) satisfying \( \varphi_0 \) has an affine horseshoe. Let \( \Lambda_0 \) be the horseshoe for \( \varphi_0 \), and let \( \Lambda_\mu \) be the maximal invariant subset for \( \varphi_\mu \) which is the continuation of \( \Lambda_0 \) as long as \( \Lambda_\mu \) is a hyperbolic set. We write

\[
\mu_0 = \sup \{\mu \in \mathbb{R} \mid \Lambda_\mu \text{ is a hyperbolic set} \} \quad \text{and} \quad \mu_1 = \inf \{\mu \in \mathbb{R} \mid \varphi_\mu \text{ has a homoclinic tangency} \}.
\]

**Problem.** In the above situation, is there a 1-parameter family of planar diffeomorphisms such that \( \mu_0 < \mu_1 \)? Moreover, is there an open set of \( C^2 \) arcs of planar diffeomorphisms such that \( \mu_0 < \mu_1 \)?

We have no answer to this question.

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REFERENCE


DEPARTMENT OF MATHEMATICAL SCIENCES, TOKYO DENKI UNIVERSITY, HATOYAMA, HIKI, SAITAMA, 350-03, JAPAN

E-mail address: ged@r.dendai.ac.jp