研究会報告

DYNAMICS OF INHIBITORY PULSE-COUPLED OSCILLATORS *

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ABSTRACT

A complete classification of dynamics of a population of a inhibitory pulse-coupled oscillators is presented. The model is based on the work of Mirollo and Strogatz, but our model has an inhibitory coupling between oscillators which makes a sharp contrast with the dynamics of the above authors' model. The main result is that for a large class of initial conditions, the population approaches a periodic state in which all the oscillators keep finite size of phase difference (we call it *"phase locking solution"* here). For the remaining class of initial data except for nongeneric ones, it evolves to a periodic state with a cluster or a synchronous state depending on a size of cluster. The criterion for the classification is explicitly given and can be judged easily only by the initial condition.

This work was motivated by the study of Mirollo and Strogatz[1] on synchronization of biological oscillators typically displayed by the flashing of fireflies in perfect unison. Their model consists of a population of identical integrate-and-fire oscillators. The coupling between oscillators is all to all and pulsatile: when a given oscillator fires, it pulls the others up by a fixed amount, or brings them to the firing threshold, whichever is less. They showed that for almost all initial conditions, the population evolves to a synchronous state. The main issue of this paper

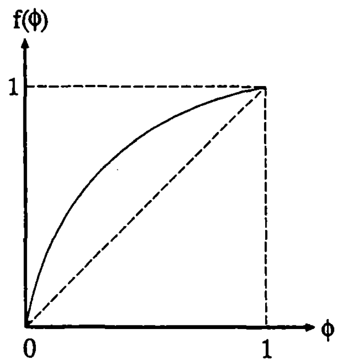

Figure 1: Functional form of $f(\phi)$

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is to study the dynamics of a population of oscillators when they interact in an *inhibitory* way, namely, when a given oscillator fires, it pulls the others *down* by a fixed amount. This type of coupling becomes important especially in models of neural oscillators[2]

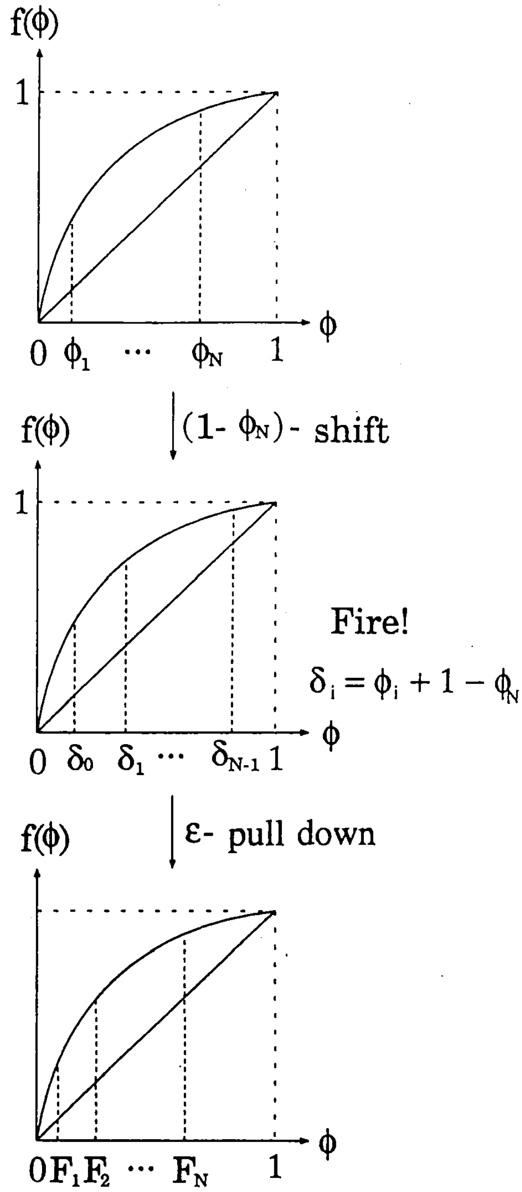


Figure 2: Firing map

In contrast to activation case, phase locking states become dominant for inhibitory case instead of synchronization. In fact generically there are three basins of attractions; phase locking, phase locking with cluster, and synchronization. The precise meaning of each state will become clear at the end of this section. A complete classification of initial data according to their asymptotic states is done by simple criterions depending only on initial condition. We consider a population of $N+1$ oscillators and each oscillator is characterized by a state variables x which is assumed to increase monotonically toward a threshold $x = 1$. When x reaches the threshold, the oscillator fires and x jumps back instantly to zero, after which the cycle repeats. Hereafter we assume that x depends only on a phase variable $\phi \in [0,1]$ and evolves according to $x = f(\phi)$, where $f : [0,1] \rightarrow [0,1]$ is a smooth function satisfying $f' > 0$, $f'' < 0$, $f(0) = 0$, and $f(1) = 1$ (see Fig. 1).

The phase variable ϕ is such that $d\phi/dt = 1/T$, where T is the cycle period. The coupling between oscillators is defined as follows. If x_i fires, then $x_j(\phi)(j \neq i)$ is pulled down instantaneously by the amount $|\epsilon|$, or to zero, whichever is more, i.e., $x_j(\phi + 0) = max(0, x_j(\phi) + \epsilon)$ $\forall j \neq i$. Note that ϵ is always a negative number. Absorption occurs when an oscillator is pulled down below zero level. Namely, when x_i fires, an oscillator $x_j(j \neq i)$ is **absorbed** by x_i if $max(0, x_j(\phi) + \epsilon) = 0$ holds. We assume that the absorbed oscillators behave in the same way as x_i thereafter. We call such a group of oscillators a **cluster**. If a cluster of k oscillators fires, it pulls all the other oscillators down by $|k\epsilon|$. When all the oscillators act as one, we call it **synchronization**. Since the interaction among oscillators is pulsatile, and when an oscillator (a cluster) fires, it instantaneously returns to zero phase, it suffices to study the following **firing map** F to know the asymptotic behavior (see Fig. 2):

$$F : \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} \longmapsto \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_N \end{pmatrix} \qquad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} \in D(0,1)$$

$$F(\Phi) = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_N \end{pmatrix} = \begin{pmatrix} g(f(1-\phi_N)+\epsilon) \\ g(f(\phi_1+1-\phi_N)+\epsilon) \\ \vdots \\ g(f(\phi_{N-1}+1-\phi_N)+\epsilon) \end{pmatrix}, \quad g \equiv f^{-1} \tag{0.1}$$

where $D(0,1)$ is the **ordered space** in $(0,1)$, i.e., $D(0,1) = \{\Phi \mid 0 < \phi_1 < \phi_2 < \cdots < \phi_N < 1\}$, and g is the inverse function of f. It is clear that F preserves order. Also note that one oscillator always sits at $\phi = 0$, so the firing map F becomes N-dimensional. $F^k(\Phi)$ stands for the k-iterations of firing map F, if it can be defined and $F_i^k = F_i^k(\Phi)(i = 1, \cdots, N)$ denotes the i-th component. $\Phi^* = {}^t(\phi_1^*, \cdots, \phi_{k-1}^*)$ is called a **k-phase locking solution** if it is a fixed point of F^k, i.e., $F^k(\Phi^*) = \Phi^*$. This notion can be easily generalized to the case where there are clusters. Our goal is to show the following theorem.

References

1. R. E. Mirollo and S. H. Strogatz, Synchronization of pulse-coupled biologincal oscillators, *SIAM J.Appl.Math.* **50** (1990), 1645-1662.

2. G. B. Ermentrout and N. Kopell, Inihibition-produced patterning in chains of coupled nonlinear oscillators, *SIAM J.Appl.Math.* **54** (1994), 478-507.

Main Theorem

Suppose an initial condition $\Phi = {}^t(\phi_1, \phi_2, \cdots, \phi_N) \in D(0,1)$ is given, then the asymptotic state is determined by the following diagram.

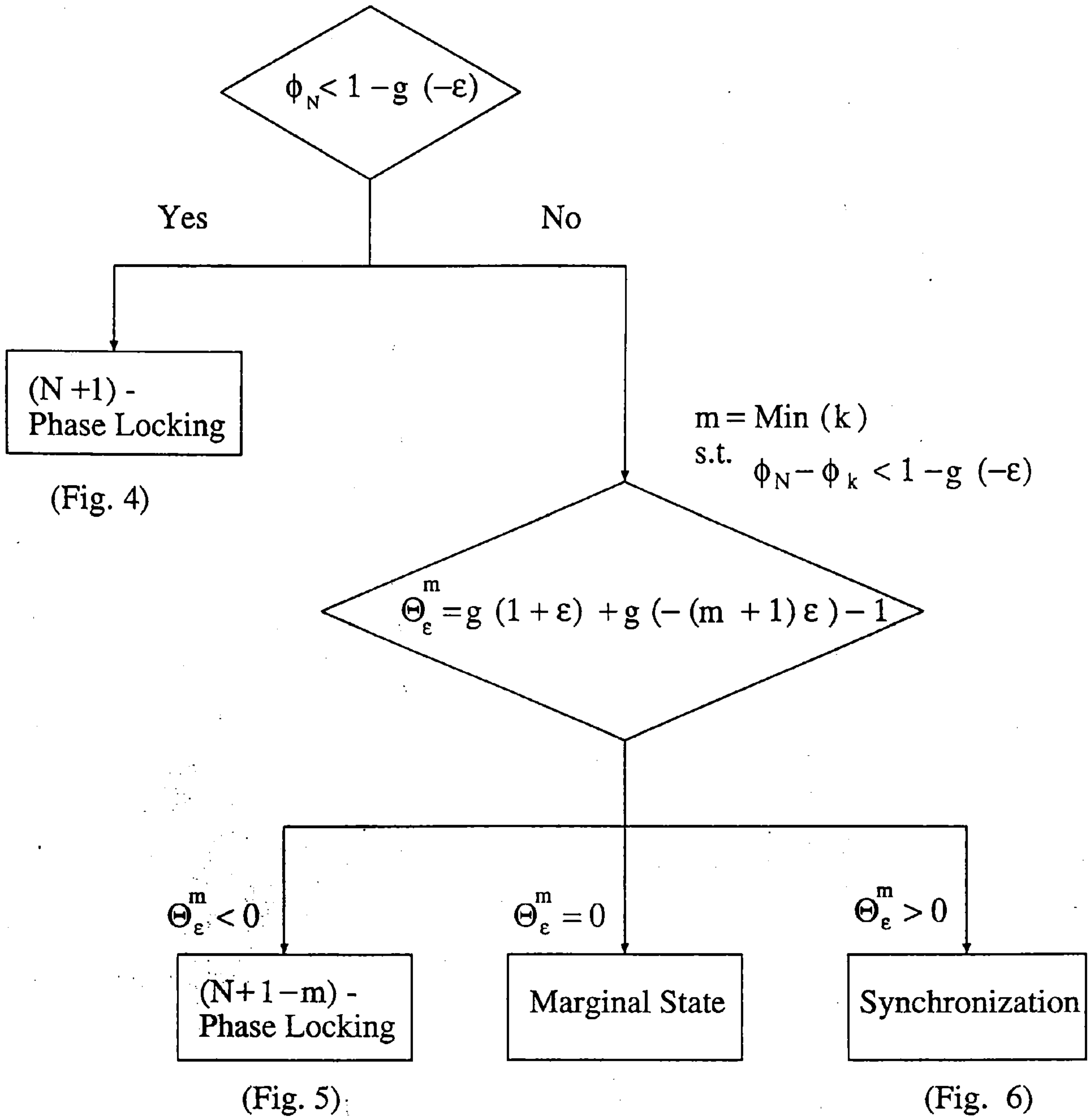
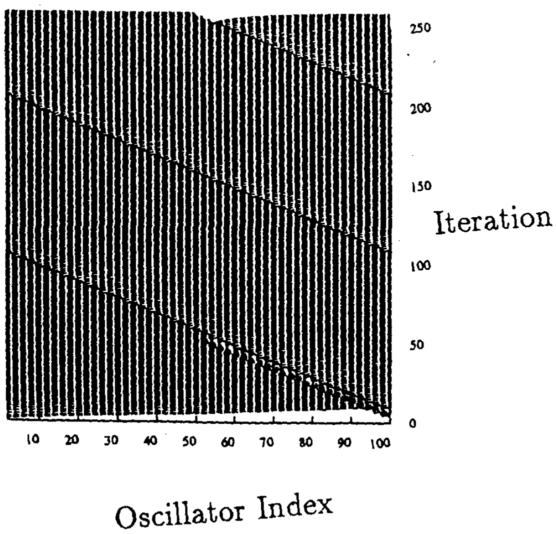


Figure 3: classification of asymptotic dynamics



The vertical direction denotes the phase axis.

Figure 4: Phase locking solution

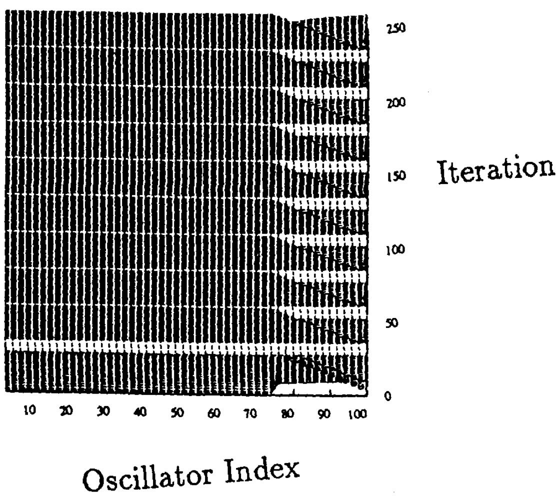


Figure 5: Phase locking solution with cluster

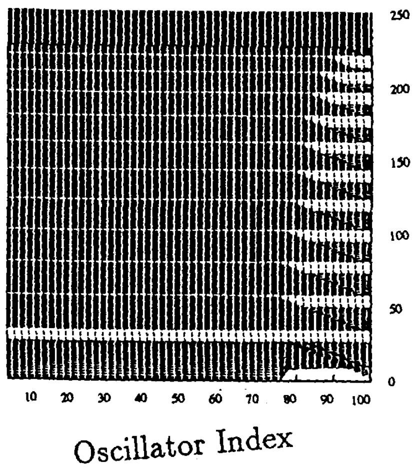


Figure 6: Synchronization