THE FIRST BIFURCATION & THE FIRST TANGENCY INSIDE THE HORSESHOE

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ABSTRACT. We introduce a general definition of *first homoclinic tangency*, and construct a 1-parameter family of planar diffeomorphisms from Smale's n-fold horseshoe diffeomorphism, which has the *first homoclinic tangency* that is a *first bifurcation* introduced by Palis and Takens. Moreover, the *limit capacity* of its maximal invariant set associated with this *first bifurcation* can be as small as is required.

1. INTRODUCTION: THE CONCEPT OF THE FIRST TANGENCY

Homoclinic tangencies play an important role in the study of bifurcations of a 1-parameter families of diffeomorphisms [2] [4] [7] [14]. Palis and Takens studied them extensively and presented many results and problems [8] [9]. Among them is the problem of clarifying the relation between the *first bifurcation* and the *first homoclinic tangency inside the horseshoe*. The purpose of this note is to give a brief description of this problem along with our motives, and give the results with the outlines of its proofs. Precise proofs are presented in [5] [6].

Here we define our general concept of the *first tangency* for a 1-parameter family of planar diffeomorphisms. Let $\mathbb{R}^2$ be a Euclidean 2-dimensional space, and $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 diffeomorphism. For each integer $n \geq 2$, we say that φ has an *affine type-n horseshoe*, e.g. $n = 3$ for the left panel in Figure 1, if there exists a square $Q \subset \mathbb{R}^2$ mapped by φ such that $\varphi(Q)$ passes through Q n times, 1, 2, 3 and 4 are mapped to $1'$, $2'$, $3'$ and $4'$, respectively, and that φ is affine and preserves both horizontal and vertical directions on $\varphi^{-1}(Q) \cap Q$ [1] [12]. We denote the *maximal invariant subset* of Q under φ by Λ, i.e. $\Lambda = \bigcap_{i \in Z} \varphi^i(Q)$, which is called *type-n horseshoe*. Specially, in the case of $n = 2$ it is called just a *horseshoe*. It is well known that $\varphi|_\Lambda$ is topologically conjugate to the shift map of Σ_n which is the set of all doubly-infinite sequences of n symbols, and that Λ is a *hyperbolic set* for φ [11] [12].

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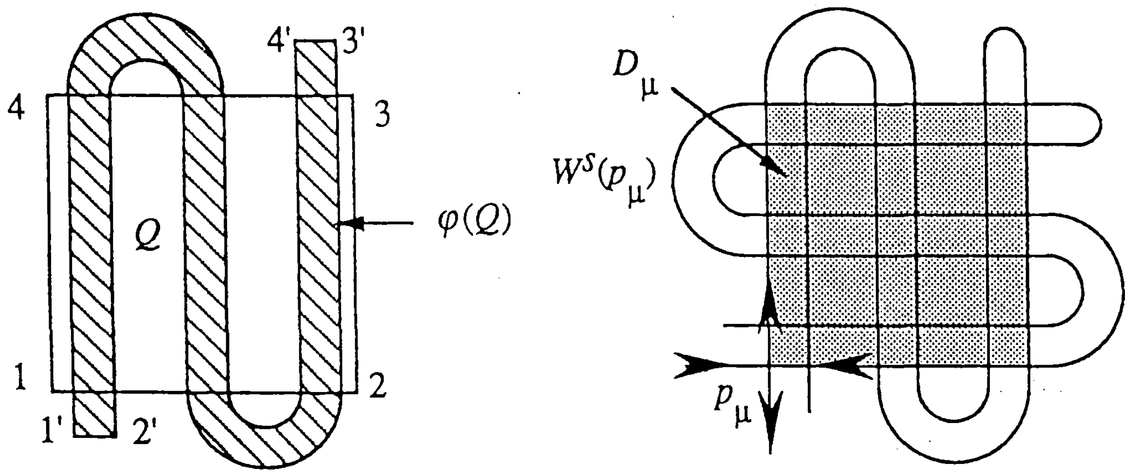

FIGURE 1

We now consider a 1-parameter family of C^1 diffeomorphisms on $\mathbb{R}^2$, denoted by $\{\varphi_\mu\ ;\ \mu \in I\}$ where $I = [0,\ 1]$, such that φ_0 has an affine type-n horseshoe. So φ_0 has n saddles in the horseshoe [11]. For $n = 3$, if we restrict our attention to one of three saddles, which is denoted by p_0, then we obtain such a configuration of *stable* and *unstable separatrices* of p_0 as shown in the left panel in Figure 1, which are denoted by $W^s(p_0)$ and $W^u(p_0)$, respectively, see [3] [9]. In this situation, each intersection between $W^s(p_0)$ and $W^u(p_0)$ is transverse, which is called *transverse homoclinic point*. When stable and unstable separatrices are tangent to each other, it is called a *homoclinic tangency* [10] [13]. We also denote the *maximal invariant subset* of Q under φ_μ by Λ_μ, and assume that, for every $\mu \in (0,\ 1]$, there exists a saddle p_μ that is the continuation of the saddle p_0 [7]. It is clear that, for μ near 0, there exists a disk $D_\mu \subset Q$ as shown in the right panel in Figure 1 such that $D_\mu \supset \Lambda_\mu$; $\partial D_\mu \subset W^u(p_\mu) \cup W^s(p_\mu)$, where ∂D_μ is the boundary of D_μ, and $\bigcap_{i \in Z} \varphi^i_\mu(D_\mu)$ is equal to Λ_μ. Let $\Omega(\varphi_\mu)$ be the nonwandering set of φ_μ. We also assume that $\Lambda_\mu = \Omega(\varphi_\mu) \cap D_\mu$, and $\Omega(\varphi_\mu)\backslash\Lambda_\mu$ consists of a finite number of hyperbolic periodic points. In this situation, we say that $\{\varphi_\mu : \mu \in I\}$ has a *first (homoclinic) tangency* at $\mu = 1$, if the family satisfies the following conditions:

(1) for any $\mu < 1$, all intersections of $W^s(p_\mu)$ and $W^u(p_\mu)$ are transversal;
(2) For $\mu = 1$, it has an orbit of contact-breaking homoclinic tangency on the boundary of D_μ and there is no other orbit of homoclinic tangency.

When $n = 2$, i.e. $\{\varphi_\mu : \mu \in I\}$ has an affine type-2 horseshoe for $\mu = 0$ and a first tangency at $\mu = 1$, the configurations of its separatrices at

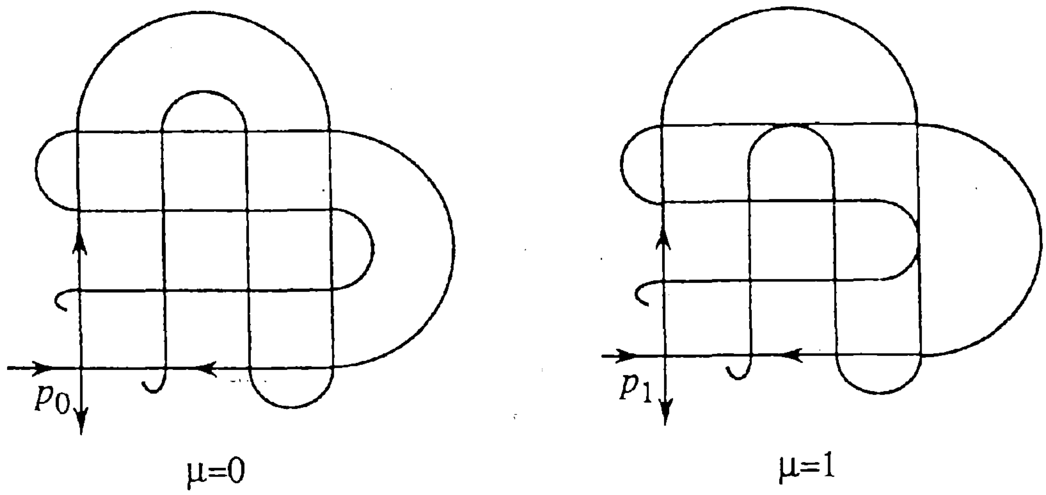

FIGURE 2

$\mu = 0,\ 1$ are as shown in Figure 2, respectively, [7] [9].

2. MOTIVES AND MAIN RESULTS

For a first tangency as defined in the previous subsection, Palis and Takens [9] posed the question:

"Whether this first tangency is a first bifurcation or not",

where the *first bifurcation* means that, for any $\mu < 1$, φ_μ satisfies Axiom A and the transversality condition, that is, it is *globally stable.* We have an answer to this problem for a special case:

Theorem 2.1 ([5]). *There exists a special 1-parameter family of C^1 diffeomorphisms near type-2 horseshoe diffeomorphism such that its first tangency is a first bifurcation.*

The above special 1-parameter family was obtained by locally modifying a planar C^1 diffeomorphism which has an affine *type-2* horseshoe with the contracting eigenvalue of $(d\varphi)_p$ which is not too small, as

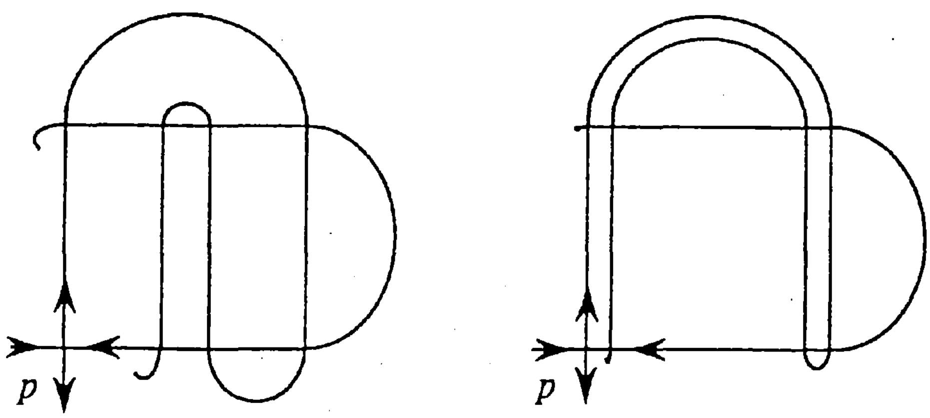

FIGURE 3

in the left panel in Figure 3. If the contracting eigenvalue is very small, as shown in the right panel in Figure 3, we do not know how to construct a 1-parameter family such that its first tangency is a first bifurcation.

We have some *fractal dimensions* available for the *Cantor set* which is defined as the intersection between the separatrix and the maximal invariant subset for the 1-parameter family, that is $W^s(p_\mu) \cap \Lambda_\mu$ or $W^u(p_\mu) \cap \Lambda_\mu$. Among them some numerical invariants distinguish between the left situation and the right situation in Figure 3, so that their values for the Cantor set in the left situation are greater than those for the right situation in Figure 3. Results by Palis and Takens [8] [9] suggest the importance of the *limit capacity*, which is one of such numerical invariants, of the Cantor set, for the study of the first bifurcation. For the Cantor set $K = W^s(p_\mu) \cap \Lambda_\mu$ or $W^u(p_\mu) \cap \Lambda_\mu$, the *limit capacity* is defined as $d(K) = \limsup_{\epsilon \to 0} \frac{\log N_\epsilon(K)}{-\log \epsilon}$, where $N_\epsilon(K)$ is the minimum number of balls of radius ϵ needed to cover K. Their results also suggest that the study of homoclinic bifurcation with small limit capacities is easier than that with large limit capacities. However, we were not able to obtain a first tangency with small limit capacities in [5]. In order to study about the bifurcations associated with the first tangency, we need such a first tangency. Therefore, we now have the motive:

> *"Is it possible to construct a 1-parameter family having the first tangency, which is also a first bifurcation, inside the maximal invariant subset with a very small limit capacity?"*

To answer this question, we construct a special 1-parameter family, for $n \geq 3$, from a *type-n horseshoe diffeomorphism* with very small limit capacities. Our second result now follows:

Theorem 2.2 ([6]). *For any $\varepsilon > 0$, there is a real number $\delta(\varepsilon) > 0$ such that, for each $0 < \delta \leq \delta(\varepsilon)$, there exists a special 1-parameter family of planar C^1 diffeomorphism $\{\varphi_\mu \ ; \ \mu \in I\}$ depending on δ, which satisfies the following two conditions:*

(1) *it has the first tangency at $\mu = 1$, which is a first bifurcation;*
(2) $d(W^s(p_1) \cap \Lambda_1) + d(W^u(p_1) \cap \Lambda_1) < \varepsilon$.

3. Outlines of the Proofs

Sketch of the proof of Theorem 2.1. First, for $0 < \delta < 1/2$ and a unit square D, we take a precise form of type-2 horseshoe C^1 diffeomorphism φ which has the saddle p such that $\lambda = \frac{1}{2} - \delta$ where λ is the

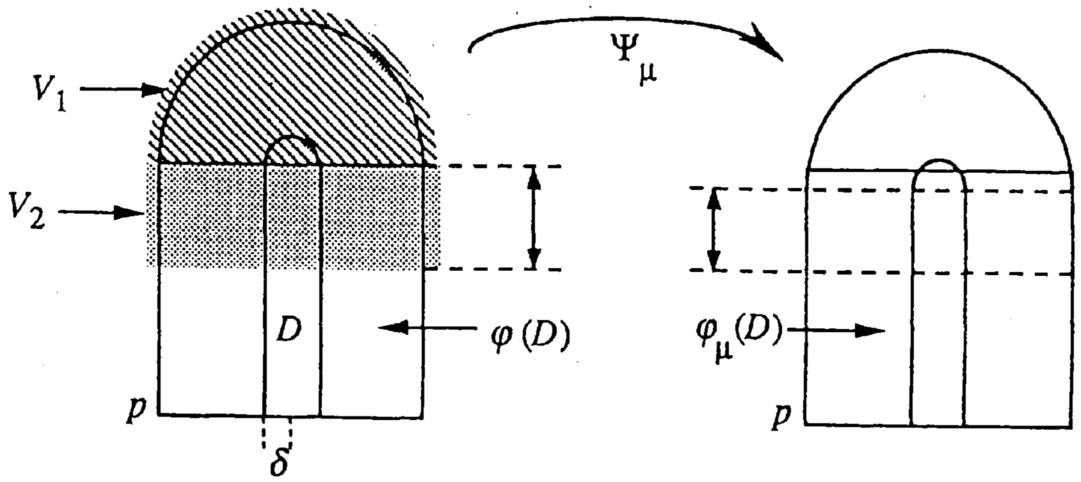

FIGURE 4

contracting eigenvalue of $(d\varphi)_p$, as shown in the left panel in Figure 4. We also define a C^1 perturbation Ψ_μ on $V = V_1 \cup V_2$ as in Figure 4 such that it *compresses* the vertical lines in V_1, and *slides* the points in V_2 *downward* as μ increases from 0 to 1. We denote $\Psi_\mu \circ \varphi$ by φ_μ which is called a *special 1-parameter family*, and denote the maximal invariant subset of D under φ_μ by Λ_μ. From the construction of this family, it also depends on δ. We write, for any $0 < \mu \leq 1$, $V_\mu = \{x = (s,\ u) \in D \mid 0 \leq s \leq 1,\ 1 - \mu\delta \leq u \leq 1\}$ which is shown in Figure 5, In the next proposition, for $0 < \mu < 1$, $x \in \Lambda_\mu \cap V_\mu$ and $n > 0$ such that $\varphi_\mu^n(x) \in V_\mu$, we construct an *unstable cone* $C_\mu(x) \subset T_x\mathbb{R}^2$ for $0 < \delta \leq \delta_h$ such that $(d\varphi_\mu^n)_x C_\mu(x) \subset C_\mu(\varphi_\mu^n(x))$ and that the length of any nonzero vector in $C_\mu(x)$ is expanded exponentially. The *stable cone field* can be constructed by taking the complement of the unstable cone field.

Proposition 3.1 ([5]). *There exists a* δ_h *such that, for* $0 < \delta \leq \delta_h$,

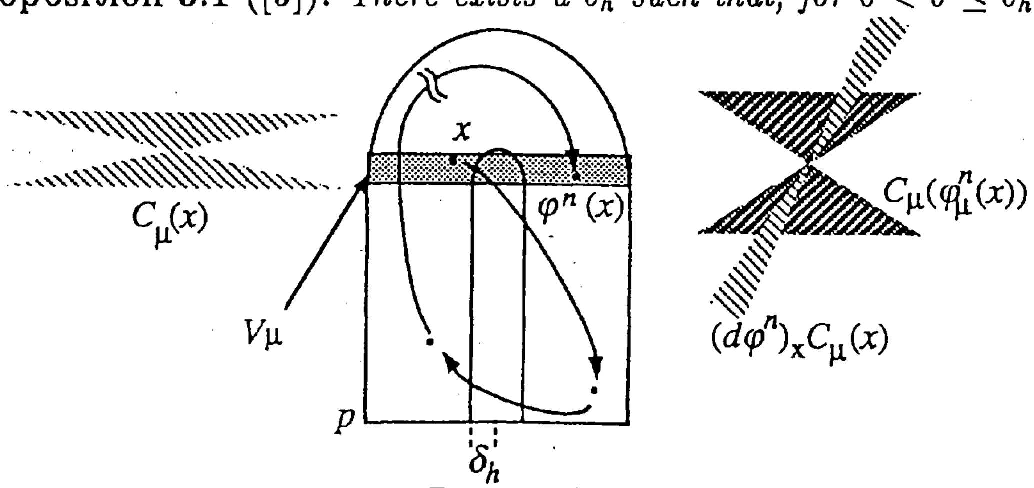

FIGURE 5

there is a continuous unstable cone field satisfying the above properties.

Next, we construct a *partition* for the above Λ_μ. The *continuous cone fields* over Λ_μ are obtained for this partition by using some *locally linear interpolation* of the angle of the cone, . These cone fields assure the hyperbolicity of this family and the transversality of the stable and unstable separatrices for $0 \leq \mu < 1$. □

Sketch of the proof of Theorem 2.2. The special 1-parameter family φ_μ $(= \Psi_\mu \circ \varphi)$ in this theorem is composed of the type-3 horseshoe C^1 diffeomorphism φ and the C^1 perturbation Ψ_μ on the open set V as shown in Figure 6. The components of $\varphi(D) \cap D$, where D is a unit square, are denoted by R_0, R_1 and R_2 as in Figure 6. We assume that $\rho(R_0,\ R_1) > \rho(R_1,\ R_2)$, and write $\delta = \rho(R_1,\ R_2)/2$, where ρ is the Euclidean metric. For some $\delta_h > 0$, similarly as in Theorem 2.1, we construct the 1-parameter family satisfying the hyperbolicity and the transversality condition. Moreover, we estimate the limit capacity of $W^s(p) \cap \Lambda_\mu$ as

$$\begin{aligned} d_\mu^s = d(W^s(p) \cap \Lambda_\mu) &\leq \lim_{n\to 0} \frac{\log 3^n}{-log(\lambda+\delta)\lambda^{n-1}} \\ &\leq \lim_{n\to 0} \frac{\log 3^n}{-log(\lambda+\delta)^n} = \frac{\log 3}{-log(\lambda+\delta)}, \end{aligned}$$

where λ is the contracting eigenvalue of $(d\varphi_\mu)_p$. Similarly, $d_\mu^u = d(W^u(p) \cap \Lambda_\mu)$ is estimated. Then we get the 1-parameter family such that, for any $\varepsilon > 0$, there is $\delta_l > 0$ depending on ε such that $d_\mu^u + d_\mu^s < \varepsilon$ for each $0 < \delta \leq \delta_l$. We now write $\delta(\varepsilon) = \min\{\delta_h,\ \delta_l\}$, and get the result. □

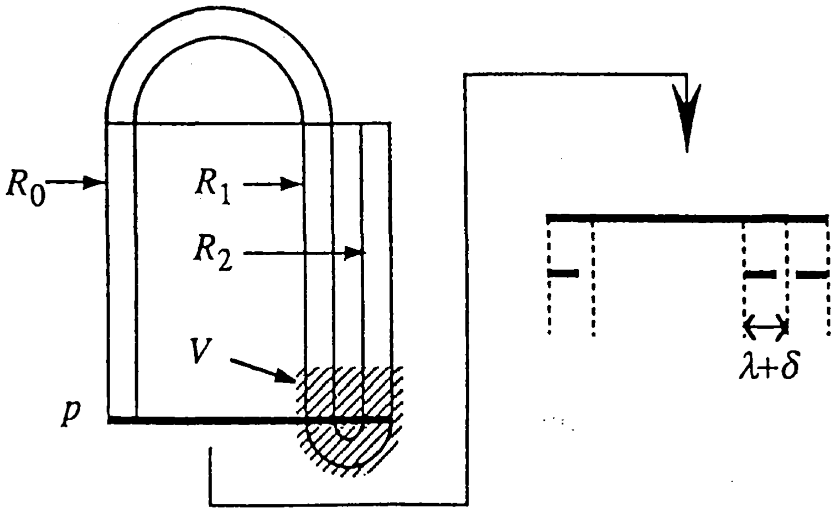

FIGURE 6

We believe that the above two results will be extended to higher dimensions, e.g. a special 1-parameter family can probably be constructed by modifying a 3-dimensinal horseshoe diffeomorphism as in Figure 7.

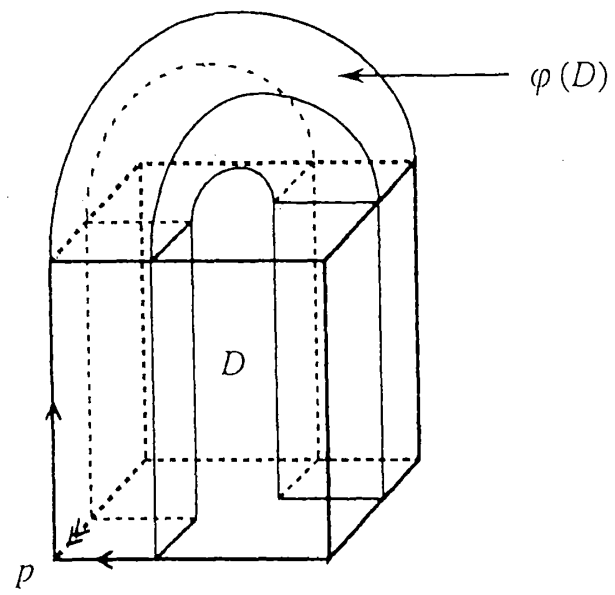

FIGURE 7

Professor F. Takens posed us a new problem of the relation between the homoclinic tangency and the first bifurcation as follows. Let $\{\varphi_\mu\ ;\ \mu \in \mathbb{R}\}$ be a 1-parameter family of diffeomorphisms on $\mathbb{R}^2$ satisfying φ_0 has an affine horseshoe. Let Λ_0 be the horseshoe for φ_0, and let Λ_μ be the maximal invariant subset for φ_μ which is the continuation of Λ_0 as long as Λ_μ is a hyperbolic set. We write

$$\mu_0 = \sup\{\mu \in \mathbb{R} \mid \Lambda_\mu \text{ is a hyperbolic set}\} \text{ and}$$

$$\mu_1 = \inf\{\mu \in \mathbb{R} \mid \varphi_\mu \text{ has a homoclinic tangency}\}.$$

Problem. *In the above situation, is there a 1-parameter family of planar diffeomorphisms such that $\mu_0 < \mu_1$? Moreover, is there an open set of C^2 arcs of planar diffeomorphisms such that $\mu_0 < \mu_1$?*

We have no answer to this question.

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REFERENCE

[1] M. Barge, *Horseshoe maps and inverse limits*, Pacific J. Math. **121** (1986), 29-39

[2] M. S. Davis, R.S. MacKay, A. Sannami *Markov shifts in the Hénon family* preprint of University of Warwick (1990)

[3] M. Hirsch, C. Pugh, M. Shub, *Invariant manifolds*, Lect. Notes in Math **583**, Springer-Verlag, 1977

[4] Y. Lai, C. Grebogi, J. A. Yorke, I. Kan, *How often are chaotic saddle nonhyperbolic?*, Nonlinearity **6** (1993), 779-797.

[5] S. Kiriki, *Hyperbolicity of the special 1-parameter family with a first homoclinic tangency*, preprint, to appear Ergod. Th. & Dynam. Sys.

[6] S. Kiriki, *The First homoclinic tangency inside the horseshoe with small fractal dimensions*, in preparation.

[7] S. Newhouse, J. Palis, *Cycles and bifurcation theory*, Astérisque **31** (1976), 44-140.

[8] J. Palis & F. Takens. Hyperbolicity and the creation of homoclinic orbits. *Annals of Math.* **125** (1987), 337-374.

[9] J. Palis & F. Takens. *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations.* Cambridge Univ. Press, 1993.

[10] R. A. Posthumus, F. Takens. *Homoclinic tangencies: moduli and topology of separatrices*, Ergod. Th. & Dynam. Sys. **13** (1993), 369-385.

[11] S. Smale, *Diffeomorphisms with many periodic points*, In *Differential and Combinatorial Topology* Princeton Univ. Press (1965), 63-80.

[12] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747-817.

[13] F. Takens, *Abundance of generic homoclinic tangencies in real-analytic families of diffeomorphisms*, Bol. Bras. Mat. Vol. **22**, No. **2** (1992), 191-214.

[14] J.C. Tatjer, C. Simó, *Basins of attraction near homoclinic tangencies*, Ergod. Th. & Dynam. Sys. **14** (1994), 351-390.

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